Exam 1

Problem 1: True or false

We are told that events A and B are conditionally independent, given a third event C, and that $\mathbf{P}(B \mid C) > 0$. For each one of the following statements, decide whether the statement is "Always true", or "Not always true."

1. A and B are conditionally independent, given the event C^c.
2. A and B^c are conditionally independent, given the event C.
3. $\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid B)$
4. $\mathbf{P}(A \mid B \cap C) = \mathbf{P}(A \mid C)$

Problem 2: A binary communication system - Part 1

A binary communication system is used to send one of two messages:

(i) message A is sent with probability $2/3$, and consists of an infinite sequence of zeroes,

(ii) message B is sent with probability $1/3$, and consists of an infinite sequence of ones.

The ith received bit is "correct" (i.e., the same as the transmitted bit) with probability $3/4$, and is "incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability $1/4$. We assume that conditioned on any specific message sent, the received bits, denoted by $Y_1, Y_2, \ldots$ are independent.

Note: Enter numerical answers; do not enter '!' or combinations.

1. Find $\mathbf{P}(Y_1 = 0)$, the probability that the first bit received is 0.

2. Given that message A was transmitted, what is the probability that exactly 6 of the first 10 received bits are ones? (Answer with at least 3 decimal digits.)

3. Find the probability that the first and second received bits are the same.

4. Given that $Y_1, \ldots, Y_5$ were all equal to 0, what is the probability that Y_6 is also zero?

5. Find the mean of K, where $K = \min\{i : Y_i = 1\}$ is the index of the first bit that is 1.

Problem 2: A binary communication system - Part 2

Note: The problem statement from part 1 has been repeated here for your convenience.

A binary communication system is used to send one of two messages:

(i) message A is sent with probability $2/3$, and consists of an infinite sequence of zeroes,

(ii) message B is sent with probability $1/3$, and consists of an infinite sequence of ones.

The ith received bit is "correct" (i.e., the same as the transmitted bit) with probability $3/4$, and is "incorrect" (i.e., a transmitted 0 is received as a 1, and vice versa), with probability $1/4$. We assume that conditioned on any specific message sent, the received bits, denoted by $Y_1, Y_2, \ldots$ are independent.

1. Is $Y_2 + Y_3$ independent of Y_1?

2. Is $Y_2 - Y_3$ independent of Y_1?

Problem 3: A six-sided die

A fair, 6-sided die is rolled 6 times independently. Assume that the results of the different rolls are independent. Let $(a_1, \ldots, a_6)$ denote a typical outcome, where each a_i belongs to $\{1, \ldots, 6\}$.

Note: Enter numerical answers; do not enter '!' or combinations. The following table for $\binom{n}{k}$ up to $n = 6$ has been provided for your convenience:

n \ k	0	1	2	3	4	5	6
1	1	1					
2	1	2	1				
3	1	3	3	1			
4	1	4	6	4	1		
5	1	5	10	10	5	1	
6	1	6	15	20	15	6	1

1. Find the probability that the results of the 6 rolls are all different. (Answer with at least 3 decimal digits.)

For any outcome $\omega = (a_1, \ldots, a_6)$, let $R(\omega)$ be the **set** $\{a_1, \ldots, a_6\}$; this is the set of numbers that showed up at least once in the different rolls. For example, if $\omega = (2, 2, 5, 2, 3, 5)$, then $R(\omega) = \{2, 3, 5\}$.

2. Find the probability that $R(\omega)$ has exactly two elements. (Answer with at least 3 decimal digits.)

3. Find the probability that $R(\omega)$ has exactly three elements.

Problem 4: Indicator random variables

Let A, B, C be three events and let $X = I_A$, $Y = I_B$, $Z = I_C$ be the associated indicator random variables. We already know that $X \cdot Y$ is the indicator random variable of the event $A \cap B$. In the same spirit, give an algebraic expression, involving X, Y, Z for the indicator random variable of the following events.

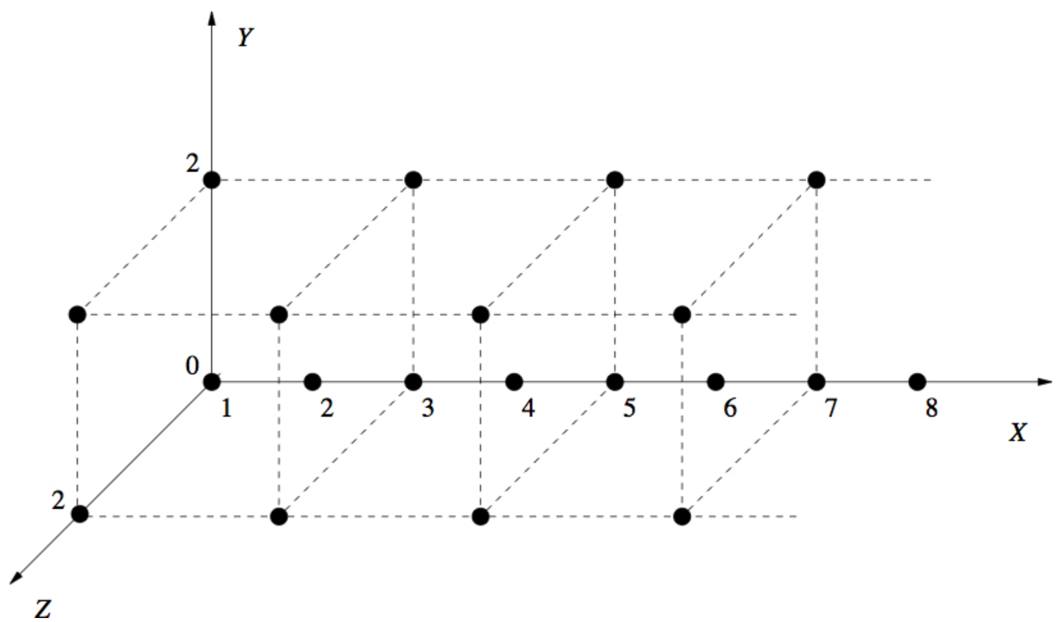
Note: Express your answers in terms of X, Y and Z (the answer box is case sensitive) using standard notation.

1. The event $A^c \cap C^c$

2. Exactly one of the events A, B, C occurred.

Problem 5: Joint PMF calculations - Part 1

Consider three random variables X, Y, and Z, associated with the same experiment. The random variable X is geometric with parameter $p \in (0,1)$. If X is even, then Y and Z are equal to zero. If X is odd, (Y, Z) is uniformly distributed on the set $S = \{(0,0),(0,2),(2,0),(2,2)\}$. The figure below shows all the possible values for the triple (X, Y, Z) that have $X \leq 8$. (Note that the X axis starts at 1 and that a complete figure would extend indefinitely to the right.)



1. Answer the following with "Yes" or "No":

   a) Are Y and Z independent?

   b)Given that $Z = 2$, are X and Y independent?

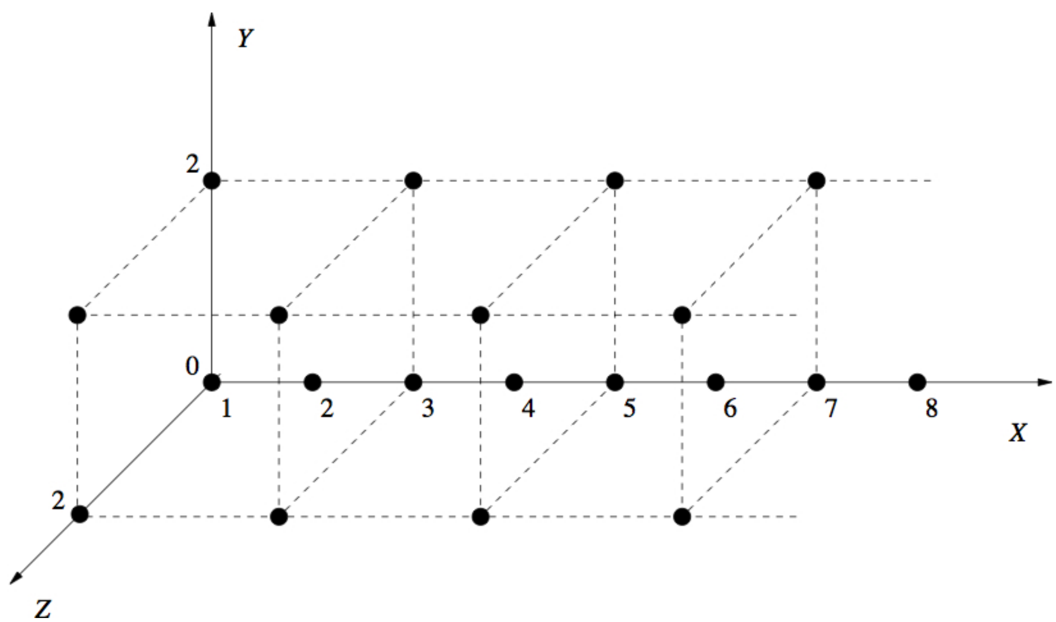
   c) Given that $Z = 0$, are X and Y independent?

   d) Given that $Z = 2$, are X and Z independent?

Problem 5: Joint PMF calculations - Part 2

Note: The problem statement from part 1 has been repeated here for your convenience.

Consider three random variables X, Y, and Z, associated with the same experiment. The random variable X is geometric with parameter $p \in (0, 1)$. If X is even, then Y and Z are equal to zero. If X is odd, (Y, Z) is uniformly distributed on the set $S = \{(0,0), (0,2), (2,0), (2,2)\}$. The figure below shows all the possible values for the triple (X, Y, Z) that have $X \leq 8$. (Note that the X axis starts at 1 and that a complete figure would extend indefinitely to the right.)



1. Find the joint PMF $p_{X,Y,Z}(x, y, z)$. Express your answers in terms of x and p using standard notation.

   If x is odd and $(y, z) \in \{(0,0), (0,2), (2,0), (2,2)\}$,
   $p_{X,Y,Z}(x, y, z) =$
   If x is even and $(y, z) = (0, 0)$,
   $p_{X,Y,Z}(x, y, z) =$

2. Find $p_{X,Y}(x, 2)$, for when x is odd. Express your answer in terms of x and p using standard notation.
   If x is odd,
   $p_{X,Y}(x, 2) =$

3. Find $p_Y(2)$. Express your answer in terms of p using standard notation.
   $p_Y(2) =$

4. Find $\text{var}(Y + Z \mid X = 5)$.