Mappings

Xiao-Ming Fu

Outlines

- Introduction
- Maintenance-based methods
- Bounded distortion methods
- Representation-based method

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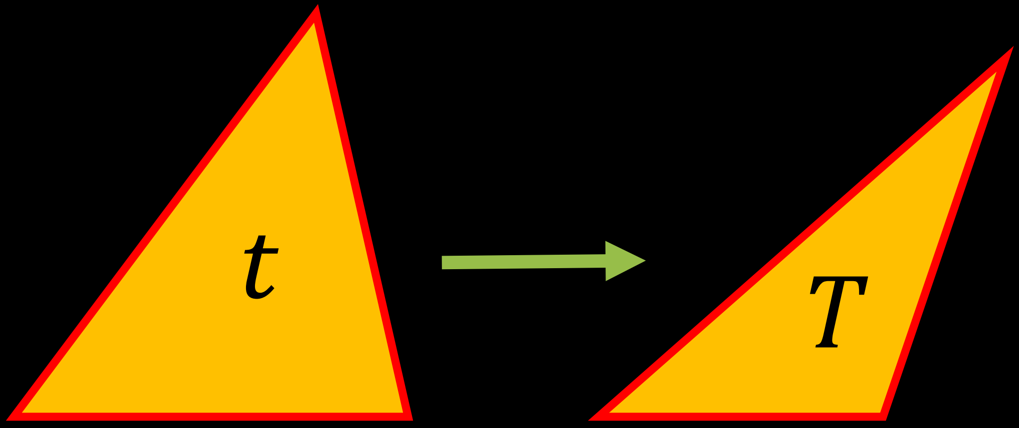
Mappings

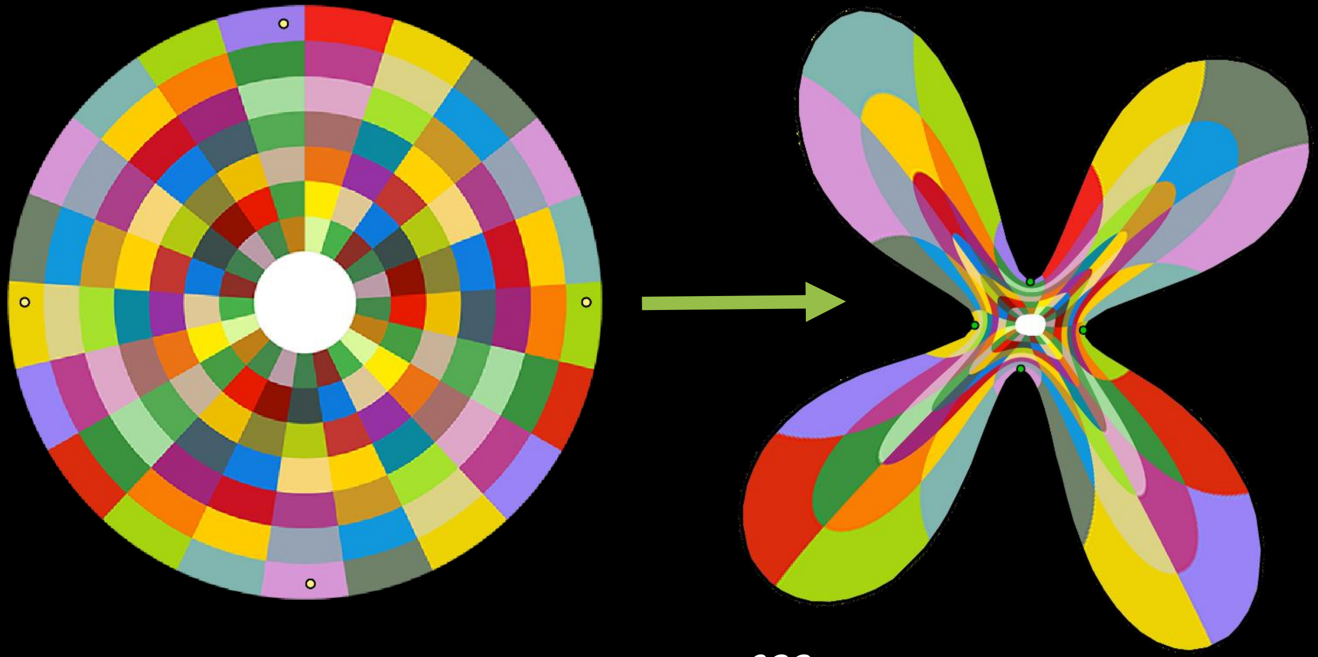
$$f: \Omega \subset R^d \rightarrow R^d$$

- mesh-based mapping



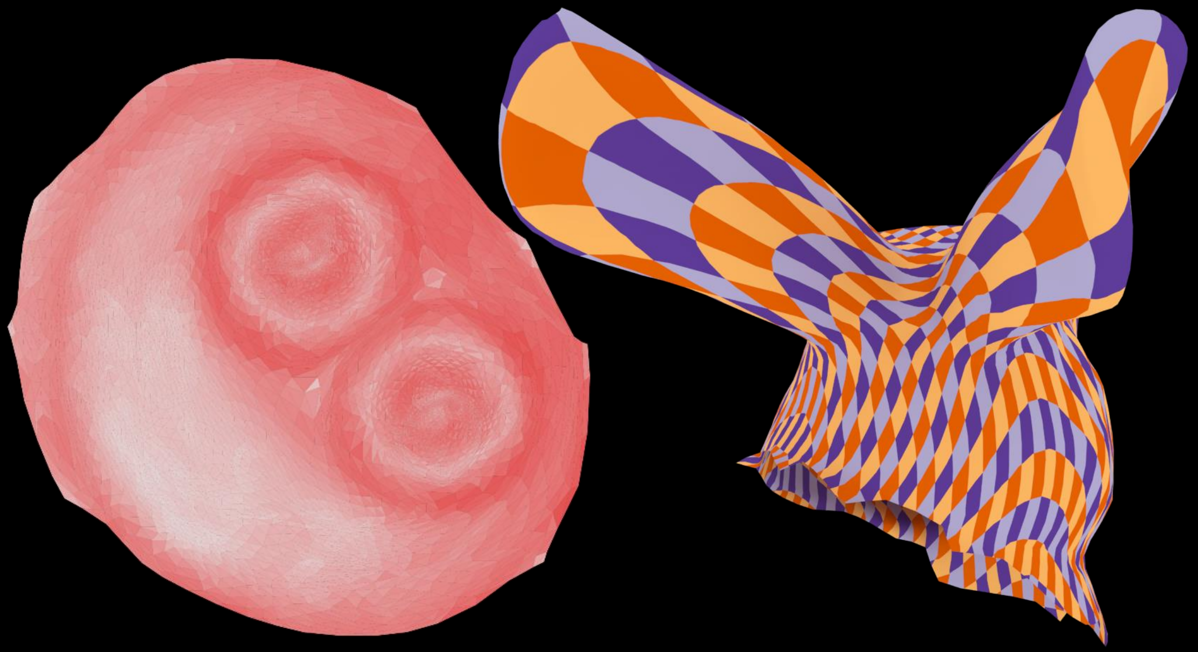
$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$

- meshless mapping

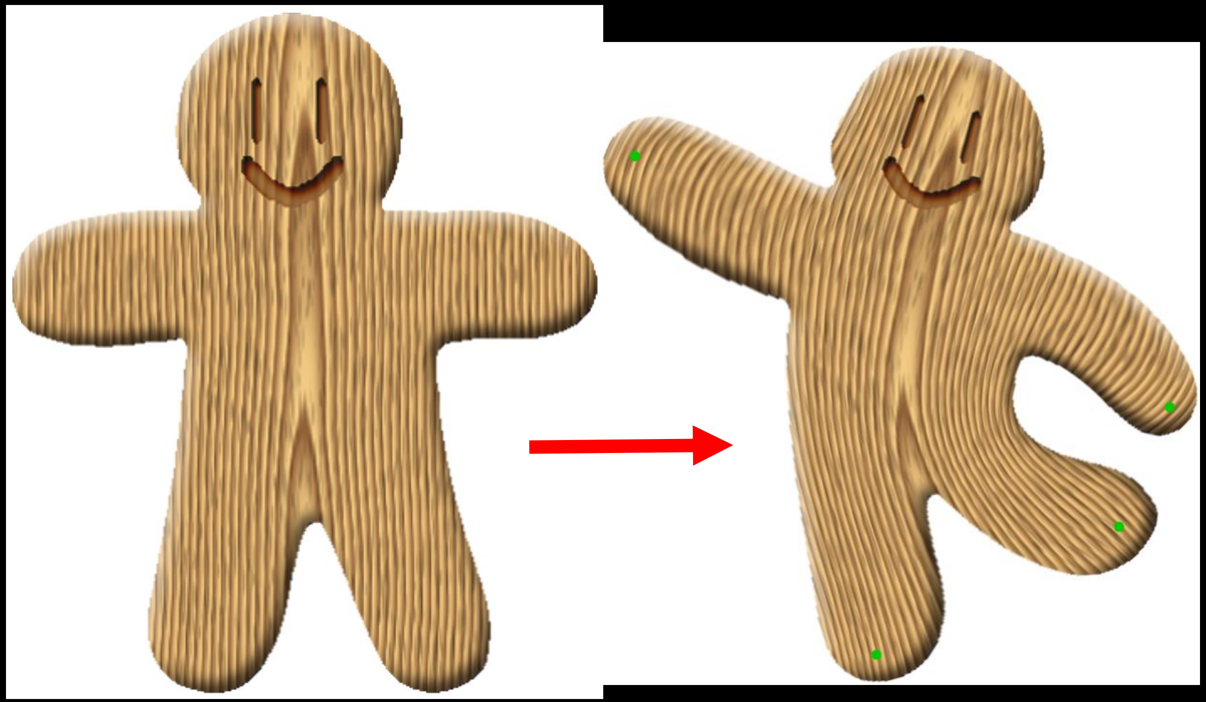


$$f(\boldsymbol{x}) = \boldsymbol{x} + \sum_{i=1}^{m} \boldsymbol{c}_i B_i(\boldsymbol{x})$$

Applications

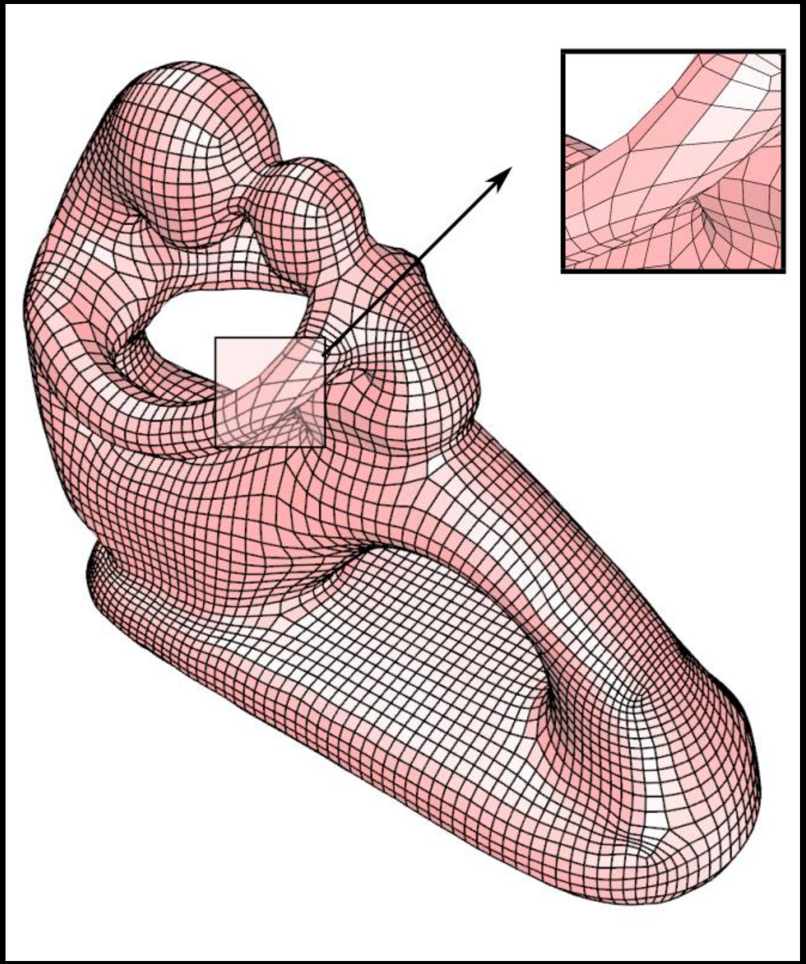


Parameterization

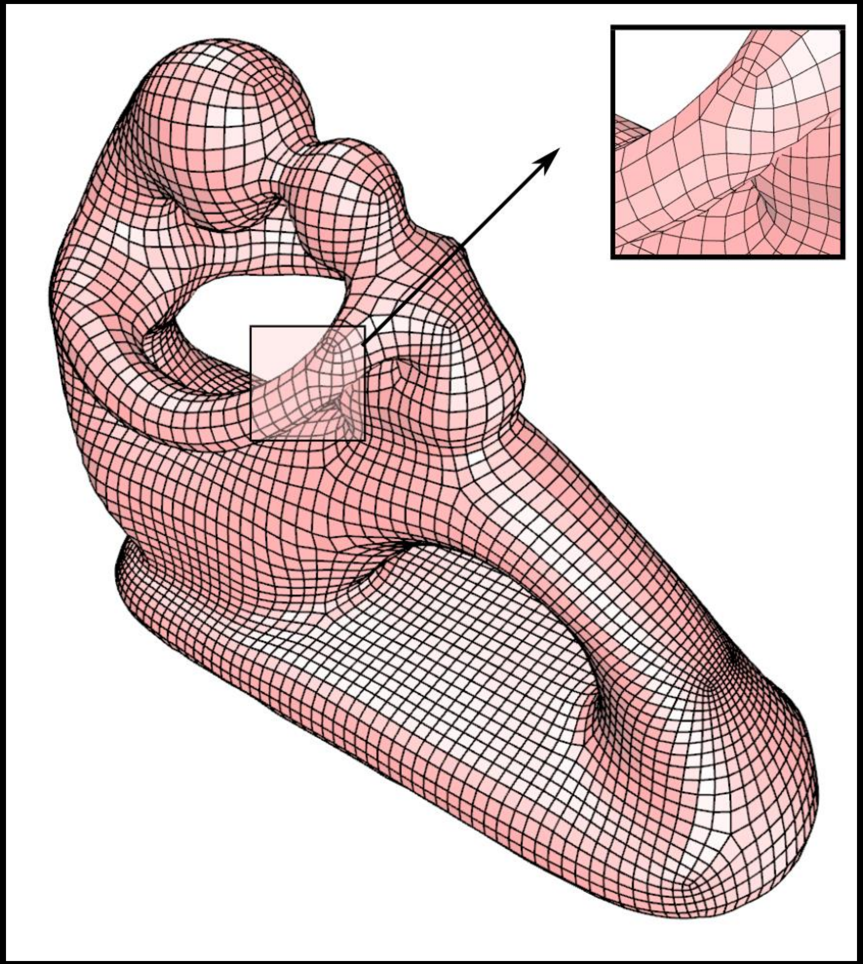


Deformation

Application - Mesh Improvement



[Gregson et al. 2011]

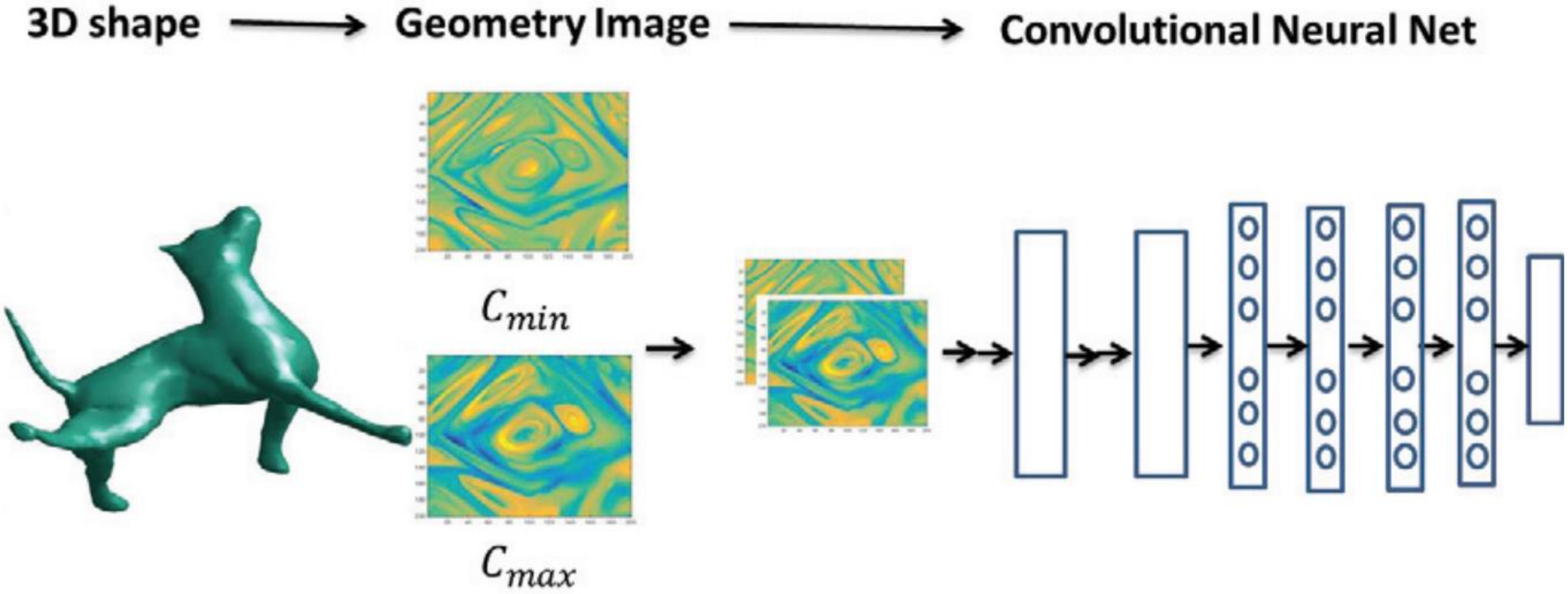


Improved results

Application – Compatible remeshing

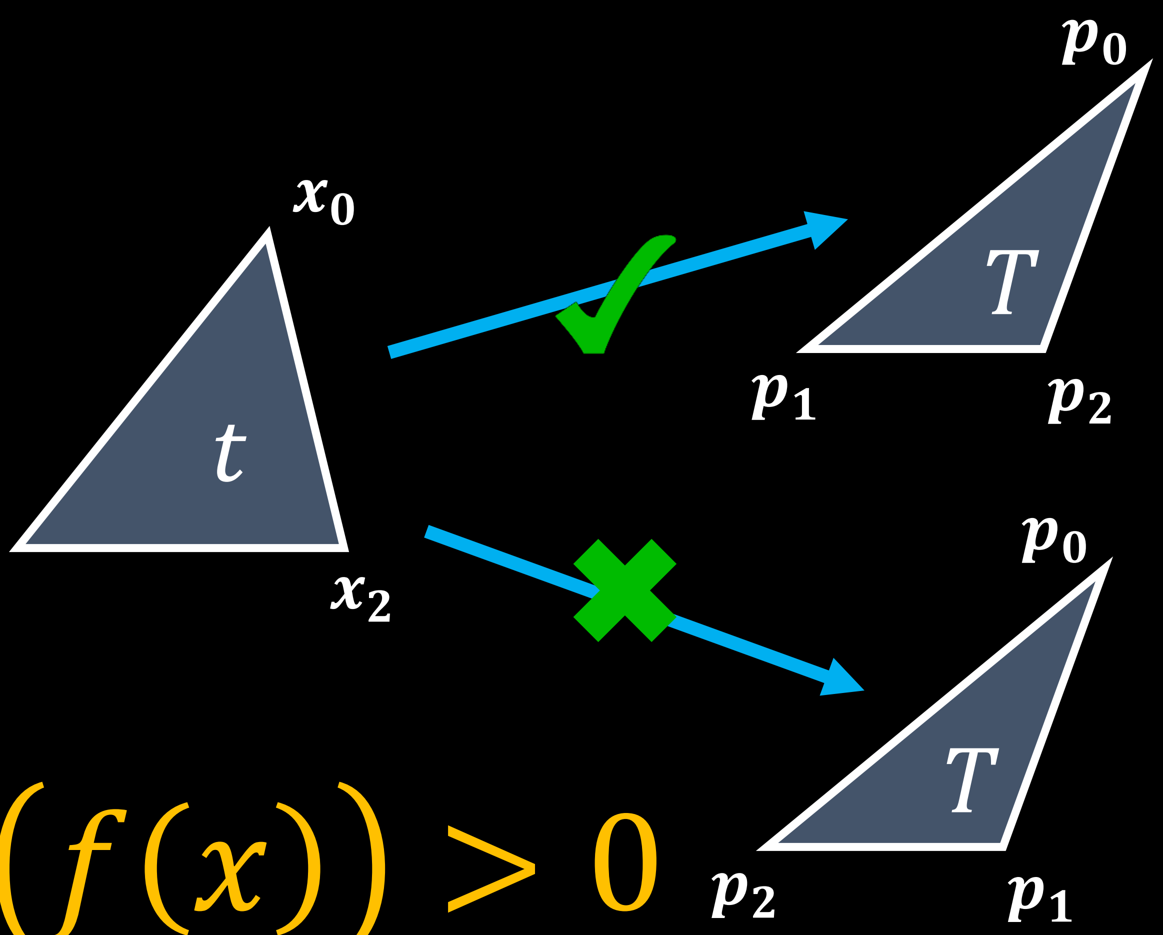
All meshes share a common connectivity.

Application - Learning



Basic requirements

- Foldover-free:
  - No realistic material can be compressed to zero or even negative volume.
  - Flipped elements correspond to physically impossible deformation.
  - Inverted elements lead invalidity for following applications, for example, remeshing.
  - ……

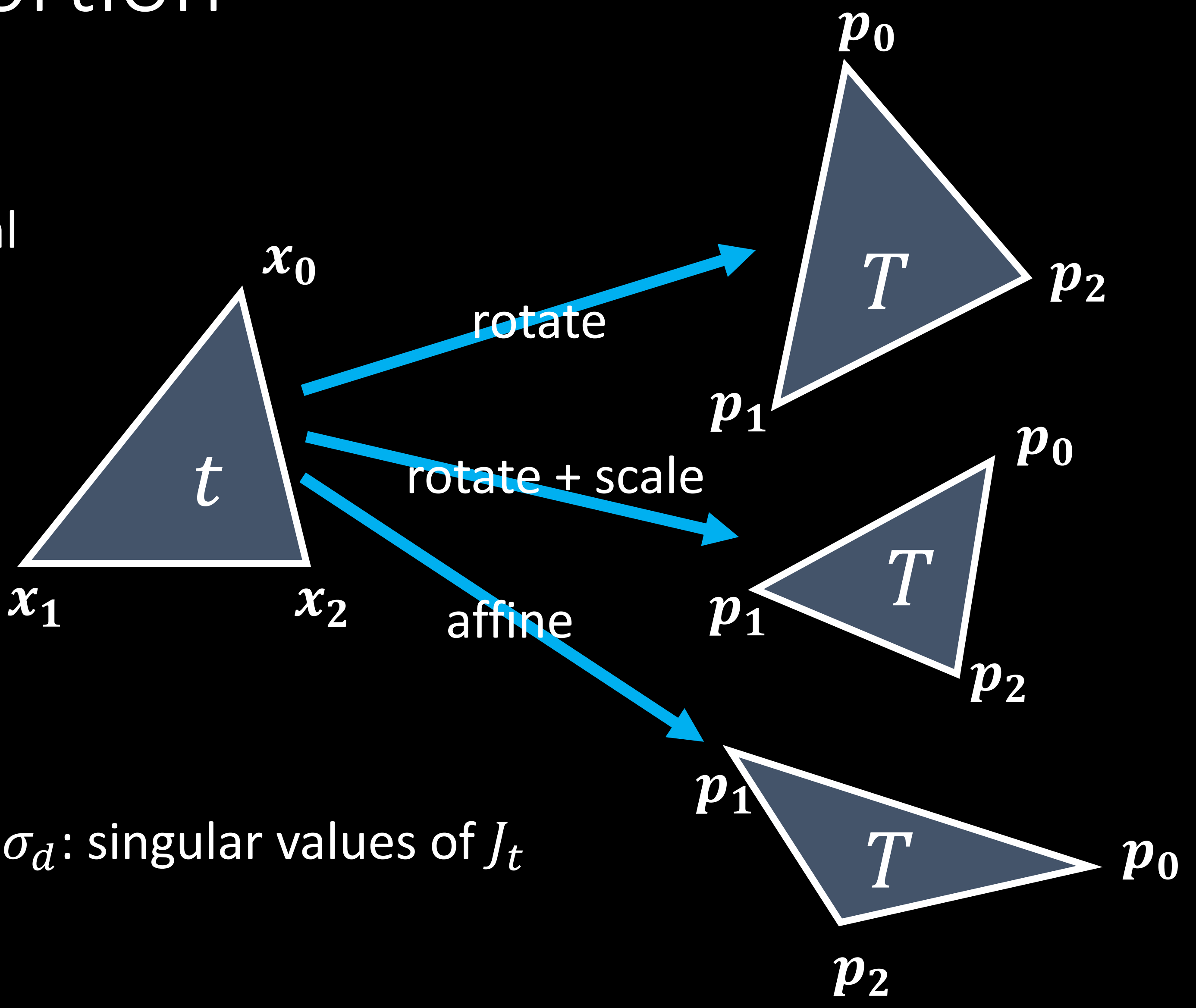


$$\det J(f(x)) > 0$$

Negative singed area

Basic goal – low distortion

- Distortion
  - Rotation: rigid transformation
    - ✓Isometric = conformal + equiareal
    - ✓$\delta^{iso} = \max\{\sigma_{max}, \frac{1}{\sigma_{min}}\}$
  - Similar transformation
    - ✓Conformal
    - ✓$\delta^{con} = \sigma_{max}/\sigma_{min}$
  - Affine transformation with positive determinant
- Our goal
  - As Rigid As Possible
  - As similar as Possible



$\sigma_1, \ldots, \sigma_d$: singular values of J_t

Common distortion metrics $D(f)$

- Common conformal distortion
  - LSCM: $\sum_t A_t(\sigma_1 - \sigma_2)^2$
  - MIPS: $\sum_t \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}$
- Common isometric distortion
  - ARAP: $\sum_t A_t\left((\sigma_1 - 1)^2 + (\sigma_2 - 1)^2\right)$
  - AMIPS: $\sum_t \left(\left(\frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1}\right) + \left(\frac{1}{\sigma_2\sigma_1} + \sigma_2\sigma_1\right)\right)$
  - Symmetric Dirichlet: $\sum_t A_t(\sigma_1^2 + \sigma_1^{-2} + \sigma_2^2 + \sigma_2^{-2})$

Formulation

$$\begin{aligned} &\min_f \; D(f) \\ &\text{s.t.} \;\; \det J\big(f(x)\big) > 0, \forall x \in M \\ &\qquad\; S(f) \le 0 \end{aligned}$$

$S(f) \le 0$: specific constraints for applications

$D(f)$: distortion metric

M: input mesh or domain

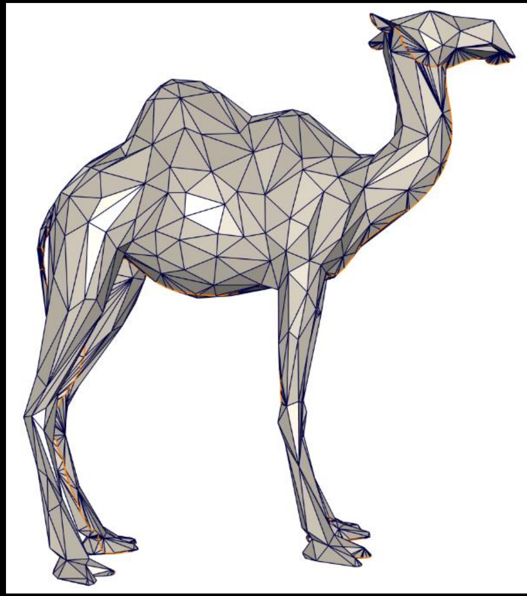
Outlines

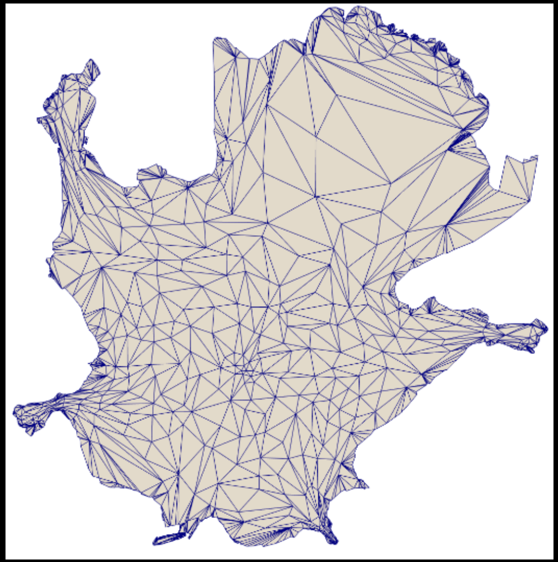
- Introduction
- Maintenance-based methods
  - MIPS: An efficient global parametrization method
- Bounded distortion methods
- Representation-based method

Most-Isometric ParameterizationS (MIPS)

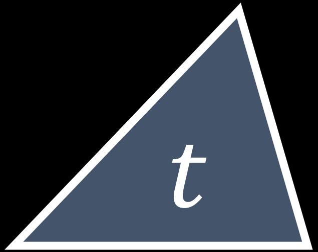
[Hormann and Greiner 2000]

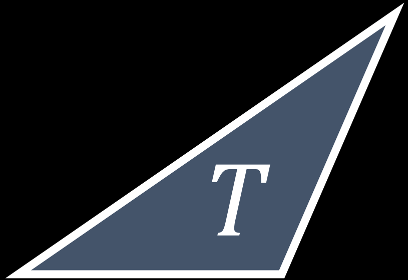
- Mapping: a triangle mesh → 2D parameterization region





$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$





MIPS energy

- MIPS energy on triangle t

$$E_{mips,t}^{2D} = \frac{\sigma_1}{\sigma_2} + \frac{\sigma_2}{\sigma_1} = \|J_t\|_F \|J_t^{-1}\|_F = \frac{trace(J_t^T J_t)}{\det(J_t)}$$

penalize degenerate triangles

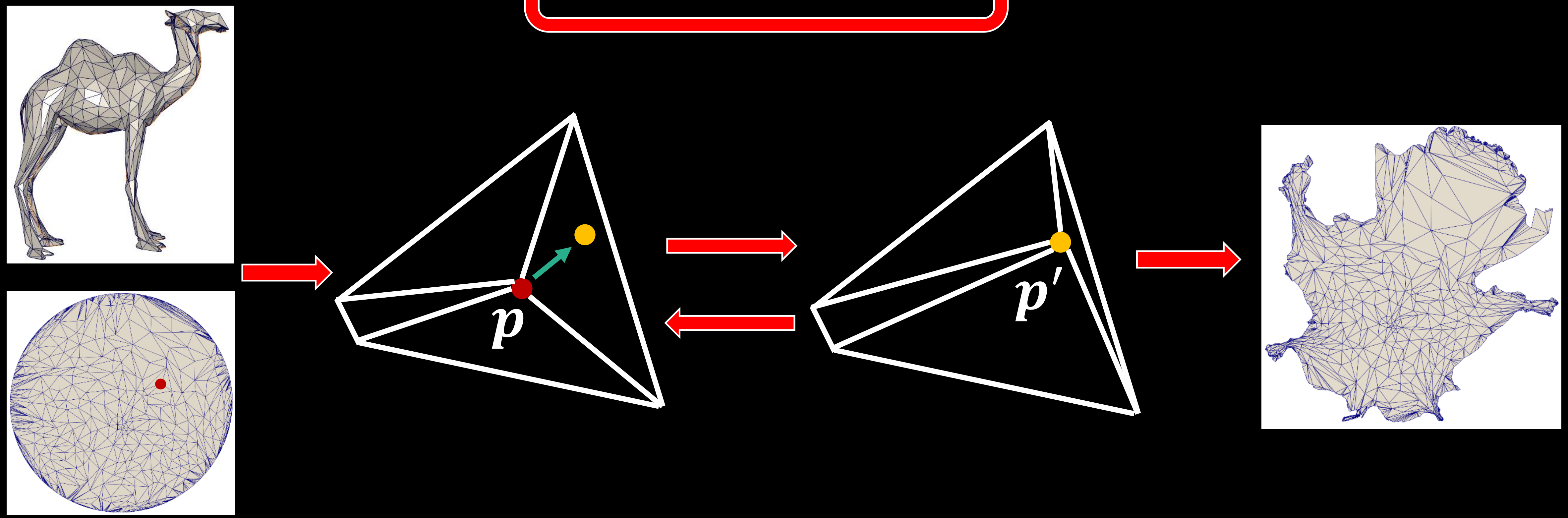
degenerate triangle t → $\det(J_t) \to 0$ → $E_{mips,t}^{2D} \to \infty$

a conformal energy

Optimal value when $\sigma_1 = \sigma_2$

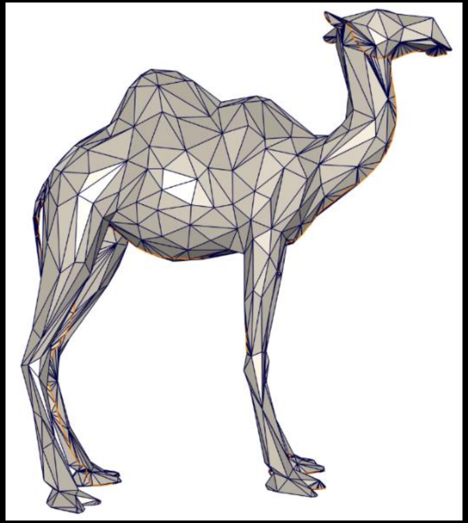
MIPS optimization – $\min \sum_t E_{mips,t}^{2D}$

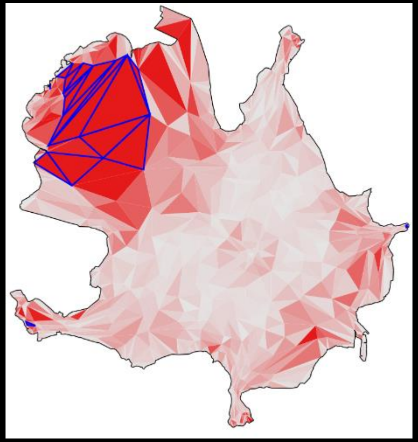
Initial valid mapping → Randomly pick a vertex $\boldsymbol{p}$ → Update $\boldsymbol{p}$ using Newton's method → Output valid mapping



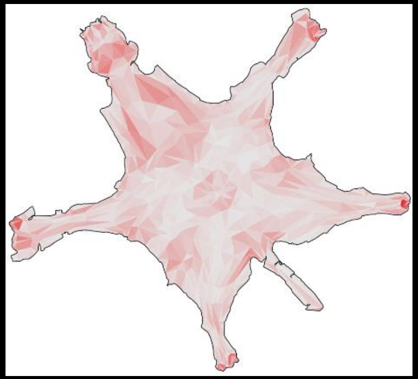
MIPS discussion

- Advantage: penalize degenerate triangles
- Disadvantages:
  - only for 2D conformal mapping
  - easily be trapped by local minimum
  - no strong penalization on maximal distortion





MIPS:
$\boldsymbol{\delta_{max}^{con} = 15.72}$
Time: 7.09s



AMIPS:
$\boldsymbol{\delta_{max}^{con} = 3.96}$
Time: 1.68s

Maintenance-based methods

- 1. An initial mapping that satisfies the constraints.
- 2. Reduce the distortion as much as possible while not violating the constraints.

- Parameterizations:
  - Initialization: Tutte’s embedding
  - distortion metrics
  - Solvers

Complex solvers

- AMIPS: Computing locally injective mappings by advanced MIPS (2015)
- AQP: Accelerated Quadratic Proxy for Geometric Optimization (2016)
- SLIM: Scalable locally injective mappings (2017)
- CM: Geometric optimization via composite majorization (2017)
- AKVF: Isometry-Aware Preconditioning for Mesh Parameterization (2017)
- ……

Outlines

- Introduction
- Maintenance-based methods
- Bounded distortion methods
  - Bounded distortion mapping spaces for triangular meshes
- Representation-based method

Bounded distortion mapping *[Lipman 2012]*

- Goal: explicitly bound the conformal distortion

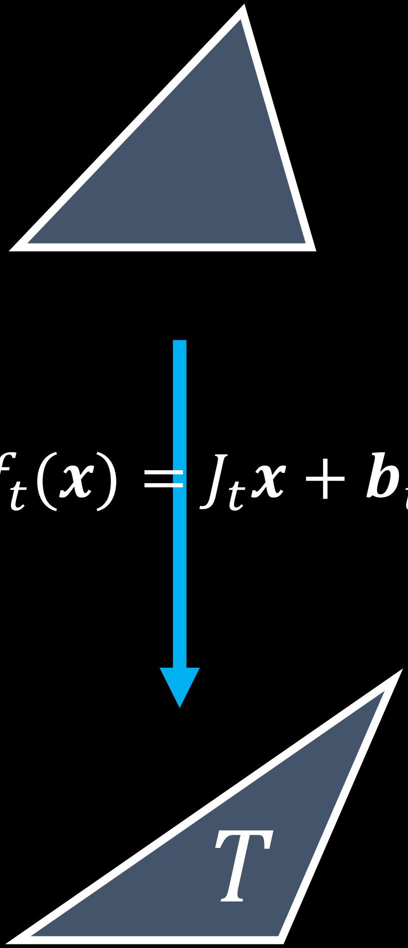
Constraints: $\delta_t^{con} < K, det\,(J_t) > 0$

non-linear and non-convex

$$J_t = \begin{pmatrix} a_t & b_t \\ c_t & d_t \end{pmatrix}$$

$$\boldsymbol{b}_t = \begin{pmatrix} b_{\boldsymbol{t},\boldsymbol{x}} \\ b_{\boldsymbol{t},\boldsymbol{y}} \end{pmatrix}$$

$$f_t(\boldsymbol{x}) = J_t\boldsymbol{x} + \boldsymbol{b}_t$$



Rewrite the constraints

$$J_t = \begin{pmatrix} a_t + c_t & d_t - b_t \\ d_t + b_t & a_t - c_t \end{pmatrix}$$

$$\sigma_{max} = \sqrt{a_t^2 + b_t^2} + \sqrt{c_t^2 + d_t^2}$$

$$\sigma_{min} = \left| \sqrt{a_t^2 + b_t^2} - \sqrt{c_t^2 + d_t^2} \right|$$

$$\boxed{\delta_t^{con} < K, det\,(J_t) > 0}$$

$$det\,(J_t) > 0 \longrightarrow \sqrt{c_t^2 + d_t^2} < \sqrt{a_t^2 + b_t^2}$$

$$\delta_t^{con} < K \longrightarrow \sqrt{c_t^2 + d_t^2} \le \frac{K-1}{K+1}\sqrt{a_t^2 + b_t^2}$$

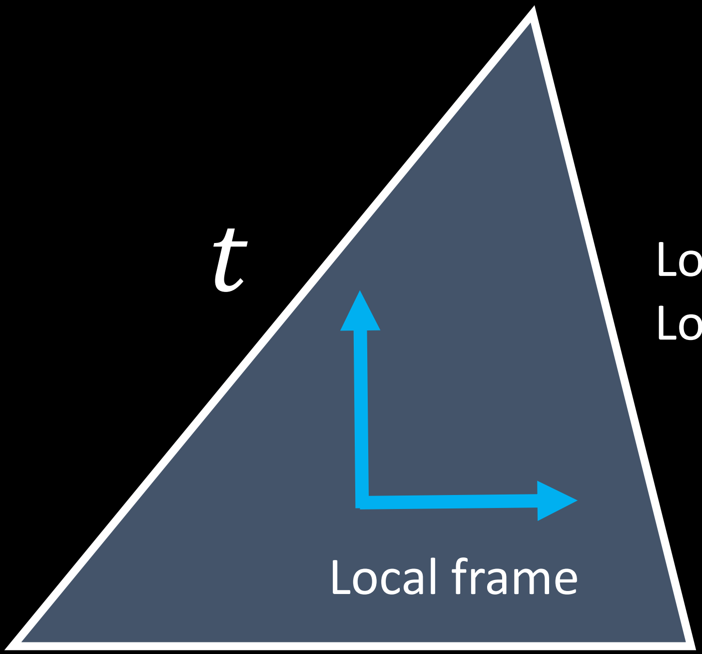
$$r_t \le \sqrt{a_t^2 + b_t^2}$$ Non-convex

$$\sqrt{c_t^2 + d_t^2} \le \frac{K-1}{K+1} r_t$$ Convex

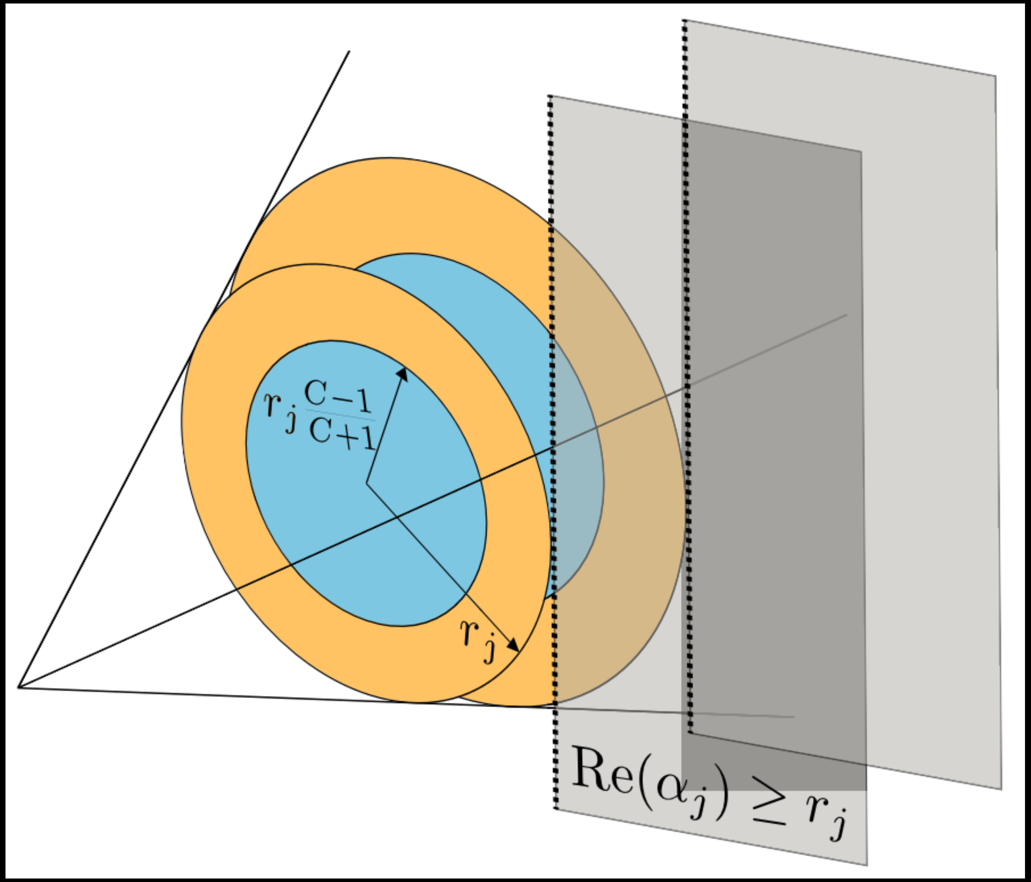
$$r_t > 0$$ Convex

maximal convex subset

$r_t \leq \sqrt{a_t^2 + b_t^2} \longrightarrow r_t \leq a_t$ Convex



Local frame changes, a_t changes.
Local frame is also a variable.



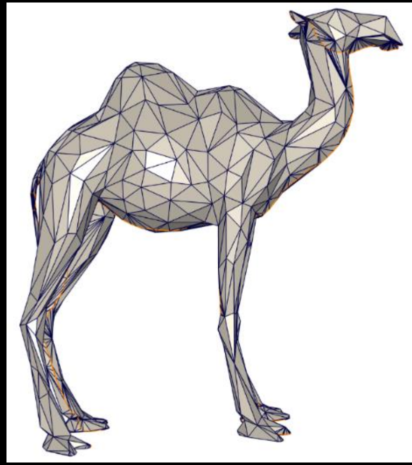
$\alpha_j = a_j + i \cdot b_j$

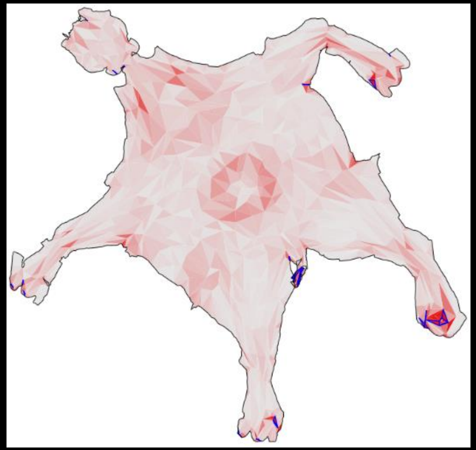
Optimization

- Objective function:
  - LSCM: $E = \sum_t Area(t) \cdot (c_t^2 + d_t^2)$
  - ARAP: $E = \sum_t Area(t) \cdot \left((a_t - 1)^2 + b_t^2 + c_t^2 + d_t^2\right)$

$$J_t = \begin{pmatrix} a_t + c_t & d_t - b_t \\ d_t + b_t & a_t - c_t \end{pmatrix}$$

- Optimization:
  - Fix the local frame on each triangle: Second-Order Cone Programming (SOCP);
  - Update local frame to let $b_t = 0$.





$\boldsymbol{\delta_{max}^{con} = 26.98}$

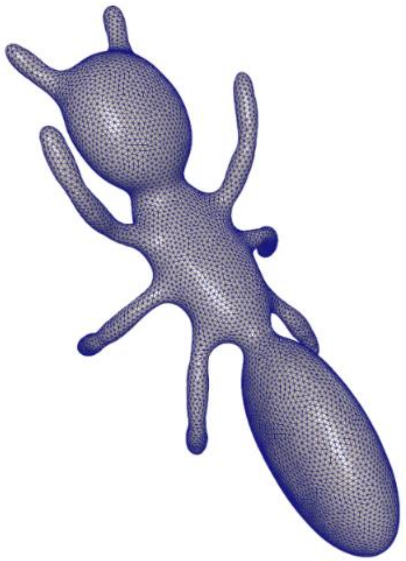
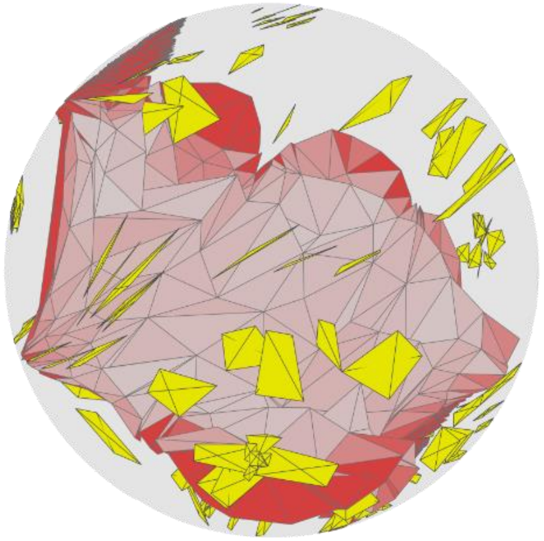
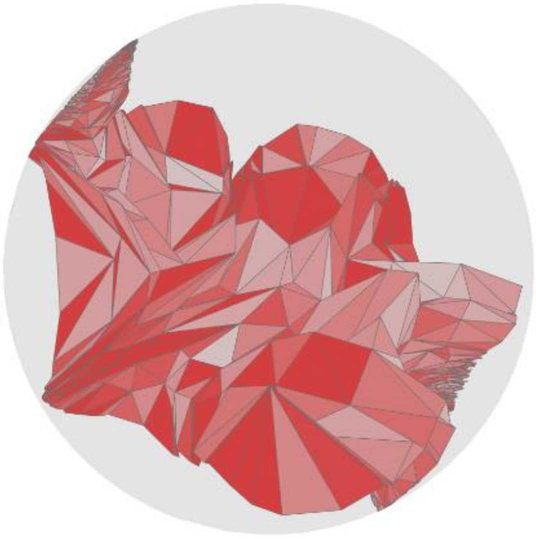
Time: 4.03s

1. How to choose K?
2. The speed is slow.

Local/global formulation

- Practical Foldover-Free Volumetric Mapping Construction (PG 2019)

Problem

Source tetrahedral mesh

Initial volumetric map

Foldover-free volumetric map

Input

Output

Preliminaries

- Signed singular value decomposition

$$J_i(\mathbf{u}) = U_i S_i V_i^T, S_i = diag(\sigma_{i,1}, \sigma_{i,2}, \sigma_{i,3})$$

$\sigma_{i,1} \geq \sigma_{i,2} \geq |\sigma_{i,3}|$.

- Foldover-free constraints

$$\det J_i(\mathbf{u}) > 0, i = 1, \cdots, N \Leftrightarrow \sigma_{i,3} > 0$$

- Conformal distortion

$$\tau\big(J_i(\mathbf{u})\big) = \sigma_{i,1}/\sigma_{i,3}$$

- Bounded conformal distortion constraints

$$1 \leq \tau\big(J_i(\mathbf{u})\big) \leq K$$

Constraints

Foldover-free constraints

$$\det J_i(\mathbf{u}) > 0$$

$$\sigma_{i,3} > 0, \tau(J_i) = \sigma_{i,1}/\sigma_{i,3}$$

$$\sigma_{i,1} \geq \sigma_{i,2} \geq |\sigma_{i,3}|$$

$$K = \max_{i=1,\cdots,N} \tau(J_i)$$

Bounded conformal distortion constraints

$$1 \leq \tau\big(J_i(\mathbf{u})\big)$$

$$\tau\big(J_i(\mathbf{u})\big) \leq K$$

$$\tau(J_i) \geq 1, \sigma_{i,3} > 0, \sigma_{i,3} > 0$$

?

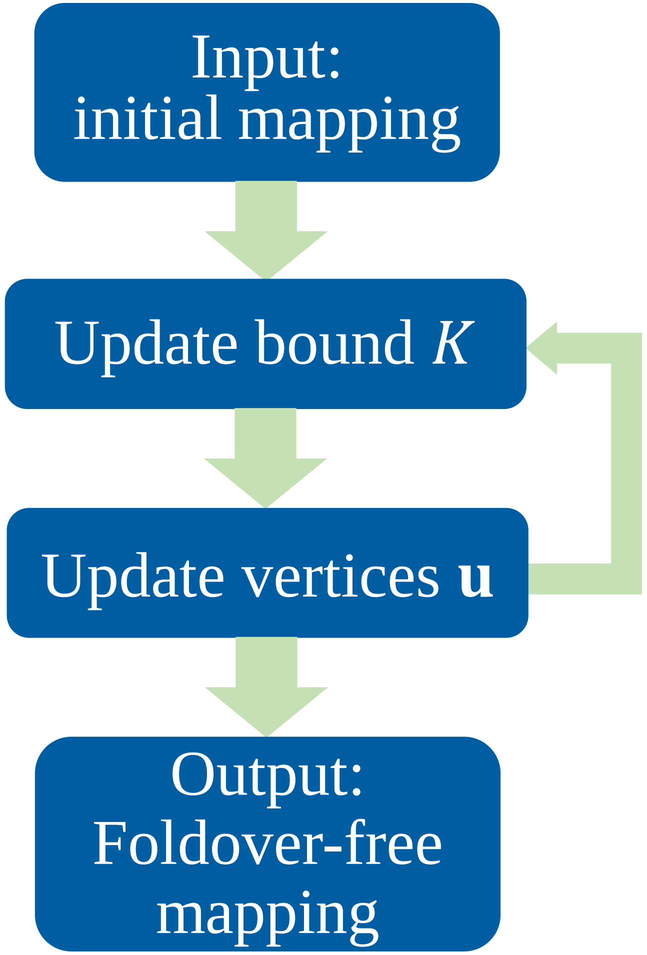
It is difficult to satisfy the constraints!

Our idea

$$1 \le \tau(J_i(\mathbf{u})) \le K$$

Alternatively solving K and $\mathbf{u}$

- Update K: generate a conformal distortion bound;
- Update $\mathbf{u}$: project the mapping into the bounded distortion space;
- If there are foldovers, go to Step 1;



Update vertices $\mathbf{u}$

- Monotone projection

$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\boxed{\mathbf{u}}) - \boxed{H_i}\|_F^2 ,$$
$$s.t. \quad H_i \in \mathcal{H}_i, i = 1, \cdots, N,$$
$$A\mathbf{u} = b.$$

$\mathcal{H}_i = \{H_i | 1 \leq \tau(H_i) \leq K\}$: bounded conformal distortion space.

Local-global solver

Update vertices $\mathbf{u}$

- Local-global solver

Local step

Fix $\mathbf{u}$ and J_i, solve H_i

$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\mathbf{u}) - H_i\|_F^2,$$

$$s.t. \quad H_i \in \mathcal{H}_i, i = 1, \cdots, N,$$

Global step

Fix H_i, solve $\mathbf{u}$

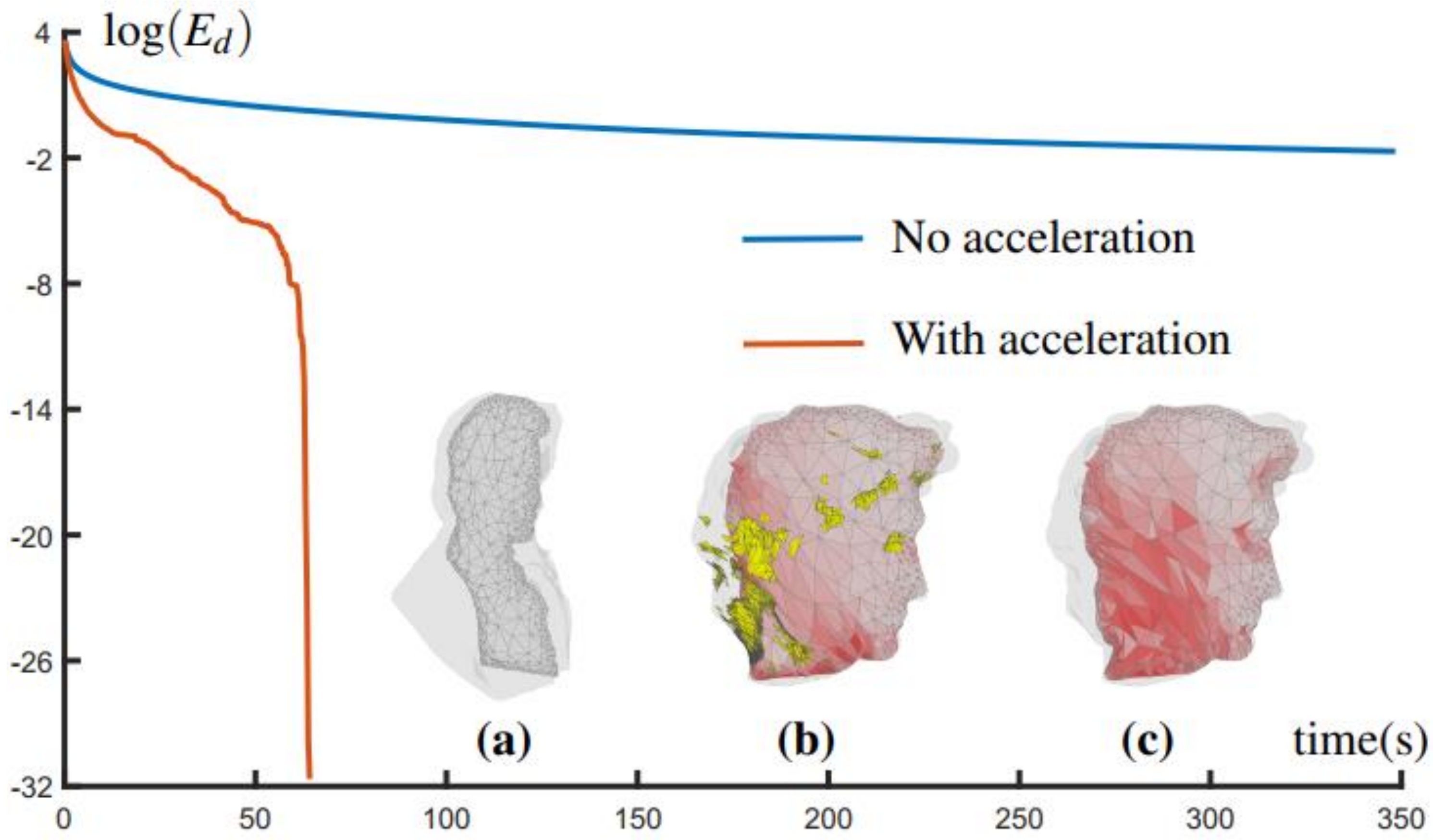
$$\min_{\mathbf{u}} \quad E_d = \sum_{i=1,\cdots,N} \|J_i(\mathbf{u}) - H_i\|_F^2,$$

$$s.t. \quad A\mathbf{u} = b$$

Very slow convergence…

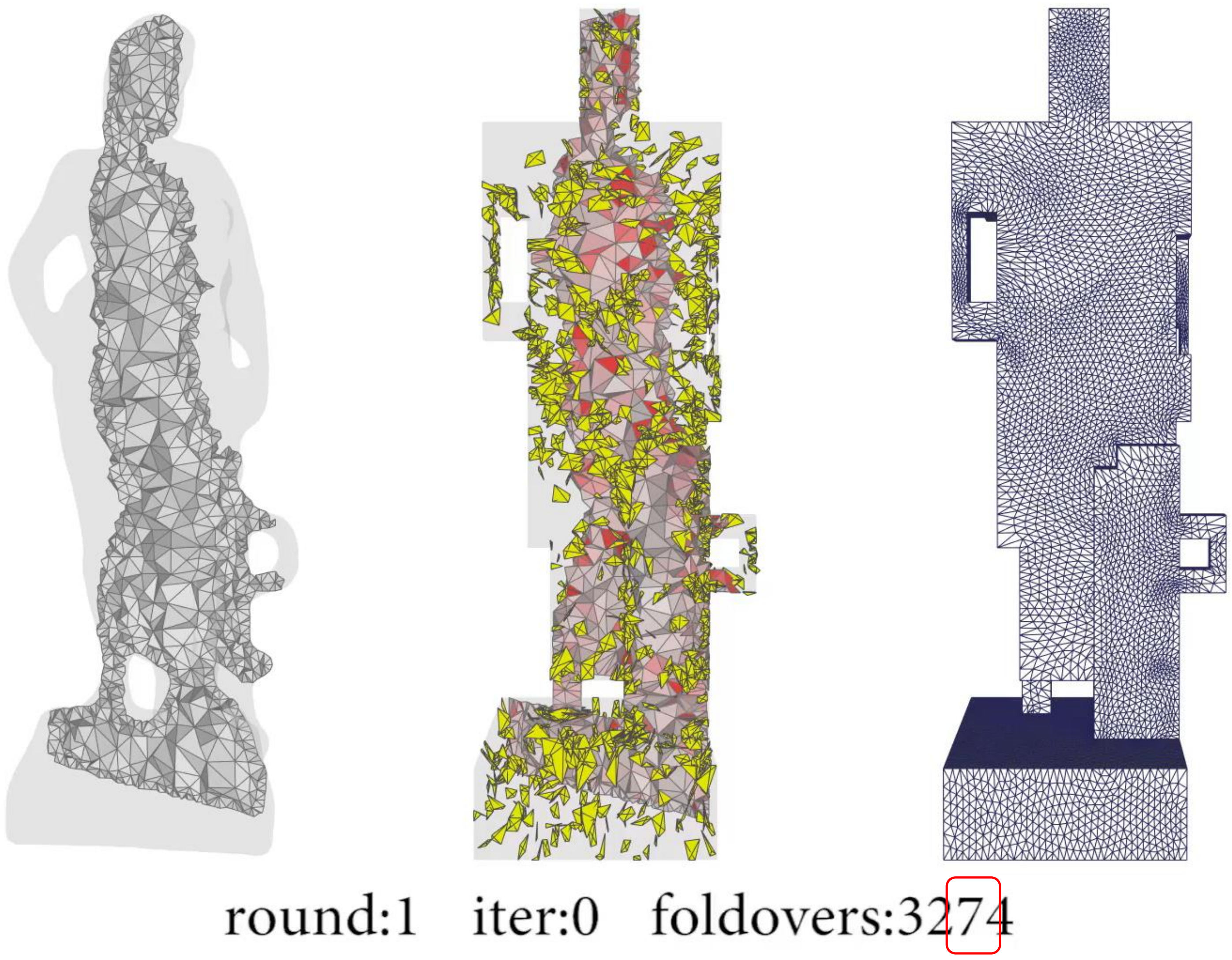
Update vertices u

- Anderson acceleration method [Peng et al. 2018]



Why update bound K?

Projection cannot eliminate all foldovers



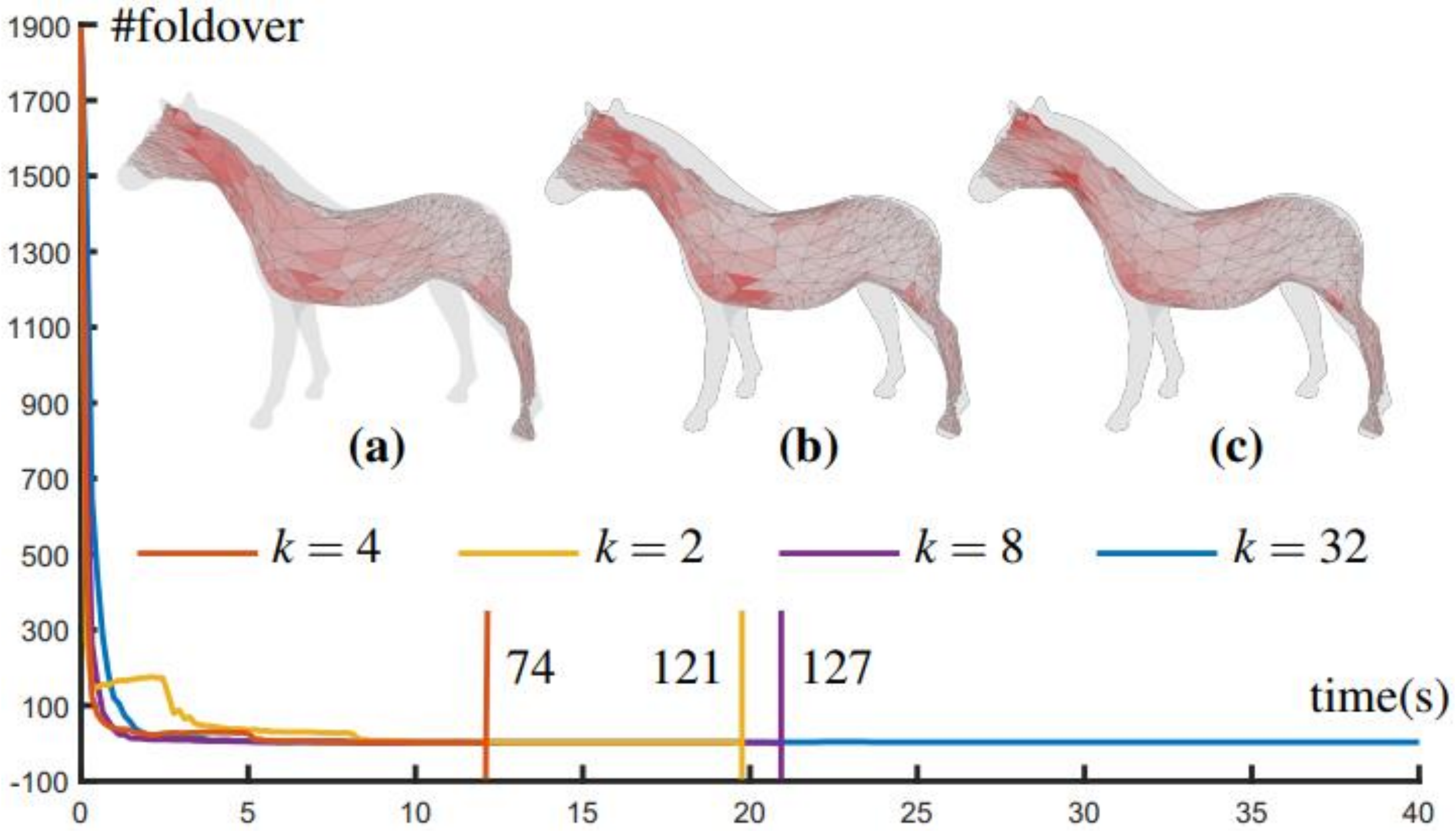
Update bound K

- Bound generation

$$K^{new} = \beta K$$

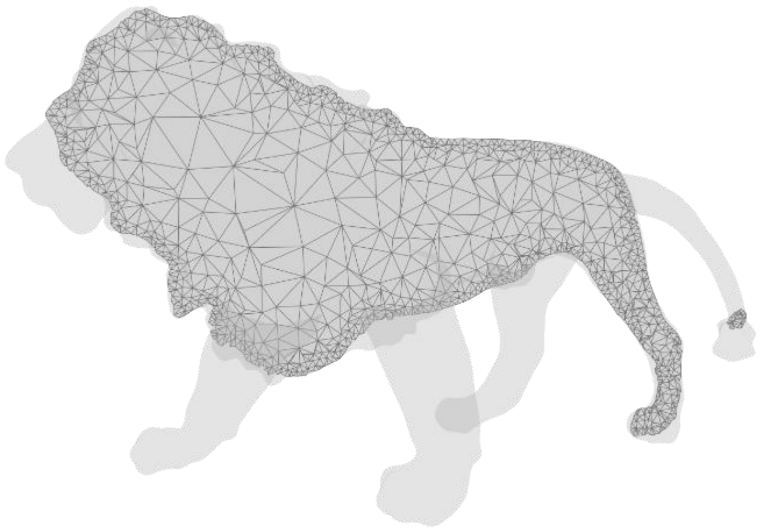
$\beta = 2$

initialize $K = 4$



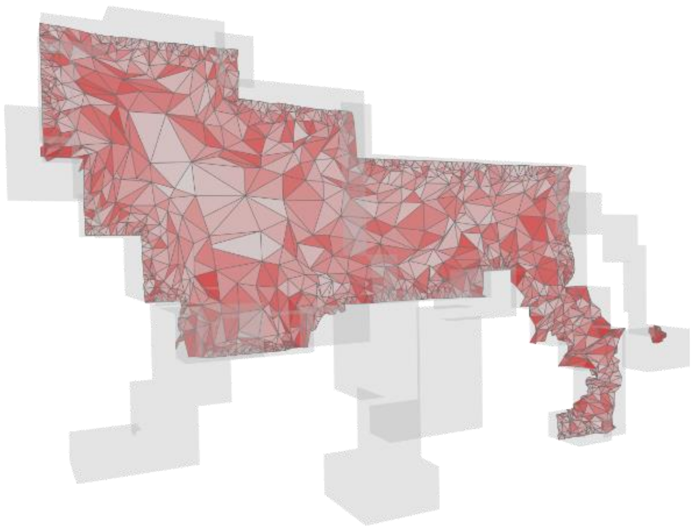
Post-optimization

- Apply a maintenance-based method



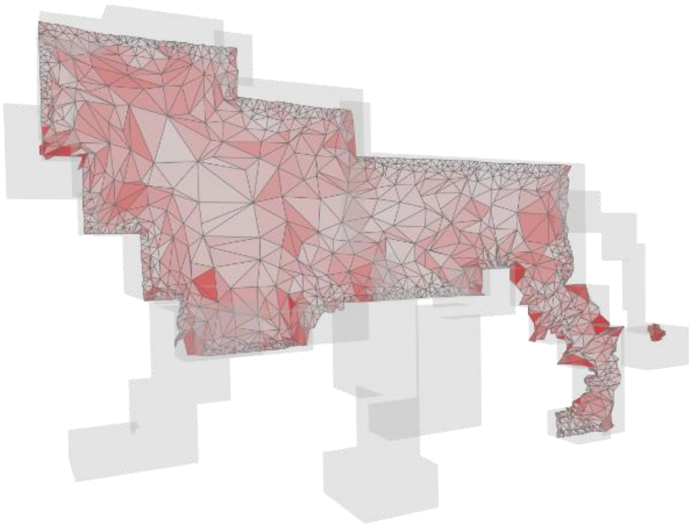
Source tetrahedral mesh

Before post-optimization



Average / maximum conformal distortion: 2.72 / 107.10

After post-optimization



Average / maximum conformal distortion: 2.08 / 22.61

Recap of our algorithm

Elimination Process :

alternates in each round:

- generate conformal distortion bound

- try to project the mapping into the bounded distortion space

Outlines

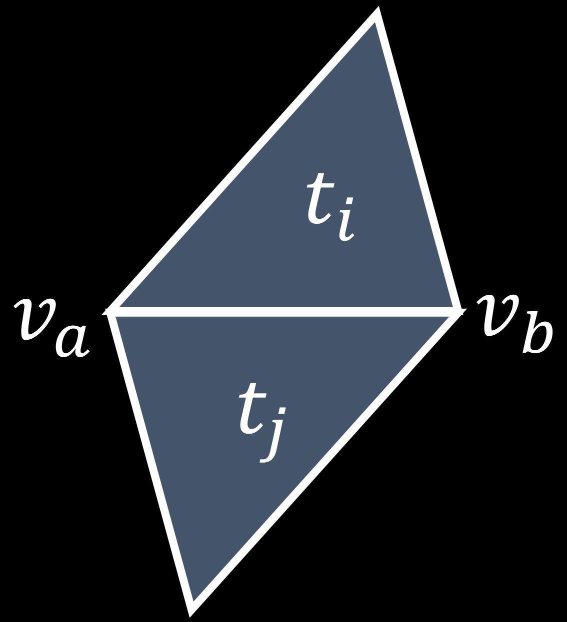
- Introduction
- Maintenance-based methods
- Bounded distortion methods
- Representation-based method
  - Computing inversion-free mappings by simplex assembly

Affine transformation

Key observation: the parameter space is a 2D triangulation, uniquely defined by all the AFFINE TRANSFORAMTIONS on the triangles.

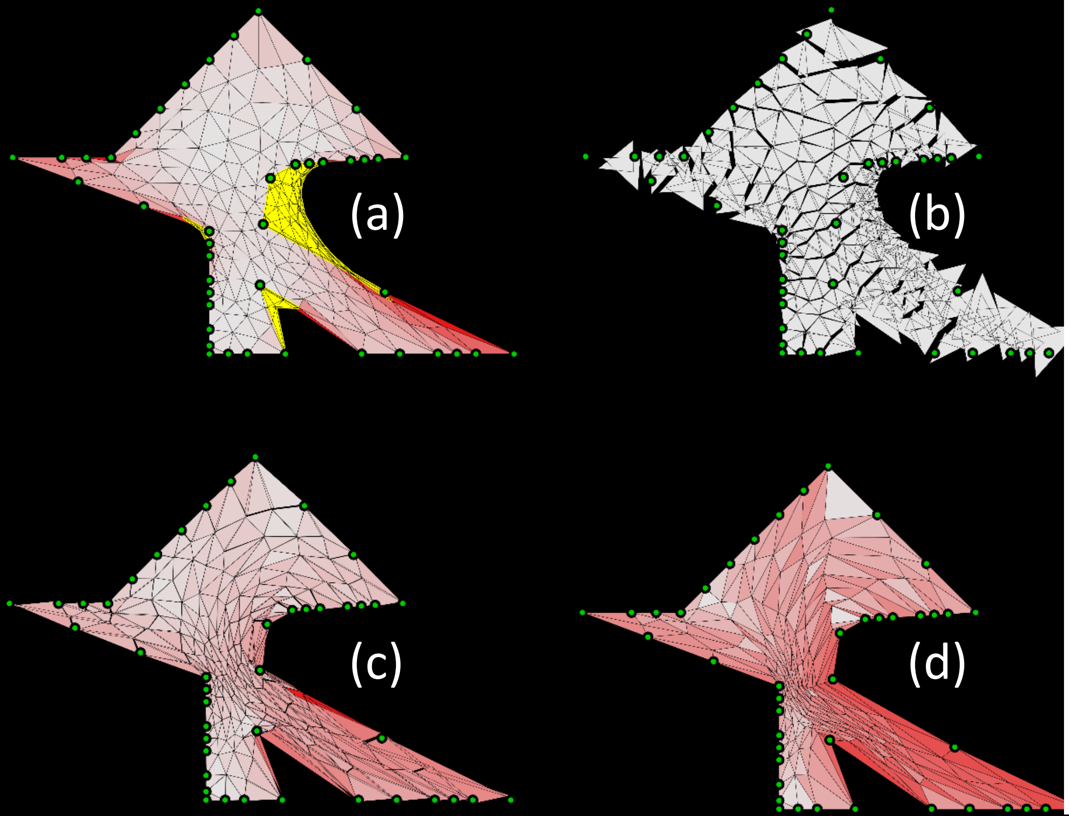
Edge assembly constraints:

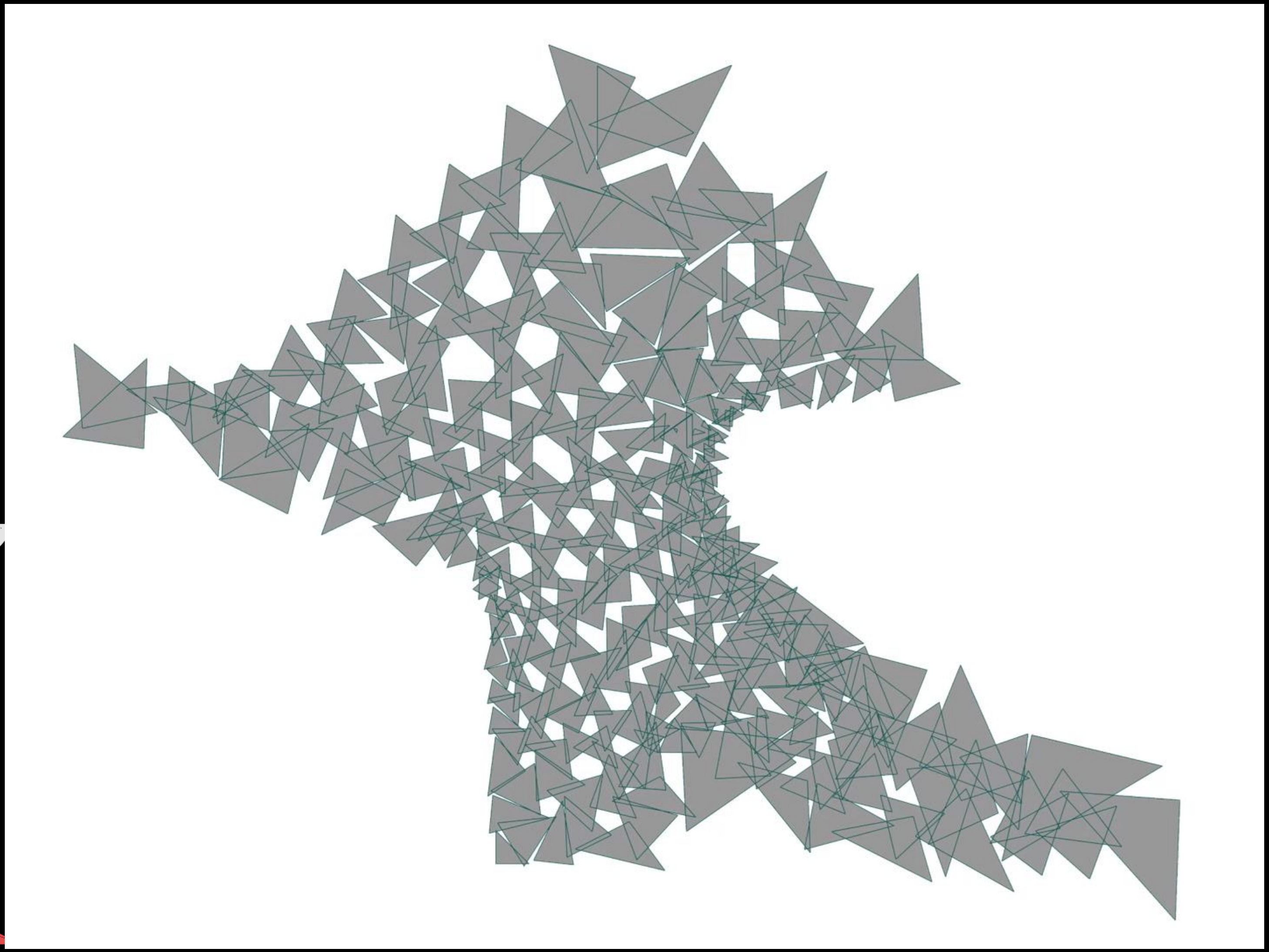
$$A_i(v_a - v_b) = A_j(v_a - v_b)$$



Key idea

- disassembly + assembly
  - Treat affine transformation as variables
  - Unconstrained optimization





Unconstrained optimization problem

Disassembly: project initial A_i^0 into feasible space.

Assembly: unconstrained optimization.

$$\min_{\substack{A_1,\ldots,A_N \\ T_1,\ldots,T_N}} \lambda E_{assembly} + E_C + \mu E_m$$

$E_{assembly}$: summation of squares of edge, assembly constraints.

E_C: Barrier function on distortion

E_m: users' designed energy

$$\lambda_{\mathrm{k+1}} = \min\left(\lambda_{\min} \cdot \max\left(\frac{E_{C,k} + \mu E_{m,k}}{E_{assembly,k}}, 1\right), \lambda_{\max}\right)$$

1. $E_{assembly}$ dominates the energy, approach zero;
2. $\lambda_{\max}$: avoid large distortion.