The Theta Correspondence Seminar

October 18, 2023

From Oxford Michaelmas Term 2023.

1 Plan for my Talk

1. Talk about theta series coming from one-variable quadratic forms. These are half-integer weight modular forms. Talk about these and give brief definition.
2. Give an example of the Shimura correspondence, perhaps from Shimura's original paper.
3. Sketch Waldspurger's results in classical form - and what its implications are (especially when combined with Gross-Zagier)
4. Argue why the representation theoretic point of view seems to be the right one for the proof of formulae like this.
5. Discuss other instances of the theta correspondence, and why they might all have a common framework
6. Say a little bit about the history, and what has been proven, any conjectures that remain

1.1 History

- [Shi73b] proof of the Shimura correspondence between half integer weight and integer weight modular forms. Uses Weil's converse theorem. The simple computation of Hecke operators tells us that there are only Hecke operators of square level, and we can compute the higher coefficients from the lower ones using this. However, the problem here is that we don't know how to determine the square-free coefficients.
- [Wal81] Here, he determines the square-free coefficients for half-integer weight forms by relating them to the value at the centre of symmetry of an L-functions attached to the associated integer form twisted by the associated character to n. He also expresses this L-function as a period. Notice that even in the case of relating the Fourier coefficients of the two forms, this is still a relation of periods, its just that which period you get changes depending on whether you go for square or non-square Fourier coefficients.
- [Wal91] does the relationship Jacquet-Langlands version of his original paper.

1.2 Theta Series and Half-Integer Weight Forms

- Give definition of original theta series, and functional equation.
- Give Shimura's example of a half-integral weight modular form, and relate the square coefficients to an integer weight form. Now ask question about the square-free coefficients.

As we should in a seminar on the theta correspondence, let me start by defining the original theta function as introduced by Euler:

$$\theta(\tau) = \sum_{n\in\mathbb{Z}} q^{n^2}, \qquad q = e^{\pi i \tau}, \tau \in \mathbb{H}.$$

By the Poisson summation formula, we get a functional equation for this of the form

$$\theta\left(-\frac{1}{\tau}\right)^2 = -i\tau\theta(\tau)^2.$$

A bit more work shows that

$$\theta\left(\frac{a\tau+b}{c\tau+d}\right) = \epsilon_d^{-1}\left(\frac{c}{d}\right)(c\tau+d)^{1/2}\theta(\tau)$$

where the square root takes the upper half plane to the positive real and imaginary part quadrant, and $\epsilon_d = 1$ if $d \equiv 1 \pmod 4$ and i otherwise. This is very close to being the definition of a modular form, as we see that if we take the 4th power we do actually get a modular form of weight 2! Magically, the function θ chooses for us a way of taking the 4-th root of the automorphy factor $(c\tau+d)^2$ in a nice way, and in fact we use this 4-th root as the definition of the automorphy factor in general (how does this choice show up in the representation theory? Is it in the additive character we choose?).

We don't particularly like this function though as it isn't a cusp form. Following Shimura, let's make it into a cusp form! Take

$$f(\tau) = \theta(\tau)^{-3}\eta(2\tau)^{12}$$

which is a generator of the space $S_{9/2}(4)$.

$$f(\tau) = q - 2q^2 + 4q^3 - 8q^4 + 4q^6 + \dots$$

As with all modular form, it is an interesting question to ask how we should interpret these Fourier coefficients, and what arithmetic significance they have. We can develop here a theory of Hecke operators $T(n)$, but due to the restrictions of the half-integer weight situation, we can only define these for square n. A simple group theoretic computation of double cosets for $\Gamma_0(N)$ allows Shimura to understand how the Hecke operators act on the Fourier coefficients. Therefore, for an eigenfunction of the Hecke operators, we can compute the Fourier coefficient $a(nm^2)$ in terms of $a(n)$ and the $T(d^2)$-eigenvalue for $d|m$. Here are some simple formulae computing the Fourier coefficients based on the eigenvalues, for odd primes,

$$a(p^2) = \left(\omega_p - \chi(p)\left(\frac{-1}{p}\right)^{(\kappa-1)/2} p^{(\kappa-3)/2}\right)a(1).$$

and in the specific example above,

$$a(9n) = \left(\omega_3 - 27\left(\frac{n}{3}\right)\right)a(n) - 3^7 a(n/9)$$

For example

$$\begin{aligned}
a(9) = -15 &\implies \omega_3 = -15 + 27 = 12 \\
a(25) = -335 &\implies \omega_5 = -335 + 125 = -210 \\
a(49) = 673 &\implies \omega_7 = 1016 \\
a(2) = -2 &\implies a(18) = (12+27)\,a(2) = -78 \\
a(1) = 1, a(9) = -15 &\implies a(81) = 12(-15) - 3^7 = -2367
\end{aligned}$$

Thus we can see it is only the Fourier coefficients for square-free values which are mysterious, as well as what interpretation the eigenvalues have. The second question is answered by Shimura, the first by Waldspurger.

1.2.1 [Shi73a]

This is where Shimura introduces these operators, and crucially tells us where to find the eigenvalues. There is an integer weight modular form that encodes them!

Theorem 1.1. Let $f(z) = \sum_{n=1}^{\infty} a(n)e^{2\pi i n z}$ be a weight $\kappa/2$ modular form of level $\Gamma_0(N)$, character χ, where $\kappa > 0$ is an odd integer, $N \equiv 0 \pmod 4$, and χ is a character modulo N, such that f is a common eigenfunction for all of the Hecke operators with

$$f|T^N_{\kappa,\chi}(p^2) = \omega_p f.$$

Then for any square-free integer $t \geq 1$,

$$\sum_{n=1}^{\infty} a(tn^2)n^{-s} = a(t)\prod_p \left(1 - \chi(p)\left(\frac{-1}{p}\right)^{\lambda}\left(\frac{t}{p}\right)p^{\lambda-1-s}\right)\left(1 - \omega_p p^{-s} + \chi(p^2)p^{\kappa-2-2s}\right)^{-1},$$

where $\lambda = (\kappa - 1)/2$. This is just a neat packaging of the formulae for the Hecke operators.

The real meat is the following result. Put $F(z) = \sum_{n=1}^{\infty} A_n e^{2\pi i n z}$ where

$$\sum_{n=1}^{\infty} A_n n^{-s} = \prod_p \left(1 - \omega_p p^{-s} + \chi(p^2)p^{\kappa-2-2s}\right)^{-1}.$$

If $\kappa \geq 3$, F is an integral modular form of weight $\kappa - 1$, level $\Gamma(N_0)$ and character χ^2, where N_0 is easily computable and usually $N/2$. It is an eigenform, and if $\kappa \geq 5$ it is a cusp form.

With our example above, we should be looking for an eigenform of weight 8 and level $\Gamma_0(2)$. There is only one of these:

$$F(z) = q - 8q^2 + 12q^3 + 64q^4 - 210q^5 - 96q^6 + 1016q^7 - ...$$

Here, we immediately see the eigenvalues! The next question is what are the square-free terms?

1.2.2 [Wal81]

Here, Waldspurger saves the day. The square-free Fourier coefficients are related to the central values of twists of the L-function for F.

Theorem 1.2. Let F be related to f as in the previous theorem.

1.3 Representation Theory

We all know that a modular form gives rise to an automorphic representation. Similarly, Weil had a way of constructing representations of the metaplectic group from a half-integer weight modular form.

Let's go back to the statement of the classical Shimura correspondence. A simple observation is that for each p, the Hecke eigenvalues of F at p only depends on the Hecke eigenvalue for $T(p^2)$, and therefore

2 Half Integer Weight Modular Forms

By Serre-Stark, the space of weight 1/2 modular forms is spanned by the collection of twisted theta series and their old forms. Thus we just consider those here. Given an even primitive character ψ of conductor r, we construct

$$\theta_\psi := \sum_{n\in\mathbb{Z}} \psi(n)q^{n^2} \in M_{1/2}\left(\Gamma_1(4r^2), \psi\right).$$

What exactly do we mean by the space of half-integer weight forms, because doesn't that mean we have to consistently choose a square root of the automorphic factor $(cz+d)$? It does, but basically in the classical setting we can do this by just choosing that square root to be

$$j(\gamma, z) = \theta(\gamma(z))/\theta(z).$$

Once we've made this choice, we can consistently define everything else. The other way to do it is to go to the metaplectic group, where you essentially carry the choice around with you of which square-root you are taking. The elements of the metaplectic group M are pairs of the form (γ, j_γ) where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $j_\gamma : \mathcal{H} \to \mathbb{C}$ such that

$$j_\gamma(\tau)^2 = c\tau + d.$$

This then gives you a reasonable action independent of choosing a square root. The reason that classically we consider only half-integer weight forms with level divisible by 4 is that the full modular group doesn't split into the metaplectic group and so there isn't a good way of making the choice such that we can look at only the $SL_2(\mathbb{R})$ action. That's why all the representation theory for these forms comes from the metaplectic group rather than SL_2.

Shimura and Buzzard both use this group $\mathscr{G}$ which includes a torus part $\mathbb{T}$, but this confuses me a little bit since they could actually ignore the $\mathbb{T}$ and simply get the metaplectic group (notice that the kernel is a two-fold cover of the t-torus). Also, Shimura assumes that the Fuchsian group that we are allowed to use has to come from the group $SL_2(\mathbb{R})$ in the sense that the map from $\mathscr{G} \to \mathrm{SL}_2(\mathbb{R})$ is injective on the Fuchsian group Δ. This also confuses me, because doing this seems to make the metaplectic cover fairly irrelevant.

The theory of Hecke operators goes in pretty much the way we would expect. For two Fuchsian groups Δ_1, Δ_2 and ξ such that $\Delta_1, \xi\Delta_2\xi^{-1}$ are commensurable, then we split up the double coset space into single cosets

$$\Delta_1\xi\Delta_2 = \bigsqcup \Delta_1\xi_\nu,$$

and define the Hecke operator to be

$$f|\left[\Delta_1\xi\Delta_2\right]_\kappa = \det(\xi)^{(\kappa/4)-1}\sum_\nu f|\left[\xi_\nu\right]_\kappa.$$

Then, Shimura considers only the θ-multiplier lift of $\Gamma_0(4)$ up to the metaplectic group. Really, the proof of the Euler product and values of the Hecke operators is nothing special at all. It just follows from explicitly working out the double coset representatives in these exact cases. There's nothing deep going on at all. Really shows how powerful the geometry of Shimura varieties are that such a thing as an Euler product just comes out of the woodwork so easily...

3 Reference Reviews

3.1 Wee Teck Gan's Notes (sent to James Newton privately)

Main phenomenon of study: Shimura correspondence between half weight and integer weight modular forms. But with new focus on local theta, see-saws, doubling and torus periods. Sections:

1. Intro

2. Local Shimura Correspondence (pg 2-10).

    The goal of the local Shimura correspondence is to paramtrize the irreducible representations of the metaplectic group. They will be parametrized by the irreducible representations of the special orthogonal groups (all forms). In the case of rank 2 metaplectic, we relate it to the SO(3) both the split and non-split versions. The representations of the split version should be related to harmonic polynomials, can we see this in the representation theory of the metaplectic group or some specific half integer weight forms?

    He produces a special family of representations of $Mp(W)$ coming from the Weil representation. Is this all of them? No, because he then produces the principal series which are different. Some of the components of the Weil representations are supercuspidal.

    Also studies the representation theory of $SO(V)$ which reduces to PGL_2 by JL. Can elementarily match up the non-supercuspidals, the hard thing is to match up the supercuspidals since we don't have a nice description of them.

    To complete this, he uses the big Weil representation of $Mp(W) \times O(V)$. In here, we get the bijection in the standard way we're used to. It matches up the local data of gamma factors, L-functions etc.

3. Global Shimura Correspondence (pg 10-14).

    Here, we expect to match the automorphic representations of the metaplectic group with those of PGL_2. The first thing we do is create a large explicit subspace of $L^2(Mp(\mathbb{A}))$ consisting of the elementary theta functions, and we want to decompose the orthogonal complement of this explicit space.

    We can simply glue together the local correspondences, but then the question is does this preserve cuspidality and how many things are in the same Waldspurger packet?

4. Global Theta Correpondence (pg 14-16).

    This is essentially the usual treatment of global theta lifts in general. There's a small section on O vs SO, and a vanishing criterion.

5. Lifts from Orthogonal to metaplectic (pg 17-23)

    An interesting way of analysing the theta lifts is to compute their periods. Here, he shows that the theta lift is almost always cuspidal.

    Also, he computes the Fourier coefficients and actually reduces them to torus periods, this is an interesting computation! In the split case, a result of Hecke-Jacquet-Langlands says that the split torus period is equal to an L-value... I guess this is inspiration for Waldspurger's result, perhaps a good practice case.

6. Lifts from metaplectic to orthogonal (pg 23-26)

    This is a very similar story to the section above, but you need to use a different model of the Weil representation to make the computations easier. The end result is that this list is non-zero iff it has a non-zero Fourier coefficient.

7. Rallis Inner Product Formula (pg 26-32)

    This is the method by which we will show non-vanishing of the theta lift, simply by computing it's Petersson norm. Here we see the doubling see-saw.

    The purpose of the doubling see-saw seems to simply be a way of giving a good setting to switch integral signs...

However, the problem is that the inside doesn't really converge and we don't know what it it. This is fixed by using Siegel-Weil to relate it to and Eisenstein series and regularising. The integral against an Eisenstein series has then been understood by Piatetskii-Shapiro and Rallis.

8. Global Shimura Correspondence (pg 32-35)

   This is pretty much just combining the previous results to prove the global correspondence now that we have the non-vanishing results and multiplicity one etc.

9. Local Torus Periods and Root numbers (pg 35-43)

   We related the non-vanishing question to torus periods, but actually solved it in the end using the Rallis Inner Product Formula which relates it to L-functions. This implies that perhaps there should be some relation between torus periods and L-functions.

   Here, he proves the theorem of Saito-Tunnell on the local non-vanishing of torus periods. It is determined precisely by the ϵ-factors.

10. Global Torus periods and central L-values (pg 43-51)

    Here, proves the global theorem. These last two sections are the most technical by the looks of it.

3.2 Shimura - On modular forms of half integral weight

This is the paper where he proves the classical Shimura correspondence. Notice that it is not an equality of Fourier coefficients but rather an equality of Hecke eigenvalues, since for the half-integer weight forms the Fourier coefficients are not equal to the eigenvalues (but they are related in a fairly simple way).

Sections:

1. Hecke operators and Euler products (10 pages)

   This is the section where he develops the theory of Hecke operators for half-integer weight, although only for $T(n^2)$, and then gets an Euler product expression related to the half-integral weight form. There's nothing in here about the integer weight forms or correspondence yet.

2. Theta series (5 pages)

   Here he constructs the theta series and proves that they are in the right spaces (i.e. uses the Poisson summation formula to get the transformation properties that he needs).

3. Main Theorem (18 pages, technical part of the paper)

   He uses the converse theorem of Weil to prove that the related object is in fact a modular form.

4. Open questions and examples

   Does a nice example with the normal theta function. Also asks some very interesting questions about the meaning of the square-free coefficients of the half-integer weight form, and in particular how they might relate to a particular grossen-character. Not sure whether these questions have been answered yet? He points out that answers to these questions might come from the representations theoretic approach similar to Jacquet-Langlands. Maybe Waldspurger has answered them then?

5. More functional equations

3.3 Waldspurger1981 - Sur les coefficients Fourier ...

Here he interprets the non-square Fourier coefficients of half integer weight forms using the central value of the L-series associated to the Shimura lift.

Sections:

1. Introduction

   Here, he mentions that the difficulty of the theorem really lies in the local problem that there doesn't seem to be a good newform theory for half-integer forms with multiplicity 1. Thus if we choose new-form spaces then we can't pick particular vectors and should study them all simultaneously, thus leading to a more representation theoretic viewpoint.

2. Statement of the Theorem

4 Questions that I have

- What is a near equivalence class? And why do the ETFs form a single one?
- How do we see that the ETFs have a single orbit of non-zero Fourier coefficients?
- How does the choice of square root corresponding to the usual theta function show up in the representation theoretic world? Maybe it is in these eighth roots of unity that Weil describes. Check it out.

5 Related Papers

- Lapid-Rallis on doubling zeta integrals.
- Stark-Heegner Points and the Shimura Correspondence, by Darmon-Tornaria.

References

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