1. We have a population of two identical individuals at time 0. At each time step, each individual has probability ϵ of dying and probability $1-\epsilon$ of surviving, and death/survival of one individual happens independently of death/survival of the other.

(a) Draw the transition diagram for a Markov chain that models the evolution of this population, and specify the initial distribution.

(b) If you did (a) correctly, the Markov chain has only one recurrent state — call that state 0. Find $E(T_0)$, the expected time of arrival in state 0, in terms of ϵ.

2. We have a population of fixed size 3. Individuals in the population are single bits. At each time step, each individual either stays the same (which happens with probability $1-p$) or flips (which happens with probability p). Assume $0 < p < 1$.

(a) Find the transition probabilities of a 4-state Markov chain that models the evolution of this population. Make sure you've covered all possibilities by verifying that $\sum_{j=1}^{4} P(i,j) = 1$ for all i, where i and j index the states. Find the transient and recurrent states.

(b) Find the probability that the population at time 43 is $\{0,0,1\}$ and the population at time 42 is $\{0,1,1\}$ given that the population at time 41 is $\{1,1,1\}$.

(c) Find the probability that the population at time 2 is $\{1,1,1\}$ given that the population at time 0 is $\{0,0,0\}$.

(d) Find the probability that the population at time 2 is $\{1,1,1\}$ for the first time given that the population at time 0 is $\{0,0,0\}$. Compare with your answer to (c) and explain what's going on.

3. This one is sort of a mixture of the previous two problems. Instead of populations of size 3, make it populations of size at most 2. Individuals are still bits. Individuals still flip with probability p at each time step, but now they can also die with probability q, so they stay the same with probability $1-p-q$. Assume p and q are positive and $0 < p+q < 1$.

(a) Model this setup with a 6-state Markov chain. Let state 6 correspond to the empty population.

(b) For every $k > 0$, find the probability that the state is 6 for the first time at time k given that the population at time 0 is $\{0\}$. (Suggestion: you make k transitions. The last one has probability q. Each of the intervening $k-1$ transitions has probability either p or $1-p-q$. It doesn't matter what order those occur in. You may use the fact that

$$\sum_{l=0}^{k-1} \binom{k-1}{l} a^l b^{k-1-l} = (a+b)^{k-1} .$$

Hope this helps.)

(c) Find the expected value of the first time of arrival in state 6 given that the population at time 0 is $\{0\}$.

4. Consider the three-state Markov chain in Diagram 1.

(a) Find the transition matrix P and the stationary distribution π^* for this Markov chain.

(b) Find labels for the arrows in Diagram 2 so that the resulting Markov chain has the same stationary distribution as the Markov chain in (a).

5. Suppose

$$\pi^* = \begin{bmatrix} \pi_1^* & \pi_2^* & \pi_3^* & \pi_4^* & \pi_5^* \end{bmatrix}$$

is a distribution — that is, the π_k^* are nonnegative and sum to 1. Suppose also that $\pi_j^* > 0$ for all j. Suppose in addition that $\pi_j^* < \pi_{j+1}^*$ for $1 \le j < 5$.

(a) Use Metropolis-Hastings to find the the transition matrix P of a Markov chain that has unique stationary distribution π^* so that $[P]_{ij} > 0$ for all i and j.

(b) Use Metropolis-Hastings to find the transition matrix P of a Markov chain that has unique stationary distribution π^* so that $[P]_{ij} = 0$ unless $|i-j| \le 1$.

6. Consider the rather trivial GA that operates on binary "strings" of length $L = 1$ and populations of size $n = 2$ with fitness function $f(1) = 5$ and $f(0) = 1$, and with $p_m = .01$ (note that crossover is somewhat meaningless here). Assume fitness-proportional selection.

(a) List all the possible populations and give each one a label.

(b) Draw the transition diagram for the Markov chain on population space corresponding to this GA.

(c) Find the transition matrix P of the Markov chain.

(d) Find (at least approximately) the stationary distribution of the Markov chain.

(e) Based on your answer to (d), find the long-term average fitness of the population, that is, find

$$\bar{f}_\infty = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} \bar{f}(t) ,$$

where $\bar{f}(t)$ is the average fitness of the population at time t. Note that $\bar{f}_\infty$ is independent of the initial population.

7. The moral of this problem is that schemas are quite special subsets of sets of strings.

(a) Let A be a set containing N elements. Show that A has exactly 2^N subsets. (Suggestion: Think of numbering the elements of A as $a_1, \ldots, a_N$. If B is a subset of A, you can associate a binary string b with B as follows: $b_i = 1$ if $a_i \in B$ and $b_i = 0$ if $a_i \notin B$. Thus every subset gives rise to a *different* binary N-string (Why different?). Similarly, every binary N-string has an associated subset B of A (Why?).)

(b) Now let A be the set of all binary strings of length L. We know from the book that there are 3^L different schemas in this context. Each schema is a special subset of A. Compare 3^L with the total number of subsets of A for $L = 2$, $L = 8$, and $L = 14$.

(c) Now let A be the set of all strings of length L whose string entries (instead of being zeroes and ones) are chosen from among the letters P, Q, R, and S. How many schemas are there now? How many total subsets of A? Compare these numbers for $L = 1$, and $L = 4$, and $L = 7$.

(d) One of Holland's rationales for using binary encoding for GAs was that encoding N individuals using binary strings made "more schemas available for testing" than did encodings of those same N individuals using strings whose entries came from larger alphabets. Have a look at (b) and (c) and explain briefly why that's the case in general. (Look at how I chose the L-values to test.)

8. You've got a box that has eight dials on it. Dials 1 and 2 have 16 possible settings; dials 3, 4, and 5 have two possible settings; and dials 6 through 8 have four possible settings.

(a) How many total dial settings are there?

(b) Design an "obvious" way to encode each dial setting as a string of eight symbols. How many schemas does your encoding give rise to?

(c) Design a scheme that encodes each dial setting as a binary string. How many schemas does your binary encoding give rise to?

9. Let A be the set of all binary strings of length L. Let H be a specific schema with k defined bits. If you pick a single string totally at random from A, what is the probability that the string will lie in H? (Suggestion: how many strings are in H?)