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BACHELOR THESIS

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Properties of Metric Spaces by Means of Convergence

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Contents

Preface

1. Role of convergence in the theory of metric spaces

1.1 Sequences in metric spaces and their convergence

We will briefly remind selected definitions and theorems, for proofs see [1] or [2].

Definition 1.1. *Let X be a non-empty set. Let $\varrho : X \times X \to \mathbb{R}$; be a mapping satisfying the following conditions:*

(M1) *For $x, y \in X : \varrho(x, y) = 0 \Leftrightarrow x = y$.*

(M2) *For $x, y, z \in X : \varrho(x, z) \leq \varrho(x, y) + \varrho(z, y)$.*

We say that ϱ is a metric on X *and (X, ϱ) is a* metric space.

The number $\varrho(x, y)$ is called the *distance between x and y*.Condition *(M1)* is called *Identity of indiscernibles* and *(M2)* is called *Triangle inequality.* It is an easy exercise to show that following holds:

(M3) For $x, y \in X : \varrho(x, y) = \varrho(y, x)$.

Sometimes *(M2)* is writen with $\varrho(y, z)$ as the last item instead of $\varrho(z, y)$ but then *(M3)* must be added to the conditions.

When we set $x = z$ in *(M2)* it follows that $\varrho(x, y) \geq 0$ for arbitrary points $x, y \in X$.

Definition 1.2. *Let (X, ϱ) be a metric space.*

(i) *For $x_0 \in X$ and $\varepsilon > 0$ the set $B(x_0, \varepsilon) := \{x \in X : \varrho(x_0, x) < \varepsilon\}$ is called the* open ball with centre x_0 and radius ε.

(ii) *Set $A \subset X$ is called* open in (X, ϱ) *if $\forall x \in A\ \exists \varepsilon > 0 : B(x, \varepsilon) \subset A$. Set $B \subset X$ is called* closed in (X, ϱ) *if $X \setminus B$ is open.*

(iii) *Let σ be a metric on X. We say ϱ and σ are* equivalent *if the family of all sets open in (X, ϱ) is the same as the family of all sets open in (X, σ).*

(iv) *For a non-empty $A \subset X$ the number* $\operatorname{diam}(A) := \sup_{x,y \in A} \varrho(x, y)$ *is called the diameter. We define* $\operatorname{diam}(\emptyset) := 0$. *The space (X, ϱ) is called* bounded *if* $\operatorname{diam}(X) < \infty$, *otherwise it is called* unbounded.

(v)

Example 1.3. TBD *Discrete space*

1.2 Some known properties of sequences in metric spaces

Definition 1.4. *Let (X, ϱ) be a metric space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. We say that $\{x_n\}_{n=1}^{\infty}$* converges to *$x \in X$ (or x is a limit of $\{x_n\}_{n=1}^{\infty}$) if $\lim_{n\to\infty} \varrho(x_n, x) = 0$ and we will denote it as $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.*

We will sometimes leave the indexes of a sequence and thus write only $\{x_n\}$.

Definition 1.5. *Let (X, ϱ) be a metric space, $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. We say that $\{x_n\}_{n=1}^{\infty}$* is Cauchy *if*

$$\forall \varepsilon\ \exists n_0 \in \mathbb{N}\ \forall n, m \in \mathbb{N}; n, m \geq n_0 : \ \varrho(x_n, x_m) < \varepsilon.$$

Theorem 1.6 (Properties of convergent sequencies)**.** *Let (X, ϱ) be a metric space, $\{x_n\}$ be a sequence in X. Then:*

(i) If $\{x_n\} \to x \in X$ and $\{x_n\} \to y \in X$ then $x = y$.

(ii) If $\{x_n\}$ is convergent then it is bounded TBD*.*

(iii) Sequence $\{x_n\}_{n=1}^{\infty}$ converges to $x \in X$ if and only if any of its subsequences $\{x_{n_k}\}_{k=1}^{\infty}$ converges to x.

(iv) Let σ be a metric on X equivalent with ϱ and $x \in X$ then $\{x_n\} \to x$ in (X, ϱ) if and only if $\{x_n\} \to x$ in(X, σ).

1.3 Historical notes

2. Sequential spaces and their properties

2.1 Family of convergent sequences on a set

Definition 2.1. *Let X be a non-empty set. A* family of convergent sequences on X TBD*(jiný název) (denoted by Ξ_X) is a set of pairs $(\{x_n\}_{n=1}^{\infty}, x)$ consisting of a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ and a point $x \in X$ satisfying the following conditions:*

(C1) If $(\{x_n\}_{n=1}^{\infty}, x) \in \Xi_X$, $(\{y_n\}_{n=1}^{\infty}, y) \in \Xi_X$ and $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ then $x = y$.

(C2) If $x \in X$ then $(\{x\}_{n=1}^{\infty}, x) \in \Xi_X$.

(C3) If $(\{x_n\}_{n=1}^{\infty}, x) \notin \Xi_X$ then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that for none of its subsequence $\{x_{n_{k_i}}\}_{i=1}^{\infty}$ is $(\{x_{n_{k_i}}\}_{i=1}^{\infty}, x) \in \Xi_X$.

In the next we will write Ξ instead of Ξ_X wherever the set X is clear from the context. Unless stated otherwise X will denote a non-empty set. When $(\{x_n\}, x) \in \Xi$ we say that $\{x_n\}$ is a *convergent sequence*, $\{x_n\}$ *converges* to x or that x is a limit of $\{x_n\}$.

The properties in definition 2.1 can be restated as follows:

(C1') A subsequence of a convergent sequence is convergent and converges to the same point.

(C2') A sequence consisting of one point (constant sequence) converges to that point.

(C3') A non-convengent sequence (that is not converging to point x) contains a subsequence such that any of its subsequences either converges to different point (not to x) or does not converge at all.

The first two properties are natural and follows from basic knowledge of convergence in metric spaces. To understand the condition *(C3)* we need to remind that a sequence $\{x_n\}$ which does not converge to x might include a subsequence $\{x_{n_k}\}$ which converges to x.

Definition 2.2. *Let (X, ρ) be a metric space. We say Ξ is* generated *by X if*

$$\Xi = \{(\{x_n\}, x) : x \in X, \forall n \in \mathbb{N}\ x_n \in X, \lim_{n\to\infty} \rho(x_n, x) = 0\}.$$

The next proposition shows that every sequence has at most one limit.

Theorem 2.3. *Let $(\{x_n\}, x) \in \Xi$. Then $\forall y \in X \setminus \{x\} : (\{x_n\}, y) \notin \Xi$.*

Proof. A sequence is a subsequence of itself. So when $(\{x_n\}, x) \in \Xi$ and $(\{x_n\}, y) \in \Xi$ than from *(C1)* we have $x = y$. □

Definition 2.4. *Let $(\{x_n\}, x) \in \Xi$ and $(\{y_n\}, y) \in \Xi$, we say that sequences $\{x_n\}$ and $\{y_n\}$ are* equivalent *if $\exists n_0, n_1 \in \mathbb{N}\ \forall n \in \mathbb{N} : x_{n_0+n} = y_{n_1+n}$. We write $\{x_n\} \sim \{y_n\}$*

Equivalence is well defined. When $\{x_n\} \sim \{y_n\}$ clearly $\{y_n\} \sim \{x_n\}$. If $\{x_n\} = \{y_n\}$ which we can rewrite as $x_n = y_n \ \forall n \in \mathbb{N}$ and we set $n_0 = n_1 = 1$. Let $\{x_n\}, \{y_n\}$ and $\{z_n\}$ be sequences such that

$$\exists\, n_0, n_1, m_0, m_1 \in \mathbb{N} \ \forall n \in \mathbb{N} : x_{n_0+n} = y_{n_1+n},\ y_{m_0+n} = z_{m_1+n}.$$

That is $\{x_n\} \sim \{y_n\}$ and $\{y_n\} \sim \{z_n\}$. To obtaint $\{x_n\} \sim \{z_n\}$ we set $k_0 = n_0 + m_0$ and $k_1 = m_1 + n_1$ and we have

$$x_{k_0+n} = x_{n_0+m_0+n} = y_{n_1+m_0+n} = z_{m_1+n_1+n} = z_{k_1+n}.$$

Theorem 2.5. *Let Ξ be a family of convergent sequences, $(\{x_n\}, x), (\{y_n\}, y) \in \Xi$ and $\{y_n\} \sim \{x_n\}$. Then $x = y$.*

Proof. We use *(C1)* twice. □

When we have $(\{x_n\}, x) \in \Xi$ and an arbitrary point $x \in X$, new sequences can be defined:

$$y_1 = x,\ y_{n+1} = x_n, \qquad z_n = x_{n+1}, \qquad \forall n \in \mathbb{N}.$$

Both of these new sequences are equivalent with $\{x_n\}$ and therefore by Theorem 2.5 they have the same limit - if present in Ξ. The definition of family of convergent sequences does not say that they do converge, it would however be odd and unnatural when removing (or adding) one member (or any finite number of members) affected the convergence of the sequence. One way to solve this would be to define something like complete family of convergent sequences i.e. when $(\{x_n\}, x) \in \Xi$ and $\{y_n\} \sim \{x_n\}$ then $(\{y_n\}, x) \in \Xi$. This would nevertheless require us to write unnecessary statements like "up to equivalence". When defined so complete family of convergent sequences would not add any new information about properties we study. To simplify the technique and the language we will develop a quotient space. That means if there is convergent sequence with limit x all its equivalent sequences with the same limit x forms one point in this space.

Definition 2.6. *Let Ξ be a family of convergent sequences. Let us denote by $\Xi/\!\sim$ the* set of all equivalence classes of *$\sim$:*

$$\begin{aligned}\Xi/\!\sim \ &:= \ \Big\{\Big[(\{x_n\}, x)\Big] : (\{x_n\}, x) \in \Xi\Big\} = \\ &= \ \Big\{\Big\{(\{y_n\}, x), \{y_n\} \sim \{x_n\}\Big\} : (\{x_n\}, x) \in \Xi\Big\}.\end{aligned}$$

Note that we do not demand $\{y_n\}$ to be convergent, but we handle it as it is. In the following we will write $[\{x_n\}, x]$ or instead of $[(\{x_n\}, x)\Big]$ and Ξ instead of $\Xi/\!\sim$.

Definition 2.7. *Let Ξ be a family of convergent sequences. We define*

$$\Xi_0 := \Big\{[\{x\}, x] : x \in X\Big\}.$$

The definitions says that Ξ_0 consists of convergent sequences which are constant up to a finite number of members. From *(C2)* if follows $\Xi_0 \subseteq \Xi$ so Ξ_0 is minimal family of convergent sequences on the set. It is wherefore interesting to see for which metric spaces $\Xi = \Xi_0$ holds.

Theorem 2.8. *Let Ξ be generated by metric space (X, ρ) then following are equivalent:*

(i) $\Xi = \Xi_0$

(ii) $\forall x \in X,\ \exists \varepsilon_x > 0,\ \forall y \in X, x \neq y : \rho(x, y) > \varepsilon_x.$

(iii) (X, ρ) *is topologically discrete.*

Proof. The space X being discrete, we know there exists $\varepsilon > 0, \forall x, y \in X, y \neq x : \rho(x, y) > \varepsilon$. When $\{x_n\}$ converges to x it has to consist only x from some index in order to meet the definition of a limit in a metric space.

Let $\Xi = \Xi_0$ and let X be not discrete, that is $\forall \varepsilon > 0\ \exists x \in X\ \exists y \in X : \rho(x, y) \leq \varepsilon$. Assume that X is complete □

Example 2.9. TBD *Difference beetween metric dicrete and topological discrete space !!!* $X = \{\frac{1}{n}, n \in N\}$, *!!!* $[0, 1)$, *!!!* $\{(n, 0), n \in N\} \cup \{(n, \frac{1}{n}), n \in N\}$

!Ekvivalence a úplnost

2.2 Open and closed sets in sequential spaces and related topological properties

We define a preclosure operator:

(PO) A point lies in the closure of a set iff there is a sequence in the set converging to the point.

2.3 Compactness

2.4 Complete spaces

2.5 Bounded and totally bounded spaces

Theorem 2.10. *When (X, ϱ) is an unbounded metric space then there exists a metric σ on X that is equivalent with ϱ and (X, σ) is bounded.*

Proof. For $x, y \in X$ we define

$$\sigma_1(x, y) := \frac{\varrho(x, y)}{1 + \varrho(x, y)}, \qquad \sigma_2(x, y) := \min\{\varrho(x, y), 1\}.$$

See [2, p. 22] for σ_1 and [4, p. 250] for σ_2 for proofs that they satisfy the conditions for a metric. Then evidently both σ_1, σ_2 are equivalent with ϱ and bounded. □

The previous theorem shows that we can not *TBD*

2.6 Connected spaces

2.7 Separable spaces

3. Convergence and mappings

3.1 Continuous mappings and homeomorphisms

3.2 Separately and jointly continuous mappings

Conclusion

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List of Abbreviations