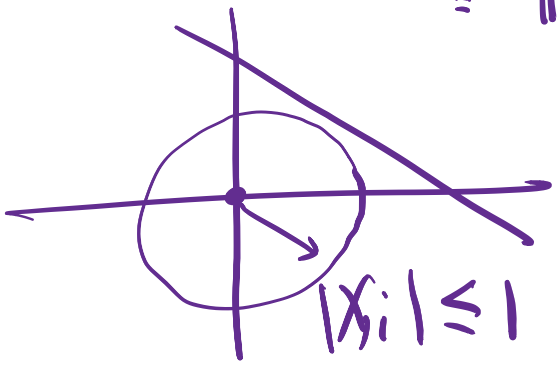
Proof of Perceptron.

Claim:- $\|t_{opt}\| \leq 1$

If $|t_{opt}| > 1$, $d(\text{origin}, (\bar{w}_{opt}, t_{opt})) = \dfrac{|\bar{w}_{opt} \cdot \bar{0} - t_{opt}|}{\|w_{opt}\|}$

$= \|t_{opt}\| \geq 1$



$|X_i| \leq 1$

For $\bar{w}, t$, let $(\bar{w}, t)$ denote a $d+1$ dimensional vector.

Claim:- $\|(\bar{w}_{opt}, t_{opt})\|_2^2 \leq 2$ —— (II)

$\|w_{opt}\|_2 = 1$, $|t_{opt}| \leq 1$

$= \|w_{opt}\|_2^2 + t_{opt}^2 \leq 2.$

Claim:- Let $(\bar{w}_j, t_j)$ denote the hyperplane after 'j' mistakes.

$(\bar{w}_0, t_0) = (\bar{0}, 0)$. Then,

$$\boxed{(\bar{w}_j, t_j) \cdot (\bar{w}_{opt}, t_{opt}) \geq j \cdot \gamma.}$$

- Look at j^{th} mistake say $(\bar{x}_i, y_i)$

$(\bar{w}_{j-1}, t_{j-1})$

$(w_j, t_j) \cdot (\bar{w}_{opt}, t_{opt})$

$= (\underline{\bar{w}_{j-1} \pm y_i \bar{x}_i}, \underline{t_{j-1} \mp y_i}) \cdot (\underline{w_{opt}}, \underline{t_{opt}})$

$= \underbrace{(\bar{w}_{j-1}, t_{j-1}) \cdot (\bar{w}_{opt}, t_{opt})} + (y_i \bar{x}_i, -y_i) \cdot (\bar{w}_{opt}, t_{opt})$

$= \quad \downarrow \quad + \underbrace{y_i(\bar{w}_{opt} \cdot \bar{x}_i - t_{opt})}_{\geq \delta}$

$\vdots$

$\geq \boxed{j\delta}$

Claim:- $\boxed{\| w_j, t_j \|_2^2 \leq 2j}$

$\|(w_j, t_j)\|_2^2 = (\bar{w}_{j-1} + y_i \bar{x}_i, t_{j-1} - y_i) \cdot (\bar{w}_{j-1} + y_i \bar{x}_i, t_{j-1} - y_i)$

$= (\bar{w}_{j-1}, t_{j-1}) \cdot (\bar{w}_{j-1}, t_{j-1})$
$+ \underbrace{(1 + \|x_i\|_2^2)}_{\leq 2} + \underbrace{\boxed{2y_i(\bar{w}_{j-1} \cdot \bar{x}_i - t_{j-1})}}_{\leq 0}$

$\vdots$

$\leq 2j.$

$\bar{u}, \bar{v}, \cos(\text{angle}(\bar{u}, \bar{v})) = \dfrac{\bar{u} \cdot \bar{v}}{\|u\| \cdot \|\bar{v}\|} \leq 1$

$\bar{u} \cdot v \leq \|u\| \cdot \|v\|.$

$(\bar{w}_j, t_j) \quad (\bar{w}_{opt}, t_{opt}) \rightarrow \geq j\delta$

$\big(\overbrace{(\bar{w}_j, t_j)(\bar{w}_{opt}, t_{opt})}\big)^2 \quad \leq 1$

$$\underline{\|\bar{w}_j, t_j\|^2}_{\le 2j} \; \underbrace{\|\bar{w}_{opt}, t_{opt}\|^2}_{\le 2}$$

$$\ge \frac{(j\gamma)^2}{(2j)\cdot 2} = \frac{j\gamma^2}{4}$$

$$\frac{j\gamma^2}{4} \le 1 \Rightarrow \boxed{j \le \frac{4}{\gamma^2}}$$