Spatial Policies in Frictional Credit and Labor Markets

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Abstract

Many business lenders specialize in giving loans to businesses in specific regions. To understand this phenomenon, I analyze a model of frictional credit and labor markets with heterogeneous locations. I find that financial market tightness (wages) is decreasing (increasing) in local productivity. Then, I use the model to understand the effects of subsidizing matches between creditors and the managers of firms. I conclude that low levels of local amenities in a location reduces its optimal subsidy.

1 Introduction

The market for business loans is geographically distributed. Recent studies have shown, for instance, that credit market characteristics such as credit scores can significantly vary across neighborhoods and regions (George et al., 2019). There is also evidence of lenders displaying a specialization in certain regions: small lenders may work especially close with nearby small businesses (Laderman, 2006).

This paper attempts to explain the geographic dispersion of credit by studying frictional credit and labor markets with heterogeneous locations. It builds on three strands of literature, the Diamond-Mortensen-Pissarides (DMP) model and its extensions (Diamond, 1982; Petrosky-Nadeau and Wasmer, 2017; Pissarides, 2000; Mortensen, 1982; Bilal, 2021; Wasmer and Weil, 2004; **?**; Pissarides, 1985; Molho, 2001; Hosios, 1990; Rubinstein, 1982; Shimer, 2005; Marinescu and Rathelot, 2018), the effects of spatial policies (Bilal and Rossi-Hansberg, 2021; Fajgelbaum et al., 2018; Roback, 1982;

Huggett and Luo, 2022; Fajgelbaum and Gaubert, 2020), and the consequences of financial frictions (Nobuhiro and Moore, 1997).

2 Model

2.1 Environment

Time is continuous. There are N firms that differ in locations with local productivities l_k where l is positive-valued and $l_i \neq l_j$ for $i, j = 1, 2, ..., N$ when $i \neq j$. For ease of notation, I drop the subscript k from l_k moving forward. Firms manage projects that require liquidity from the credit market. Creditors are homogeneous with a mass of measure N and are freely mobile across locations when searching for projects. The number of projects in need of liquidity and the number of creditors in each location are denoted as $\mathcal{P}(l)$ as $\mathcal{B}(l)$ respectively. I assume that the mass of projects cannot exceed measure N, i.e. $\sum_l \mathcal{P}(l) \leq N$.

Financial market tightness is defined as $\phi(l) = \mathcal{P}(l)/\mathcal{B}(l)$. Matches between creditors and firms are described by the Cobb-Douglas matching function $M_c(\mathcal{P}(l), \mathcal{B}(l)) = m_c \mathcal{P}(l)^{\eta_c} \mathcal{B}(l)^{1-\eta_c}$. The rate at which projects find credit is $p(\phi(l)) = M_c(\mathcal{P}(l), \mathcal{B}(l))/\mathcal{P}(l)$. Similarly, the rate at which creditors find projects is $c(\phi(l)) = M_c(\mathcal{P}(l), \mathcal{B}(l))/\mathcal{B}(l)$. By construction, we have $\phi p(\phi(l)) = c(\phi(l))$. Furthermore, the definition of the matching function implies that $p(\phi(l)) = m_c \phi(l)^{-\eta_c}$ and $c(\phi(l)) = m_c \phi(l)^{1-\eta_c}$.

Similar to creditors, workers are homogeneous with a mass of measure N. Let $\mathcal{U}(l)$ and $\mathcal{V}(l)$ represent the mass of unemployed workers and job vacancies respectively. Then labor market tightness is $\theta(l) = \mathcal{U}(l)/\mathcal{V}(l)$. Workers also have a similar matching function to creditors: $M_j(\mathcal{U}(l), \mathcal{V}(l)) = m_j \mathcal{U}(l)^{\eta_j} \mathcal{V}(l)^{1-\eta_j}$. The job-finding rate $f(\theta(l))$ and the vacancy-filling rate $q(\theta(l))$ are similarly defined to the case with credit markets: $f(\theta(l)) = M_j(\mathcal{U}(l), \mathcal{V}(l))/\mathcal{U}(l)$ and $q(\theta(l)) = M_j(\mathcal{U}(l), \mathcal{V}(l))/\mathcal{V}(l)$. It follows that $\theta(l)q(\theta(l)) = f(\theta(l))$. In the remainder of the paper, we simplify the notation by writing expressions like $\phi(l)$ as ϕ.

2.2 Value Functions

The spatial credit and labor markets have three distinct stages. Creditors in each stage derive Al in utility from the amenities of their location where A is the baseline level of amenities. In the first stage c, creditors search to match with projects and vice versa. Search costs associated with entering the market are k_B for creditors and k_I for projects. I refer to matches between creditors and projects as funded projects. During this search process, The government may implement a subsidy H encourage matches.Once the project is funded, the second stage v begins and the funded project looks for matches with job seekers. The creditor pays a vacancy flow cost of γ. The third stage π consists of a matched creditor, project, and worker producing output. In this stage, the creditor gains an income of ψ. The project pays workers the wage w and produces output xl where x is the baseline productivity of a match.

The destruction of matches happens exogenously. There are two exogenous separation shocks. The first shock s^L destroys the match between a worker and a funded project. The second shock s^C can occur in either the profit or vacancy stage and brings creditors and projects back into the vacancy stage.

The creditor value B in each stage is governed by the following equations where r is the real interest rate:

$$rB_c = -k_B + Al + \phi p(\phi)(B_v - B_c) \tag{1}$$

$$rB_v = -\gamma + Al + q(\theta)(B_\pi - B_v) + s^C(B_c - B_v) \tag{2}$$

$$rB_\pi = \psi + Al + s^C(B_c - B_\pi) + s^L(B_v - B_\pi) \tag{3}$$

There is a corresponding set of equations for project value E:

$$rE_c = -k_I + p(\phi)(E_v - E_c + H) \tag{4}$$

$$rE_v = q(\theta)(E_\pi - E_v) + s^C(E_c - E_v) \tag{5}$$

$$rE_\pi = xl - w - \psi + s^C(E_c - E_\pi) + s^L(E_v - E_\pi) \tag{6}$$

I assume that projects and creditors have free entry into the credit market, so $rE_c = 0$ and $rB_c = 0$. Notice that these assumption imply that the value of creditors and projects in the vacancy stage are normalized across locations. Using these assumptions, I obtain the following equations for the manager and creditor values in the vacancy stage:

$$E_v = \frac{k_I}{p(\phi)} - H \tag{7}$$

$$B_v = \frac{k_B - Al}{\phi p(\phi)} \tag{8}$$

I define the value of a funded project as $J = B + E$. Immediately we find that the value of the creditor-project match J_v equals the sum of frictional costs which we denote as K for convenience:

$$J_v = \frac{k_I}{p(\phi)} - H + \frac{k_B - Al}{\phi p(\phi)} = K \tag{9}$$

Creditors and projects conduct a Nash bargaining game to determine the size of the creditor flow value ψ. The bargaining power of a creditor is denoted by α_c. It follows that ψ maximizes the surplus

$$S_c = (B_v - B_c)^{\alpha_c}(E_v - E_c)^{1-\alpha_c} \tag{10}$$

The first order conditions of the maximization problem imply that

$$(1-\alpha_c)(B_v - B_c) = \alpha_c(E_v - E_c). \tag{11}$$

Moving to the labor market, workers earn a wage w when employed and receive unemployment benefits equal to bl where b represents the baseline level of unemployment benefits. The value functions for employed and unemployed workers respectively are:

$$rW_n = w + (s^L + s^C)(W_u - W_n) \tag{12}$$

$$rW_u = bl + \theta q(\theta)(W_n - W_u) \tag{13}$$

Wages are determined through Nash bargaining between the workers and the funded project. Let α_l represent the bargaining power of a worker. We have that

$$w = (J_\pi - J_v)^{1-\alpha_l}(W_n - W_u)^{\alpha_l}, \tag{14}$$

which leads to the following first-order condition:

$$(1-\alpha_l)(W_n - W_u) = \alpha_l(J_\pi - J_v). \tag{15}$$

2.3 Equilibrium

In this section, I derive the key equations of the model. First, I find the following job creation condition:

$$\frac{\gamma_k}{q(\theta)} = \frac{x_l^{CL} - w}{r + s^C + s^L} + Al\left(\frac{1}{q(\theta)} + \frac{1}{r + s^C + s^L}\right) \tag{16}$$

where $x_l^{CL} = xl - (r + s^C)K$. This equation suggests that the discounted vacancy cost equals the discounted value of profits and local amenities. Second, we find the following solutions to ψ and w:

$$\psi = \alpha_c(xl - w) - \frac{1-\alpha_c}{q}((Al - \gamma)(r + s^C + s^L) + Alq) \tag{17}$$

$$w = (1-\alpha)bl + \alpha(x^{CL} + \gamma_K\theta) + \alpha Al(1-\theta) \tag{18}$$

From these solutions, I find that wages are increasing in local productivity. Workers are paid more in higher productivity locations. On the other hand, the effects of local productivity on creditor repayments is ambiguous.

Finally, we derive credit market tightness. Denote $\phi_0(l)$ as the value of $\phi(l)$ when $H = 0$. In other words, $\phi_0(l)$ is the tightness of the credit market in location l without government intervention. Then we have:

$$\phi_0 = \frac{z + k_B - Al}{k_I}\frac{1-\alpha_c}{\alpha_c} \tag{19}$$

It follows that financial market tightness is decreasing in local productivity. Hence, creditors disproportionally favor enter credit markets in locations with high local productivities.

3 Efficiency

3.1 The Social Planner Problem

I study the efficiency implications of implementing hiring subsides H. The social planner solves the problem

$$\max \quad \Omega = \int e^{-rt}\sum_l((xl - s^C H)(1-\mathcal{U}) + bl\mathcal{U} - \gamma\mathcal{V} - (k_B + Al)\mathcal{B} - k_I\mathcal{N}) \tag{20}$$

$$\text{s.t.} \quad \dot{\mathcal{U}} = (s^C + s^L)(1-\mathcal{U}) - M_j(\mathcal{U},\mathcal{V}) \tag{21}$$

$$\dot{\mathcal{V}} = M_c(\mathcal{N},\mathcal{B}) + s^L(1-\mathcal{U}) - M_j(\mathcal{U},\mathcal{V}) - s^C\mathcal{V} \tag{22}$$

The solution to the social planner problem can be described by constructing the Hamiltonian and taking first-order conditions. I do these derivations in the appendix.

The solution to the social planner problem implies the following job creation condition:

$$\frac{\gamma_k^*}{q(\theta)} = \frac{(x_l^{CL*} - s^C H - bl)(1-n_l) - n_l\gamma_k^*\theta}{r + s^C + s^L} \tag{23}$$

where $\gamma*_k = \gamma + (r+s^C)(K+H)$ and $x_l^{CL*} = xl - (r+s^C)(K+H)$. Unlike other search models, the Hosios condition does not guarantee that decentralized and social planner equilibriums are identical. The reason why is because the presence of local amentities does not direct enter the job creation condition for the social planner in contrast to the decentralized equilibrium. However, the Hosios condition can be used to show that the socially optimal value of financial market tightness is identical to the decentralized value in a special case. Suppose that the the social planner is unable to implement a subsidy, so $H = 0$. The social planner problem implies that financial market tightness in this special case is

$$\phi_0^{SP} = \frac{k_B - Al}{k_I}\frac{1-n_c}{n_c} \tag{24}$$

Hence, $\phi_0^{SP} = \phi_0$ when $\alpha_c = n_c$.

To uncover the optimal subsidy, I find the value of H that equalizes the social planner and decentralized job creation condition:

$$H = (\frac{1}{t})(\frac{(\alpha_l - n_l)}{r+s^C+s^L}(x_l^{CL} + \gamma_k\theta - bl) + Ald \tag{25}$$

where $t = (\frac{r+s^C}{q(\theta)} + \frac{(1-n_l)r + n_l\theta(r+s^C)}{r+s^C+s^L})$ and $d = \frac{1}{q(\theta} + \frac{1+\alpha_l(\theta-1)}{r+s^C+s^L}$. Using this equation, I answer the question of how the social planner should subsidize locations. The two terms on the right-hand side present two distinct channels for how local productivity influences the optimal subsidy. The first term on the right, Ald, shows how the presence of local amenities affects creditor-project matches. Since high productivity locations can provide higher levels of amenities, the social planner will subsidize these regions more so than low productivity regions.

The second term on the right reveals how the bargaining power of workers influence the subsidy. First, suppose that $\alpha_l = n_l$. In this case, the second term has no effect on the subsidy. Now, suppose $\alpha_l \neq n_l$. In this case, the optimal subsidy depends on whether or not the worker's bargaining power is higher or lower than n_l. When $\alpha_l > n_l$, the social planner is incentivized to subsidize matches.

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Appendix A Derivations

A.1 Job Creation Condition

By definition, I have

$$(r+s^C)J_v = -\gamma + Al + q(\theta)(J_\pi - J_v)$$

$$(r+s^C)J_\pi = xl - w + Al + s^L(J_v - J\pi)$$

Hence, doing some algebra gives the following identities

$$J_\pi = \frac{xl - w + Al + s^L K}{r + s^C + s^L}$$
$$J_\pi = \frac{(r+s^C)K + \gamma - Al}{q(\theta)} + K$$

Setting both expressions for J_π equal to each other gives

$$K\left(\frac{r+s^C}{q(\theta)} + 1\right) + \frac{\gamma - Al}{q(\theta)} = \frac{xl - w + Al + s^L K}{r + s^C + s^L}$$

I can then simplify this equation as

$$\frac{\gamma_k}{q(\theta)} + \frac{-Al}{q(\theta)} = \frac{x_l^{CL} - w + Al}{r + s^C + s^L}$$

Some more algebra gives

$$\frac{\gamma_k}{q(\theta)} = \frac{x_l^{CL} - w}{r + s^C + s^L} + Al(\frac{1}{q(\theta)} + \frac{1}{r + s^C + s^L})$$

A.2 Psi Derivation

By construction, the forward values of the vacancy stages for creditors and projects is

$$(r+s^C)E_v = q(\theta)(E_\pi - E_v) = q(\theta)(\frac{xl + s^L E_v - w - \psi}{r + s^C + s^L} - E_v)$$
$$(r+s^C)B_v = -\gamma + Al + q(\theta)(B_\pi - B_v)$$

Doing some algebra gives us

$$(r + s^C + q)E_v = q(\theta)(\frac{xl + s^L E_v - w - \psi}{r + s^C + s^L})$$
$$(r + s^C + q)B_v = -\gamma + Al + q(\theta)(\frac{\psi + Al + s^L B_v}{r + s^C + s^L})$$

More gives us

$$(r + s^C + q(\theta) - \frac{q(\theta)s^L}{r + s^C + s^L})E_v = q(\theta)(\frac{xl - w - \psi}{r + s^C + s^L})$$
$$(r + s^C + q(\theta) - \frac{q(\theta)s^L}{r + s^C + s^L})B_v = -\gamma + Al + q(\theta)(\frac{\psi + Al}{r + s^C + s^L})$$

Then the Nash bargaining surplus identity gives

$$q(\frac{xl - w - \psi}{r + s^C + s^L})\frac{\alpha_c}{1 - \alpha_c} = -\gamma + Al + q(\theta)(\frac{\psi + Al}{r + s^C + s^L})$$

Doing some further simplification gives:

$$\frac{xl - w - \psi}{r + s^C + s^L}(\alpha_c) = (1 - \alpha_c)(\frac{-\gamma + Al}{q} + \frac{\psi + Al}{r + s^C + s^L})$$

Isolating ψ produces

$$\frac{\psi}{r + s^C + s^L} = (1 - \alpha_c)(-\frac{-\gamma + Al}{q} - \frac{Al}{r + s^C + s^L}) + \frac{xl - w}{r + s^C + s^L}\alpha_c$$

Multiplying $r + s^C + s^L$ on both sides gets

$$\psi = \alpha_c(xl - w) - (1 - \alpha_c)(\frac{(-\gamma + Al)(r + s^C + s^L)}{q} + Al)$$

Finally, combining like terms again gives

$$\psi = \alpha_c(xl - w) - (1 - \alpha_c)(\frac{-\gamma(r + s^C + s^L)}{q} + \frac{Al(r + s^C + s^L + q)}{q})$$

Rearranging terms gets

$$\psi = \alpha_c(xl - w) - \frac{1 - \alpha_c}{q(\theta)}((Al - \gamma)(r + s^C + s^L) + Alq)$$

A.3 Wage Derivation

Using the workers' value functions gives

$$rW_n - rW_u = w + (s^L + s^C)(W_u - W_n) - rW_u$$

Rearranging terms implies that

$$(r + s^L + s^C)(W_n - W_u) = w - rW_u,$$

so

$$(W_n - W_u) = \frac{w - rW_u}{r + s^L + s^C}$$

Turning to the funded project, we have the following expression:

$$rJ_\pi = xl - w + Al + s^L(J_v - J_\pi) + s^C(J_c - J_\pi)$$

We subtract both sides by rJ_v to gt

$$rJ_\pi - rJ_v = xl - w + Al + s^L(J_v - J_\pi) + s^C(J_c - J_\pi) - rJ_v$$

And then rearrange terms on the right-hand side

$$rJ_\pi - rJ_v = xl - w + Al + s^L(J_v - J_\pi) + s^C(J_v - J_\pi) - rJ_v - s^C(J_v)$$

Adding terms to the left-hand side gives

$$(r + s^C + s^L)(J_\pi - J_v) = xl - w + Al - rJ_v - s^C(J_v)$$

We divide both sides by $r + s^C + s^L$

$$(J_\pi - J_v) = \frac{xl - w + Al - rJ_v - s^C(J_v)}{r + s^C + s^L}$$

Then, I have

$$(1-\alpha)w = \alpha_l(xl - w + Al - (r + s^C)J_v) + (1-\alpha)rW_u,$$

which simplifies to

$$w = \alpha_l(xl + Al - (r + s^C)J_v) + (1-\alpha)rW_u$$

Knowing that $rW_u = bl + \theta q(\theta)(W_n - W_u)$ gives

$$rW_u = bl + \theta q(\theta)(\alpha/(1-\alpha))(J_\pi - J_v)$$

I write this as

$$rW_u = bl + \theta q(\theta)(\alpha/(1-\alpha))(\frac{k_\phi + \gamma - Al}{q})$$

so I have (where $k_\phi = (r + s^C)K)$)

$$w = \alpha(xl - k_\phi + Al) + (1-\alpha)bl + \alpha\theta(k_\phi + \gamma - Al)$$

Doing some more algebra provides

$$w = \alpha(xl + \theta\gamma) + (1-\alpha)bl + \alpha(\theta - 1)(k_\phi - Al)$$

Separating terms gets

$$w = \alpha(xl + \theta\gamma) + (1-\alpha)bl + \alpha(\theta - 1)(k_\phi) + \alpha(\theta - 1)(-Al)$$

Finally, we end up with an analytical solution for w

$$w = (1-\alpha)bl + \alpha(x^{CL} + \gamma_K\theta) + \alpha(\theta - 1)(-Al)$$

Some simplification gives

$$w = (1-\alpha)bl + \alpha(x^{CL} + \gamma_K\theta) + \alpha Al(1-\theta)$$

A.4 Financial Market Tightness

Let us assume the government does not implement a matching subsidy. Then the Nash bargaining condition implies that

$$\frac{k_B - Al}{\phi_0 p(\phi_0)} = \frac{\alpha_c}{1-\alpha_c}(\frac{k_I}{p(\phi_0)})$$

which gives us the following expression:

$$\phi_0 = \frac{k_B - Al}{k_I}\frac{1-\alpha_c}{\alpha_c}$$

Appendix B Efficiency

The social planner problem is

$$\Omega = \int e^{-rt}\sum_l((xl - s^C H)(1-\mathcal{U}) + bl\mathcal{U} - \gamma\mathcal{V} - (k_B + Al)\mathcal{B} - k_I\mathcal{N})$$

s.t.

$$\dot{\mathcal{U}} = (s^C + s^L)(1-\mathcal{U}) - M_j(\mathcal{U},\mathcal{V})$$

$$\dot{\mathcal{V}} = M_c(\mathcal{N},\mathcal{B}) + s^L(1-\mathcal{U}) - M_j(\mathcal{U},\mathcal{V}) - s^C\mathcal{V}$$

which is associated with the Hamiltonian

$$H = e^{-rt}((x - s^C H)(1-\mathcal{U}) + bl\mathcal{U} - \gamma\mathcal{V} - (k_B + Al)\mathcal{B} - k_I\mathcal{N})$$
$$+ \Psi_U((s^C + s^L)(1-\mathcal{U}) - M_j(\mathcal{U},\mathcal{V})) + \Psi_V(M_c(\mathcal{N},\mathcal{B}) + s^L(1-\mathcal{U}) - M_j(\mathcal{U},\mathcal{V}) - s^C\mathcal{V})$$

The first-order conditions of the Hamiltonian are

$$\frac{\partial H}{\partial \mathcal{B}} = e^{-rt}(-k_B + Al) + \Psi_V \phi p(\phi) n_c = 0$$

$$\frac{\partial H}{\partial \mathcal{N}} = e^{-rt}(-k_I) + \Psi_V p(\phi)(1-n_c) = 0$$

$$\dot{\Psi}_U = -\partial H/\partial \mathcal{U} = e^{-rt}(xl - s^C H - bl) + \Psi_U(s^C + s^L + n_l\theta q(\theta)) + \Psi_V(n_l\theta q(\theta) + s^L)$$

$$\dot{\Psi}_V = -\partial H/\partial \mathcal{V} = e^{-rt}\gamma + \Psi_U((1-n_l)q(\theta)) + \Psi_V((1-n_l)q(\theta) + s^C)$$

We now proceed to derive the socially optimal credit market tightness. First, we have

$$\Psi_V p(\phi) = e^{-rt}(k_I/(1-n_c))$$

Then using the FOC conditions gives

$$e^{-rt}(-k_B + Al) + e^{-rt}(k_I/(1-n_c))\phi n_c = 0$$

Simplifying this expression gives

$$\phi = \frac{k_B - Al}{k_I}\frac{1-n_c}{n_c}$$

I now proceed to derive the job market condition. We have that the following two equations hold:

$$\Psi_V = e^{-rt}\frac{k_B - Al}{\phi p(\phi) n_c}$$

$$\Psi_V = e^{-rt}\frac{k_I}{p(\phi)(1-n_c)}$$

so

$$\Psi_V = e^{-rt}(\frac{k_B - Al}{\phi p(\phi)} + \frac{k_I}{p(\phi)}) = e^{-rt}(K + H) = e^{-rt}K^*$$

$$\dot{\Psi}_V = -re^{-rt}(\frac{k_B - Al}{\phi p(\phi)} + \frac{k_I}{p(\phi)}) = -re^{-rt}(K + H) = -re^{-rt}K^*$$

These equations imply that

$$\dot{\Psi}_U = e^{-rt}(xl - s^C H - bl) + \Psi_U(s^C + s^L + n_l\theta q(\theta)) + e^{-rt}K^*(n_l\theta q(\theta) + s^L)$$

$$e^{-rt}\gamma + \Psi_U(1 - n_l)q(\theta) + e^{-rt}K^*((1 - n_l)q(\theta) + s^C) = -re^{-rt}K^*$$

then

$$\Psi_U = -e^{-rt}(\frac{\gamma + k^*_\phi}{(1 - n_l)q(\theta)} + K^*)$$

$$\dot{\Psi}_U = re^{-rt}(\frac{\gamma + k^*_\phi}{(1 - n_l)q(\theta)} + K^*)$$

plugging these equations in, with $\gamma^*_k = \gamma + k^*_\phi$ and $x^{CL*}_l = xl - k^*_\phi$, gives

$$r(\frac{\gamma + k^*_\phi}{(1 - n_l)q(\theta)} + K^*) = (xl - s^C H - bl) - (\frac{\gamma + k^*_\phi}{(1 - n_l)q(\theta)} + K^*)(s^C + s^L + n_l\theta q(\theta)) + K^*(n_l\theta q(\theta) + s^L)$$

Rearranging terms gives

$$(xl + s^C H - bl) = (\frac{\gamma + k^*_\phi}{(1 - n_l)q(\theta)} + K^*)(s^C + s^L + n_l\theta q(\theta) + r) - K^*(n_l\theta q(\theta) + s^L)$$

Cleaning the left-hand side terms means that

$$(xl - s^C H - bl) = (\frac{\gamma + k^*_\phi}{(1 - n_l)q(\theta)} + K^*)(s^C + s^L + r) + \frac{n_l\gamma^*_k\theta}{1 - n_l} + K^*(r + s^C + s^L + n_l\theta q(\theta)) - K^*(n_l\theta q(\theta) + s^L)$$

Subtracting terms on both sides implies

$$\frac{\gamma^*_k}{q(\theta)}(r + s^C + s^L) = (xl - s^C H - k^*(\phi) - bl)(1 - n_l) - n_l\gamma^*_k\theta$$

Them we have

$$\frac{\gamma^*_k}{q(\theta)} = \frac{(xl^{CL*} - s^C H - bl)(1 - n_l) - n_l\gamma^*_k\theta}{r + s^C + s^L}$$

Recall that we derived the job creation condition

$$\frac{\gamma_k}{q(\theta)} = \frac{(1 - \alpha)(x^{CL} - bl) - \alpha\gamma_k\theta}{r + s^C + s^L} + Al(\frac{1}{q(\theta)} + \frac{1 + \alpha(\theta - 1)}{r + s^C + s^L})$$

so now the thing left to do is to derive H using these conditions. note that

$$k_\phi^* = (r+s^C)(K+H)$$

$$\gamma_k^* = \gamma_k + (r+s^C)H$$
$$x^{CL*} = x^{CL} - (r+s^C)H$$

lets simplify the social planner job creation condition

$$\frac{\gamma_k}{q(\theta)} = \frac{(1-n_l)(x_l^{CL} - (r+s^C)H + s^C H - bl) - n_l\theta(\gamma_k + (r+s^C)H)}{r+s^C+s^L} - \frac{(r+s^C)H}{q(\theta)}$$

then by plugging in the job creation conditions together gives that H must satisfy

$$\begin{aligned}\frac{\gamma_k}{q(\theta)} &= \frac{(1-n_l)(x_l^{CL} - (r+s^C)H + s^C H - bl) - n_l\theta(\gamma_k + (r+s^C)H)}{r+s^C+s^L} - \frac{(r+s^C)H}{q(\theta)} \\ &= \frac{(1-\alpha)(x_l^{CL} - bl) - \alpha\gamma_k\theta}{r+s^C+s^L} + Al(\frac{1}{q(\theta)} + \frac{1+\alpha(\theta-1)}{r+s^C+s^L})\end{aligned}$$

then we have

$$\begin{aligned}\frac{(1-\alpha_l)(x_l^{CL} - bl) - \alpha_l\gamma_k\theta}{r+s^C+s^L} + Al(\frac{1}{q(\theta)} + \frac{1+\alpha_l(\theta-1)}{r+s^C+s^L}) \\ -\frac{(x_l^{CL} - bl)(1-n_l) - n_l\theta(\gamma_k)}{r+s^C+s^L} \\ = -(\frac{r+s^C}{q(\theta)} + \frac{(1-n_l)r + n_l\theta(r+s^C)}{r+s^C+s^L})H\end{aligned}$$

lets call $t = (\frac{r+s^C}{q(\theta)} + \frac{(1-n_l)r+n_l\theta(r+s^C)}{r+s^C+s^L})$. Then

$$\begin{aligned}-\frac{1}{t}(\frac{(1-\alpha_l)(x_l^{CL} - bl) - \alpha_l\gamma_k\theta}{r+s^C+s^L} + Al(\frac{1}{q(\theta)} + \frac{1+\alpha_l(\theta-1)}{r+s^C+s^L}) \\ -\frac{(x_l^{CL} - bl)(1-n_l) - n_l\theta(\gamma_k)}{r+s^C+s^L}) \\ = H\end{aligned}$$

we can simplify this as (with $d = \frac{1}{q(\theta)} + \frac{1+\alpha_l(\theta-1)}{r+s^C+s^L}$)

$$H = (\frac{1}{t})(\frac{(\alpha_l - n_l)}{r+s^C+s^L}(x_l^{CL} + \gamma_k\theta - bl) + Ald$$