The proof equivalence problem for multiplicative linear logic is PSPACE-complete v0.3

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Abstract

MLL proof equivalence is the problem of deciding whether two proofs are related by a series of rule permutations. Previous work has shown the problem to be equivalent to a rewiring problem on proof nets, which are not canonical for full MLL due to the presence of the two units. Drawing from recent work on reconfiguration problems, in this paper it is shown that MLL proof equivalence is PSPACE-complete, using a reduction from Nondeterministic Constraint Logic.

$$\frac{\Gamma}{\Gamma, \bot}\,\bot \qquad \frac{}{1}\,1 \qquad \frac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr \qquad \frac{\Gamma, A \qquad \Delta, B}{\Gamma, \Delta, A \otimes B}\,\otimes$$

Figure 1: Inference rules for unit-only MLL

$$\frac{\dfrac{\Gamma}{\Gamma, \bot^a}\,\bot}{\Gamma, \bot^a, \bot^b}\,\bot \sim \frac{\dfrac{\Gamma}{\Gamma, \bot^b}\,\bot}{\Gamma, \bot^a, \bot^b}\,\bot \qquad \frac{\dfrac{\Gamma, A, B}{\Gamma, A \parr B}\,\parr}{\Gamma, A \parr B, \bot}\,\bot \sim \frac{\dfrac{\Gamma, A, B}{\Gamma, A, B, \bot}\,\bot}{\Gamma, A \parr B, \bot}\,\parr$$

$$\frac{\Gamma, A \qquad \Delta, B}{\dfrac{\Gamma, \Delta, A \otimes B}{\Gamma, \Delta, A \otimes B, \bot}\,\bot}\,\otimes \sim \frac{\dfrac{\Gamma, A}{\Gamma, A, \bot}\,\bot \qquad \Delta, B}{\Gamma, \Delta, A \otimes B, \bot}\,\otimes \sim \frac{\Gamma, A \qquad \dfrac{\Delta, B}{\Delta, B, \bot}\,\bot}{\Gamma, \Delta, A \otimes B, \bot}\,\otimes$$

$$\frac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A \parr B, C, D}\,\parr}{\Gamma, A \parr B, C \parr D}\,\parr \sim \frac{\dfrac{\Gamma, A, B, C, D}{\Gamma, A, B, C \parr D}\,\parr}{\Gamma, A \parr B, C \parr D}\,\parr$$

$$\frac{\dfrac{\Gamma, A \qquad \Delta, B, C, D}{\Gamma, \Delta, A \otimes B, C, D}\,\otimes}{\Gamma, \Delta, A \otimes B, C \parr D}\,\parr \sim \frac{\Gamma, A \qquad \dfrac{\Delta, B, C, D}{\Delta, B, C \parr D}\,\parr}{\Gamma, \Delta, A \otimes B, C \parr D}\,\otimes$$

$$\frac{\Gamma, A \qquad \dfrac{\Delta, B, C \qquad \Lambda, D}{\Delta, \Lambda, B, C \otimes D}\,\otimes}{\Gamma, \Delta, \Lambda, A \otimes B, C \otimes D}\,\otimes \sim \frac{\dfrac{\Gamma, A \qquad \Delta, B, C}{\Gamma, \Delta, A \otimes B, C}\,\otimes \qquad \Lambda, D}{\Gamma, \Delta, \Lambda, A \otimes B, C \otimes D}\,\otimes$$

Figure 2: Permutations

1 MLL

The formulae of unit-only multiplicative linear logic are given by the following grammar.

$$A, B, C := \bot \mid 1 \mid A \parr B \mid A \otimes B$$

The connectives $\otimes$ and $\parr$ will be considered up to associativity, and *duality* $A^\star$ is via DeMorgan. A *sequent* Γ, Δ will be a multiset of formulae. Within a sequent, connectives and units will be *named* with distinct elements from an arbitrary set of names N, e.g. $1^a \parr^b 1^c, \bot^d \otimes^e \bot^f$. This allows to 1) avoid using the notion of *occurrence*, and instead refer to subformulae by the name of their root connective, as e.g. A^b, 2) distinguish the two proofs of the above sequent while using standard multiset sequents, and 3) easily extract proof nets, as graphs using the names of connectives as vertices. Names will mostly be left implicit.

Proofs are constructed from the inference rules in Figure 1. The names of connectives are preserved through inferences. Only cut-free proofs are considered, and no cut-rule is added. *Permutations* of inference rules are displayed in Figure 2; the symmetric variants of the last two permutations, *par-tensor* and *tensor-tensor*, have been omitted.

Definition 1. *Equivalence* of proofs in (cut-free, unit-only) multiplicative linear logic ($\sim$) is the congruence generated by the permutations given in Figure 2. *MLL proof equivalence* is the problem of deciding whether two given proofs are equivalent.

The motivation to consider proofs up to equivalence is three-fold. Firstly, there is the strong intuition that the order of permutable inferences does not contribute to the essential content of the proof. Secondly, a technical motivation is that cut-elimination in MLL incorporates permutation steps, and composition via cut-elimination is only associative up to permutations. Thirdly, equivalent proofs are identified in natural models of multiplicative linear logic such as coherence spaces, and in the categorical semantics of MLL, $\star$-autonomous categories.

In one of several possible definitions, a *$\star$-autonomous category* (Barr, 1979) is a symmetric monoidal category $(\mathcal{C}, \otimes, 1)$ with:

- a *duality*, a contravariant functor $-^\star$ such that $A \cong A^{\star\star}$, and

- *closure*, an adjunction $- \otimes B \dashv (B \otimes -^\star)^\star$ for any object B,

satisfying natural coherence conditions. The category $\textsc{mll}(\varnothing)$ of unit-only MLL-formulae and equivalence classes of proofs is a $\star$-autonomous category. The formulation used induces various forms of *strictness*, instances where isomorphisms of the definition are identities: DeMorgan duality means $A = A^{\star\star}$, one-sided sequents mean the closure adjunction is an equivalence of categories, and the associativities are identities by decree. Modulo strictness, $\textsc{mll}(\varnothing)$ is the *free* $\star$-autonomous category over the empty category $\varnothing$. This means that *any* $\star$-autonomous category is a model of the logic, and that MLL proof equivalence is the *word problem* for $\star$-autonomous categories, the problem of deciding when two representations of morphisms denote the same morphism.

1.1 Proof nets

A partial solution to the MLL proof equivalence problem is provided by proof nets.

Definition 2. For a sequent Γ,

- a *linking* ℓ is a function from the names of $\bot$-subformulae to the names of 1-subformulae,
- a *switching graph* for ℓ is an undirected graph over the names of Γ, with for every subformula $A^a \otimes^c B^b$ the edges $a - c$ and $b - c$, for every subformula $A^a \parr^c B^b$ either the edge $a - c$ or the edge $b - c$, and for every subformula $\bot^a$ the edge $a - \ell(a)$,
- a *proof net* ℓ or (Γ, ℓ) is a linking ℓ such that every switching graph is acyclic and connected.

An edge $a - \ell(a)$ in a proof net or switching graph is a *link* or *jump*.

Definition 3. A *permutation* between proof nets is the redirection of exactly one link. *Equivalence* $(\sim)$ of proof nets over a sequent Γ is the congruence generated by permutations.

There is no canonical interpretation of a proof as a proof net, since the introduction rule for $\bot$ in proofs joins a $\bot$-formula to a sequent, rather than a formula.

Definition 4. The relation $(\vDash\!\!\Rightarrow)$ interprets a proof by a linking ℓ as follows: for each $\bot^a$, if Γ is the context of the inference introducing $\bot^a$, as illustrated below, then $\ell(a)$ is the name of some 1 in Γ.

$$\frac{\Gamma}{\Gamma, \bot^a}\,\bot$$

Proposition 5 (Danos and Regnier, 1989). *For a proof* Π *with conclusion* Γ*, if* $\Pi \vDash\!\!\Rightarrow \ell$ *then* ℓ *is a proof net for* Γ*. For a net* ℓ *for* Γ*, there is a proof* Π *of* Γ *such that* $\Pi \vDash\!\!\Rightarrow \ell$ *(*sequentialisation*).*

Proof nets are canonical representations of proofs in the absence of units: they factor out the permutations among tensor- and par-inferences, which are the last three permutations in Figure 2. Equivalence of proof nets is generated by the remaining equations, the permutations on $\bot$-introduction.

Proposition 6 (Hughes, 2012). *For proofs* Π*,* Π' *and proof nets* ℓ*,* ℓ' *such that* $\Pi \vDash\!\!\Rightarrow \ell$ *and* $\Pi' \vDash\!\!\Rightarrow \ell'$*,* $\Pi \sim \Pi'$ *if and only if* $\ell \sim \ell'$*.*

MLL proof equivalence is the problem of deciding equivalence of proof nets.

2 Equivalence in the absence of ⅋

Let a *1-alternation* sequent be one over formulae of the form 1 or $\bot \otimes \ldots \otimes \bot$, where the number of $\bot$-subformulae is at least 2. Such a sequent is inhabited exactly when the number of formulae in the sequent is one greater than the total number of $\bot$-subformulae it contains. An inhabited 1-alternation sequent with only one tensor-formula, i.e. a sequent of the form $1, \ldots, 1, \bot \otimes \ldots \otimes \bot$ with n $\bot$-subformulae and n 1-subformulae, will admit $n!$ different proof nets, each with n links. Since no link can re-attach, its equivalence classes are singletons.

Proposition 7. *For a 1-alternation sequent with at least two tensor-formulae there are at most two equivalence classes of proof nets.*

Proof. It will be shown by induction on the number of $\bot$-formulae in Γ that every proof net for Γ belongs to one of two equivalence classes. For the base case, the smallest inhabited sequent with two tensor-formulae is the following.

$$1, 1, 1, \bot \otimes \bot, \bot\otimes, \bot$$

It has two equivalence classes of 12 proof nets each.

For the inductive step, let Γ be the following sequent.

$$\Delta, A \otimes \bot^a, 1^x$$

There are two cases: 1) where A is a tensor-formula, and 2) where A is $\bot$ and where, for the induction hypothesis to apply, Δ contains at least two tensor-formulae. For both cases, it will be shown that any net ℓ for Γ is equivalent to a net ℓ' where $\bot^a$ connects to 1^x, and is the only link to do so. This reduces equivalence on Γ to equivalence on Δ, A in case 1, and on Δ in case 2, so that the induction hypothesis applies.

Let $\bot^a$ connect to 1^y in Δ, and let $\bot^c$ be a subformula of A, which means that for case 2, $A = \bot^c$. Then ℓ' is obtained by adjusting ℓ as follows.

- Let $c - z$, i.e. 1^z is the target of the jump from $\bot^c$. Ensure that 1^z is the only target shared between jumps in A and in Δ, by moving any other such jump from Δ to 1^z.
- If there are multiple links connecting to 1^x, select one $b - x$ for some $\bot^b$ in a tensor-formula B. Re-attach the others to the target of another jump out of B, of which there must be at least one.
- Since A is only connected via 1^z, there is a jump $d - z$ connecting B to A (though $\bot^d$ is not necessarily a subformula of B). Re-attach $\bot^d$ to 1^y, then $\bot^a$ to 1^x, and $\bot^b$ to 1^z.

□

Consider the following naming scheme for the units in a 1-alternation sequent Γ with tensor-formulae $A_1, \ldots, A_n$.

- The first $\bot$ in each A_i is named $b(i)$, while the remaining $\bot$-subformulae in Γ are named $b(n+1), \ldots, b(m)$.
- The first 1 in Γ is named by the set $A = \{1, \ldots, n\}$, the remaining ones are named $n+1, \ldots, m$.

The naming scheme suggests a linking for Γ, where the first $\bot$-subformula of each tensor-formula connects to 1^A, while other $\bot$-subformulae connect uniquely to the remaining 1-subformulae; i.e. $b(i) - A$ when $i \leq n$ and $b(i) - i$ otherwise. A net for Γ can be interpreted as a permutation (an automorphism on $\{1, \ldots, m\}$) as follows.

Definition 8. To a proof net ℓ for a 1-alternation sequent Γ named as above, associate the *permutation* $p_\ell : \{1, \ldots, m\} \to \{1, \ldots, m\}$ given by:

$$p_\ell(i) = \begin{cases} j & \text{if } b(i) \text{ may connect to } A \text{ and } \bot^{b(i)} \text{ is a subformula of } t(j) \\ \ell(b(i)) & \text{otherwise.} \end{cases}$$

The *parity* of ℓ is the parity of its permutation.

Proposition 9. *Re-attaching a jump in a net ℓ preserves its parity.*

Proof. TO DO □

References

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