# ON ROTATION SYSTEMS AND THE SYMMETRIC DIFFERENCE OF DISJOINT CYCLE COVERS.

Abstract. This appendix to *Hamiltonicity of the Cayley Digraph on the Symmetric Group Generated by* (1 2) *and* (1 2 $\cdots$ $n$) formalizes the bijection between faces in a rotation system and the directed cycles of a disjoint cycle cover in a more general setting.

In Section 4 of the arxiv paper we used a rotation system to model the size of disjoint cycle cover in $\overrightarrow{\mathsf{Cay}}(\mathbb{S}_n, (\sigma, \tau))$. In this appendix, we establish this strategy as a general approach for arbitrary directed graphs. First, we examine the relationship between the symmetric difference of two disjoint cycle covers and alternating cycles of a certain type. Second, we show that the size of these disjoint cycle cover can be determined from the number of faces in a rotation system.

0.1. ***S*-Alternating Cycles.** An *$FB$-alternating cycle* for $F, B \subseteq E$ is a closed trail

$$u_1, f_1, v_1, b_1, u_2, f_2, v_2, b_2, \ldots, u_k, f_k, v_k, b_k$$

with the following conditions

- the forward edges are $f_i = u_i v_i \in F \backslash B$ for $1 \leq i \leq k$;
- the backward edges are $b_i = u_{i+1} v_i \in B \backslash F$ for $1 \leq i \leq k$ (where $u_{k+1} = u_1$);
- $u_i \neq u_j$ and $v_i \neq v_j$ for $1 \leq i < j \leq k$.

The first two points ensure that the edges of the trail alternate between forward edges in $F \backslash B$, and backward edges in $B \backslash F$. The third point ensures that each vertex is incident with either zero or two outgoing edges, and either zero or two incoming edges. Two $FB$-alternating cycles are *disjoint* if they do not have an edge in common. Lemma 1 is illustrated by Figure 1.

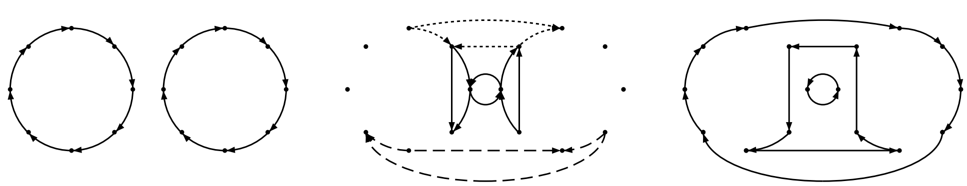

Figure 1. Disjoint cycle cover $F$ of size two (left) and $B$ of size three (right), and their symmetric difference $A = F \oplus B$ (middle). The edges of $A$ uniquely partition into three $FB$-alternating cycles, as illustrated by the three different line styles.

**Lemma 1.** *Suppose $F \subseteq E$ is a disjoint cycle cover, $B \subseteq E$, and $C = F \oplus B$. Then $B$ is a disjoint cycle cover if and only if $C$ is the union of edges in disjoint $FB$-alternating cycles.*

*Proof.* Partition $C$ into $C_F = F \backslash B$ and $C_B = B \backslash F$ for both directions of the proof.

For the first direction, suppose $B$ is a disjoint cycle cover. For each $u \in V$,

$$|\{uv \in F\}| = |\{uv \in B\}| = 1 \text{ and } |\{vu \in F\}| = |\{vu \in B\}| = 1. \tag{1}$$

The equalities in (1) imply the following for each $u \in V$,

$$|\{uv \in C_F\}| = |\{uv \in C_B\}| \in \{0,1\} \text{ and } |\{vu \in C_F\}| = |\{vu \in C_B\}| \in \{0,1\}. \tag{2}$$

Consider an edge $u_1v_1 \in C_F$. By (2), we can uniquely extend a trail starting from $u_1$ by alternately adding forward edges $u_iv_i \in C_F$ and backward edges $u_{i+1}v_i \in C_B$ for $i = 1, 2, \ldots, k$, until the trail is closed with backward edge $u_1v_k \in C_B$. Each trail created in this way is an $FB$-alternating cycle. Thus, the edges of $C$ partition uniquely into disjoint $FB$-alternating cycles.

For the second direction, suppose $C = C_1 \cup C_2 \cup \cdots \cup C_p$, where the union is over the edges of disjoint $FB$-alternating cycles. Therefore, the following holds for each $u \in V$,

$$|\{uv \in C_F\}| = |\{uv \in C_B\}| \text{ and } |\{vu \in C_F\}| = |\{vu \in C_B\}|. \tag{3}$$

Since $B = F \oplus C$, (3) implies the following for each $u \in V$

$$|\{uv \in B\}| = |\{uv \in F\}| \text{ and } |\{vu \in B\}| = |\{vu \in F\}|. \tag{4}$$

Since $F$ is a disjoint cycle cover, the two values in (4) equal 1, and hence $B$ is a disjoint cycle cover. $\square$

Given a disjoint cycle cover $F$ containing directed cycles $F_1, F_2, \ldots, F_n$ and a set of disjoint $F$-alternating cycles $C = C_1 \cup C_2 \cup \cdots \cup C_m$, we create a rotation system $\mathsf{Rot}(F, C)$ as follows

- The vertices are $\{F_1, F_2, \ldots, F_n\} \cup \{C_1, C_2, \ldots, C_m\}$.
- If the $x$th forward edge of $C_i$ is the $y$th edge of $F_j$, then $C_iF_j$ is the $x$th edge around $C_i$ and the $y$th edge around $F_j$.

(Well-defined since looking at the cycles, and not the edges.) Figure 2 illustrates this definition, and Figure 4 illustrates Lemma **??**.

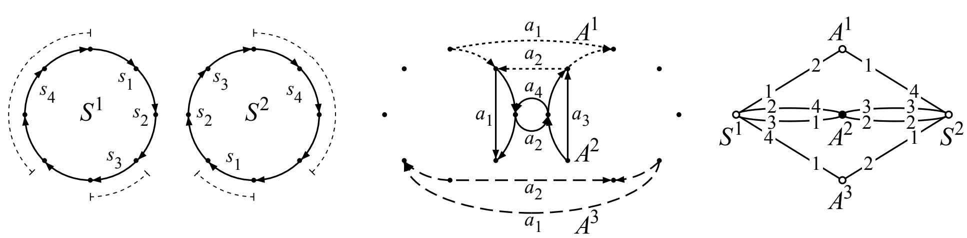


FIGURE 2. A disjoint cycle cover $S^1 \cup S^2$ (left), three disjoint $S$-alternating cycles $A^1, A^2, A^3$ (middle), and their rotation system $\mathsf{Rot}(S, A)$ (right). Superscripts are omitted from each directed path and edge above since the corresponding directed cycle is clear.

Figure 3 illustrates the three faces of the rotation system in Figure 2.

**Lemma 2.** *If $D$ is a disjoint cycle cover, and $A$ is a set of vertex-disjoint alternating cycles with respect to $D$, then the number of directed cycles in $D \oplus A$ equals the number of faces in* $\mathsf{Rot}(D, A)$.

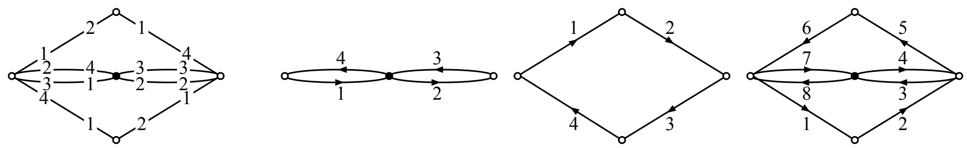

Figure 3. A rotation system (left) and its three faces (right). The edges along each face are numbered in increasing order. By convention, the white vertices and black vertices are drawn with clockwise and counter-clockwise edge orders, respectively.

*Proof.* Let the cycles in $D$ be $D_1, D_2, \ldots, D_m$ and the cycles in $A$ be $A_1, A_2, \ldots, A_k$. Need to define segments etc ... maybe do the figures first?

Consider a face in $\mathsf{Rot}(D, A)$: $d_1, a_1, d_2, a_2, \ldots$. Bang, this is a directed cycle.

Consider a directed cycle in $D \oplus A$. Bang, this is a face. □

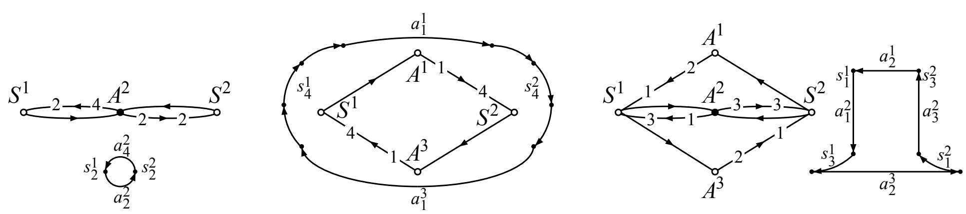

Figure 4. Left-to-right: The three faces of $\mathsf{Rot}(S, A)$ and the corresponding directed cycles in the disjoint cycle cover $T = S \oplus A$. For example, the facial edge $A^3S^2$ with label $2 \to 1$ (right) corresponds to the directed cycle edge $a_2^3$ and directed path $s_1^2$ (right).