Master Thesis - notes

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Products of sequential spaces

Definition 1. Let (X, Ξ_X) and (Y, Ξ_Y) be sequential spaces. We say that $\{(x_n, y_n)\} \subset X \times Y$ *converges* to $(x, y) \in X \times Y$ if $(\{x_n\}, x) \in \Xi_X$ and $(\{y_n\}), y) \in \Xi_Y$. This defines the (coordinatewise) *product sequential structure* on $X \times Y$.

Sequential semigroups and groups

Definition 2. A *right sequential semigroup* is a semigroup S equipted with a sequential structure Ξ such that for all $a \in S$, the right action ϱ_a of a on S is a continuous mapping of the space S to itself. That is

$$x_n \to x \Rightarrow \varrho_a(x_n) = x_n a \to xa = \varrho_a(x).$$

A *left sequential semigroup* is a semigroup S equipted with a sequential structure Ξ such that for all $a \in S$, the left action λ_a of a on S is a continuous mapping of the space S to itself. That is

$$x_n \to x \Rightarrow \lambda_a(x_n) = ax_n \to ax = \lambda_a(x).$$

A *semisequential semigroup* is a right sequential semigroup which is also a left sequential semigroup.

A *sequential semigroup* is a semigroup S equipted with a sequential structure Ξ such that the multiplication in S, as a mapping of $S \times S$ to S is (jointly) continuous. That is

$$x_n \to x, y_n \to y \Rightarrow x_n y_n \to xy.$$

A *right sequential (left sequential, semisequential) monoid* is a right sequential (left sequential, semisequential) semigroup with identity e.

A *right (left) sequential group* is a right (left) sequential semigroup whose underlying semigroup is a group, and a *semisequential group* is a right sequential group which is also a left sequential group.

A *parasequential group* is a group G equipted with a sequential structure Ξ such that the multiplication in G, as a mapping of $G \times G$ to G is (jointly) continuous. That is

$$x_n \to x, y_n \to y \Rightarrow x_n y_n \to xy.$$

A *quasisequential group* is a semisequential group G such that the inverse mapping $^{-1} : G \to G$ is continuous. That is

$$x_n \to x \Rightarrow x_n^{-1} \to x^{-1}.$$

A *sequential group* is a parasequential group G such that the inverse mapping $^{-1} : G \to G$ is continuous.

Theorem 3. *A group G with a sequential structure Ξ is a sequential group iff $(x, y) \mapsto xy^{-1}$ is a continuous mapping of $G \times G$ to G, that is*

$$(\{x_n\}, x), (\{y_n\}, y) \in \Xi \Rightarrow x_n y_n^{-1} \to xy^{-1}.$$

Example 4. A group (semigroup) with discrete sequential strunture is a sequential group (semigroup).

As we will ask when a group admits a sequential structure, to exclude this trivial solution we should look for non-dictrete sequential structures, also called *non-trivial*.

Example 5. Let $S = \mathbb{R} \cup \{\alpha\}$ be the Alexandroff compactification of the space of real numbers. Define multiplication on S by

$$xy = \begin{cases} x + y, & \text{if } x, y \in \mathbb{R}; \\ \alpha, & \text{othervise.} \end{cases}$$

With this operation S is a semisequential semigroup; however, it is not a sequential semigroup as the multiplication is not continuous at (α, α).

Definition 6. Fox a sequential space X, let $S_p(X, X)$ be the semigroup of all mappings of the set X to X, taken with the topology of pointwise convergence. That is

$$f_n \in S_p(X, X) \to f \in S_p(X, X) \Rightarrow \forall x \in X, f_n(x) \to f(x).$$

Let $C_p(X, X)$ be the semigroup of all continuous mapping of X to X, taken as a subsemigroup of $S_p(X, X)$.

Theorem 7. *For any sequential space X, $S_p(X, X)$ is a right sequential semigroup.*

Proof. Let $f_n, f, g \in S_p(X, X)$, $f_n \to f$ and $x \in X$, then $g(x) \in X$ and since

$$(f_n g)(x) = f_n(g(x)) \to f(g(x)) = (fg)(x),$$

we have $\varrho_g(f_n) \to \varrho_g(f)$. Mapping $g \in S_p(X, X)$ was arbitrary, therefore $S_p(X, X)$ is a right sequential semigroup. $\square$

Theorem 8. *For any sequential space X, $C_p(X,X)$ is a semisequential semigroup.*

Proof. From Theorem 7 it follows that $C_p(X,X)$ if a right sequential semigroup.

To show the statenemt of the theorem, let $g_n, g, f \in C_p(X,X)$, $g_n \to g$ and $x \in X$. Then $g_n(x) \to g(x)$ in X and since f is continuous we obtain

$$(fg_n)(x) = f(g_n(x)) \to f(g(x)) = (fg)(x).$$

That is the left action λ_f is continuous. Therefore $C_p(X,X)$ is a left sequential semigroup. □

Note. For $f \in S_p(X,X) \setminus C_p(X,X)$ discontinuous, the left action λ_f in discontinuous in $S_p(X,X)$.

Proof. As f is discontinuous there exists a sequence y_n in X converging to $y \in X$, such that $f(y_n) \not\to f(y)$ (in X).

Let $g_n \in S_p(X,X) : x \mapsto y_n$. Then it is easy to see that $g_n \to g$, where $g : x \mapsto y$. But the left action λ_f is not continuous at g because

$$(fg_n)(x) = f(g_n(x)) = f(x_n) \not\to f(x) = f(g(x)) = (fg)(x).$$

□

Homomorphisms on sequential groups

For our convenience in the following we formulate and prove theorems for right sequential groups. It is easy to modify them form left sequential groups.

Theorem 9. *Let G be a right sequential group and $g \in G$ be an arbitrary element. Then the right action ϱ_g is a homeomorphism of the space G into itself.*

Proof. Clearly in a right sequential group, the right action ϱ_g is a continuous bijection. Since $\varrho_g \circ \varrho_{g^{-1}}$ is the identity mapping, it follows that the inverse mapping $\varrho_g^{-1} = \varrho_{g^{-1}}$ is also continuous. □

Theorem 10. *Let $f : G \to H$ be a homomorphism of right sequential groups. Then f is continuous iff for every sequence $\{x_n\}$ in G, such that $x_n \to e_G$, $f(x_n) \to f(e_G)$ in H (we say f is continuos at point e_G).*

Proof. Clearly any continuous function is continuous at all point, hence also at e.

Let f be continuous at e_G and $\{x_n\}$ be a sequence in G converging to a point $x \in G$. We show that $f(x_n) \to f(x)$ in H. Because G is a right sequential group $\varrho_{x^{-1}}$ is continuous, so $x_n x^{-1} \to x x^{-1} = e_G$. Then $f(x_n x^{-1}) \to f(e_G)$. Using the properties of homomorphism f

$$f(x_n x^{-1}) = f(x_n) f(x^{-1}) = f(x_n) f(x)^{-1} \to f(e_G) = e_H.$$

In right sequential group H the right action $\varrho_{f(x)}$ is continuous, which finishes the proof. □

Definition 11. A sequential space X is said to be *homogeneous* if for arbitrary two points $x, y \in X$, there exists a homeomorphism f of space X onto itself such that $f(x) = y$.

Theorem 12. *Every right sequential group is a homogeneous space.*

Proof. For any two points x, y in the group put $z = x^{-1}y$. Then from Theorem 9 the right action ϱ_z is a homeomorphism and the following holds

$$\varrho_z(x) = xz = xx^{-1}y = y.$$

□

From Theorem 12 it follows that for a group G to make G into a sequential group, we can only use homogeneous sequential structures.

A convenient property of homogeneous spaces is that they behave in the same way at any point. When we know property of a sequential structure at a certain point of such space we can deduce it properties everywhere. The point we will examine the convergence at will be the identity of the group. Then using left and right translations we will move the defined sequential behavior around.

Let $\Theta = \{\{x_n^\alpha\}, \alpha \in A\}$ be a family of sequences in a group G. We say a sequence $\{x_n\}$ of points in G Θ-converges to identity e if there exist $\alpha_0 \in A$ and subsequences $\{x_{i_n}^{\alpha_0}\}$ of $\{x_n^{\alpha_0}\}$ and $\{x_{j_n}\}$ of $\{x_n\}$ such that $x_{i_n}^{\alpha_0} = x_{j_n}$ for every n. We will write $x_n \overset{\Theta}{\to} e$.

Theorem 13. *Let $\Theta = \{\{x_n^\alpha\} : \alpha \in A\}$ be a family of sequences in a group G such that for every $\alpha \in A$ the sequence $\{(x_n^\alpha)^{-1}\}$ of inverses Θ-converges to identity e. Then $\Xi := \{\{x_n a\} : x_n \overset{\Theta}{\to} e, a \in G\}$ is a sequential structure turning G into a right sequential group.*

Proof. ??? □

Complete sequential groups

Definition 14. Sequencies $\{x_n\}$ and $\{y_n\}$ in a quasisequential group, satisfying $x_{i_n} y_{j_n}^{-1} \to e$ for arbitrary subsequences $\{x_{i_n}\}$ of $\{x_n\}$ and $\{y_{j_n}\}$ of $\{y_n\}$ will be denoted by $\{x_n\} \sim \{y_n\}$.

Theorem 15. *Relation $\sim$ is a uniformly sequential structure on quasisequential group G.*

Proof. As the group has continuous inverse, $x_{i_n} y_{j_n}^{-1} \to e$ implies that

$$y_{j_n} x_{i_n}^{-1} = (x_{i_n} y_{j_n}^{-1})^{-1} \to e^{-1} = e.$$

So the relation is symmetric. The rest is evident. □

Now we can define the important term of *Cauchy sequence* in a quasisequential group as a sequence $\{x_n\}$ such that $\{x_n\} \sim \{x_n\}$; cf. [My Bachelor Thesis]. Naturally *complete (quasi)sequential group* is a (quasi)sequential group such that all Cauchy sequencies in it are convergent.

From [Novák 1972; Theorem 1] we have that every commutative sequential group G has at least one completion, i.e. a complete sequential group in which G is dense (in certain meaning).