Concentration Inequalities

ECE 4200

Introduction

- “Concentration” – probabilistic phenomenon
- A “well-behaved” random variable concentrates around its mean
- Game:
  - I want a non-negative random variable X with a specific E[X].
  - Assign maximum probability to $\Pr(X \geq a)$

Markov's inequality

For a non-negative random variable *X* let a>0 :

$$\Pr(X \geq a) \leq \frac{E[X]}{a}$$

Proof of Markov's inequality

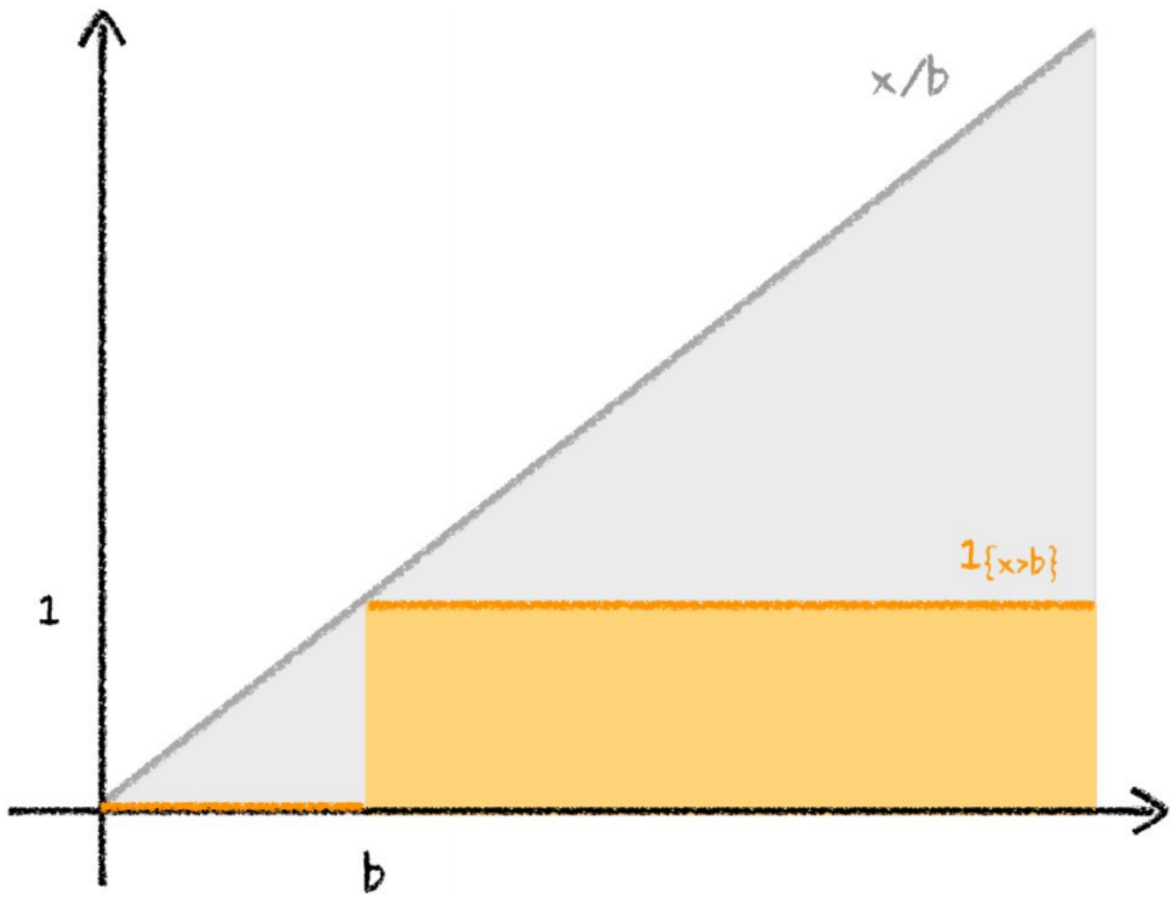
For a non-negative random variable X let a>0 :

$$\Pr(X \geq a) \leq \frac{E[X]}{a}$$

$$E[X] = \underbrace{E[X \mid X \geq a]}_{\geq a} P(X \geq a) + \underbrace{E[X \mid X < a]}_{\geq 0} \underbrace{P(X < a)}_{\geq 0}$$

$$E[X] \geq a\, P(X \geq a) + 0.$$

Pictorial proof (Prof. Roch – UW Madison)



Example

A coin is tossed 100 times. Give an upper bound on the probability that the Head occurs:

(a) at least 90 times

(b) at most 10 times

(c) at least 70 times

Example

(a) **Let N be the number of occurrences of `Head`.**

We know $E[N] = 50.$

$$P[N \geq 90] \leq E[N] \; / \; 90 \; = 5/9$$

(b) $P[N \leq 10] \leq P[100 - N \geq 90]$

$$\leq E[100 - N] \; / \; 90 = 5/9$$

(c) $P[N \geq 70] \leq E[N] \; / \; 90 \leq 7/9$

Chebyshev's inequality

Let X be a random variable with finite mean μ and finite variance σ^2 and let t>0 :

$$P(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}$$

Proof of Chebyshev's inequality

Let X be a random variable with finite mean μ and finite variance σ^2 and let t>0 :

$$P(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}$$

$$P(|X - \mu| \geq t\sigma) = P\left((X - \mu)^2 \geq t^2\sigma^2\right)$$

$$= \frac{\mathrm{E}[(X-\mu)^2]}{t^2\sigma^2} = \frac{1}{t^2}$$

Example

$E[N] = 100/2 = 50$ $\mu = 50$

$Var[N] = 100 * 0.5 * 0.5$ $\sigma = 5$

(a) $P(X \geq 90) \leq 0.5 * P(|X - \mu| \geq 8\sigma)$

$\leq 1/128 = 0.0078125$

(b) $P(X \leq 10) \leq 0.5 * P(|X - \mu| \geq 8\sigma)$

$\leq 1/128 = 0.0078125$

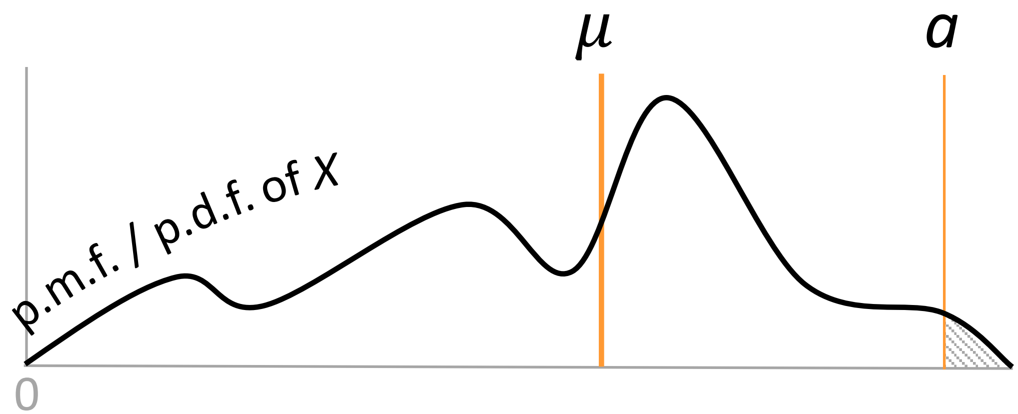
(c) $P(X \geq 70) \leq 0.5 * P(|X - \mu| \geq 4\sigma)$

$\leq 1/32 = 0.03125$

An illustration

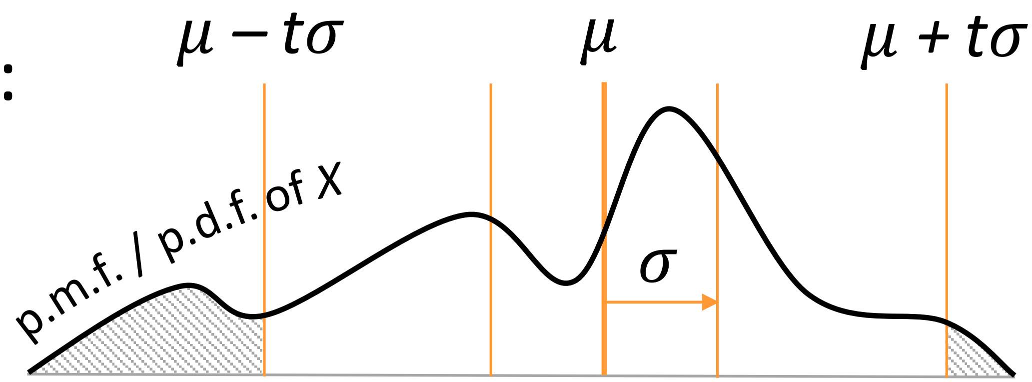
Markov's inequality:

$$\Pr(X \geq a) \leq \frac{E[X]}{a}$$



Chebyshev's inequality:

$$P(|X - \mu| \geq t\sigma) \leq \frac{1}{t^2}$$



Chernoff bound

For every t>0,

$$P(X \geq a) = P(e^{tX} \geq e^{ta}) \leq E[e^{tX}] \;/\; e^{ta}$$

Similarly, for t>0,

$$P(X \leq a) = P(e^{-tX} \geq e^{-ta}) \leq E[e^{-tX}] \;/\; e^{-ta}$$

Sum of iid random variables – Chernoff

- When Z = $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{X_i}$ -> sum of n iid random variables

$$P(Z \geq a) = P(e^{tZ} \geq e^{ta}) \leq e^{-ta} E[\prod e^{tXi}]$$

- Chernoff bound for binomial Z = $\sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{X_i}$ where $\mathrm{X_i}$ ~ Ber(p)