

# Everything I know about Subset Sum Reconfiguration

Robin Houston · July 17, 2013

This is a brief summary of everything I know about the subset sum reconfiguration problem as of four o'clock in the afternoon of Monday the 15th of July. It is brief by necessity, because I know so little. **Update on Tuesday:** I found out another thing.

**Proposition 1.** *Subset sum reconfiguration is weakly NP-hard. (Demaine–Ito)*

*Proof.* The ordinary subset sum problem may be reduced to subset sum reconfiguration. Given elements  $a_1, \dots, a_m$  with target sum  $t$ , build an SSR instance with elements  $a_1, \dots, a_m$  having the corresponding weights,  $t_1$  and  $t_2$  having weight  $t$ , threshold values  $k = t$  and  $c = 2t$ , starting configuration  $t_1$  and ending configuration  $t_2$ . At some point element  $t_1$  must move from inside to outside: the first time it does, the inside set during the move must consist of some collection of  $a$  elements that sum to  $t$ .  $\square$

**Proposition 2.** *For all  $n > 1$ , there is an  $n$ -element SSR instance whose configuration graph has diameter  $2n - 2$ .*

*Proof.* We exhibit in terms of a parameter  $m$ :

- an instance with  $2m + 2$  elements and a shortest solution with  $4m + 2$  steps;
- if  $m > 0$ , an instance with  $2m + 1$  elements and a shortest solution with  $4m + 1$  steps.

The instance with  $2m + 2$  elements has elements  $a_1, \dots, a_m, b_1, \dots, b_m, x$  and  $y$ . For the weights, let  $w(a_i) = m + 1$  and  $w(b_i) = m + 2$  for all  $i$ . Let  $w(x) = w(y) = m(m + 2)$ . The starting configuration is  $a_1, \dots, a_m, x$  and the target configuration is  $a_1, \dots, a_m, y$ . For threshold values let  $k = m(m + 2)$  and  $c = 2m(m + 2)$ .

The first time  $x$  moves, the weights of the remaining elements inside must sum to  $m(m + 2)$ . In particular, these weights must sum to  $m$  modulo  $m + 1$ , hence the inside elements at this point must include all the  $b$ 's. Since the weight of  $x$  plus the  $b$ 's equals the capacity  $c$ , the elements inside at this point must therefore be precisely all the  $b$ 's. So all the  $a$ 's and  $b$ 's have to move twice, and  $x$  and  $y$  once each, for a total of  $4m + 2$  moves.

To obtain the instance with  $2m + 1$  elements, remove the element  $a_1$ .  $\square$

**Remark 3.** *The lower bound exhibited in Proposition 2 is optimal for  $n < 8$ . I have verified this by exhaustive enumeration using the program `ssr_graphs.py`.*

**Proposition 4.** *For all  $m$ , there is a  $(5m + 3)$ -element SSR instance that has an element that must move at least  $2m + 1$  times in any solution sequence.*

*Proof.* The elements are  $x_1, \dots, x_{2m+1}$  of weight 3,  $a_1, \dots, a_{3m+1}$  of weight 2, and  $b$  of weight 1. Let  $k = 6m$  and  $c = 6m + 3$ . The starting configuration is  $x_1, \dots, x_{2m+1}$ , and the target is  $a_1, \dots, a_{3m+1}, b$ . For  $i = 0, 1, \dots, 2m$ , consider the first move after which there are precisely  $2m - i$  of the  $x$  elements inside. That means the  $a$ 's and possibly  $b$  inside at this point must sum to  $3i$ . If  $i$  is odd then, since the  $a$ 's all have even weight,  $b$  must be inside at this point; similarly if  $i$  is even then  $b$  must be outside. So element  $b$  moves at least  $2m + 1$  times.  $\square$

## New finding

The construction of Proposition 4 can be modified to give a quadratic lower bound for the diameter of the reconfiguration graph.

**Proposition 5.** *For all  $m$ , there is a  $(6m+2)$ -element SSR instance that requires  $2m^2+6m+2$  moves.*

*Proof.* The elements are  $x_1, \dots, x_{2m+1}$  of weight  $3m$ ,  $a_1, \dots, a_{3m+1}$  of weight  $2m$ , and  $b_1, \dots, b_m$  of weight 1. Let  $k = 6m^2$  and  $c = 6m^2 + 3m$ . The starting configuration is  $x_1, \dots, x_{2m+1}$ , and the target is  $a_1, \dots, a_{3m+1}, b_1, \dots, b_m$ . For  $i = 0, 1, \dots, 2m$ , consider the first move after which there are precisely  $2m - i$  of the  $x$  elements inside. That means the  $a$ 's and  $b$ 's inside at this point must sum to  $3im$ . If  $i$  is odd then, since the  $a$ 's all weigh  $2m$ , all the  $b$ 's must be inside at this point; similarly if  $i$  is even then the  $b$ 's must all be outside. So the  $b$  elements each move at least  $2m + 1$  times. Therefore the total number of moves must be at least  $(2m + 1) + (3m + 1) + (2m + 1)m$ , which is equal to  $2m^2 + 6m + 2$  as required.  $\square$