Basics of Shimura Varieties

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The collection of modular curves has a Hecke symmetry which is a really topological property of quotient spaces. There is also a Galois symmetry, but this is a harder property to deal with.

Suppose that G is a reductive algebraic group, and $G(\mathbb{R})$ acts on some space D. Then for each open compact subgroup we get

$$X_{\mathcal{U}} := G(\mathbb{Q})\backslash(D \times G(\mathbb{A}^\infty))/\mathcal{U}$$

where the group on the left acts diagonally. The action of $G(\mathbb{A}^\infty)$ on the $\mathcal{U}$'s gives us an action on the class of such $X_{\mathcal{U}}$, and this gives rise to Hecke symmetries. What D's should we consider?

In the case of $\mathrm{SL}_n(\mathbb{R})$, this acts on the space of real symmetric positive definite matrices of determinant 1 by

$$X \mapsto g^t X g$$

with transitive action and the stabiliser of the identity is $SO_n(\mathbb{R})$. For general n it might be the case that this space is odd dimensional, so cannot be an algebraic variety.

Note that the double coset spaces

$$G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/\mathcal{U}$$

is finite, and so we can rewrite the double coset as quotients of D by arithmetic subgroups. We want these quotients to be algebraic varieties so that we get Galois symmetry and congruences.

We want D to be Hermitian symmetric domains. Note that $\mathrm{SL}_n(\mathbb{R})$ doesnt act 'nicely' on any Hermitian symmetric domain and so doesn't have a Shimura variety!

One example of a Hermitian symmetric domain comes from the Siegel case, $G = \mathrm{Sp}_{2n}(\mathbb{R})$ which acts on

$$\mathbb{H}_n = \{z \in \mathrm{Sym}_n(\mathbb{C}) : \mathrm{Im}(z) > 0\}$$

which is called Siegel symmetric space. These conditions are equivalent to asking that $(z, 1)$ is isotropic and has negative pairing with its complex conjugate. Therefore it is acted on by the symplectic group. This is a transitive action and the stabiliser is isomorphic to $U_n(\mathbb{R})$! So

$$\mathbb{H}_n = \mathrm{Sp}_{2n}(\mathbb{R})/U_n(\mathbb{R}).$$

This is cool! Note that U_n is the centraliser of the map

$$h_0 : U_1(\mathbb{R}) \to \mathrm{Sp}_{2n}(\mathbb{R})$$

So $\mathbb{H}_n$ is in fact the orbit $G(\mathbb{R}) \cdot h_0$ of this nice homomorphism. Consider $J = h_0(i)$ which defines a complex structure on $\mathbb{R}^{2n}$. And therefore defines a complex torus on $\mathbb{R}^{2n}/\mathbb{Z}_{2n}$, which is in fact an abelian variety with a polarisation, and so the Siegel upper hand space is a moduli space!

The Satake topology is relevant.

When we go to the general symplectic group we can actually embed in $\mathbb{C}^\times$ rather than $U_1(\mathbb{R})$ since that is better from the point of view of variations of Hodge structures.

The unitary groups have bounded realisations which are somehow nicer. Elements in unitary groups sit inside symplectic groups. There is also some unbounded domains.

1 Background on Abelian Varieties

We know the singular cohomology of complex tori, if $X = V/U$ where $V \cong \mathbb{C}^n$ and U is a lattice, then

$$\mathrm{H}^k_{sing}(X, \mathbb{Z}) \cong \bigwedge^k \mathrm{Hom}(U, \mathbb{Z}) = \mathrm{Hom}(\bigwedge^k, \mathbb{Z}) \cong \mathbb{Z}^{\binom{n}{k}}.$$

The space of line bundle admits a map (the Chern map), to $\mathrm{H}^2(X, \mathbb{Z})$ which associates to any line bundle an alternative 2-form on U. Such a form coming from an ample line bundle is called a polarisation. An alternating form on U can be represented by a matrix $\begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}$ where D is a diagonal matrix where each entry divides the next. This is the theory of elementary divisors. D is known as the type of the polarisation. A principal polarisation is one where the matrix D is the identity, and therefore it induces an isomorphism of X with its dual.

We could attempt to find a moduli space for the abelian varieties with this type of polarisation. It is not too hard to show:

Lemma 1.1. There is a canonical bijection between the set of polarized abelian varieties of type D with a symplectic basis to the Siegel upper half space:

$$\mathfrak{H}_n = \left\{ Z \in \mathrm{M}_n(\mathbb{C}) | Z^{\mathrm{T}} = Z, \mathrm{im} Z > 0 \right\}.$$

2 Matsushima's Formula

This relates the cohomology of locally symmetric spaces with automorphic representations.