Exam 2

Problem 1: Independent normal random variables

Let U, V, and W be independent standard normal random variables (that is, independent normal random variables, each with mean 0 and variance 1), and let $X = 3U + 4V$ and $Y = U + W$. Give a numerical answer for each part below. You may want to refer to the standard normal table.

1. What is the probability that $X \geq 8$? $\mathbf{P}(X \geq 8) =$
2. $\mathbf{E}[XY] =$
3. $\text{var}(X + Y) =$

Problem 2: Calculation with PDFs

Let X be a random variable that takes non-zero values in $[1, \infty)$, with a PDF of the form

$$f_X(x) = \begin{cases} \frac{c}{x^3}, & \text{if } x \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let U be a uniform random variable on $[0, 2]$. Assume that X and U are independent.

1. What is the value of the constant c? $c =$

2. $\mathbf{P}(X \leq U) =$

3. Find the PDF of $D = 1/X$. Express your answer in terms of d using standard notation. For $0 \leq d \leq 1$, $f_D(d) =$

Problem 3: Sums of a random number of random variables

Let $N, X_1, Y_1, X_2, Y_2, \ldots$ be independent random variables. The random variable N takes positive integer values and has mean a and variance r. The random variables X_i are independent and identically distributed with mean b and variance s, and the random variables Y_i are independent and identically distributed with mean c and variance t. Let

$$A = \sum_{i=1}^{N} X_i \quad \text{and} \quad B = \sum_{i=1}^{N} Y_i.$$

1. Find $\text{cov}(A, B)$. Express your answer in terms of the given means and variances using standard notation. $\text{cov}(A, B) =$

2. Find $\text{var}(A+B)$. Express your answer in terms of the given means and variances using standard notation. $\text{var}(A + B) =$

Problem 4: Manhole covers

Manhole explosions (usually caused by gas leaks and sparks) are on the rise in your city. On any given day, the manhole cover near your house explodes with some unknown probability, which is the same across all days. We model this unknown probability of explosion as a random variable Q, which is uniformly distributed between 0 and 0.1. Let X_i be a Bernoulli random variable that indicates whether the manhole cover near your house explodes on day i (where today is day 1).

Give numerical answers for parts (1) and (2).

1. $\mathbf{E}[X_i] =$

2. $\mathrm{var}(X_i) =$

3. Let A be the event that the manhole cover did not explode yesterday (i.e., $X_0 = 0$). Find the conditional PDF of Q given A. Express your answer in terms of q using standard notation. For $0 \le q \le 0.1$, $f_{Q|A}(q) =$

Problem 5: Fire alarm

Consider a fire alarm that senses the environment constantly to figure out if there is smoke in the air and hence to conclude whether there is a fire or not. Consider a simple model for this phenomenon. Let Θ be the unknown true state of the environment: $\Theta = 1$ means that there is a fire and $\Theta = 0$ means that there is no fire. The signal observed by the alarm at time n is $X_n = \Theta + W_n$, where the random variable W_n represents noise. Assume that W_n is Gaussian with mean 0 and variance 1 and is independent of Θ. Furthermore, assume that for $i \neq j$, W_i and W_j are independent. Suppose that Θ is 1 with probability 0.1 and 0 with probability 0.9.

Give numerical answers for all parts below.

1. Given the observation $X_1 = 0.5$, calculate the posterior distribution of Θ. That is, find the conditional distribution of Θ given $X_1 = 0.5$. $\mathbf{P}(\Theta = 0 \mid X_1 = 0.5) =$

   $\mathbf{P}(\Theta = 1 \mid X_1 = 0.5) =$

2. What is the LMS estimate of Θ given $X_1 = 0.5$? $\widehat{\theta}_{LMS} =$

3. What is the resulting conditional mean squared error of the LMS estimator given $X_1 = 0.5$?

Problem 6: Two measurement instruments

Let Θ be an unknown random variable that we wish to estimate. It has a prior distribution with mean 1 and variance 2. Let W be a noise term, another unknown random variable with mean 3 and variance 5. Assume that Θ and W are independent.

We have two different instruments that we can use to measure Θ. The first instrument yields a measurement of the form $X_1 = \Theta + W$, and the second instrument yields a measurement of the form $X_2 = 2\Theta + 3W$. We pick an instrument at random, with each instrument having probability 1/2 of being chosen. Assume that this choice of instrument is independent of everything else. Let X be the measurement that we observe, without knowing which instrument was used.

Give numerical answers for all parts below.

1. $\mathbf{E}[X] =$

2. $\mathbf{E}[X^2] =$

3. The LLMS estimator of Θ given X is of the form $aX + b$. Give the numerical values of a and b.
   $a =$

   $b =$