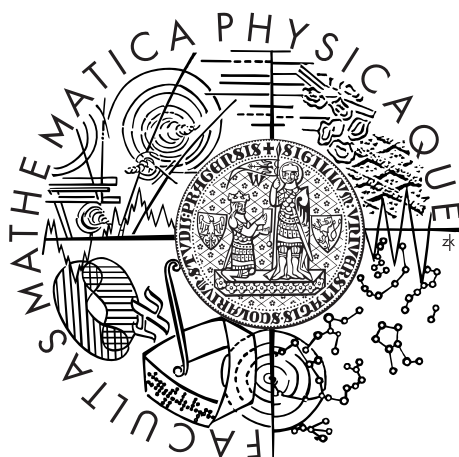


Charles University in Prague  
Faculty of Mathematics and Physics

## BACHELOR THESIS



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## Properties of Metric Spaces by Means of Convergence

Department of Mathematical Analysis

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Study programme: Mathematics

Specialization: General Mathematics

Prague 2012

Acknowledgement (?).

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Název práce: Vlastnosti metrických prostorů pomocí konvergence

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Abstrakt:

Klíčová slova:

Title: Properties of metric spaces by means of convergence

Author: Robin Pokorný

Department: Department of Mathematical Analysis

Supervisor: prof. RNDr. Miroslav Hušek, DrSc., Department of Mathematical Analysis

Abstract:

Keywords:

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# Preface

# 1. Role of convergence in the theory of metric spaces

- 1.1 Series in metric spaces and their convergence
- 1.2 Some known properties of series in metric spaces
- 1.3 Historical notes

## 2. Sequential spaces and their metric properties

### 2.1 Family of convergent sequences on a set

**Definition 2.1.** Let  $X$  be a set. A family of convergent sequences on  $X$  (denoted by  $\Xi_X$ ) is a set of pairs  $(\{x_n\}_{n=1}^\infty, x)$  consisting of a sequence  $\{x_n\}_{n=1}^\infty \subseteq X$  and a point  $x \in X$  satisfying the following conditions:

- (C1) If  $(\{x_n\}_{n=1}^\infty, x) \in \Xi_X$ ,  $(\{y_n\}_{n=1}^\infty, y) \in \Xi_X$  and  $\{y_n\}_{n=1}^\infty$  is a subsequence of  $\{x_n\}_{n=1}^\infty$  then  $x = y$ .
- (C2) If  $x \in X$  then  $(\{x\}_{n=1}^\infty, x) \in \Xi_X$ .
- (C3) If  $(\{x_n\}_{n=1}^\infty, x) \notin \Xi_X$  then there exists a subsequence  $\{x_{n_k}\}_{k=1}^\infty$  such that for none of its subsequence  $\{x_{n_{k_i}}\}_{i=1}^\infty$  is  $(\{x_{n_{k_i}}\}_{i=1}^\infty, x) \in \Xi_X$ .

In the next we will write  $\Xi$  instead of  $\Xi_X$  wherever the set  $X$  is clear from the context; we will also sometimes leave the indexes of a sequence and thus write only  $\{x_n\}$ . Unless stated otherwise  $X$  will denote a non-empty set. When  $(\{x_n\}, x) \in \Xi$  we say that  $\{x_n\}$  is a *convergent sequence*,  $\{x_n\}$  *converges to*  $x$  or that  $x$  is a *limit* of  $\{x_n\}$ .

The properties in definition 2.1 can be restated as follows:

- (C1') A subsequence of a convergent sequence is convergent and converges to the same point.
- (C2') A sequence consisting of one point (constant sequence) converges to that point.
- (C3') A non-convergent sequence TBD

The first two properties are natural and follows from basic knowledge of convergence in metric spaces. To understand the condition (C3) we need to remind that a sequence  $\{x_n\}$  which does not converge to  $x$  might include a subsequence  $\{x_{n_k}\}$  which converges to  $x$ . The next proposition shows that every sequence has at most one limit.

**Theorem 2.2.** Let  $(\{x_n\}, x) \in \Xi$ . Then  $\forall y \in X \setminus \{x\} : (\{x_n\}, y) \notin \Xi$ .

*Proof.* TBD □

**Definition 2.3.** Let  $(\{x_n\}, x) \in \Xi$  and  $(\{y_n\}, y) \in \Xi$ , we say that  $\{x_n\}$  and  $\{y_n\}$  are equivalent if  $\exists n_0, n_1 \in \mathbb{N} \forall n \in \mathbb{N} : x_{n_0+n} = y_{n_1+n}$ . We write  $\{x_n\} \sim \{y_n\}$

**Definition 2.4.** Let  $\Xi$  be a family of convergent sequences. We define

$$\Xi_0 := \{(\{x_n\}, x) \in \Xi : \exists y \in X \{x_n\} \sim \{y\}\}.$$

Evidently  $\Xi_0 \subseteq \Xi$ . The definitions says that  $\Xi_0$  consists of convergent sequences which are constant up to a finite number of members.



**Definition 2.5.** Let  $(X, \rho)$  be a metric space. We say  $\Xi$  is generated by  $X$  if

$$\Xi = \{(\{x_n\}, x) : x \in X, \forall n \in \mathbb{N} \ x_n \in X, \lim_{n \rightarrow \infty} \rho(x_n, x) = 0\}.$$

**Theorem 2.6.** Let  $\Xi$  be generated by metric space  $(X, \rho)$  than following are equivalent:

- (i)  $\Xi = \Xi_0$
- (ii)  $X$  is finite or  $(X, \rho)$  is discrete.

*Proof.* **TBD**

□

!Ekvivalence a úplnost  
!Preclosure operator

## 2.2 Open and closed sets in sequential spaces and related topological properties

## 2.3 Compactness

## 2.4 Complete spaces

## 2.5 Bounded and totally bounded spaces

## 2.6 Connected spaces

## 2.7 Separable spaces

## 3. Convergence and mappings

3.1 Continuous mappings and homeomorphisms

3.2 Separately and jointly continuous mappings

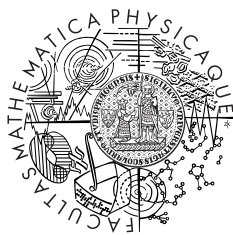


Figure 3.1: Logo MFF UK

### 3.2.1 Sample of $\text{\LaTeX}$

*V této krátké části ukážeme použití několika základních konstrukcí  $\text{\LaTeX}$ u, které by se vám mohly při psaní práce hodit.*

*Třeba odrážky:*

- *Logo Matfyzu vidíme na obrázku 3.1.*
- *Tato subsekce má číslo 3.2.1.*
- *Odkaz na literaturu [2].*

*Druhy pomlček: červeno-černý (krátká), strana 16–22 (střední), 45–44 (minus), a toto je — jak se asi dalo čekat — vložená věta ohraničená dlouhými pomlčkami. (Všimněte si, že jsme za **a** napsali vlnovku místo mezery: to aby se tam nemohl rozdělit řádek.)*

**Definition 3.1.** *Strom je souvislý graf bez kružnic.*

**Theorem 3.2.** *Tato věta neplatí.*

*Proof.* Neplatné věty nemají důkaz. □

# Conclusion

# Bibliography

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- [2] ENGELKING, *Ryszard*. *General Topology. Revised and completed edition.* Berlin: Heldermann Verlag, 1989. 529 p. *Sigma Series in Pure Mathematics*; Vol. 6. ISBN 3-8538-006-4.

# List of Abbreviations