1. Now for some combinatorial counting. Mitchell asserts in Chapter 4 that the number of possible populations of size n for a standard GA operating on binary strings of length L is given by

$$\binom{2^L+n-1}{2^L-1}.$$

This problem steps you through a derivation of that fact.

(a) Show that there are 2^L populations in which only one string is represented and $\binom{2^L}{n}$ populations in which exactly n strings are represented.

(b) Show that there are $\binom{2^L}{2}\binom{n-1}{1}$ populations in which exactly two strings are represented. (Suggestion: first you need to pick which two strings are represented; then you have to pick a "dividing line" in the population so that one of the strings (say x) is to the left of the line and the other (say y) is to the right of the line:

$$xxx\cdots xxx|yyy\cdots yy \quad \longleftarrow n \text{ spots altogether.}$$

(c) Show that there are $\binom{2^L}{k}\binom{n-1}{k-1}$ populations in which exactly k strings are represented.

(d) Conclude that the total number of populations is

$$\sum_{k=1}^{n}\binom{2^L}{k}\binom{n-1}{k-1}.$$

(e) Complete the derivation by applying the well known formula

$$\sum_{k=0}^{n}\binom{a}{k}\binom{b}{n-k}=\binom{a+b}{n}.$$

(Recall also that $\binom{n-1}{k-1}=\binom{n-1}{n-k}$.)

2. Consider the following lightwall setup. The wall features a red light and a green light. The supervisor flips a fair coin to decide which light flashes at time $n=0$. To decide what light flashes at time $n=1$, the operator of the light that flashes at time $n=0$ flips a fair coin. For $n \geq 1$, the light operators proceed as follows:

- If the red light flashes at time n, the red-light operator flips a fair coin to determine which light flashes at time $n+1$.
- If the green light flashes at time n, the green-light operator flashes his light again at time $n+1$ if he flashed his light at time $n-1$. If the red light flashed at time $n-1$, the green-light operator tells the red-light operator to flash his light at time $n+1$.

(a) Show for $0 \leq n \leq 2$ that

$$\text{Prob}\{\text{red light flashes at time n}\} = \frac{1}{2}$$

and

$$\text{Prob}\{\text{green light flashes at time n}\} = \frac{1}{2}$$

It turns out that these identities actually hold for all $n \geq 0$.

(b) Show for $0 \le n \le 2$ that

$$\text{Prob}\{\text{red light flashes at time n} + 1 \text{ given green light flashes at time n}\} = \frac{1}{2}$$

$$\text{Prob}\{\text{green light flashes at time n} + 1 \text{ given green light flashes at time n}\} = \frac{1}{2}$$

$$\text{Prob}\{\text{green light flashes at time n} + 1 \text{ given red light flashes at time n}\} = \frac{1}{2}$$

$$\text{Prob}\{\text{red light flashes at time n} + 1 \text{ given red light flashes at time n}\} = \frac{1}{2}$$

It turns out that these identities hold for all $n \ge 0$.

(c) Show that the process is not a Markov chain.

(d) "Run" the process for about 37 time steps and see how many reds and greens you get.

(e) "Run" for 37 or so time steps the Markov chain you get by having the operator of the currently flashing light always choose the next light to flash by flipping a fair coin. See how many reds and greens you get.

3. Suppose i is such that $P(i, j) > 0$ for some $j \neq i$, where we're talking about a Markov chain, its states, and its one-step transition probabilities.

(a) Show that

$$\text{Prob}_i\{\text{the chain leaves state } i \text{ in finite time}\} = 1\ .$$

That is, if you can leave state i, you will leave state i. (Suggestion: consider separately the cases $P(i,i) = 0$ and $P(i,i) = p > 0$.)

(b) Find $E_i(T)$, where T is the first positive time that the state of the Markov chain isn't i.

4. The Rangers and Canadiens are set to play a 3-game series where the first team to win two games wins the series. The outcomes of the games are independent, and for each game the Rangers win with probability $p \in (0, 1)$. Ties are impossible.

(a) Draw the transition diagram for a 6-state Markov chain that models this situation. What are the transient and recurrent states?

(b) What's the only initial distribution $\pi(0)$ that makes sense?

(c) Find Prob{Rangers win the series}.

5. As it happens, the series in the preceding problem goes to a third game, and the game is tied after regulation time. The teams play a sequence of sudden-death overtime periods — first team to score a goal wins instantly. The probability that the Rangers score in any given overtime period is q, the probability that the Canadiens score is r, and the probability that neither team scores is $1 - q - r$, where all three probabilities are positive and less than 1.

(a) Draw the transition diagram for a 3-state Markov chain that models the play in overtime.

(b) Without doing any calculation, explain why the probability that the game never ends is zero. (Consider one of the previous problems on this assignment.)

(c) Find Prob{Rangers win the game} and Prob{Canadiens win the game}.

6. For the Markov chain in Figure 1, determine the following items. (Note: the picture should say that p, q, and r are all strictly positive rather than nonnegative.)

(a) $P^{(2)}(i,j)$ for every i and j.

(b) $f_{13}^{(4)}$, the probability that you reach state 3 for the first time after four steps given that you start in state 1.

(c) For each pair of states i and j, $\lim_{n\to\infty} \frac{E_i(N_j(n))}{n}$, the limiting average fraction of time you spend in state j given that you start in state i.,

(d) For each state j, $m_j = E_j(T_j)$, the average first-return time for state j.

7. For the Markov chain in Figure 2, determine the following items.

(a) The set S_T of transient states and the set S_R of recurrent states.

(b) The different recurrence classes.

(c) For each transient state i and each recurrence class C, the probability r_i^C that the state of the Markov chain enters C in finite time given that it starts in state i at time 0. (Suggestion: Note that

- r_{41} is the probability, given you start in 4, of following a path that hits 3 in finite time and then following a path starting from 3 that hits 1 in finite time, and
- r_{31} is the sum of the probabilities of two disjoint events: (1) following a path starting from 3 that hits 1 in finite time and never hits 4 and (2) following a path starting from 3 that hits 4 in finite time and then following a path starting from 4 that hits 1 in finite time.)