Cancellation of Resonance for Elastically Supported Beams Subjected to Successive Moving Loads: Optimal Design Condition for Bridges

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Abstract

This paper investigates comprehensively the resonance and cancellation conditions for the free vibration of elastically-supported (ES) beams subjected to successive moving loads. Focus is placed on application of the cancellation condition to minimize bridge vibrations, considering particularly the effect of elastic supports. In terms of the modal amplitude R of free vibration of the ES beam, both resonance and cancellation conditions are identified. This paper is featured by the fact that the cancellations are classified into two types as the external (load-related) and internal (structure-related) ones. Through the (internal) cancellation function, the criterion for selecting the optimal support stiffness ratio (SSR) is derived for the first time for suppressing the resonance of short to medium-span railway bridges. It depends solely on the bridge/vehicle length ratio L/d, and can be utilized to achieve near-perfect cancellation. The theoretical findings are validated by the finite element method (FEM) for various parameters. The results reveal that for beams with lengths in the ranges of $(0.5d,\ d]$ and $(1.5d, 2d]$, an SSR closer to the lower bound of the acceptable range should be selected to achieve the best effect. And for beams with lengths in the range of $(d,\ 1.5d]$, the SSR should be selected as close to the optimal value as possible. Besides, it was found that damping in the beam and supports contributes to further suppression of vibration for bridges designed with the optimal SSR.

Keywords: Bridge, Cancellation, Elastically supported beam, Free vibration, Resonance, Support optimization

1. Introduction

High-speed rail (HSR) has emerged as an important mode of transportation worldwide, distinguished by its high speed, efficiency, and passenger capacity. Elevated multi single-span bridges represent an essential structural component for the HSR system, pivotal to ensure the right-of-way and passengers' riding comfort. Train loads exhibit characteristics of continuity and periodicity, resulting in the possible accumulation of bridge vibrations [1-4]. These vibrations may even trigger the fatigue problem in track compositions and sensing devices, threating the safety and comfort of train operations [5,6]. The challenges related to the train-induced resonance and the pursuit of optimal design to minimize vibrations in bridges have remained at the forefront of transportation engineering research. This becomes more prominent as the HSR continues to advance in terms of speed and train load [7-9].

In the previous theoretical studies on bridge vibrations induced by moving loads, most of the bridges were modelled as a simply supported (SS) beam, with limited attention paid to the effect of elastic supports [1-5,10]. Using the SS beam model as the technical tool, various aspects of vehicle-bridge interaction (VBI) have been investigated, covering topics such as impact factors [11-13], ride comfort [14-16], optimal design [17-19], and vehicle scanning methods [20-24]. These topics, along with their applications to railways, have been meticulously documented and expanded upon in the 2004 monograph by Yang et al. [2].

Resonance and cancellation are of paramount concern in the study of beam vibrations caused by moving loads, as they crucially link to the operational safety of vehicles and bridges [25]. Resonance signifies the most unfavorable vibration scenarios, which should be avoided in bridge design. Conversely, cancellation symbolizes the most desirable vibration conditions that bridge designers endeavor to attain. Along the lines of research on resonance and cancellation, the SS boundary has also been employed for the beam for the easiness in theoretical formulations. For instance, Frýba [1] demonstrated that resonance occurs when the time interval between successive load passages matches one of the natural periods of the beam. Yang et al. [5] firstly derived the resonance and cancellation conditions of SS beams traveled by high-speed trains. Subsequently, Li and Su [25] theoretically analyzed the VBI effect on resonant speeds, and Ju and Lin [26] numerically studied the high-speed train-induced resonant characteristics of bridges. Xia et al. [27] explored the resonance mechanism of railway train-bridge systems both numerically and experimentally. Museros et al. [28] studied the maximum resonance, cancellation and resonant vertical acceleration of simply supported bridges under moving loads. Kumar et al. [29] conducted a detailed investigation into the cancellation phenomenon for SS beams subjected to a single moving load, with the damping effect included. Li et al. [30] investigated the cancellation speeds for simple beams utilizing a moving load amplitude spectrum in frequency domain. Recently, Wang et al. [31] conducted an extensive analysis of the cancellation conditions for SS beams in time domain, categorizing them as the *external* (load-related) cancellation and *internal* (structure-related) cancellation. Such an idea will be revisited and further explored for the elastically supported (ES) beams to optimize the support stiffness in the present study.

In particular, the concept of cancellation and resonance has been widely applied to minimizing the bridge vibration. Yang et al. [5] were the pioneers in proposing the bridge design criteria that the bridge span length should be 1.5 times the car length for minimization of bridge vibrations. The effectiveness of the *1.5-times rule* for the ICE railway lines in Europe has been affirmed by Museros and Alarcón [32]. They reported a significant reduction in resonance for 40 m bridges, where the bridge/coach length ratio (L/D = 1.52) closely aligns with Yang's 1.5-times rule. According to Cho et al. [33], the 1.5-times rule for span length has also been applied to the Korea HSR bridges for vibration reduction. To break the mechanism for resonance generation, Shin et al. [34] proposed a vibration reduction scheme by inserting size-adjusted vehicle(s) into the existing train arrangement to generate out-of-phase loading, thereby suppressing the resonance phenomenon. Inspired by the cumulative amplification of resonance, Matsuoka [35] introduced a technique for detecting the resonance in bridges by utilizing the amplified variations in track irregularities measured between the first and last vehicles. Recently, a historical review on the resonance and cancellation of SS beams subjected to moving train loads was conducted by Yang et al. [4], providing a complete coverage of the related theories and applications mainly based on the 1.5-times rule.

In practical design, elastic bearings are often inserted at the beam ends to meet the needs such as seismic isolation, thermal expansion, etc. [36,37]. This suggests that the use of elastically supported (ES) beams is a closer approximation to the real bridges, compared with the SS beams. Previously, numerous researches have been conducted on the effect of elastic supports on the dynamic response of the VBI system. It was found that elastic supports can amplify the time-history vibrations, while decreasing the natural frequencies of the beams [38-43]. The phenomena of resonance and cancellation in ES beams have also been investigated by researchers. Yau et al. [38,44] proposed the use of a combined flexural sine mode (for beam) and rigid displacement mode (for bearings) to approximate the mode shape of the ES beam. Such an engineering approach facilities the derivation of the resonance and cancellation conditions for the ES beam. The same mode shape was adopted by Museros et al. [28] in investigating the resonant acceleration response of ES beams. In the study by Ma et al. [45], the dynamic impact factor was utilized to estimate the resonance amplification of ES beams.

The literature reviewed above highlights the significance of elastic supports on beam's vibration. Although limited in number, some studies have also identified the resonance and cancellation conditions for ES beams. To the best knowledge of the authors, however, the cancellation conditions for ES beams have *not* been fully explored with respect to the stiffness of elastic bearings. In most cases, bridge designers consider mainly the bridge structure itself, but not the elastic supports, when coming to the problem of vibration reduction. Whenever the elastic support is of concern, attention was paid mainly the seismic or vibration isolation of the beam structure [46-48], with basically no attention paid to the role of elastic supports in mitigating train-induced vibrations on the beam. Moreover, the 1.5-times rule proposed by Yang et al. [5] for optimal design of bridges works mainly for the simplified SS beams, but not for the more realistic ES beams.

To fill the above gap, this paper is aimed at unveiling the resonance and cancellation mechanism for ES beams under successive moving loads. Specifically, focus is placed on the *supplementation* of the 1.5-times rule to ES beams for optimal design of practical bridges. This paper is organized as follows: In Section 2, concise analytical solutions for free vibration of ES beam are derived in terms of the modal amplitude R. In Section 3, both the resonance and cancellation conditions are identified for ES beams, with emphasis on the cancellations associated with elastic supports. In Section 4, a criterion for selecting the bridge *support stiffness ratio* (SSR) is proposed for suppressing the resonance of short to medium-span railway bridges. Sections 5 to 7 present numerical validation and parametric analysis related to the cancellation conditions and the SSR selection criteria. Conclusions are given in Section 8.

2. Theoretical Formulation

The objective of this paper is to investigate the effect of *elastic supports* on the dynamic behavior of railway bridges (which often appear in a series of identical spans) under the moving train loads. Focus will be placed on derivation of the resonance and cancellation conditions and the optimal design criteria for railway bridges with elastic supports. The following are the assumptions adopted [44]:

(1) The railway bridge is modeled as an elastically supported (ES) beam, i.e., a flexible beam supported by elastic springs of equal stiffness at both ends.

(2) The vibration shape of the beam is approximated by the combination of a *flexural sine mode* of the corresponding simply supported (SS) beam and a *rigid* displacement mode by the elastic supports, as shown in Fig. 1.

(3) Due to the transient nature of the moving loads, focus is placed solely on the first mode of vibration, disregarding the damping effect of the beam.

(4) The train traveling over the bridge is modeled by a sequence of N concentrated loads of magnitude p spaced at interval d and moving at constant speed v. The load interval d may also be interpreted as the interval between two front (or rear) bogies of two consecutive vehicles, or the vehicle length [5].

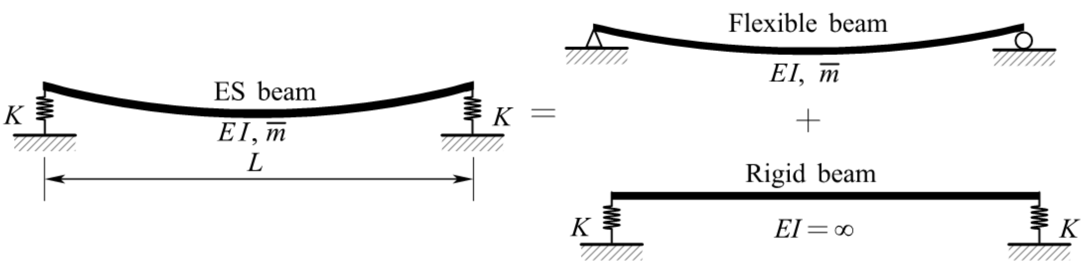


Fig. 1. Elastically supported (ES) beam modeled as the superposition of an elastic beam with pinned supports and rigid beam with elastic supports.

2.1 Assumed modal shape of vibration

Based on the aforementioned assumptions, the vibration shape of the ES beam can be expressed as follows [44]:

$$\phi(x) \approx \sin\frac{\pi x}{L} + \kappa, \tag{1}$$

where $\kappa = EI\pi^3/(L^3K)$ denotes the ratio of the flexural rigidity EI of the beam to the stiffness K of the elastic support, referred to as the *support stiffness ratio* (SSR), with L denoting the bridge span length. Here a higher κ means a softer elastic support, and $\kappa = 0$ represents the case of simply-supported (SS) beam. Previously, it has been confirmed that the mode shape given in Eq. (1) closely resemble the exact theoretical mode shape presented in Refs. [38,41,49] under various SSRs. The resemblance is also true for the modal frequencies, as will be demonstrated in Section 2.2.

For an elastic beam considering only the first mode of vibration, the vertical displacement $u(x,t)$ of the beam can be approximated as

$$u(x,t) \approx \phi(x)q(t) = q(t) \times \left(\sin\frac{\pi x}{L} + \kappa\right), \tag{2}$$

where $q(t)$ represents the modal coordinate of the first mode, the primary factor influencing the vibration amplitude of the ES beam.

2.2 ES beam subjected to successive loads

As shown in Fig. 2, the ES beam is subjected to a series of N moving loads of magnitude p spaced at interval d and moving at constant speed v.

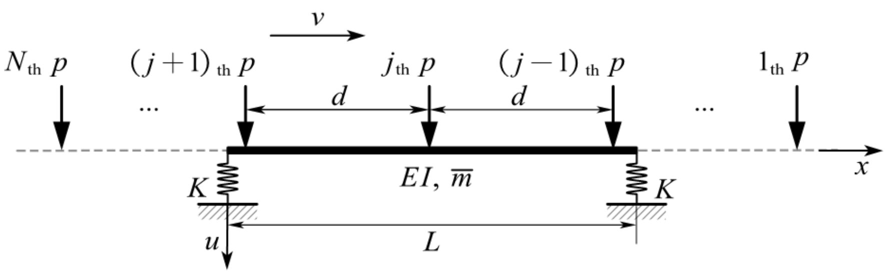


Fig. 2. ES beam under successive equidistant moving loads.

The equation of motion for the vertical vibration of the ES beam is

$$\bar{m}\ddot{u} + c\dot{u} + EIu'''' = p\sum_{j=1}^{N} \delta\left[x - v\left(t - t_j\right)\right] \times \left[H\left(t - t_j\right) - H\left(t - t_j - \Delta_{\mathrm{T}}\right)\right], \tag{3}$$

where for the bridge, $\bar{m}$ is the mass per unit length, c the damping coefficient, E the elastic modulus, and I the moment of inertia; $(\ )' = \frac{\mathrm{d}(\ )}{\mathrm{d}x}$, $(\dot{\ }) = \frac{\mathrm{d}(\ )}{\mathrm{d}t}$, $\delta(x)$ the Dirac's delta function, $H(t)$ the unit step function, $t_j = (j-1)d/v$ the time for the j_{th} load to enter the beam, and $\Delta_T = L/v$ the travel time for each load over the beam.

By substituting Eq. (2) into Eq. (3), multiplying both sides by $\phi(x)$, integrating over the length L, one can obtain from Eq. (3) the modal coordinate $q(t)$, as

$$\ddot{q}(t) + 2\xi\omega\dot{q}(t) + \omega^2 q(t) = \frac{1}{\bar{m}\chi} p\sum_{j=1}^{N} \delta\left[x - v\left(t - t_j\right)\right] \times \left[H\left(t - t_j\right) - H\left(t - t_j - \Delta_{\mathrm{T}}\right)\right]\phi(vt), \tag{4}$$

in which $H\left(t - t_j\right) - H\left(t - t_j - \Delta t\right)$ represents the action of the jth moving load on the bridge, and ω is the first natural frequency of the ES beam, given by

$$\omega^2 = \epsilon\omega_0^2, \tag{5}$$

Here $\omega_0 = \sqrt{\pi^4 EI/(L^4 \bar{m})}$ is the natural frequency of the SS beam, and ϵ the contribution factor of the elastic support to the bridge frequency, expressed by

$$\epsilon = \frac{1}{1 + \frac{4\kappa + 2\pi\kappa^2}{\pi + 4\kappa}}. \tag{6}$$

Clearly, the contribution factor ϵ is less than 1 since κ is always positive, namely, the elastic supports will reduce the beam frequency compared with the SS beams [38,44].

For zero initial boundary conditions, the modal coordinate $q(t)$ can be solved from Eq. (4) as

$$q(t) = \Delta_{st}[Q_1(t) + Q_2(t)] \tag{7}$$

where $\Delta_{st} = P/\bar{m}L\omega^2$ represents the static deflection of the beam; Q_1 denotes the flexural vibration contributed by the SS beam, and Q_2 the rigid displacement by the elastic supports, respectively, as

$$Q_1(t) = \frac{1}{1 - S^2} \sum_{j=1}^{N} \left\{ \begin{array}{c} [\sin \Omega(t - t_j) - S\mathrm{e}^{-\xi\omega(t-t_j)} \sin \omega(t - t_j)]\mathrm{H}(t - t_j) + \\ [\sin \Omega(t - t_j - \Delta_T) - S\mathrm{e}^{-\xi\omega(t-t_j-\Delta_T)} \sin \omega(t - t_j - \Delta_T)] \\ \mathrm{H}(t - t_j - \Delta_T) \end{array} \right\}, \tag{8a}$$

$$Q_2(t) = \kappa \sum_{j=1}^{N} \left\{ \begin{array}{c} [1 - \mathrm{e}^{-\xi\omega(t-t_j)} \cos \omega(t - t_j)]\mathrm{H}(t - t_j) + \\ [1 - \mathrm{e}^{-\xi\omega(t-t_j-\Delta_T)} \cos \omega(t - t_j - \Delta_T)] \\ \mathrm{H}(t - t_j - \Delta_T) \end{array} \right\}, \tag{8b}$$

where $\Omega = \frac{\pi v}{L}$ denotes the *driving frequency*, and $S = \frac{\pi v}{L\omega}$ the *speed parameter*.

2.3 Concise analytical solution for free vibration of ES beam

For railways supported by a series of bridges, it is common to maintain a vehicle length d that does not exceed the span length L of the beam, i.e., $d < L$. This ensures continuous loading on the bridge from the moment the first load enters the bridge until the departure of the last load. Prior to departure of all loads from the bridge ($t \leq t_N + \Delta_T$), the beam undergoes *forced vibration*. After this ($t > t_N + \Delta_T$), the beam transits into *free vibration*. This paper focuses mainly on the free vibration since it may be accumulated by the passing loads and create the phenomenon of *resonance*. Herein, the damping of the beam is temporarily neglected, which will be studied in Section 5.3.

Substituting Eq. (8) into Eq. (7) and letting $t > t_N + \Delta_T$ yields the generalized coordinate $q(t)$ for free vibration as:

$$q(t) = \frac{-S\Delta_{st}}{1-S^2}\sum_{j=1}^{N}\left\{\begin{matrix}\left(\sin\omega t\cos\omega t_j - \cos\omega t\sin\omega t_j\right) + \\ cos\,\omega t \times \left[\begin{matrix}\left(-\cos\omega t_j\sin\omega\Delta_t - \cos\omega\Delta_t\sin\omega t_j\right) + \\ \sin\omega t\ \left(\cos\omega\Delta_t\cos\omega t_j - \sin\omega\Delta_t\sin\omega t_j\right)\end{matrix}\right]\end{matrix}\right\}$$

$$-\Delta_{st}\kappa\sum_{j=1}^{N}\left\{\begin{matrix}\left(\cos\omega t\cos\omega t_j + \sin\omega t\sin\omega t_j\right) \\ -\left[\begin{matrix}\cos\omega t\left(\cos\omega\Delta_t\cos\omega t_j - \sin\omega\Delta_t\sin\omega t_j\right) \\ +\sin\omega t\left(\cos\omega t_j\sin\omega\Delta_t + \cos\omega\Delta_t\sin\omega t_j\right)\end{matrix}\right]\end{matrix}\right\}. \quad (9)$$

In order to obtain a concise analytical solution that eliminates the summation sign Σ contributed by the N moving loads, the following two parameters are introduced:

$$\vartheta_1 = \sum_{j=1}^{N}\sin\omega t_j = -\frac{\sin\frac{\omega d}{v} + \sin\frac{(N-1)\omega d}{v} - \sin\frac{N\omega d}{v}}{2\left(\cos\frac{\omega d}{v} - 1\right)} \quad (10a)$$

$$\vartheta_2 = \sum_{k=1}^{N}\cos\omega t_j = 1 - \frac{\cos\frac{\omega d}{v} - 1 + \cos\frac{(N-1)\omega d}{v} - \cos\frac{N\omega d}{v}}{2\left(\cos\frac{\omega d}{v} - 1\right)}. \quad (10b)$$

Then, by substituting Eq. (10) into Eq. (9) and performing some mathematical operations, one obtains the following concise analytical expression:

$$q(t) = \Delta_{st}R\sin(\omega t + \psi), \qquad (t > t_N + \Delta_T), \quad (11)$$

in which the *modal amplitude* R is

$$R = \sqrt{\frac{(\sin\frac{N\pi d}{2SL})^2}{(\sin\frac{\pi d}{2SL})^2}}\sqrt{\frac{2S^2}{(1-S^2)^2}\left(1+\cos\frac{\pi}{S}\right) - \kappa\left[\frac{4S}{1-S^2}\sin\frac{\pi}{S} - 2\kappa\left(1-\cos\frac{\pi}{S}\right)\right]}, \quad (12)$$

and the phase angle ψ of the free vibration response is

$$\psi = \tan^{-1}\frac{\left\{\begin{matrix}\frac{S}{1-S^2}\left[\vartheta_1 + (\vartheta_1\cos\omega\Delta_t + \vartheta_2\sin\omega\Delta_t)\right] - \\ \kappa\left[\vartheta_2 - (\vartheta_2\cos\omega\Delta_t - \vartheta_1\sin\omega\Delta_t)\right]\end{matrix}\right\}}{\left\{\begin{matrix}\frac{S}{1-S^2}\left[-\vartheta_2 + (\vartheta_1\sin\omega\Delta_t - \vartheta_2\cos\omega\Delta_t)\right] - \\ \kappa\left[\vartheta_1 - (\vartheta_2\sin\omega\Delta_t + \vartheta_1\cos\omega\Delta_t)\right]\end{matrix}\right\}} \quad (13)$$

As evident from Eq. (12), the modal amplitude R is comprised of two terms each enclosed by a square root. Aside from the speed parameter S and beam length L, the *first term* within the first square root encompasses parameters related to the external loads' quantity N and interval d, that indicates the influence of external moving loads on the modal amplitude, herein called the *external* (load-related) *term*. Particularly, the *vehicle/bridge length ratio* d/L between the load interval d and beam length L is crucial to the present investigation.

The *second term* within the second square root in Eq. (12) pertains exclusively to the SSR (support stiffness ratio) κ, defined as $\kappa = EI\pi^3/(L^3K)$, which represents an inherent characteristic of the bridge structure. This term is therefore referred to as the *internal* (structure-related) *term*. The internal term will also be referred to as the *cancellation function C*, which will be elucidated in detail in Sections 3.2 and 4.

The cancellation function C can be broken down into two distinct parts. The first part, denoted as I_b, that excludes the SSR (κ), indicates the contribution of the elastic beam and is expressed as

$$I_b = \frac{2S^2}{(1-S^2)^2}\left(1+\cos\frac{\pi}{S}\right), \tag{14}$$

where the subscript '*b*' signifies *beam*. Certainly, I_b is always greater than zero for $S < 1$, indicating the beam's magnifying effect on the overall beam vibration. The second part, denoted as I_s, that includes the SSR (κ), represents the contribution of the elastic supports and is given by

$$I_s = -\kappa\left[\frac{4S}{1-S^2}\sin\frac{\pi}{S} - 2\kappa\left(1-\cos\frac{\pi}{S}\right)\right], \tag{15}$$

where the subscript 's' signifies the elastic *support*. As was shown above, the cancellation function C can be divided into two parts as $C = I_b + I_s$, with I_b in Eq. (14) denoting the contribution of the elastic beam, and I_s in Eq. (15) the effect of the elastic supports.

For the specific case where $\kappa = 0$, I_s reduces to zero. Consequently, the ES beam reduce to the SS beam with the following modal amplitude:

$$R = \sqrt{(\sin\frac{N\pi d}{2SL})^2 \Big/ (\sin\frac{\pi d}{2SL})^2}\sqrt{I_b}, \tag{16}$$

identical to the one given in Ref. [31].

It is important to note that the parameter R in Eq. (12) plays a crucial role in determining the resonance and cancellation conditions, and further the optimal criterion of railway bridges.

Then, by substituting Eq. (11) into Eq. (2), one can derive the analytical solution for the vertical displacement of ES beam in free vibration as follows:

$$u(x,t) = \Delta_{st}R\left(\sin\frac{\pi x}{L} + \kappa\right)\sin(\omega t + \psi), \tag{17}$$

which can be twice differentiated to yield the acceleration as

$$\ddot{u}(x,t) = -\Delta_{st} R \omega^2 \left(\sin\frac{\pi x}{L} + \kappa \right) \sin(\omega t + \psi) . \tag{18}$$

3. Resonance and Cancellation Conditions

Concerning the train-induced bridge vibrations, the resonance and cancellation phenomena have garnered significant attention. Resonance poses a continuous threat to the operational functionality and durability of train-bridge coupling systems, whereas cancellation is regarded as a beneficial phenomenon. In this section, we will delve into the cancellation and resonance conditions of railway bridges, focusing on the effect of elastic supports.

3.1 Resonance condition

The resonance condition occurs when the external (first) term in Eq. (12) reaches its maximum (local) when the denominator $\sin \pi d/(2SL)$ equals zero, i.e., when $\pi d/(2SL) = 2n\pi$, where $n \in Z^+$, or in terms of the speed parameter S as

$$S_r = \frac{1}{2n}\frac{d}{L}, \quad (\mathrm{n} \in \mathrm{Z}^+), \tag{19}$$

in which the subscript 'r' signifies *resonance*. The resonance speed parameter S_r given above for the ES beam is consistent with that of the SS beam, as documented in Ref. [5]. The preceding expression underscores the fact that S_r depends solely upon the load spacing (*d*) and beam length (*L*), irrespective of the elastic supports, as has been demonstrated in previous works [4,5,28,31,44,50] In short, the resonance speed parameter S_r in Eq. (19) represents a universal resonance condition applicable to beams of length *L* when exposed to a series of moving loads with a spacing of *d*.

Furthermore, when the resonance condition in Eq. (19) is met, the fraction for the external term in Eq (12) reaches an indeterminate form of $0/0$. This can be resolved by employing the L'Hospital's rule, namely,

$$\lim_{\sin\frac{Nd\pi}{2L\,S}\to 0} \frac{\sin\dfrac{Nd}{2L}\dfrac{\pi}{S}}{\sin\dfrac{d}{2L}\dfrac{\pi}{S}} = N. \tag{20}$$

This condition implies that under the resonance condition, the modal amplitude *R* is positively related to the load quantity *N*. In other words, the resonant response of the bridge will be built up as there are more loads passing the bridge.

It should be noted that the cancellation function *C* (i.e., internal term) within the second square root in Eq. (12) does not imply any possibility of resonance, since it grows continuously as the speed parameter S increases from 0 to 1 for the typical range of high-speed railways. Some fluctuation in C may indeed arise due to the trigonometric terms involved; however, such fluctuations are quite minor and will not affect the global result.

3.2 Cancellation condition

Unlike resonance, cancellation is a favorable phenomenon since it may help in suppressing oscillations under certain conditions, which directly relates to the optimal design of railway bridges. As revealed by Eq. (12), the modal amplitude R is the product of the external term and internal term (or cancellation function C). To achieve the condition of cancellation with $R = 0$, one may set either the external or internal term equal to 0. Correspondingly, there are two types of cancellation conditions for the ES beam, i.e., external and internal cancellations.

3.2.1 External cancellation

The external cancellation may occur by setting the first square root of Eq. (12) equal to 0, i.e., $\sin N\pi d/(2SL) = 0$ or $N\pi d/(2SL) = 2n\pi$, where $n \in Z^{+}$. It can be expressed in term of the speed parameter S as

$$S_c^E = \frac{Nd}{2nL}, \qquad (N \geq 2 \,|\, n \neq N * i \,|\, n, i \in Z^{+}), \tag{21}$$

where the subscript 'c' signifies *cancellation*, and the superscript 'E' *external*. For the condition to be meaningful, the load quantity N should have a minimum of 2. Otherwise, the external term, i.e., the first term within the first square root in Eq. (12), will always be greater than unity, which does not lead to cancellation. Moreover, the integer n must not be a multiple of the load quantity N (e.g., $n \neq N, 2N, 3N$ ...), in order not to make the denominator equal to zero, which actually means the condition of resonance.

Aside from the beam length L, the external cancellation condition S_c^E relates only to the external load properties, i.e., spacing d and quantity N, and is irrespective of the elastic supports. Namely, the external cancellation condition is valid for SS or ES beams of length L under the moving loads with spacing d and quantity N. To the authors' knowledge, the external cancellation condition S_c^E has *never* been studied in the literature. All the cancellation conditions studied previously relate to the internal cancellation of the SS beam, a special case of the ES beam to be discussed below.

3.2.2 Internal cancellation

By letting the term inside the second square root of Eq. (12) equal to 0, one can express the *cancellation function C* for the ES beam as follows:

$$C = \frac{2{S_c^I}^2}{\left(1-{S_c^I}^2\right)^2}\left(1+\cos\frac{\pi}{S_c^I}\right) - \kappa\left[\frac{4S_c^I\kappa}{1-{S_c^I}^2}\sin\frac{\pi}{S_c^I} - 2\kappa\left(1-\cos\frac{\pi}{S_c^I}\right)\right] = 0, \qquad (22)$$

in which the superscript I denotes '*internal'*. Since Eq. (22) is an implicit function of the speed parameter S_c^I, to obtain a concise analytical solution for the internal cancellation condition can be challenging; however, to find a numerical solution instead is relatively easy. On the other hand, one observes that the cancellation function *C* in Eq. (22) is linked to the SSR, $\kappa = EI\pi^3/(L^3K)$, this indicates the need to include the support stiffness K in the design of railway bridges, especially when it comes to cancellation design.

For the special case when κ equals 0, the internal cancellation condition in Eq. (22) reduces to the following for the SS beam:

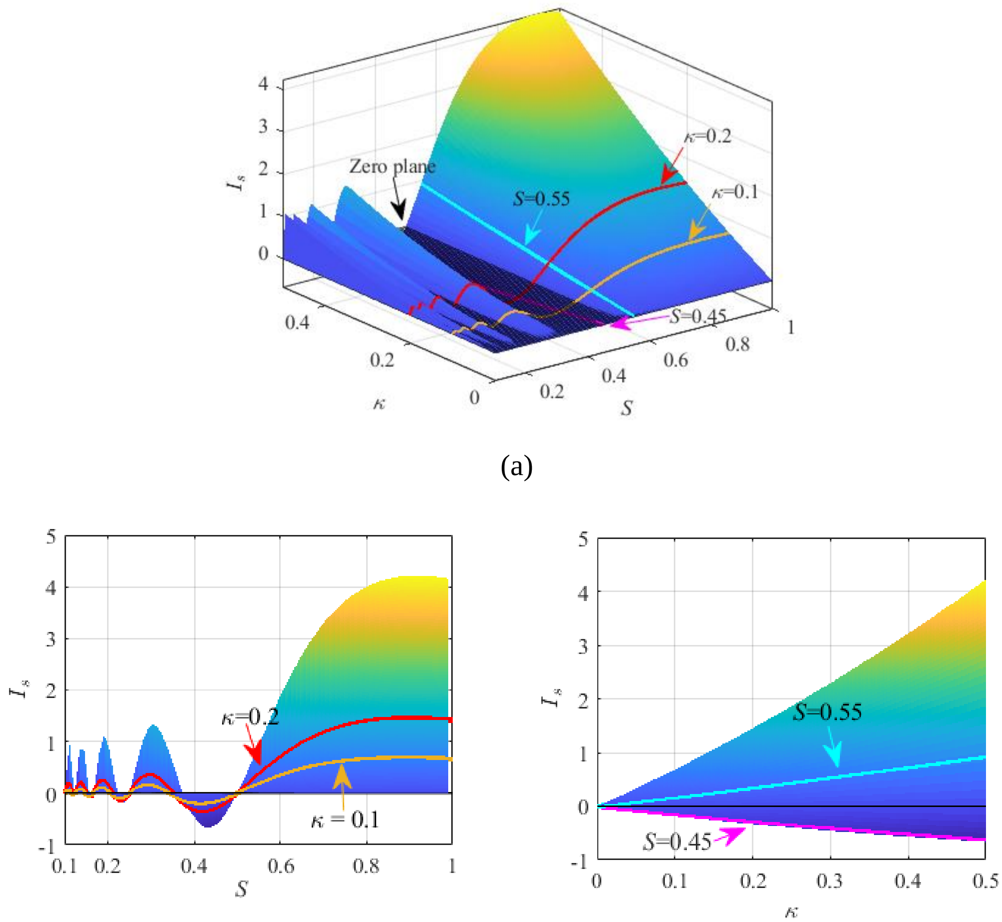
$$S_c^I = \frac{1}{2j-1}. \quad j = \in \mathbb{Z}^+ \qquad (23)$$

Previously, the cancellation condition presented in Eq. (23) was noted as an optimal condition of railway bridges. In reality, it is difficult to use, since the speed parameter *S* involves both the load speed *v* and bridge frequency ω. Besides, it failed to consider the support stiffness, which does exist in reality. The main focus of this study is to address this limitation and offer a more precise approach to optimizing the design of railway bridges, as will be elaborated in the following section.

4. Criterion for Selecting Support Stiffness Ratio (SSR)

In practice, achieving precise railway bridge design requires not only the proper selection of the beam structure, but also the elastic supports, i.e., finding the optimal solution for SSR (κ) under favorable conditions. Prior to this, it is necessary to revisit the cancellation function *C* (internal term) to elucidate the effect of elastic supports on bridge vibration.

As was mentioned, the cancellation function *C* can be divided into two parts: $C = I_b + I_s$, with I_b for the elastic beam and I_s for elastic supports. To unveil the effect of elastic supports on the vibration of the beam, a three-dimensional (3D) graph showing the variation of I_s with respect to κ and S has been given in Fig. 3(a), accompanied by a two-dimensional (2D) profile for $\kappa = 0.1$ and 0.2 in Fig. 3(b), and another one for $S = 0.45$ and 0.55 in Fig. 3(c). As can be seen from Figs. 3(a) and (b), for a constant κ, I_s exhibits oscillatory behavior around zero versus the speed parameter S. This suggests the dynamic effect of the elastic supports on the vibration of the beam, which can either enhance or suppress the vibration depending on the speed parameter S.



(a)

(b) (c)

Fig. 3. Variations in I_s with respect to κ and S: (a) $I_s - S - \kappa$: 3D graph; (b) $I_s - S$: 2D graph for $\kappa = 0.1$ and 0.2; (c) $I_s - \kappa$: 2D graph for $S = 0.45$ and 0.55.

In contrast, for a fixed speed parameter S as in Fig. 3(c), I_s shows a monotonic variation with respect to the SSR (κ). Particularly, for $S = 0.55$, increasing the κ value can amplify the beam's vibration, and the magnification effect intensifies with higher κ values. Such a phenomenon is undesirable in bridge design. Conversely, for $S = 0.45$, an increase in the κ value leads to a reduction in the beam's vibration, and the suppression effect becomes more pronounced for larger κ values. Such a phenomenon is desirable in the cancellation design of bridges. The above analysis underscores the significance of selecting appropriate elastic supports in the vibration mitigation of railway bridges.

4.1 Perfect cancellation

As was said previously, I_b is always greater than zero, while I_s exhibits oscillatory behavior around zero. With this, *perfect cancellation* can be achieved by letting the cancellation function C equal to zero, i.e., $C = I_b + I_s = 0$, which can be met by adjusting the I_s value that is a function of κ. On the other hand, as revealed by Figs. 3(a) and (b), I_s also varies with the speed parameter S. Thus, it is difficult to find a suitable κ for the perfect cancellation that also works for various speed parameters.

To tackle the above problem, the concept of cancellation will be directly applied to the resonance condition, by utilizing the fact that the resonance condition S_r in Eq. (19) is solely determined by the bridge/vehicle length ratio L/d, which is a constant and unaffected by the support stiffness. Then, by *replacing* the cancellation parameter S_c^I in Eq. (22) by the resonance parameter S_r, one can rewrite the cancellation function $C(\kappa)$ in Eq. (22) in terms of the κ value as

$$C(\kappa) = \frac{2{S_r}^2}{\left(1-{S_r}^2\right)^2}\left(1+\cos\frac{\pi}{S_r}\right) - \frac{4S_r\kappa}{1-{S_r}^2}\sin\frac{\pi}{S_r} + 2\kappa^2\left(1-\cos\frac{\pi}{S_r}\right) = 0. \quad (24)$$

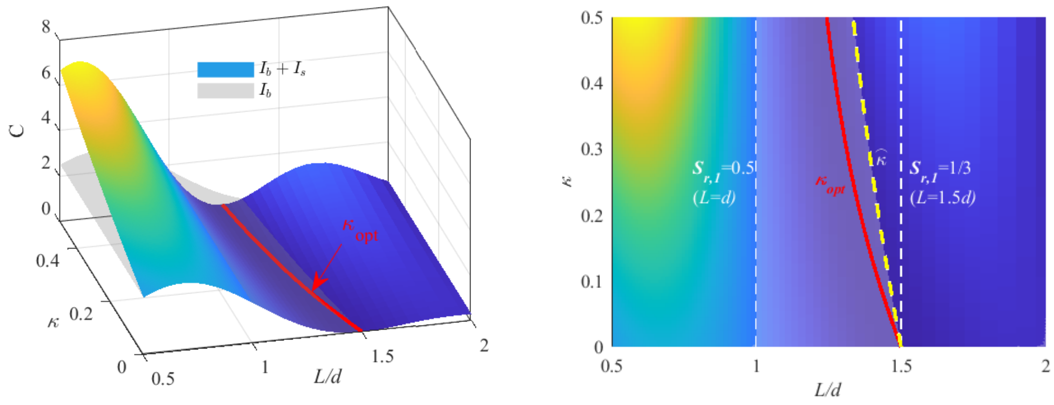
By so doing, the resonance (via S_r) has been *absorbed* by the cancellation condition. In the above enforcement, it is realized that the cancellation condition is *more decisive* than the resonance condition. This can be easily explained by the modal amplitude R in Eq. (12), in that *once the second square root is zero (cancellation), the first square root (resonance) becomes nothing*, no matter how large it is.

In practice, only the first resonance condition $S_{r,1}$ is of primary importance, since it is the one most likely to be encountered. By substituting Eq. (19) (with n = 1) into Eq. (24), one can solve from the quadratic equation $C(\kappa)$ for the optimal SSR (κ) as

$$\kappa_{opt} = \frac{\frac{2d}{L}\sin\frac{L}{d}2\pi}{\left(4-\frac{d^2}{L^2}\right)\left(1-\cos\frac{L}{d}2\pi\right)}, \quad (25)$$

where the subscript 'opt' signifies *optimal*. Of interest is that the optimal κ_{opt} used for cancellation of resonance in Eq. (25) is valid regardless of the speed parameter S, and depends solely on the bridge/vehicle length ratio L/d. Such a simple formula proves to be convenient for engineering applications. In high-speed railways, the most commonly used bridges are of small to medium spans, with length L ranging from 20 to 50 m. The load spacing or vehicle length d adopted is generally around 25 m. Hence, the range of bridge/vehicle length ratios, $L/d \in [0.5, 2]$, that encompasses the majority of railway systems will be adopted in the following analysis.

To visualize the effect of the optimal κ_{opt}, the cancellation function ($C = I_b + I_s$) versus the SSR (κ) and bridge/vehicle length ratio L/d were plotted in the 3D graph in Fig. 4(a), along with the elastic beam part I_b for reference. Figure 4(b) provides the top view of the 3D graph. The points that meet the condition of perfect cancellation, $I_b + I_s = 0$, have been marked in red, appearing as a line. These points represent exactly the optimal κ_{opt} given by Eq. (25).



(a) (b)

Fig. 4. Plots for the cancellation function ($C = I_b + I_s$) and the elastic beam part I_b versus SSR (κ) and length ratio L/d: (a) 3D plot; (b) Top view

On the other hand, the selection of SSR (κ) for the bridge requires a comprehensive consideration from the engineering points of view, as the selection of elastic supports is subject to constraints by factors such as material properties, seismic isolation and resistance, stability, and temperature, etc. As revealed by Fig. 4(a), the range of length ratios that satisfy the perfect cancellation condition is quite narrow, i.e., with $L/d \in [1.25, 1.50]$, which is generally restrictive for the selection of the SSR, $\kappa \in [0, 0.5]$. In contrast, for the remaining range, perfect cancellation cannot be achieved, since $C > 0$, which will be referred to as *normal cancellation*. That is to say, when the bridge does not meet the condition $L/d \in [1.25, 1.50]$, normal cancellation, rather than perfect cancellation can be achieved instead, even though it does not satisfy the condition $C = 0$, In such a circumstance, the κ value will deviate from the optimal one given by Eq.

(25). More discussion will be presented in the following for selection of the κ value in the normal cancellation range.

4.2 Normal cancellation

Based on the magnitude of the cancellation function C with respect to the part I_b of the elastic beam, as shown in Fig. 4(a), the range of the length ratio L/d can be categorized into three regions.

- Region I encompasses $L/d \in [0.5, 1]$ where $I_b + I_s > I_b$;
- Region II applies to $L/d \in (1, 1.5]$ where $I_b + I_s \leq I_b$ or $I_b + I_s > I_b$;
- Region III pertains to $L/d \in (1.5, 2]$ where $I_b + I_s > I_b$.

4.2.1 Region I: $L/d \in [0.5, 1]$

Referring to Fig. 4, it is evident that for beams falling within the range of length ratio $L/d \in [0.5, 1]$, the cancellation function $C = I_s + I_b$ is always larger than I_b for any value of κ, due to the fact that $I_s > 0$. This implies that the elastic support amplifies the beam's vibration, and the overall beam vibration increases as κ increase. For vibration reduction, the SSR (κ) within this region should be chosen to be as *small* as possible to minimize amplification of the elastic supports on the beam's vibration, meaning that the elastic supports should be designed as *rigid* as possible. In this regard, one should bear in mind that the selection of support stiffness must also meet other engineering concerns, such as the needs for isolation, stability, etc.

Of interest is that the cancellation function $C = I_s + I_b$ intersects with I_b for the line with $L = d$, as shown in Figs. 4(a) and (b), for which $I_s = 0$. That is, *when the beam length is equal to the vehicle length, for $L = d$, the elastic supports have minimal impact on the beam's vibration, irrespective of the value of κ.* This condition can be regarded as the *critical point* where the effect of elastic supports on beam vibrations shifts *from enhancing to suppressing*.

4.2.2 Region II: $L/d \in (1, 1.5]$

When the length ratio falls in $L/d \in (1, 1.5]$, as can be seen from Figs. 4(a) and (b), the cancellation function $C = I_s + I_b$ turns out to be smaller than I_b, implying that $I_s < 0$, namely, the elastic support suppresses the beam's vibration. Particularly, when the κ value and the length ratio L/d precisely satisfy the optimal condition in Eq. (25), the cancellation function $C = I_s + I_b$ will reach the *minimum* of zero value, shown as the red line in Fig. 4. Nonetheless, there also exists a *critical point* within this range where the effect of elastic supports on beam vibrations shifts *from suppressing to enhancing.* This critical point can be determined by substituting Eq. (19) into Eq. (15) and letting Eq. (15) equal to zero. The result is

$$\hat{\kappa} = \frac{4L/d}{4(L/d)^2 - 1} \tan\left(\frac{L}{d}\pi - \frac{2i-1}{2}\pi\right), \quad i = 1,2,3, \dots \tag{26}$$

which has been indicated by the dashed yellow line in Fig. 4(b). More precisely, when the SSR (κ) is related to the bridge/vehicle length ratio L/d by Eq. (26), the elastic support will have *no effect* on the beam's vibration ($I_s = 0$). When the SSR (κ) falls on the lefthand side of the line, it will exhibit a suppressive effect ($I_s < 0$), while it demonstrates an enhancing effect on the righthand side ($I_s > 0$). Clearly, *perfect cancellation* is achievable in Region II, where the cancellation condition $C = I_s + I_b = 0$ exists, as indicated by the red line in Fig. 4(b).

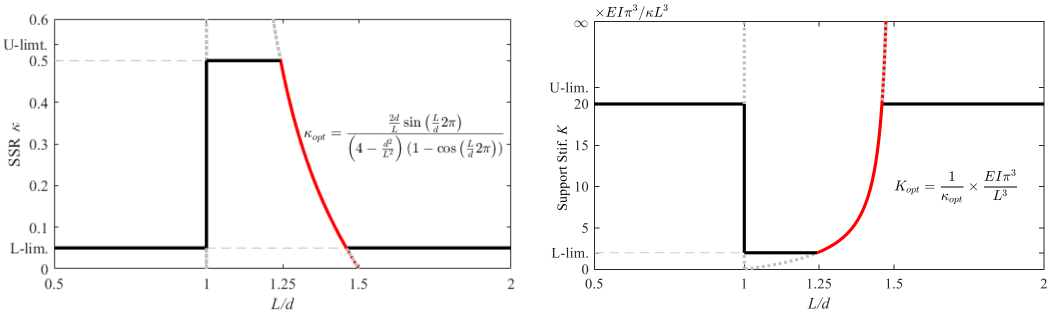
In particular, when the beam length is equal to 1.5 times the load spacing, i.e., $L/d = 1.5$, the optimal SSR, κ_{opt}, calculated by Eq. (25), will be exactly 0. This simply creates the *SS beam configuration* for achieving perfect cancellation. Such a criterion for selecting the span of a bridge has been recommended by Yang et al. [4] and adopted elsewhere [28,31-33,37,44,50], which is a classic criterion for railway bridge design. However, it is challenging to fully satisfy the pin-pin support condition in practical applications due to considerations such as seismic and vibration isolation. Therefore, it is suggested that when $L/d = 1.5$ is met, the SSR (κ) should be chosen as *small* as possible to minimize the amplification of elastic supports on the beam's vibration. In other words, the elastic supports should be designed as *rigid* as possible to meet practical needs, while striving to maintain the ideal criterion.

4.2.3 Region III: $L/d \in (1.5, 2]$

When the length ratio falls within the range $L/d \in (1.5, 2]$, as shown in Figs. 4(a) and (b), the cancellation function $C = I_s + I_b$ generally exceeds the value of I_b, due to the persistent positive value of I_s. In this region, the cancellation function C increases with the κ value. To minimize the effect of elastic supports on beam's vibrations, the SSR (κ) should be chosen as close as possible to the *lower bound* of the acceptable κ range, while meeting the requirements of engineering practice.

4.3 Overall criterion for selecting SSR

When aiming to minimize beam vibrations for short to medium-span railway bridges, the choice of the SSR (κ) or inversely the support stiffness K can be guided by the design criteria presented in Fig. 5. The following is the summary: (1) For beams of lengths falling within the ranges of $(0.5d, d)$ and $(1.5d, 2d)$, the elastic supports tend to amplify the beam vibration. Therefore, the SSR (κ) should be selected closer to the lower limit of the range to mitigate the amplification effect. Or inversely, the support stiffness K should be selected closer to the upper limit. (2) For beams of length $L = d$, the elastic supports have a minimal impact on beam's vibration, allowing the SSR (κ) to be selected based on practical considerations. (3) For beams of lengths between $(d, 1.5d)$, the elastic supports can have a suppressing effect on beam's vibration. There exists the possibility that the SSR (κ) be selected relative to the length ratio L/d to precisely satisfy the optimal condition in Eq. (25), resulting in minimal or zero beam vibration, as for perfect cancellation. Here, it is reiterated that the final choice of the support stiffness K should also conform to the requirements of other concerns in practical engineering.



(a) (b)

Fig. 5. Elastic support design criteria for beams of length L=0.5d ~ 2d: (a) SSR κ; (b) Support stiffness K.

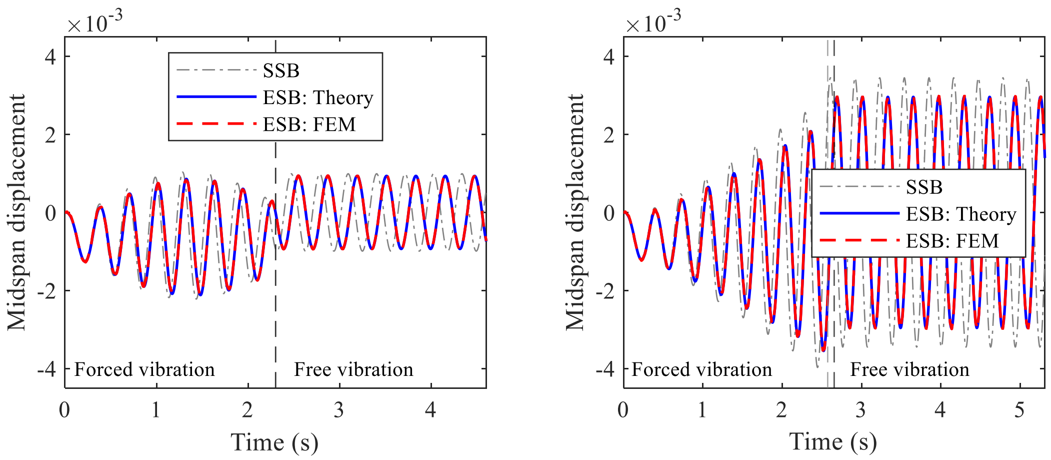
5. Validation of Analytical Solutions for Free Vibration of ES Beam

As indicated above, the concise analytical solution outlined in Eq. (12), which is newly introduced herein, is pivotal for establishing the resonance and cancellation conditions, as well as the criteria for selecting the SSR (κ). It is imperative to first evaluate the accuracy of this expression before delving into discussions on related points.

The properties close to those used in engineering practice are adopted for the beam: length $L = 32$ m, elastic modulus $E = 2.75\times10^{10}$ N/m^2, moment of inertia $I = 0.4$ m^4, per-unit-length mass $m = 2{,}500$ kg/m. A total of 8 loads is considered, each with weight of $P = 12{,}000$ kN, evenly distributed at the interval $d = 25$ m. The stiffness K of the elastic support is set to 2.08×10^8 N/m, equivalent to an SSR value of $\kappa = 0.05$. As indicated by Eqs. (5) and (6), the frequency of the ES beam is lower than that of the SS type, which currently is 3.12 Hz for the ES beam and 3.22 Hz for the SS type. Consequently, for the ES and SS beams to be subjected to the same speed parameter S, the train speed will exhibit slight variations, and the time required for the train to cross the two beams will also differ accordingly.

To assess the validity of the simple solution in Eq. (11), the mid-span displacements of the ES beam (with $\kappa = 0.05$) subjected to 8 constant loads moving at various speeds were plotted in Fig. 6. In the figure, the theoretical results for the ES beam are represented by the blue line, which are calculated by substituting Eq. (11) into Eq. (2) and letting $x = L/2$, while those of the finite element method (FEM) [41] by the red dashed line. For the sake of comparison, the results for the SS beam (where $\kappa = 0$) are also included in Fig. 6. Herein, only the free vibration is compared and analyzed since it is related to the cancellation phenomenon of concern. For all the speeds considered in Fig. 6, it is evident that the theoretical solution agrees excellently with the FEM result. This indicates clearly the suitability and high accuracy of the current solution for describing the free vibration of ES beams under successive moving loads.

For the *normal* speed parameter of $S = 0.42$ presented in Fig. 6(a), the loads travel on the SS and ES beams at the speeds of 87.23 and 84.54 m/s, respectively, the free vibration for both beams remain stationary with basically no phenomena of cancellation or resonance.



(a) (b)

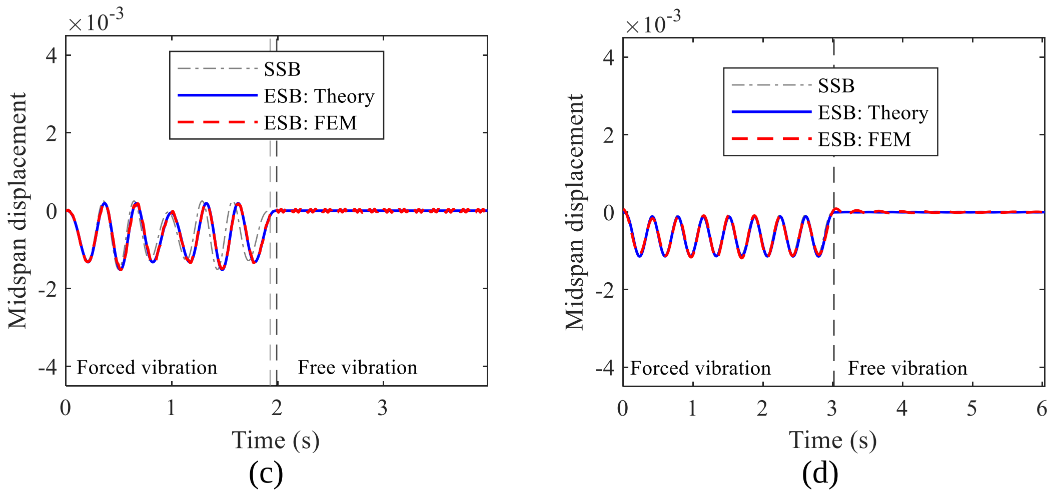


Fig. 6. Midspan responses of beams traversed by 8 loads at different speeds: (a) Normal speed (S = 0.42, for SS beam v = 87.23 m/s, for ES beam v = 84.54 m/s); (b) Resonance speed ($S_{r,1}$ = 0.39, for SS beam v = 80.44 m/s, for ES beam v = 77.97 m/s); (c) External cancellation speed (S_c^E = 0.52, for SS beam v = 107.25 m/s, for ES beam v = 103.95 m/s); (d) Internal cancellation speed (for SS beam: S_c^I = 0.33, v = 68.64 m/s; for ES beam: S = 0.3423, v = 68.32 m/s).

For the *resonant* speed parameter S_r= 0.39 presented in Fig. 6(b), the loads travel over the SS and ES beams at the speeds of 80.44 and 77.97 m/s, respectively. The forced vibration on both beams amplifies considerably due to the continuous action of the moving loads, resulting in the free vibration with an amplitude nearly three times greater than that for the normal speed condition. Moreover, the mid-span displacement of the ES beam is smaller than that of the SS beam. This is due to the suppression effect of the elastic supports on the beam's vibration, under the condition (L/d = 1.28 and κ = 0.05), as illustrated in Fig. 4.

For the loads moving at the *external cancellation* speed S_c^E= 0.52 (107.25 m/s for the SS beam and 103.95 m/s for the ES beam), as presented in Fig. 6(c), the free vibration of the ES beam vanishes entirely.

For the loads moving at the *internal cancellation* speed S_c^I = 0.33 and v = 68.64 m/s the SS beam, S_c^I = 0.34 and v = 68.32 m/s for the ES beam, as calculated by Eqs. (22) and (23), the results have been plotted in Fig. 6(d). It is observed that the free vibration also vanishes, same as the one for the *external cancellation* in Fig. 6(c). However, it should be noted that the internal cancellation condition for the ES beam differs from that of the SS beam due to inclusion of the elastic support effect.

It should be noted that the FEM results in Figs. 6(c) and (d) for the cancellation conditions do not show exactly zero free vibrations, but with some minor fluctuations. This is due to the fact that the cancellation conditions derived theoretically were based merely for the first vibration mode of the beam, which are approximate in nature. However, the FEM results have been obtained with virtually no assumption, representing the near-exact results for multi vibration modes. In spite of this, the theoretical results obtained demonstrate a high level of accuracy.

6. Factors Affecting Cancellation Conditions

As mentioned in Section 3.2, the external cancellation condition of the ES beam is completely related to the external moving load, while the internal cancellation condition is affected by both the speed parameter S and the SSR κ. In this section, a numerical study of factors affecting the cancellation conditions, both internal and external, will be performed to evaluate this effect.

6.1 Number of moving loads

To assess the effect of the number of moving loads N on the cancellation conditions, the modal amplitude R of the ES beam, subjected to either 1 or 8 loads, was calculated using Eq. (12) and presented for comparison in Fig. 7. Besides, the speed parameters for both the internal and external cancellations were also indicated in Fig. 7 and listed in Table 1.

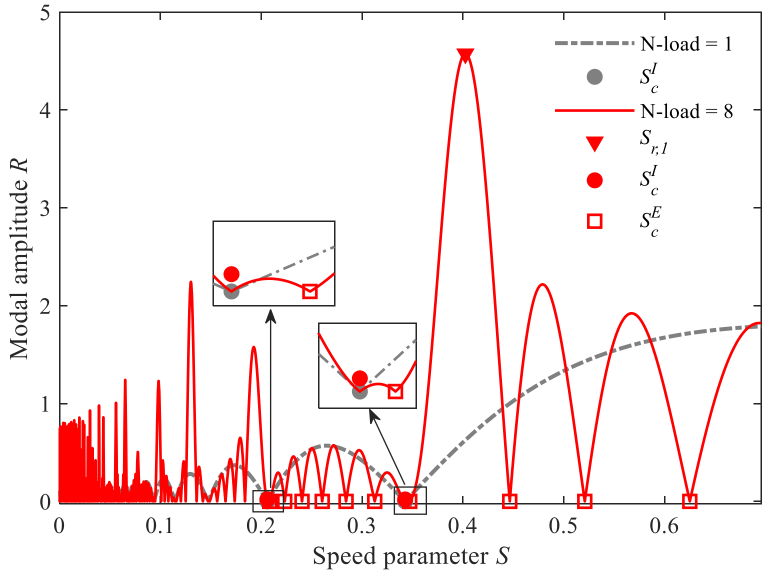


Fig. 7. Modal amplitude R of free vibration for ES beam with SSR κ = 0.05 traversed by different number of loads.

Table 1. Cancellation speed parameters for ES beam traversed by different numbers of loads.

Load quantity	Internal cancellation S_c^I	External cancellation S_c^E
1	0.343, 0.206	-
8	0.343, 0.206	0.625 (n=5→0.625); 0.521 (n=6→0.521) 0.446 (n=7→0.446); 0.347 (n=9→0.347)

Note: The values outside the parentheses are the cancellation speed parameters extracted from Fig. 7, and the parenthesized n-value is the one used in Eq. (21) to compute the external cancellation speed parameter close to the extracted one.

Internal cancellation conditions will first be discussed. It is evident from Fig. 7 and Table 1 that the internal cancellation speed parameters S_c^I remain consistent for both quantities of loads. They are represented in the figure by the grey and red solid dots, which overlap in the same position, as shown in the two enlarged views of the figure. In other words, once the SSR (κ) is determined, the internal cancellation speed parameters, as given by Eq. (22), that is an inherent characteristic of the system won't be affected by the number of external moving loads.

Regarding the external cancellation speed parameter S_c^E, denoted by red hollow squares, it is clear from Fig. 7 and Table 1 that it varies with the number of moving loads. In particular, no external cancellation speed parameter exists for a beam subjected to a single moving load. However, when the number of loads increases to 8, several external cancellation conditions emerge, marked with red hollow squares in Fig. 7. These points align closely with the theoretical values derived from Eq. (21). Notably, the number of the external cancellation points increases with the increasing quantity of moving loads, which is physically justified.

6.2 Bridge/vehicle length ratio L/d

The bridge/vehicle length ratio L/d is another factor that can affect the cancellation conditions. Typically, the vehicle length (d) or load interval has been designed to be around 25 m, as in China high-speed railways. To investigate the effect of the length ratio, one may fix the load spacing at 25 m and vary the beam length for numerical testing. For the present purposes, three different beam lengths: L = 32, 37.5, and 40 m, are considered. Both 32 m and 40 m beams have been widely used in Chinese railways. The 37.5 m beam is the one satisfying the ratio of $L = 1.5d$ and is therefore the optimal span length [5].

The modal amplitude R of the three beams traversed by 8 uniform moving loads has been illustrated in Fig. 8, in which the resonance conditions, internal cancellation conditions, and external cancellation conditions are denoted by different colored symbols. Of interest to note is that the cancellation speed parameters extracted from the figure align closely with the parameters calculated according to Eqs. (21) and (22), which have been listed in Table 2. Such an alignment serves as a testament to the reliability of the proposed cancellation conditions.

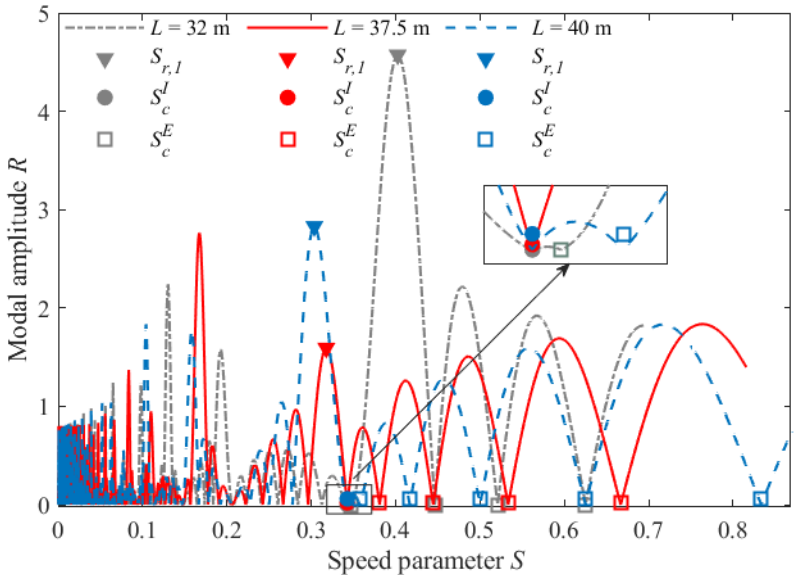


Fig. 8. Modal amplitude R of free vibration for ES beams of three different lengths with SSR $\kappa = 0.05$, traversed by 8 uniform moving loads.

Table 2 Cancellation speed parameters for beams of three different lengths traversed by 8 loads.

Beam length L (m)	Internal cancellation S_c^I	External cancellation S_c^E
32	0.343	0.625 ($n \to 5$, 0.625); 0.521 ($n \to 6$, 0.521) 0.446 ($n \to 7$, 0.446); 0.347 ($n \to 9$, 0.347)
37.5	0.343	0.667 ($n \to 4$, 0.667); 0.533 ($n \to 5$, 0.533) 0.444 ($n \to 6$, 0.444); 0.381 ($n \to 7$, 0.381)
40	0.343	0.625 ($n \to 4$, 0.625); 0.500 ($n \to 5$,0.500) 0.417 ($n \to 6$, 0.417); 0.357 ($n \to 7$, 0.357)

Note: The values outside the parentheses are the cancellation speed parameters extracted from Fig. 8, and the parenthesized n-value is the one used in Eq. (21) to compute the external cancellation speed parameter close to the extracted one.

Noteworthy is that the internal cancellation condition remains constant, $S_c^I = 0.343$, for all the beam lengths considered, a fact evident from the enlarged inset in Fig. 8, where the internal cancellation conditions for the three beams meet at the same point.

This observation confirms that the internal cancellation S_c^I depends exclusively on the SSR (κ) and is not affected by the bridge/vehicle length ratio (L/d), as indicated in Eq. (22).

Regarding the external cancellation condition, it actually fluctuates with the beam length, as depicted in Fig. 8. For a constant value of 'n' as listed in Table 2, the external cancellation parameter decreases as the beam length increases. This observation reveals the inverse relationship between the external cancellation parameter and the length ratio (L/d), as implied by Eq. (21). Particularly, the external cancellation fluctuates with the value of 'n', resulting in many external cancellation points that cannot be easily utilized in practice. In comparison, the internal cancellation point appears only once and remains constant, and thus is easy to use.

6.3 Bridge support stiffness ratio (SSR)

The analysis above reveals that neither the bridge/vehicle length ratio L/d nor the load quantity N can affect the internal cancellation of the beam. As revealed by Eq. (22), the internal cancellation speed parameter of the ES beam depends solely on the SSR (κ). To investigate such an effect, the modal amplitude R for the free vibration of the ES beam of 32 m has been plotted in Fig. 9 for different SSR values: κ = 0.05, κ = 0.2, and κ_{opt} = 0.381 (optimal value calculated using Eq. (25)). The other properties remain identical to those outlined in Section 5.

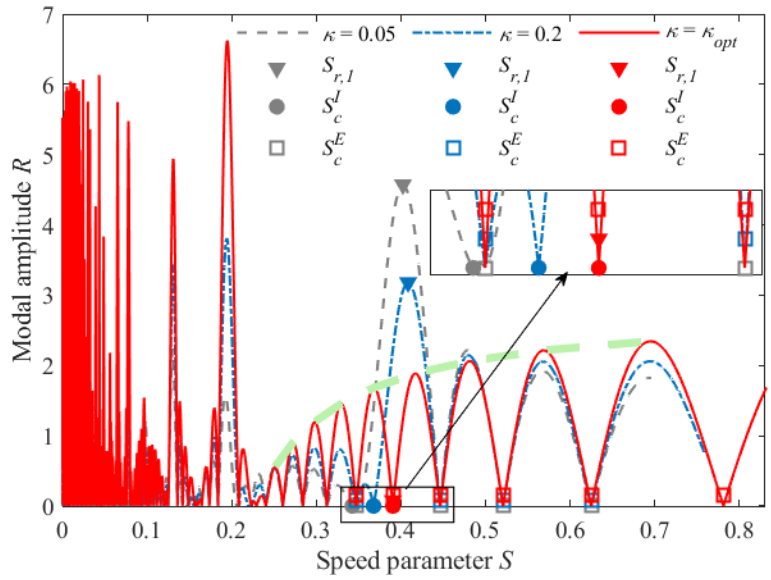


Fig. 9. Modal amplitude R of free vibration for ES beams with various SSRs κ traversed by 8 uniform moving loads.

Again, one observes from Fig. 9 that the external cancellation parameters S_c^E

remain unchanged as κ increases. These values conform to those calculated by substituting different 'n' values into Eq. (21), as listed in Table 3. From Fig. 9, one also observes that the internal cancellation speed parameter S_c^I progressively increases as κ rises, as indicated by various filled colored dots. Noteworthy is that when κ equals the optimal value κ_{opt}, the internal cancellation speed parameter S_c^I matches the first resonance speed parameter $S_{r,1}$. Consequently, the resonance phenomenon is entirely suppressed, as indicated by the red line at the cancellation point. This serves as an illustration of the perfect cancellation outlined in Section 4.1. The observation made herein for the exemplar 32 m beam remains valid for beams of other lengths.

Table 3 Cancellation speed parameters for 32 m ES beams with different SSR κ.

SSR κ	Internal cancellation S_c^I	External cancellation S_c^E
0.05	0.343	0.625 ($n \rightarrow 5$, 0.625); 0.521 ($n \rightarrow 6$, 0.521) 0.446 ($n \rightarrow 7$, 0.446); 0.347 ($n \rightarrow 9$, 0.347)
0.2	0.368	0.625 ($n \rightarrow 5$, 0.625); 0.521 ($n \rightarrow 6$, 0.521) 0.446 ($n \rightarrow 7$, 0.446); 0.347 ($n \rightarrow 9$, 0.347)
0.38 (κ_{opt})	0.390	0.781 ($n \rightarrow 4$, 0.781); 0.625 ($n \rightarrow 5$, 0.625); 0.521 ($n \rightarrow 6$, 0.521); 0.446 ($n \rightarrow 7$, 0.446); 0.347 ($n \rightarrow 9$, 0.347)

Note: The values outside the parentheses are the cancellation speed parameters extracted from Fig. 9, and the parenthesized n-value is the one used in Eq. (21) to compute the external cancellation speed parameter close to the extracted one.

7. Application of the Criterion to Selecting SSR

In Section 4, it has been demonstrated that the SSR (κ) of elastic bearings can be manipulated to either enhance or mitigate the first resonance ($S_{r,1}$) of beams of varying lengths. Of special interest is the possibility to achieve complete "resonance cancellation", as illustrated in Fig. 5 under optimal SSR conditions.

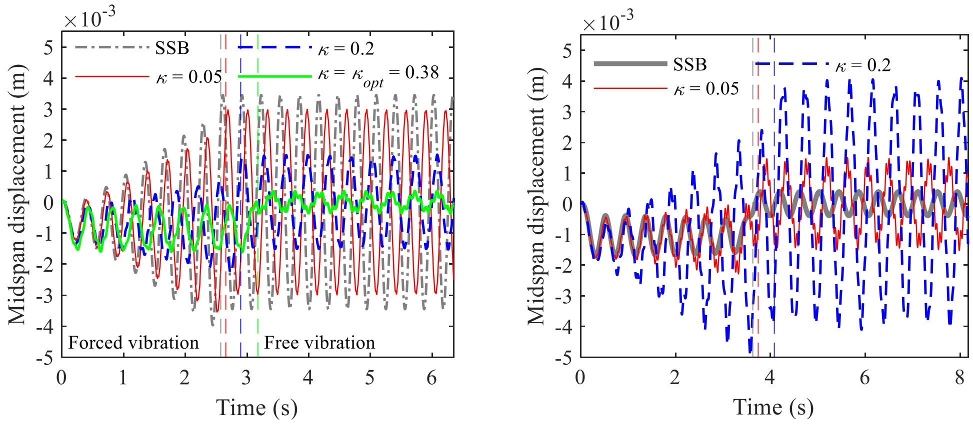
In this section, the efficacy of the proposed criterion for selecting the SSR will be assessed by the FEM [41] since it does not rely only on the first mode of the beam. Furthermore, an in-depth analysis of the damping effect will be carried out for both the beam and elastic supports on the applicability of the SSR selection criterion.

7.1 Validation of the criterion for selecting SSR

Herein, two railway bridge spans, 32 and 37.5 m, will be employed to ascertain the effectiveness of the SSR selection criterion. The 32 m beam has been widely used in

Chinese in railways, while the 37.5 m beam, being 1.5 times the vehicle length, has been recognized as the optimal beam for minimizing vibrations of resonance. In addition, three SSRs, including κ = 0 (SSB), 0.05, and 0.2, are used to simulate the different degrees of support stiffness. Besides, the optimal SSR cases are also investigated: $\kappa_{opt} = 0.38$ for the 32 m beam, and $\kappa_{opt} = 0$ for the 37.5 m beam, which simply represents the SSB case.

Figure 10(a) shows the midspan displacements of the 32 m beam subjected to 8 moving loads at the first resonance condition $S_{r,1}$ = 0.39 calculated by Eq. (19). Obviously, for the non-optimal SSR cases, the forced vibration of the beam progressively increases when exposed to consecutive moving loads, signifying that no use has been made of the elastic supports to suppress the beam's resonance vibration. Nevertheless, it was observed that the amplitude of free vibration decreases with an increasing SSR (for softer bearings). This suggests that a *softer* elastic support by itself can help reducing the vibration for the beam. When the SSR is increased to the optimal value of κ_{opt} = 0.38, calculated by Eq. (25), the forced vibration is suppressed, and the beam remains in a state of stationary oscillation. Consequently, the free vibration reaches its *minimum* with minor fluctuations. The free vibration does not reduce to zero because the optimal SSR calculated by Eq. (25) considers only the first mode of the beam, resulting in some deviations in the FEM result that considers all the vibration modes. Nonetheless, the cancellation efficacy remains substantial.



(a) (b)

Fig. 10. Midspan displacements of ES beams with various SSRs (κ) of span length: (a) L = 32 m; (b) L = 37.5 m.

For the 37.5 m beam under its first resonance condition ($S_{r,1}$ = 0.33) in Fig. 10(b), a striking different vibrational behavior can be observed, when compared with the 32 m beam in Fig. 10(a). Specifically, both the forced vibration and free vibrations exhibit a diminishing trend for smaller SSRs (i.e., for stiffer bearings). This implies that employing a stiffer elastic support leads to a more effective suppression of resonance for the 37.5 m beam. Notably, the lowest vibration level is achieved for the special case when κ_{opt} equals zero, known as the SSB case. This phenomenon underscores Yang' criterion [5] that the optimal span length for effectively mitigating resonance in SS beams is 1.5 times the vehicle length. However, it is realized that most railway brides do not appear in the simple form of SS beams. Rather, they often appear as ES beams due to considerations beyond train-induced vibrations. Consequently, when designing a beam with a length 1.5 times that of the vehicle length, such as the 37.5 m beam discussed herein, employing a support that is stiffer and closer to the hinge support proves to be more effective in suppressing the resonance.

In summary, the criteria laid out in Section 4.3 for choosing the SSR have been substantiated by the finite element analyses. This holds particularly true for the widely used 32 m beam, where the optimal κ value, as determined by Eq. (25), plays a pivotal role in the effective cancellation of resonance.

7.2 Damping effect of the whole ES beam

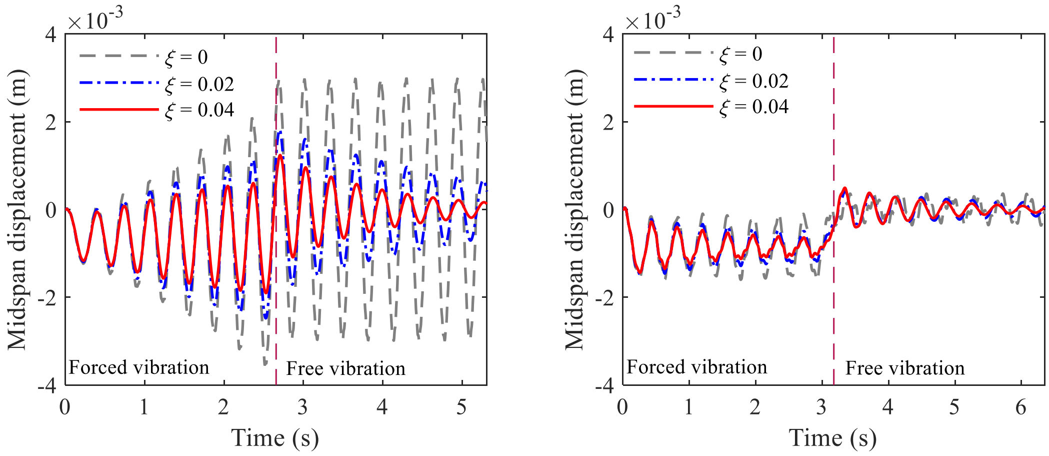
In practice, damping exists both in the beam and elastic supports, which plays an important role in vibration attenuation [29,31,42]. This section will numerically assess the damping effect on the efficacy of the proposed criterion for selecting the SSR.

7.2.1 Damping of the beam

Let us examine first the damping effect of the beam by considering three different Rayleigh damping ratios, i.e., ξ= 0, 0.02 and 0.04. No damping is considered of the elastic supports. The midspan displacements of the 32 m beam subjected to eight moving loads under the first resonance condition have been plotted in Fig. 11, with (a) indicating a regular SSR of $\kappa = 0.05$, and (b) for the optimal case of $\kappa_{opt} = 0.38$.

As revealed by Fig. 11(a), the inclusion of damping for the beam has resulted in a noticeable reduction in free vibration. The higher the damping ratio, the more pronounced the reduction in vibration. This confirms the role of damping of the beam in mitigating resonance vibrations, particularly for beams with a non-optimal SSR. In contrast, for the optimal case with κ_{opt} = 0.38 in Fig. 11(b), the resonance vibration has been drastically suppressed by the optimal elastic supports. A comparison between

Figs. 11(a) and (b) indicates that both the forced and free vibrations in the optimal SSR case are considerably smaller than those in the non-optimal case, even for various levels of damping in the beam. The above observation confirms the efficacy of the proposed SSR selection criterion for application to damped bridges.

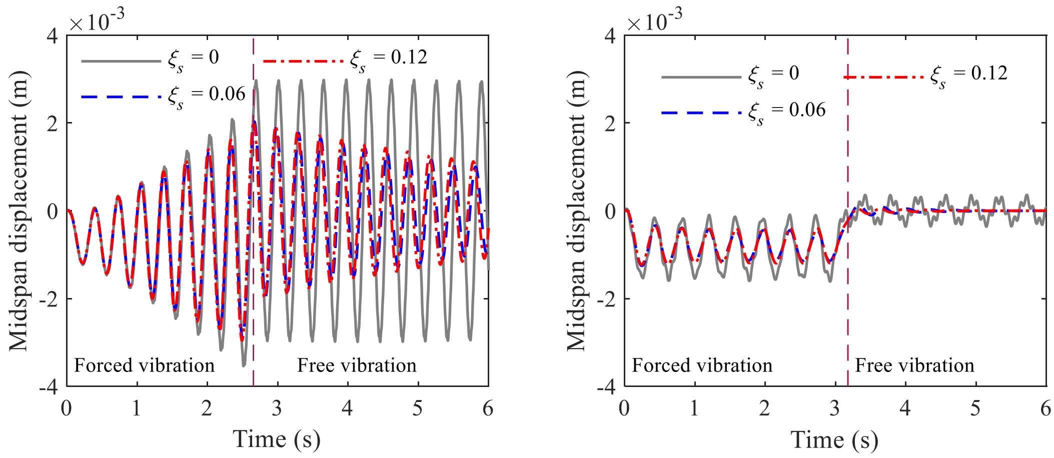


(a) (b)

Fig. 11. Midspan displacement of the 32 m beam for various damping ratios with SSR: (a) $\kappa = 0.05$; (b) $\kappa = \kappa_{opt} = 0.38$.

7.2.2 Damping of the elastic supports

Let us now investigate the damping effect of the elastic supports. Likewise, the damping is considered only for the supports. Three damping ratios, ξ_s = 0, 0.06 and 0.12, will be investigated, among which ξ_s= 0.12 represents bearings of high damping. The midspan displacements of the 32 m beam under the first resonance condition have been plotted in Fig. 12, with (a) indicating a regular SSR case with $\kappa = 0.05$, and (b) the optimal case with $\kappa_{opt} = 0.38$.



(a) (b)

Fig. 12. Midspan displacement of the 32 m beam with various support damping ratios and SSR: (a) $\kappa = 0.05$; (b) $\kappa = \kappa_{opt} = 0.38$.

Overall, support damping plays a crucial role in reducing the resonance vibration of the beam for both non-optimal SSR cases and optimal cases. Noticeable reduction in vibration can be observed for damped supports (ξ_s= 0.06 and 0.12) compared with the undamped case (ξ_s= 0). However, the effect of support damping has its limit, as little reduction in vibration can be achieved by increasing the support damping ratio from 0.06 to 0.12. It is interesting to note from Fig. 12(b) that the free vibration quickly diminishes to zero due to the contribution of support damping. This suggests that support damping aids in further suppressing the cancellation of resonance for bridges designed with the optimal SSR. In other words, the effectiveness in implementing the bearings with optimal SSR is enhanced by the inclusion of support damping.

8. Conclusion

This paper presents a comprehensive investigation into the resonance and cancellation conditions for the free vibration of elastically-supported (ES) beams subjected to successive moving loads. Considering particularly the effect of elastic supports, the cancellation condition is studied thoroughly for minimization of bridge vibrations. The (internal) cancellation function is employed to derive the optimal support stiffness ratio (SSR) for suppressing the resonance of railway bridges. Theoretical findings are validated by FEM for various factors. Based on the theory, numerical analysis and data adopted, the following conclusions are drawn:

(1) The resonance speed parameter S_r for the ES beam depends solely upon the bridge/vehicle length ratio L/d, irrespective of the elastic supports, making it consistent with that of the SS beam.

(2) The cancellations for the ES beam can be categorized into two types: external (load-related) and internal (structure-related). The external cancellation fluctuates and manifests in multiple points that cannot be easily utilized in practice. In contrast, the internal cancellation appears only once, remains constant, and thus is easy to use.

(3) The proposed criterion for selecting the optimal SSR depends solely on the bridge/vehicle length ratio L/d, regardless of the speed parameter S. This proves to be convenient for engineering applications.

(4) For minimizing bridge vibrations, an SSR closer to the lower bound of the

acceptable range should be selected for beams of lengths in the ranges $(0.5d, d]$ and $(1.5d, 2d]$, while for beams of lengths in the range $(d, 1.5d]$, the SSR should be selected as close to the optimal value as possible.

(5) Damping in the beam and supports contributes to further suppression of vibration for bridges designed with the optimal SSR.

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