Final Exam

Problem 1: Independent uniform random variables

Let X and Y be independent random variables, each uniformly distributed on the interval $[0, 2]$.

1. Find the mean and variance of XY.
   $\mathbf{E}[XY] =$

   $\text{var}[XY] =$

2. Find the probability that $XY \geq 1$. Enter a numerical answer.
   $\mathbf{P}(XY \geq 1) =$

Problem 2: Busy signal

Suppose that you move to a new house and you are 10% sure that your new house's phone number is 561290. To verify this, you use the house's phone to dial 561290, obtain a busy signal, and conclude that this is indeed your phone number. (Suppose that you were incorrect, and that this is somebody else's phone number. In that case, you will receive a busy signal if and only if that person is making a phone call at the same time. But if you call your own number, you will always receive a busy signal.)

Assume that the probability of a random six-digit phone number being in use at any given time is 0.001. What is the probability that you will be correct in concluding that 561290 is indeed your own phone number? Your answer should be accurate to 3 decimal places.

Problem 3: Weight fluctuation

Uncle Henry has been having trouble keeping his weight constant. In fact, during each week, his weight changes from the beginning of the week to the end of the week by a random amount, uniformly distributed between -0.5 and 0.5 pounds. Assuming that his weight change during any given week is independent of his weight change during any other week, approximate the probability that at the end of 50 weeks Uncle Henry will have had a net change in weight of at least $+3$ pounds. You may want to refer to the standard normal table.

Problem 4: Maximum likelihood estimation

Let θ be an unknown constant. Let $W_1, \ldots, W_n$ be independent exponential random variables each with parameter 1. Let $X_i = \theta + W_i$.

1. What is the maximum likelihood estimate of θ based on a single observation $X_1 = x_1$? Enter your answer in terms of x_1 (enter as `x_1`) using standard notation.
   $\hat{\theta}_{ML}(x_1) =$

2. What is the maximum likelihood estimate of θ based on a sequence of observations $(X_1, \ldots, X_n) = (x_1, \ldots, x_n)$?
   $\hat{\theta}_{ML}(x_1, \ldots, x_n) =$
   - $(x_1 x_2 \cdots x_n)^{1/n}$
   - $\dfrac{x_1 + \cdots + x_n}{n}$
   - $\dfrac{1}{\frac{1}{x_1} + \cdots \frac{1}{x_n}}$
   - $\min\limits_i x_i$
   - $\max\limits_i x_i$
   - None of the above

3. You have been asked to construct a confidence interval of the particular form $[\hat{\Theta} - c, \hat{\Theta}]$, where $\hat{\Theta} = \min_i\{X_i\}$ and c is a constant that we need to choose. For $n = 10$, how should the constant c be chosen so that we have a 95% confidence interval? (Give the smallest possible value of c.) Your answer should be accurate to 3 decimal places.
   $c =$

Problem 5: Office hours

A dedicated professor has been holding infinitely long office hours. Undergraduate students arrive according to a Poisson process at a rate of $\lambda_u = 3$ per hour, while graduate students arrive according to a second, independent Poisson process at a rate of $\lambda_g = 5$ per hour. An arriving student receives immediate attention (the previous student's stay is immediately terminated) and stays with the professor until the next student arrives. (Thus, the professor is always busy, meeting with the most recently arrived student.)

(1) What is the probability that exactly three undergraduates arrive between 10:00 pm and 10:30 pm?

(2) What is the expected length of time in hours that the 10th arriving student (undergraduate or graduate) will stay with the professor?

(3) Given that the professor is currently talking with an undergraduate, what is the expected number of subsequent student arrivals up to and including the next graduate student arrival?

(4) Given that the professor is currently talking with an undergraduate, what is the probability that 5 of the next 7 students to arrive will be undergraduates?

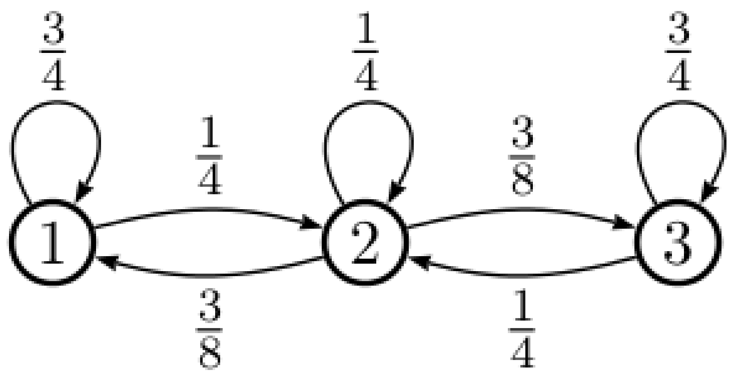
As rumors spread around campus, a worried department head drops in at midnight and begins observing the professor.

(5) Beginning at midnight, what is the expected length of time until the next student arrives, conditioned on the event that the next student will be an undergraduate?

(6) What is the expected time that the department head will have to wait until the set of students he/she has observed meeting with the professor (including the student who was meeting the professor when the deparment head arrived) include both an undergraduate and a graduate student?

Problem 6: Markov chain

Consider a Markov chain $X_0, X_1, X_2, \ldots$ described by the transition probability graph shown below. The chain starts at state 1; that is, $X_0 = 1$.



1. Find the probability that $X_2 = 3$.
   $\mathbf{P}(X_2 = 3) =$

2. Find the probability that the process is in state 3 immediately after the second change of state. (A "change of state" is a transition that is not a self-transition.)

3. Find (approximately) $\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001})$. $\mathbf{P}(X_{1000} = 2 \mid X_{1000} = X_{1001}) \approx$

4. Let T be the first time that the state is equal to 3. $\mathbf{E}[T] =$

5. Suppose for this part of the problem that the process starts instead at state 2, i.e., $X_0 = 2$. Let S be the first time by which both states 1 and 3 have been visited. $\mathbf{E}[S] =$