Charles University in Prague
Faculty of Mathematics and Physics

BACHELOR THESIS

Robin Pokorný

Properties of Metric Spaces by Means of Convergence

Department of Mathematical Analysis

Supervisor of the bachelor thesis: prof. RNDr. Miroslav Hušek, DrSc.
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In .... date .... Signature

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Autor: Robin Pokorný

Katedra: Katedra matematické analýzy

Vedoucí bakalářské práce: prof. RNDr. Miroslav Hušek, DrSc., Katedra matematické analýzy

Abstrakt:

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Author: Robin Pokorný

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Miroslav Hušek, DrSc., Department of Mathematical Analysis

Abstract:

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Preface

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1.1 Series in metric spaces and their convergence

1.2 Some known properties of series in metric spaces

1.3 Historical notes

2. Sequential spaces and their metric properties

2.1 Family of convergent sequences on a set

Definition 2.1. *Let X be a set. A family of convergent sequences on X (denoted by Ξ_X) is a set of pairs $(\{x_n\}_{n=1}^{\infty}, x)$ consisting of a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ and a point $x \in X$ satisfying the following conditions:*

(C1) *If $(\{x_n\}_{n=1}^{\infty}, x) \in \Xi_X$, $(\{y_n\}_{n=1}^{\infty}, y) \in \Xi_X$ and $\{y_n\}_{n=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=1}^{\infty}$ then $x = y$.*

(C2) *If $x \in X$ then $(\{x\}_{n=1}^{\infty}, x) \in \Xi_X$.*

(C3) *If $(\{x_n\}_{n=1}^{\infty}, x) \notin \Xi_X$ then there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ such that for none of its subsequence $\{x_{n_{k_i}}\}_{i=1}^{\infty}$ is $(\{x_{n_{k_i}}\}_{i=1}^{\infty}, x) \in \Xi_X$.*

In the next we will write Ξ instead of Ξ_X wherever the set X is clear from the context; we will also sometimes leave the indexes of a sequence and thus write only $\{x_n\}$. Unless stated otherwise X will denote a non-empty set. When $(\{x_n\}, x) \in \Xi$ we say that $\{x_n\}$ is a *convergent sequence*, $\{x_n\}$ *converges to x or that x is a limit of $\{x_n\}$.*

The properties in definition 2.1 can be restated as follows:

(C1') *A subsequence of a convergent sequence is convergent and converges to the same point.*

(C2') *A sequence consisting of one point (constant sequence) converges to that point.*

(C3') *A non-convengent sequence* TBD

The first two properties are natural and follows from basic knowledge of convergence in metric spaces. To understand the condition (C3) we need to remind that a sequence $\{x_n\}$ which does not converge to x might include a subsequence $\{x_{n_k}\}$ which converges to x. The next proposition shows that every sequence has at most one limit.

Theorem 2.2. *Let $(\{x_n\}, x) \in \Xi$. Then $\forall y \in X \setminus \{x\} : (\{x_n\}, y) \notin \Xi$.*

Proof. A sequence is a subsequence of itself. So when $(\{x_n\}, x) \in \Xi$ and $(\{x_n\}, y) \in \Xi$ than from *(C1) we have $x = y$.* □

Definition 2.3. *Let $(\{x_n\}, x) \in \Xi$ and $(\{y_n\}, y) \in \Xi$, we say that $\{x_n\}$ and $\{y_n\}$ are equivalent if $\exists\, n_0, n_1 \in \mathbb{N}\ \forall n \in \mathbb{N} : x_{n_0+n} = y_{n_1+n}$. We write $\{x_n\} \sim \{y_n\}$*

Definition 2.4. *Let Ξ be a family of convergent sequences. We define*

$$\Xi_0 := \{(\{x_n\}, x) \in \Xi : \exists y \in X\ \{x_n\} \sim \{y\}\}.$$

Evidently $\Xi_0 \subseteq \Xi$. The definitions says that Ξ_0 consists of convergent sequences which are constant up to a finite number of members.

Definition 2.5. *Let (X, ρ) be a metric space. We say Ξ is generated by X if*

$$\Xi = \{(\{x_n\}, x) : x \in X, \forall n \in \mathbb{N}\ x_n \in X, \lim_{n \to \infty} \rho(x_n, x) = 0\}.$$

Theorem 2.6. *Let Ξ be generated by metric space (X, ρ) then following are equivalent:*

(i) $\Xi = \Xi_0$

(ii) $\forall x \in X\ \exists \varepsilon_x > 0\ \forall y \in X, x \neq y : \rho(x, y) > \varepsilon_x$.

(iii) *(X, ρ) is discrete or it is not complete.*

Proof. The space X being discrete, we know there exists $\varepsilon > 0, \forall x, y \in X, y \neq x : \rho(x, y) > \varepsilon$. When $\{x_n\}$ converges to x it has to consist only x from some index in order to *TBD* splnit the definition of a limit in a metric space.

Let $\Xi = \Xi_0$ and let X be not discrete, that is $\forall \varepsilon > 0\ \exists x \in X\ \exists y \in X : \rho(x, y) \leq \varepsilon$. Assume that X is complete

TBD !!! $X = \{\frac{1}{n}, n \in N\}$, !!! $[0, 1)$ □

$\Xi = \Xi_0 \Rightarrow$ *not complete, totaly bounded??*
discrete $\Rightarrow \Xi = \Xi_0$

!Ekvivalence a úplnost
!Preclosure operator

2.2 Open and closed sets in sequential spaces and related topological properties

2.3 Compactness

2.4 Complete spaces

2.5 Bounded and totally bounded spaces

2.6 Connected spaces

2.7 Separable spaces

3. Convergence and mappings

3.1 Continuous mappings and homeomorphisms

3.2 Separately and jointly continuous mappings

Figure 3.1: Logo MFF UK

3.2.1 Sample of LaTeX

V této krátké části ukážeme použití několika základních konstrukcí LaTeXu, které by se vám mohly při psaní práce hodit.

Třeba odrážky:

- *Logo Matfyzu vidíme na obrázku 3.1.*
- *Tato subsekce má číslo 3.2.1.*
- *Odkaz na literaturu [2].*

Druhy pomlček: červeno-černý (krátká), strana 16–22 (střední), $45-44$ *(minus), a toto je — jak se asi dalo čekat — vložená věta ohraničená dlouhými pomlčkami. (Všimněte si, že jsme za* `a` *napsali vlnovku místo mezery: to aby se tam nemohl rozdělit řádek.)*

Definition 3.1. *Strom je souvislý graf bez kružnic.*

Theorem 3.2. *Tato věta neplatí.*

Proof. Neplatné věty nemají důkaz. □

Conclusion

Bibliography

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[2] Engelking, *Ryszard.* General Topology. *Revised and completed edition. Berlin: Heldermann Verlag, 1989. 529 p. Sigma Series in Pure Mathematics; Vol. 6. ISBN 3-8538-006-4.*

List of Abbreviations