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Spring 2022

ECE4271 - HW II

1. a) If only 1 string represented, then every string within a given population must be identical.

I.E: Pop = $\{S_1, S_2, S_3, \ldots S_n\}$, $S_1 = S_2 = S_3 = \ldots S_n$.

Since each string is of length L, there are 2^L possible strings (distinct).

Since $S_1 = S_2 = \ldots S_n$, then there are 2^L possible populations of size n, each being the same string.

If exactly n distinct strings represented:

There are 2^L possible strings (distinct), as each position in the string has 2 possibilities (0 or 1). We want populations where there are n distinct strings. Thus, we need to find every possible way of selecting n strings from a pool of 2^L strings. The total number of ways to do this is:

$$\binom{2^L}{n} \rightarrow \text{choose } n \text{ strings from } 2^L \text{ candidates.}$$

total # of ways to choose n strings from 2^L candidates.

b) There are two strings represented in the population. There are $\binom{2^L}{2}$ ways to select the 2 strings to be represented. The next step is determine how many occurrences of the first string (x) and how many occurrences of second string (y).

$\Rightarrow$ Represent population as: $\underbrace{xxx \ldots xxx\, \dot{y}yy \ldots y}_{n \text{ strings in population.}}$

The divider determines the distribution of x and y:
i.e. how many x and how many y such that $n_x + n_y = n$ ← total pop. size

$$\underbrace{xxx\cdots xxx}_{\text{string 1}} \;\underset{\text{divider}}{\Big|}\; \underbrace{yyy - yyy\,y}_{\text{string 2}}\,{}^{n}$$

There are $n-1$ slots to place the divider b/c we need at least 1 "x" & 1 "y". The # of ways to place divider is:

$$\binom{n \text{ slots}}{1} = \binom{n-1}{1}$$

# of slots

Applying product rule:

$$\begin{pmatrix}\text{\# of ways to select} \\ \text{2 strings from } 2^L \\ \text{samples}\end{pmatrix} \times \begin{pmatrix}\text{\# of ways to} \\ \text{place divider}\end{pmatrix}$$

$$= \boxed{\binom{2^L}{2}\binom{n-1}{1}}$$

c) $\binom{2^L}{k}$ # of ways to select k distinct strings from a pool of 2^L strings.

Since we have K distinct strings represented, we need to divide our population $x_1 x_2 x_3 \ldots x_n$ into k distinct parts. This requires k-1 dividers. The total # of ways to split up the population is:

$$\underbrace{x_1 x_2 x_3 | \ldots | \ldots | x_{n-2} | x_{n-1} x_n}_{k-1 \text{ dividers}}$$

ways to pick dividers $= \binom{n-1}{k-1}$

of slots available

# of slots to choose

Using product rule:

$\left(\text{\# of ways to select } k \text{ strings from } 2^L \text{ samples}\right) \times \left(\text{\# of ways to place } k-1 \text{ dividers}\right)$

$$= \binom{2^L}{k} \cdot \binom{n-1}{k-1}$$

d) <u>Total # of reproductions</u>

$$n_{total} = \underbrace{\frac{\#\text{ of rep w/}}{1 \text{ string reproduced}} + \frac{\#\text{ of rep w/}}{2 \text{ strings rep}} + \dots + \frac{\#\text{ of rep w/}}{n \text{ strings rep.}}}_{\text{disjoint}}$$

$$= \sum_{k=1}^{n} \underbrace{\binom{2^L}{k}\binom{n-1}{k-1}}_{\text{part (c)}}$$

e) $$n_{total} = \sum_{k=1}^{n} \binom{2^L}{k}\binom{n-1}{k-1}$$

$$= \sum_{k=1}^{n} \binom{2^L}{k}\binom{n-1}{n-k} \quad \left[\binom{n-1}{k-1} \overset{\text{using}}{=} \binom{n-1}{n-k}\right]$$

$$= \sum_{k=0}^{n} \binom{2^L}{k}\binom{n-1}{n-k} - \cancel{\binom{2^L}{0}\binom{n-1}{n}} \to 0$$

$$= \sum_{k=0}^{n} \binom{2^L}{k}\binom{n-1}{n-k} = \binom{2^L+n-1}{n} \overset{\text{using}}{=} \boxed{\binom{2^L+n-1}{2^L-1}} \quad \left[\binom{n-1}{k-1} = \binom{n-1}{n-k}\right]$$

∴ QED.

2(a) $\underline{n=0}$:

Fair coin toss: $P(R \text{ at } n=0) = \boxed{1/2}$

$P(G \text{ at } n=0) = \boxed{1/2}$

Red at time $n=0$

$\underline{n=1}$:

$P(R \text{ at } n=1) = P(R_0)P(R_1|R_0) + P(G_0)P(R_1|G_0)$

$= 1/2 \times 1/2 + 1/2 \times 1/2 = \boxed{\frac{1}{2}}$

$P(G_1) = P(R_0)P(G_1|R_0) + P(G_0)P(G_1|G_0)$

$= 1/2 \times 1/2 + 1/2 \times 1/2 = \boxed{\frac{1}{2}}$

$\underline{n=2}$: $P(R_2) = P(R_0R_1) \cdot P(R_2|R_0R_1) + P(G_0G_1)P(R_2|G_0G_1) +$

$P(R_0G_1)P(R_2|R_0G_1) + P(G_0R_1)P(R_2|G_0R_1)$

$= \frac{1}{2} \times 1/2 \times 1/2 + \cancel{1/2 \times 1/2 \times 0} + 1/2 \times 1/2 \times 1 + 1/2 \times 1/2 \times 1/2$

$= 1/8 + 1/8 + 1/4 = 1/4 + 1/4 = \boxed{\frac{1}{2}}$

$P(G_2) = P(R_0R_1) \cdot P(G_2|R_0R_1) + P(G_0G_1)P(G_2|G_0G_1) +$

$P(R_0G_1)P(G_2|R_0G_1) + P(G_0R_1)P(G_2|G_0R_1)$

$= 1/2 \times 1/2 \times 1/2 + 1/2 \times 1/2 \times 1 + 1/2 \times 1/2 \times 0 + 1/2 \times 1/2 \times 1/2$

$= 1/8 + 1/8 + 1/2 = \boxed{1/2}$

$\therefore$ QED.

b) Show for $0 \le n \le 2$:

find (R_{n+1} given G_n):

$n=0$: $P(R_1 \mid G_0) = \boxed{1/2}$ [from opposite coin flip]

$n=1$: $P(R_2 \mid G_1) = \dfrac{P(R_2 \cap G_1)}{P(G_1)}$

$$= \frac{\underbrace{\tfrac12 \times \tfrac12 \times 1}_{R_0G_1R_2} + \underbrace{\cancel{\tfrac12 \times \tfrac12 \times 0}}_{G_0G_1R_2} \; 0}{\underbrace{\tfrac12 \times \tfrac12}_{R_0G_1} + \underbrace{\tfrac12 \times \tfrac12}_{G_0G_1}}$$

$$= \frac{1/4}{1/2} = \boxed{1/2}$$

$n=2$: $P(R_3 \mid G_2) = \dfrac{P(R_3 \cap G_2)}{P(G_2)}$

$$= \frac{P(R_0R_1G_2R_3) + P(G_0G_1G_2R_3) + P(R_0G_1G_2R_3) + P(G_0R_1G_2R_3)}{1/2 \text{ (from part (a))}}$$

$$= \frac{\tfrac12 \times \tfrac12 \times \tfrac12 \times 1 + \tfrac12 \times \cancel{\tfrac12 \times 1 \times 0} \; 0 + \cancel{\tfrac12 \times \tfrac12 \times 0} \; 0 + \tfrac12 \times \tfrac12 \times \tfrac12 \times 1}{1/2}$$

$$= \frac{1/8 + 1/8}{1/2} = \frac{1/4}{1/2} = \boxed{\frac{1}{2}}$$

$\underline{P(G_{n+1} \mid G_n):}$

$\underline{n=0:}$

$P(G_1 \mid G_0) = \boxed{1/2}$ [Green operator flips coin]

$\underline{n=1:}$

$$P(G_2 \mid G_1) = \frac{P(G_2 \cap G_1)}{P(G_1)}$$

$R_0G_1G_2$

$$G_0G_1G_2 = \frac{1/2 \times 1/2 \times 1 + \cancel{1/2 \times 1/2 \times 0}}{1/2 \times 1/2 + 1/2 \times 1/2}$$

$$= \frac{1/4}{1/2} = \boxed{1/2}$$

$\underline{n=2:}$

$$P(G_3 \mid G_2) = \frac{P(G_2 \cap G_3)}{P(G_2)}$$

$$= \frac{P(G_0G_1G_2G_3) + P(R_0R_1G_2G_3) + P(R_0G_1G_2G_3) + P(G_0R_1G_2G_3)}{1/2}$$

$$= \frac{1/2 \times 1/2 \times 1 \times 1 + \cancel{1/2 \times 1/2 \times 1/2 \times 0} + \cancel{1/2 \times 1/2 \times 0} + \cancel{1/2 \times 1/2 \times 1/2 \times 0}}{1/2}$$

$$= \frac{1/4}{1/2} = \boxed{1/2}$$

$\underline{P(G_{n+1} \mid R_n):}$

$\underline{n=0:}$ $P(G_1 \mid R_0) = \boxed{1/2}$ (coin flip).

$\underline{n=1:}$

$$P(G_2 \mid R_1) = \frac{P(R_1 \cap G_2)}{P(R_1)}$$

$$= \frac{1/2 \times 1/2 \times 1/2 + 1/2 \times 1/2 \times 1/2}{1/2} = \frac{1/4}{1/2} = \boxed{1/2}$$

$\underline{n=2:}$

$$P(G_3 \mid R_2) = \frac{P(R_2 \cap G_3)}{P(R_2)}$$

$$= \frac{P(R_0R_1R_2G_3) + P(G_0G_1R_2G_3) + P(R_0G_1R_2G_3) + P(G_0R_1R_2G_3)}{1/2}$$

$$= \frac{\frac12\times\frac12\times\frac12\times\frac12 + \cancel{\frac12\times\frac12\times 0} + \frac12\times\frac12\times 1\times\frac12 + \frac12\times\frac12\times\frac12\times\frac12}{1/2}$$

$$= \frac{1/16 + 1/16 + 1/8}{1/2} = \frac{1/4}{1/2} = \boxed{1/2}$$

$\underline{P(R_{n+1} \mid R_n):}$

$\underline{n=0:}$ $P(R_1 \mid R_0) = \boxed{1/2}$ (coin flip)

$\underline{n=1:}$ $$P(R_2 \mid R_1) = \frac{P(R_1 \cap R_2)}{P(R_1)}$$

$$= \frac{\frac12\times\frac12\times\frac12 + \frac12\times\frac12\times\frac12}{1/2}$$

$$= \frac{1/4}{1/2} = \boxed{1/2}$$

$\underline{n=2:}$ $$P(R_3 \mid R_2) = \frac{P(R_3 \cap R_2)}{P(R_2)}$$

$$= \frac{P(R_0R_1R_2R_3) + P(G_0G_1R_2R_3) + P(R_0G_1R_2R_3) + P(G_0R_1R_2R_3)}{P(R_2)}$$

$$= \frac{\frac12\times\frac12\times\frac12\times\frac12 + \cancel{\frac12\times\frac12\times 0} + \frac12\times\frac12\times 1\times\frac12 + \frac12\times\frac12\times\frac12\times\frac12}{1/2}$$

$$= \frac{1/16 + 1/16 + 1/8}{1/2} = \boxed{1/2}$$

c) A discrete time Markov chain satisfies the Markov property:

$$P(X_{n+1} = x \mid X_0 = x_0, X_1 = x_1, \ldots X_n = x_n) = P(X_{n+1} = x \mid X_n = x_n)$$

Counter-example: $X_0 = G,\ X_1 = G,\ X_2 = G;$

$$P(X_3 = R \mid X_0 = G, X_1 = G, X_2 = G) = 0$$

$n=1$ is G

$n-1$ is G, so

$n+1$ is G

However, $P(X_3 = R \mid X_2 = G) = 1/2$ as shown in part (b).

$$\Rightarrow P(X_3 = R \mid X_2 = G) \neq P(X_3 = R \mid X_0 = G, X_1 = G, X_2 = G)$$

$\therefore$ Not Markov Chain.

d) See Simulation Code.

Result: Red: 26
Green: 11

e) See Simulation Code.

Result: Red: 18
Green: 19

```python
from random import randint
from typing import List


def update(state_history: List[int]) -> List[int]:
    new_state_history = state_history.copy()
    if not state_history or len(state_history) == 1 or state_history[-1] == "R":
        new_state = "G" if randint(0, 1) == 0 else "R"
        new_state_history.append(new_state)

    elif state_history[-1] == "G":
        new_state = "G" if state_history[-2] == "G" else "R"
        new_state_history.append(new_state)

    else:
        raise ValueError("Unexpected condition.")

    return new_state_history


def main():
    STEPS = 37
    history = []
    for i in range(STEPS):
        history = update(history)

    print(history)

    RED_COUNT = history.count("R")
    GREEN_COUNT = history.count("G")

    print(f"RED COUNT {RED_COUNT}")
    print(f"GREEN COUNT {GREEN_COUNT}")


if __name__ == "__main__":
    main()
```

```python
from random import randint
from typing import List


def update(state_history: List[int]) -> List[int]:
    new_state_history = state_history.copy()

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def main():
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if __name__ == "__main__":
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```

3. a) show $P_i\{\text{chain leaves state } i \text{ in finite time}\} = 1$

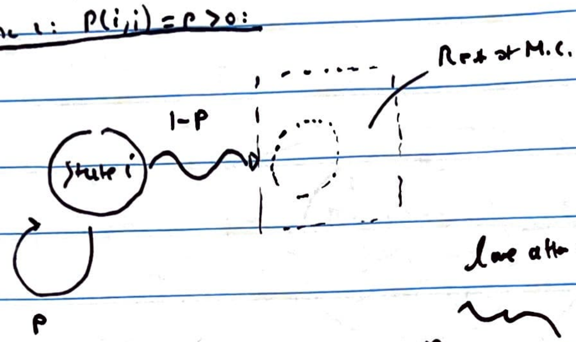
Case 1: $P(i,i) = 0$

At time 0, if at

$P(i,i) = 0 \Rightarrow$ State i, it will not remain at state i.

Since $\sum_{j \in S} P(i,j) = 1$ (i.e it must go somewhere), and $P(i,i) = 0$, there is 100% probability it will leave state i in finite time (as it can never stay at state i).

Case 2: $P(i,i) = p > 0$:



leave after $n=1, n=2, n=3, \ldots$

$$P_i\{\text{chain leaves state } i \text{ in finite time}\} = \sum_{n=0}^{\infty} p^n(1-p)$$

$$= \sum_{n=0}^{\infty} (p^n - p^{n+1})$$

$$= \frac{1}{1-p} - \frac{p}{1-p}$$

$$= \cancel{\frac{1-p}{1-p}} = \boxed{1}$$

100% prob it will leave in finite time.

$\therefore$ 100% prob in both cases.

b) $E_i(T)$: **PMF of T:**

$T=1$: $1-p$

$T=2$: $p(1-p)$

$T=3$: $p^2(1-p)$

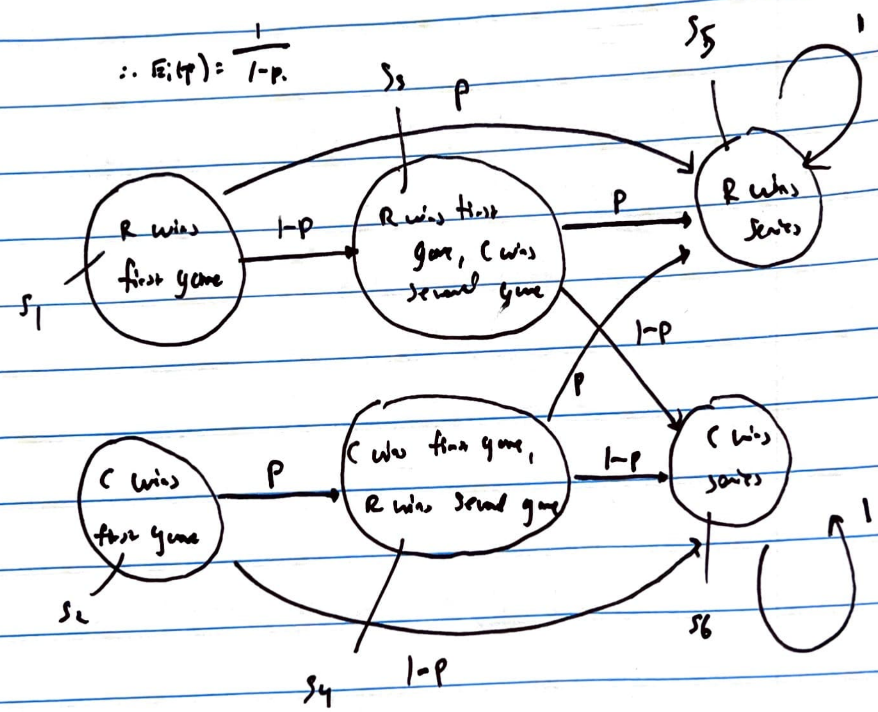
$\vdots$

$$E_i(T) = \sum_{n=1}^{\infty} p^{n-1}(1-p)\cdot n = \sum_{n=0}^{\infty} p^{n-1}\cdot(1-p)\cdot n$$

$$= \boxed{\frac{1}{1-p}} \quad \text{for } |p|<1$$

$\therefore E_i(T) = \frac{1}{1-p}$.

4a)



Transient States:

$\{S_1, S_2, S_3, S_4\}$

Recurrent States:

$\{S_5, S_6\}$

b) The only initial distribution $\pi(0)$ that makes sense is:

$$\pi(0) = \begin{bmatrix} p \\ 1-p \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{matrix} \leftarrow \text{state 1} \\ \leftarrow \text{state 2} \\ \leftarrow \text{state 3-6} \end{matrix}$$

This is because Rangers has Prob p of winning 1st game and Canadiens has Prob $1-p$ of winning 1st game.

c) **Prob{Rangers win the series}:**

$$\text{Prob\{Rangers win the series\}} = P(S_1 \to S_5) + P(S_1 \to S_3 \to S_5) + P(S_2 \to S_4 \to S_5)$$

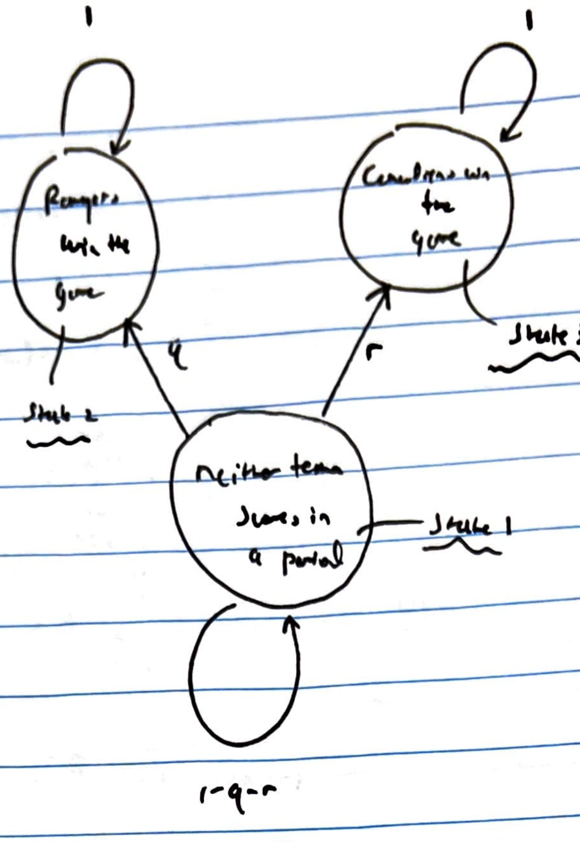
$$= p \cdot p + p \cdot (1-p) \cdot p + (1-p) \cdot p \cdot p$$

$$= p^2 + p^2 - p^3 + p^2 - p^3$$

$$= \boxed{3p^2 - 2p^3}$$

$\therefore$ Prob{Rangers win the series} $= 3p^2 - 2p^3$.

5. a)



$$\pi(0) = \begin{bmatrix} 1-q-r & q & r \end{bmatrix}^T$$

$$\begin{array}{ccc} \text{State 1} & \text{State 2} & \text{State 3} \end{array}$$

b) ~~Why probability of never ending game is zero:~~

In question 3(a), we showed that in a M.C with $P(i,j) > 0$ for some $i \neq j$,

$\text{Prob}_i\{\text{the chain leaves the state } i \text{ in finite time}\} = 1.$

Case #1: If we start in State 2 or State 3, we are done because there is a winner.

Case #2: If start in State 1, $P(S_1, S_2) = q > 0$ and $p(S_1, S_3) = r > 0$. Thus,

$\text{Prob}_{S_1}\{\text{leave } S_1 \text{ in finite time}\} = 1$

$\Rightarrow$ We will definitely enter S_2 or S_3.

$\therefore$ There will definitely be a winner.

1) Prob {Rangers win the series}:

$$P(\text{Rangers series}) = q + \sum_{n=1}^{\infty} (1-q-r)^n \cdot q$$

$$= \sum_{n=0}^{\infty} (1-q-r)^n \cdot q$$

$$= q \sum_{n=0}^{\infty} (1-q-r)^n \qquad \text{Using } \left\{ \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}, \ |a|<1 \right\}$$

$$= q \cdot \frac{1}{1-(1-q-r)}$$

$$= q \cdot \frac{1}{q+r}$$

$$= \boxed{\frac{q}{q+r}}$$

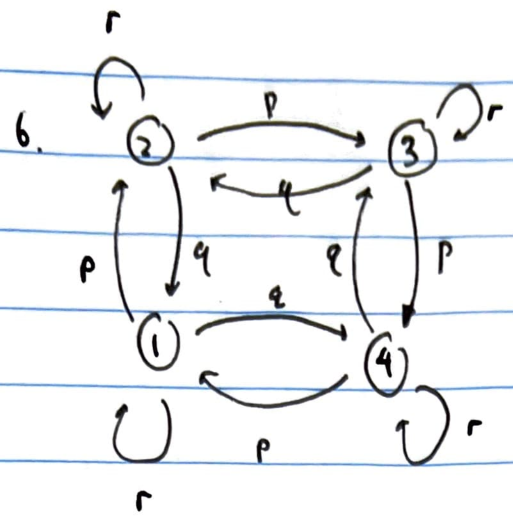
Prob {Canadiens win the series}:

$$P\{\text{Canadiens win}\} = \sum_{n=0}^{\infty} (1-q-r)^n \cdot r$$

$$= r \sum_{n=0}^{\infty} (1-q-r)^n$$

$$= \frac{r}{1-(1-q-r)} = \boxed{\frac{r}{q+r}}$$

6.



$$\left[\begin{array}{l} 0 < p,q,r < 1 \\ p+q+r=1 \end{array}\right]$$

Figure 1

a) $P^{(2)}(i,j)$ for every i,j

$P^{(2)}(1,1)$: Walks: $\cdot\ 1 \xrightarrow{r} 1 \xrightarrow{r} 1$

$\cdot\ 1 \xrightarrow{q} 4 \xrightarrow{p} 1$

$\cdot\ 1 \xrightarrow{p} 2 \xrightarrow{q} 1$

$P^{(2)}(1,1) = r\cdot r + p\cdot q + q\cdot p$

$\boxed{= r^2 + 2pq}$

$P^{(2)}(1,2)$: Walks: $\cdot\ 1 \xrightarrow{r} 1 \xrightarrow{p} 2$

$\cdot\ 1 \xrightarrow{p} 2 \xrightarrow{r} 2$

$P^{(2)}(1,2) = r\cdot p + p\cdot r = \boxed{2rp}$

$P^{(2)}(1,3)$: walks: $\cdot\ 1 \xrightarrow{p} 2 \xrightarrow{p} 3$

$\cdot\ 1 \xrightarrow{q} 4 \xrightarrow{q} 3$

$P^{(2)}(1,3) = \boxed{p^2 + q^2}$

$P^{(2)}(1,4)$: walks: $\cdot\ 1 \xrightarrow{q} 4 \xrightarrow{r} 4$

$\cdot\ 1 \xrightarrow{r} 1 \xrightarrow{q} 4$

$P^{(2)}(1,4) = \boxed{2qr}$

$p^{(2)}(2,1)$: walks: $\cdot\ 2 \xrightarrow{q} 1 \xrightarrow{r} 1$

$\cdot\ 2 \xrightarrow{r} 2 \xrightarrow{q} 1$

$p^{(2)}(2,1) = \boxed{2qr}$

$p^{(2)}(2,2)$: walks: $\cdot\ 2 \xrightarrow{r} 2 \xrightarrow{r} 2$

$\cdot\ 2 \xrightarrow{q} 1 \xrightarrow{p} 2$

$\cdot\ 2 \xrightarrow{p} 3 \xrightarrow{q} 2$

$p^{(2)}(2,2) = \boxed{r^2 + 2pq}$

$p^{(2)}(2,3)$: walks: $\cdot\ 2 \xrightarrow{p} 2 \xrightarrow{p} 3$

$\cdot\ 2 \xrightarrow{p} 3 \xrightarrow{r} 3$

$p^{(2)}(2,3) = \boxed{2pr}$

$p^{(2)}(2,4)$: walks: $\cdot\ 2 \xrightarrow{q} 1 \xrightarrow{q} 4$

$\cdot\ 2 \xrightarrow{p} 3 \xrightarrow{p} 4$

$p^{(2)}(2,4) = \boxed{p^2 + q^2}$

$p^{(2)}(3,1)$: walks: $\cdot\ 3 \xrightarrow{p} 4 \xrightarrow{p} 1$

$\cdot\ 3 \xrightarrow{q} 2 \xrightarrow{q} 1$

$p^{(2)}(3,1) = \boxed{p^2 + q^2}$

$p^{(2)}(3,2)$: walks: $\cdot\ 3 \xrightarrow{q} 2 \xrightarrow{r} 2$

$\cdot\ 3 \xrightarrow{r} 3 \xrightarrow{q} 2$

$p^{(2)}(3,2) = \boxed{2rq}$

$p^{(2)}(3,3)$:   walks: $3 \xrightarrow{r} 3 \xrightarrow{r} 3$

$3 \xrightarrow{p} 4 \xrightarrow{q} 3$

$3 \xrightarrow{q} 2 \xrightarrow{p} 3$

$p^{(2)}(3,3) = \boxed{r^2 + 2pq}$

$p^{(2)}(3,4)$:   walks: $3 \xrightarrow{r} 3 \xrightarrow{p} 4$

$3 \xrightarrow{p} 4 \xrightarrow{r} 4$

$p^{(2)}(3,4) = \boxed{2rp}$

$p^{(2)}(4,1)$:   walks: $4 \xrightarrow{r} 4 \xrightarrow{p} 1$

$4 \xrightarrow{p} 1 \xrightarrow{r} 1$

$p^{(2)}(4,1) = \boxed{2rp}$

$p^{(2)}(4,2)$:   walks: $4 \xrightarrow{q} 3 \xrightarrow{q} 2$

$4 \xrightarrow{p} 1 \xrightarrow{p} 2$

$p^{(2)}(4,2) = \boxed{p^2 + q^2}$

$p^{(2)}(4,3)$:   walks: $4 \xrightarrow{r} 4 \xrightarrow{q} 3$

$4 \xrightarrow{q} 3 \xrightarrow{r} 3$

$p^{(2)}(4,3) = \boxed{2rq}$

$p^{(2)}(4,4)$:   walks: $4 \xrightarrow{r} 4 \xrightarrow{r} 4$

$4 \xrightarrow{p} 1 \xrightarrow{q} 4$

$4 \xrightarrow{q} 3 \xrightarrow{p} 4$

$p^{(2)}(4,4) = \boxed{r^2 + 2pq}$

b) $\underline{f_{13}^{(4)}}$: prob. reach S_3 for first time after 4 steps given start in S_1.

Possible walks:

$$\left.\begin{array}{l}
\cdot\ 1,1,1,2,3 \\
\cdot\ 1,4,1,2,3 \\
\cdot\ 1,2,1,2,3 \\
\cdot\ 1,1,2,2,3 \\
\cdot\ 1,2,2,2,3 \\
\cdot\ 1,1,1,4,3 \\
\cdot\ 1,2,1,4,3 \\
\cdot\ 1,4,1,4,3 \\
\cdot\ 1,4,4,4,3 \\
\cdot\ 1,1,4,4,3
\end{array}\right\}$$

possible 4-step transitions starting at S_1 and hitting S_3 for first time

$$f_{13}^{(4)} = P(11123) + P(14123) + P(12123) + P(11223) + P(12223) + P(11143) + P(12143) + P(14143) + P(14443) + P(11443)$$

$$= r^2p^2 + p^3q^1 + p^3q^1 + r^2p^2 + r^2p^2 + r^2q^2 + q^3p + q^3p + r^2q^2 + r^2q^2$$

$$= 3r^2p^2 + 2p^3q + 3r^2q^2 + 2q^3p$$

$$\boxed{= 3(r^2p^2 + r^2q^2) + 2(p^3q + q^3p)}$$

c) $\lim_{n\to\infty} \frac{E_i(N_j(n))}{n}$

All states are positively-recurrent. The state space forms a single recurrence class.

<u>For i, j and i=j:</u>

$$\lim_{n\to\infty} \frac{E_i(N_j(n))}{n}\Bigg|_{i=j} = \lim_{n\to\infty} \frac{E_j(N_j(n))}{n} = \frac{1}{m_j} \quad \text{(Theorem 2 in handout)}$$

For ergodic Markov chain, the expected first return time m_j when we only have <u>positively recurrent</u> states and <u>irreducible</u> $(i \leftrightarrow j)$ for any pair i, j:

$$m_j = 1/\pi_j^* \quad \text{where} \quad \underbrace{\pi^* = [\pi_1^* \; \pi_2^* \dots \pi_j^* \dots \pi_n^*]}_{\text{unique stationary distribution}}$$

<u>Find π^*:</u>

Eigenvalue problem:

$$[\pi_1^* \; \pi_2^* \; \pi_3^* \; \pi_4^*] \begin{bmatrix} r & p & 0 & q \\ q & r & p & 0 \\ 0 & q & r & p \\ p & 0 & q & r \end{bmatrix} = [\pi_1^* \; \pi_2^* \; \pi_3^* \; \pi_4^*]$$

Solution for π^*: $\quad \pi^* = [1/4 \; 1/4 \; 1/4 \; 1/4]$

$$[1/4 \; 1/4 \; 1/4 \; 1/4] \begin{bmatrix} r & p & 0 & q \\ q & & & \\ 0 & & \ddots & \\ p & & & r \end{bmatrix} = [1/4 \underbrace{\cancel{(r+p+q)}}_{r+p+q=1} \; 1/4\cancel{[r+p+q]} \; 1/4\cancel{(r+p+q)} \; 1/4\cancel{(r+p+q)}]$$

$$= \boxed{[1/4 \; 1/4 \; 1/4 \; 1/4]}$$

The stationary distribution is: $\pi^* = [\,1/4 \;\; 1/4 \;\; 1/4 \;\; 1/4\,]$.

Regardless of where you start, the M.C spends in each state j a limiting fraction proportional to the stationary distribution (π^*).

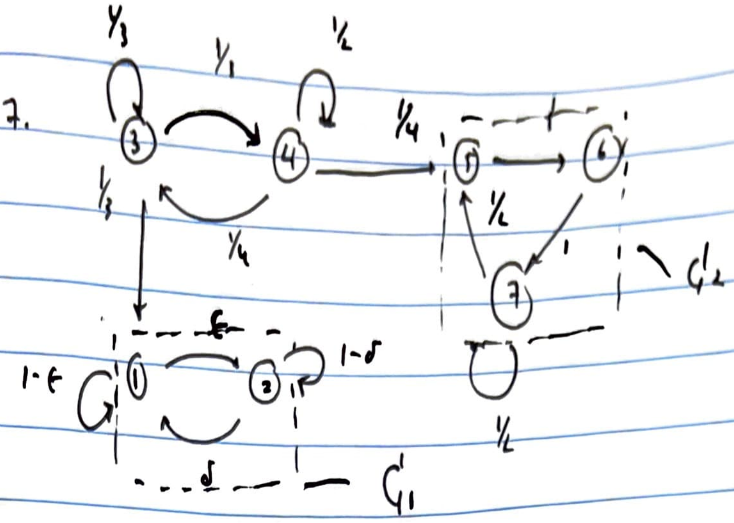
$\therefore$ For all pairs (i,j), $\displaystyle\lim_{n\to\infty} \frac{E_i(N_j(n))}{n} = \boxed{1/4}$.

d) ~~Show $m_j = 1/\pi_j^*$ for $j = 1 \ldots 4$:~~

$$m_j = \frac{1}{1/4} = \boxed{4} \quad \text{for } j = 1 \ldots 4$$

$\therefore$ For $j = 1 \ldots j$, avg first-return time for state j is 4.

7.



$0 < \varepsilon, \delta < 1$

a) $S_T = \{3,4\}$ (Transient)

$S_R = \{1,2,5,6,7\}$ (Recurrent)

b) $G_1' = \{1,2\}$

$G_2' = \{5,6,7\}$

c) <u>Compute r_i^{G}:</u>

$r_4^{G_1}$: Prob. start at S_4 and enter G_1 in finite time

$$r_4^{G_1} = \underbrace{\left\{\sum_{n=0}^{\infty} \left(1/2\right)^n \cdot \left(1/4\right)\right\}}_{r_{43}} \cdot r_{31} \qquad = \underbrace{r_{43}}_{4\to 3 \text{ path}} \cdot \underbrace{r_{31}}_{3\to 1 \text{ path}} = \underbrace{r_{41}}_{4\to 1 \text{ path}}$$

- For r_{43}, the state 3 is reached last and for the first time.
- For r_{31}, the state 1 is reached last and for the first time.

$r_{41} = r_4^{C_1} = \left(\frac{1}{1-1/2}\right)^2 \cdot 1/4 \cdot r_{31}$

$= \underline{\underline{1/2 \cdot r_{31}}}$

Path from 3 that hits 1 in finite time and never hits 4

$\Rightarrow r_{31} = \text{Prob}\{\underbrace{\text{Directly from 3 to 1 (without visiting 4)}}\} +$

$\text{Prob}\{\underbrace{\text{go from 3 to 4 and to 1}}\}$

Path starting from 3 that hits 4 in finite time and then following a path starting from 4 that hits 1 in finite time

$= \sum_{n=0}^{\infty} (1/3)^n \cdot 1/3 + \left(\sum_{n=0}^{\infty} (1/3)^n \cdot 1/3\right) r_{41}$

$= 1/2 + 1/2\, r_{41}$

Thus,

$r_{41} = 1/2 \cdot r_{31} = 1/2\left[1/2 + 1/2\, r_{41}\right]$

$r_{41} = 1/4 + 1/4\, r_{41}$

$r_{41}(3/4) = 1/4$

$r_{41} = 1/4 \cdot 4/3 = \boxed{1/3}$

$\therefore r_4^{C_1} = 1/3.$

$r_3^{C_1}$: $r_3^{C_1} = r_{31} = \text{Prob}\{\text{directly from 3 to 1 (w/o going to 4)}\} +$

$\text{Prob}\{\text{go from 3 to 4 and then to 1}\} = 1/2 + 1/2 \cdot r_{41}$

$= 1/2 + 1/2 \cdot 1/3 = 1/2 + 1/6 = 4/6 = \boxed{2/3}$

$\therefore r_3^{C_1} = 2/3.$

$r_4^{C_2}$:

$r_4^{C_2} = r_{45} = \text{Prob}\{\text{directly from 4 to 5 (w/o hitting 3)}\} +$

$\text{Prob}\{\text{go from 4 to 3 then to 5}\}$

$$= \underbrace{\sum_{n=0}^{\infty} (1/2)^n \cdot 1/4}_{4 \to 5 \text{ w/o hitting } 3} + \underbrace{\sum_{n=0}^{\infty} (1/2)^n \cdot 1/4 \cdot r_{35}}_{4 \to 3}$$

$$= \cancel{\left(\frac{1}{1-1/2}\right)}^{1/1/2} 1/4 + \cancel{\left(\frac{1}{1-1/2}\right)}^{1/1/2} 1/4 \cdot r_{35}$$

$$= 1/2 + 1/2\, r_{35} = 1/2 + 1/2\,(1/3) = 1/2 + 1/6 = 4/6 = \boxed{2/3}$$

$$\therefore r_4^{G_2} = 2/3.$$

$\underline{\underline{r_3}}^{G_2}:$

$$r_3^{G_2} = \left\{\sum_{n=0}^{\infty} (1/3)^n \cdot 1/3\right\} \cdot r_{45} = r_{35}$$

$$= \frac{1}{1-1/3} \cdot 1/3 \quad \text{or } r_{45} = \underline{\underline{1/2\, r_{45}}}$$

$$\Rightarrow r_{45} = 1/2 + 1/2\, r_{35}$$

Thus,

$$r_{35} = 1/2\left[1/2 + 1/2\, r_{35}\right]$$

$$r_{35} = 1/4 + 1/4\, r_{35}$$

$$r_{35}\,(3/4) = 1/4$$

$$r_{35} = \boxed{1/3}$$

$$\therefore r_{35} = r_3^{G_2} = \underline{\underline{1/3}}.$$

In summary:

- $r_3^{C_1} = 2/3$
- $r_3^{C_2} = 1/3$

$r_3^{C_1} + r_3^{C_2} = \boxed{1}$

- $r_4^{C_1} = 1/3$
- $r_4^{C_2} = 2/3$

$r_4^{C_1} + r_4^{C_2} = \boxed{1}$

This makes sense because states 3 and 4 are transient. Thus they must leave and enter one of the recurrence classes. There are two recurrence classes C_1 and C_2. The sum of probabilities add up to 1.