For what follows, we're going to consider the set of real numbers to be the universe of discourse.

Convex Sets[1]

A **convex combination** is a linear combination of points where all coefficients are non-negative and sum to one.

Consider points (possibly vectors) $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. A general convex combination, which can be denoted $\mathbf{w}$, is

$$\mathbf{w} = k_1\mathbf{x} + k_2\mathbf{y} + k_3\mathbf{z}$$

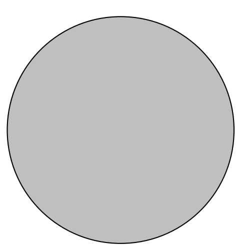
where $k_1 + k_2 + k_3 = 1$ and $k_i \geq 0, i = 1, 2, 3$.

The convex combination we are going to use most is:

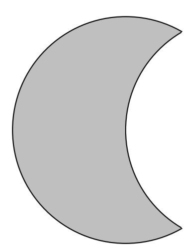
$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \qquad \alpha \in [0,1]$$

Think of it like a weighted average between two points (or vectors), where α determines the weight. The convex combinations made by all possible values of α will be a line between the two points.

$A \subseteq \mathbb{R}^n$ is a **convex set** iff $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in A \quad \forall\, \mathbf{x}, \mathbf{y} \in A, \alpha \in [0,1]$



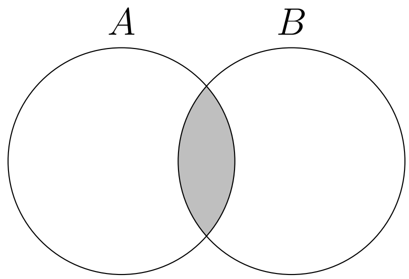
A Convex Set



A Non-Convex Set

[1]Prepared by Sarah Robinson

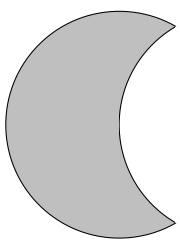
If A and B are both convex sets in $\mathbb{R}^n$, then $A \cap B$ is a convex set.



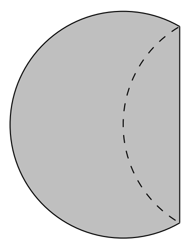
Intersection: $A \cap B$

Is $A \cup B$ a convex set?

The **convex hull** of set $B \subseteq \mathbb{R}^n$ is the smallest convex set containing B (the set of all convex combinations of points in B).



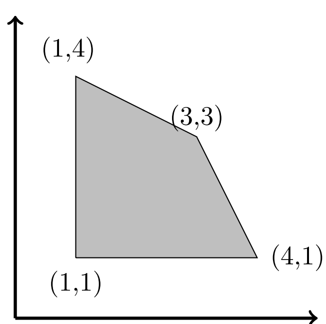
A Non-Convex Set



The Convex Hull

Example: A two-player prisoners' dilemma from game theory and the convex hull of the payoff profiles:

	C	D
C	(3, 3)	(1, 4)
D	(4, 1)	(1, 1)



Example: Consider set S:

$$S = \{x \mid x \in \mathbb{R} \wedge -1 \leq x \leq 1\}$$

Show that S is a convex set.

- $A \subseteq \mathbb{R}^n$ is a **convex set** iff $\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in A \quad \forall\, \mathbf{x}, \mathbf{y} \in A, \alpha \in [0,1]$

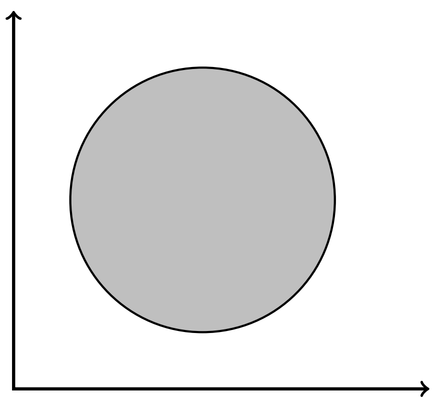
To Show:

Proof:

Closed Sets

A set $A \subseteq \mathbb{R}^n$ is **closed** iff for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in A$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in A$

- $\approx$ set A also includes its boundaries



A Closed Set in $\mathbb{R}^2$

A set is an **open set** if and only if its complement is a closed set.

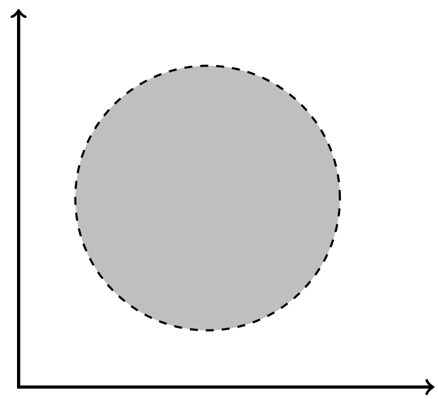
The following sets in $\mathbb{R}^n$ are open sets:

- The empty set $\emptyset$
- The entire space $\mathbb{R}^n$
- The union of any number of open sets
- The intersection of any finite number of open sets

The following sets in $\mathbb{R}^n$ are closed sets:

- The empty set $\emptyset$
- The entire space $\mathbb{R}^n$
- The union of any finite number of closed sets
- The intersection of any number of closed sets

We could also define open sets using the notion of an epsilon-neighborhood (a ball with radius ε). A set A is open if and only if for all $\mathbf{x} \in A$, there exists some $\varepsilon > 0$ such that the ε-ball centered at $\mathbf{x}$ is contained in A.



An Open Set in $\mathbb{R}^2$

For any point in an open set, we can always draw a tiny circle around the point that lies entirely within the set. I bring up this definition because ε-balls will come up in other contexts.

Example: Consider set S:

$$S = \{(x, y) \mid (x, y) \in \mathbb{R}^2 \wedge x^2 + y^2 \leq 1\}$$

Show that S is closed.

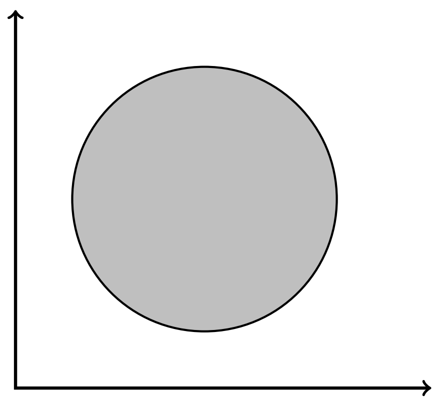
- $A \subseteq \mathbb{R}^n$ is **closed** iff for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in A$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in A$
- Theorem 1: If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$ and $a_n b_n \to ab$
- Theorem 2: If $a_n \to a$, then $a_n \leq b$ for all n implies $a \leq b$.

<u>To Show:</u>

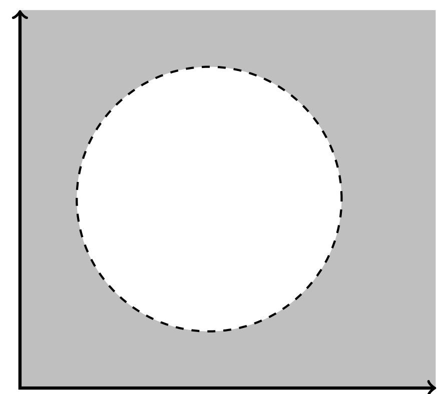
<u>Proof:</u>

Bounded Sets

A set $A \subseteq \mathbb{R}^n$ is **bounded** if and only if there exists an M and a point $\mathbf{x} \in \mathbb{R}^n$ such that the M-ball centered at $\mathbf{x}$ contains all of A.



A Bounded (Closed) Set



A Non-Bounded (Open) Set

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded:

- Pick a radius M
- Let $\mathbf{x} \in A$
- Show that $-M \leq x_i \leq M \quad \forall i = 1, \ldots, n$
- (This is for an M-ball centered at zero. You could also define a center point $\mathbf{c}$ and show that $c_i - M \leq x_i \leq c_i + M \quad \forall i = 1, \ldots, n$)

A set $A \subseteq \mathbb{R}^n$ is **compact** if and only if it is closed and bounded.

Example: Consider set S:

$$S = \{(x, y) \mid (x, y) \in \mathbb{R}^2 \wedge x^2 + y^2 \leq 1\}$$

Show that S is bounded.

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded:

- Pick a radius M
- Let $\mathbf{x} \in A$
- Show that $-M \leq x_i \leq M \quad \forall i = 1, \ldots, n$

To Show:

Proof: