1. Construct a 4-state Markov chain (transition diagram and initial distribution) that "models" the non-Markov random process in Problem 2 of Homework II in the sense that the two processes have the same one-step transition probabilities and initial distribution.

2. Find the unique stationary distribution π^* for the Markov chain in Problems 6 of Homework Assignment II.

3. A random walk on M states is a Markov chain with a transition diagram such as that in Figure 1. Find the unique stationary distribution π^* when $M = 3$ and $p_1 = p_2 = p$, $q_2 = q_3 = q$, and $r_1 = r_3 = r$. Assume $p + q < 1$.

4. You have two boxes and M identical marbles. Each marble is in one box or the other at any given time. At each integer time, you pick a marble randomly (i.e., uniformly over all M marbles) and move it to the other box.

(a) Model this situation as a Markov chain. (Suggestion: call the boxes A and B and let state i be "i balls are in A.") Find all the transition probabilities $P(i, j)$.

(b) If $M = 3$ and you start the chain off with initial probability distribution $\pi(0) = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \end{bmatrix}$, find $\pi(1)$ and $\pi(2)$.

(c) Find the transient and recurrent states and the recurrence classes.

5. Suppose we have a Markov chain with M states and suppose that j is a state for which $r_{ij} = 1$ for every state i. It turns out that for every state i we have

$$E_i(T_j) = 1 + \sum_{q \neq j} P(i, q) E_q(T_j) \ .$$

Let's step through the proof.

(a) Note that $f_{ij}^{(1)} = P(i, j)$. Note also that

$$f_{ij}^{(2)} = \sum_{q \neq j} P(i, q) f_{qj}^{(1)} \ .$$

Similarly, for every $k > 0$,

$$f_{ij}^{(k+1)} = \sum_{q \neq j} P(i, q) f_{qj}^{(k)} \ .$$

(b) Show from the identities in (a) that

$$E_i(T_j) = P(i, j) + \sum_{k=1}^{\infty} (k + 1) \left(\sum_{q \neq j} P(i, q) f_{qj}^{(k)} \right) \ .$$

Conclude from this that

$$E_i(T_j) = P(i,j) + \sum_{q \neq j} P(i,q) \left(E_q(T_j) + \sum_{k=1}^{\infty} f_{qj}^{(k)} \right).$$

(Suggestion: flip the order of summation.)

(c) The conclusion of the problem follows from this last identity. Fill in the details.

6. The result of the preceding problem is useful when a Markov chain has a hierarchical structure of sorts. Here's one example. We have a doubly infinite line of small boxes. We also have a hungry spider and a fly hopping between the boxes. At each time step, the spider moves one box in the direction of the fly. Meanwhile, the fly moves one box to the left with probability p and one box to the right with probability p while staying put with probability $1-2p$. Assume $0 < p < 1/2$. When the fly and the spider land in the same box, game over, at least for the fly.

(a) Model this situation with a Markov chain whose state space is $\{0, 1, 2, 3, \ldots\}$ with state 0 corresponding to game over. Draw the part of transition diagram that includes states 0, 1, 2, 3, 4 and all their outgoing arrows.
(b) What is the most general form of a closed set of states for this Markov chain?
(c) Find $E_1(T_0)$ in the "standard" way involving a geometric series.
(d) Find $E_1(T_0)$ and $E_2(T_0)$ using the result of the previous problem.
(e) Write a recursion you could use to generate $E_i(T_0)$ for every $i > 2$ after initializing the recursion with the results of (d).

7. A real $(M \times M)$ matrix P is a stochastic matrix if and only if $[P]_{ij} \geq 0$ for all i and j and, in addition,

$$\sum_{j=1}^{M} [P]_{ij} = 1$$

for all i.

(a) Show that if P and Q are stochastic $(M \times M)$ matrices, then so is PQ.
(b) Show by induction that if P is stochastic, then so is P^k for every $k > 0$.
(c) Suppose P is the matrix of one-step transition probabilities for some M-state Markov chain that has only one recurrence class. Suppose that for some i and j we have $\left[P^k\right]_{ij} = 0$ for every $k \geq 0$. Show that j is a transient state.
(d) Suppose again that P is the matrix of one-step transition probabilities for some M-state Markov chain. Suppose that for some $K > 0$ we have $\left[P^K\right]_{ij} > 0$ for all i and j. Show that $\left[P^k\right]_{ij} > 0$ for all i and j and all $k > K$. Show also that the Markov chain is irreducible and that every state is recurrent. (It turns out that the converse of this last statement is also true under the additional condition that the Markov chain is *aperiodic*, which means that for every i and j the greatest common divisor of all $k > 0$ such that $\left[P^k\right]_{ij} > 0$ is 1.)