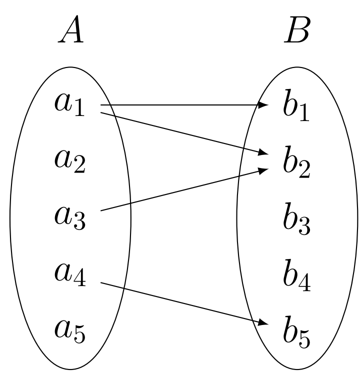
FUNCTIONS[1]

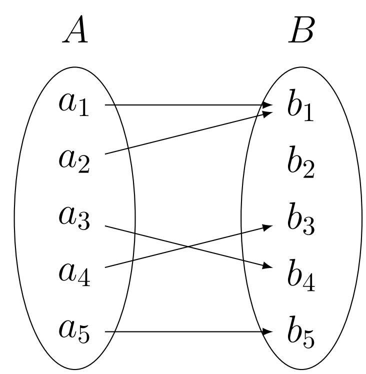
The reference document on relations is much more rigorous with terminology and definitions. In class, I'm going to focus on describing (not necessarily defining) the most important aspects. Use the reference document if there's something in Math Camp or first year sequences that you need clarification on (not a signal of what you need to know).

We will start with functions, which are a type of relation that you're very familiar with.

Let A and B be sets. A **function** f maps A to B such that, for every element $a \in A$, there is exactly one mapped element $b \in B$.



Not a function!

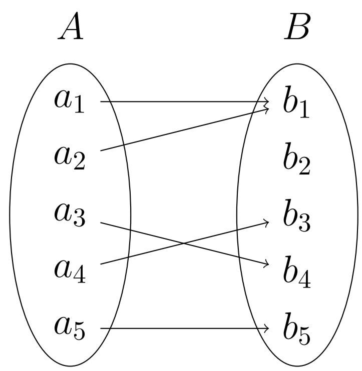


Function!

We call set A the **domain**. It is the set of values "getting mapped from."

We call set B the **codomain**. It is the set of values that *could be* "mapped to." Note that in the function on the right, b_2 is an element of the codomain but isn't mapped to. We call the set of "actually mapped to" elements $\{b_1, b_3, b_4, b_5\}$ the **range**.

[1] Prepared by Sarah Robinson



Function!

We can think of f as a set of ordered pairs (the set of arrows). In the above example, $(a_1, b_1) \in f$ but $(a_2, b_2) \notin f$.

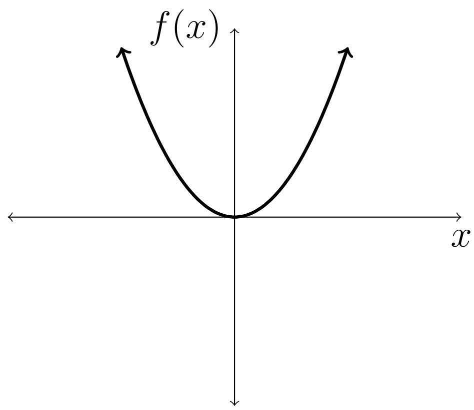
$f : A \to B$	Function f maps A to B
$f \subseteq A \times B$	f is a subset of the cross product of A and B
$(a, b) \in f$	f maps element a to element b

Let's consider the function $y = f(x) = x^2$ where $x \in \mathbb{R}$.

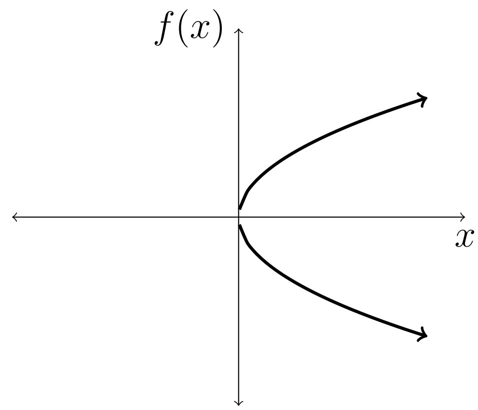
$f : \mathbb{R} \to \mathbb{R}$	The potential y values are also in $\mathbb{R}$
$f \subseteq \mathbb{R}^2$	The set f of ordered pairs (x, y) are a subset of $\mathbb{R}^2$
$(x, y) \in f$	f maps element x to element y

We call function f **real-valued** if the codomain is $\mathbb{R}$. f is **vector-valued** if the codomain is $\mathbb{R}^n$ for $n > 1$.

When you first learned about functions, you probably learned about the vertical line test, where f is a function if any possible vertical line only crosses the function once. This is the same as the requirement that each element $a \in A$ cannot map to more than one element $b \in B$. (And we didn't worry too much then about making sure that every element $a \in A$ actually mapped to something.)

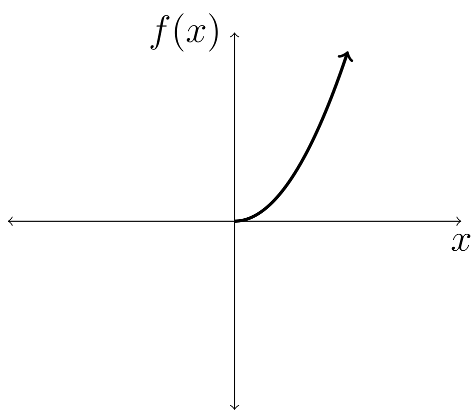


Function!



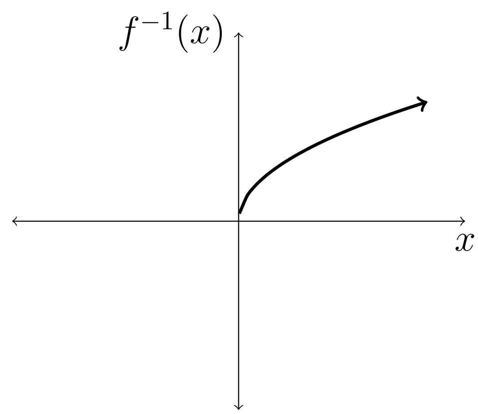
Not a function!

When we think about defining **inverse functions** (whether they exist), we need to be really careful about how we've defined the domain and codomain (they need to be exactly switched).



$$f : \mathbb{R}^+ \to \mathbb{R}^+$$
$$f(x) = x^2$$

NOT $f : \mathbb{R} \to \mathbb{R}$



$$f : \mathbb{R}^+ \to \mathbb{R}^+$$
$$f^{-1}(x) = \sqrt{x}$$

Let f be a real-valued function whose domain includes interval I:

- f is **increasing** on I iff $\forall\ x, y \in I,\ x \leq y \Rightarrow f(x) \leq f(y)$
- f is **decreasing** on I iff $\forall\ x, y \in I,\ x \leq y \Rightarrow f(x) \geq f(y)$
- For strictly increasing and strictly decreasing, replace the weak inequalities with strict inequalities

For functions where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, we need some more notation:

$\mathbf{x} \geq \mathbf{y}$ $\quad x_i \geq y_i \ \ \forall i = 1, \ldots, n$

$\mathbf{x} \gg \mathbf{y}$ $\quad x_i > y_i \ \ \forall i = 1, \ldots, n$

Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$. Then f is:

- **increasing** iff $f(\mathbf{x}_0) \geq f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \geq \mathbf{x}_1$.
- **strictly increasing** iff $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \gg \mathbf{x}_1$
- **strongly increasing** iff $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 \geq \mathbf{x}_1$

- **decreasing** iff $f(\mathbf{x}_0) \geq f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \leq \mathbf{x}_1$
- **strictly decreasing** iff $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \ll \mathbf{x}_1$
- **strongly decreasing** iff $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 \leq \mathbf{x}_1$

Sequences

A function x with domain $\mathbb{N}$ is called an **infinite sequence**.

Example: Let $x : \mathbb{N} \to \mathbb{R}$ where $x_n = \dfrac{1}{n+1}$

We also refer to infinite sequences using set notation $\{x_n\}_{n=1}^{\infty}$

- Note that in an infinite sequence, items can repeat and order matters (unlike sets in general); elements are indexed by n

For a sequence x of real numbers and a real number L, x has a **limit** L (or x **converges** to L) if and only if for all $\varepsilon > 0$, there exists a natural number N such that if $n > N$, then $|x_n - L| < \varepsilon$.

$$\lim_{n\to\infty} x_n = L \quad \text{or} \quad x_n \to L$$

If no number L exists, then x does not converge (or the limit does not exist). If x converges, then its limit is unique.

Example:

- $\left\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \dots\right\}$
- $y_n = (-1)^n$
- $\{a_n\}_{n=1}^{\infty} = \{1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \dots\}$
- $x_n = \dfrac{3n^2}{n^2+1}$

Outline of how to prove that $x_n \to L$:

- On scratch paper, rearrange $|x_N - L| = \varepsilon$ to $N = f(\varepsilon)$
- Let $\varepsilon > 0$ – arbitrary ε
- Let $N > f(\varepsilon)$ – using the $f(\cdot)$ you found
- Let $n > N$ – arbitrary n
- Show that $|x_n - L| < \varepsilon$ using $n > f(\varepsilon)$

Example: Show that this sequence converges to zero.

$$x_n = \frac{(-1)^n}{n+1}$$

$$|x_N - L| = |x_N| = \left|\frac{(-1)^N}{N+1}\right| = \frac{1}{N+1} = \varepsilon \quad \Longrightarrow \quad N = \frac{1}{\varepsilon} - 1$$

Illustration

Example: Show that this sequence converges to zero.

$$x_n = \frac{(-1)^n}{n+1}$$

$$|x_N - L| = |x_N| = \left|\frac{(-1)^N}{N+1}\right| = \frac{1}{N+1} = \varepsilon \quad \implies \quad N = \frac{1}{\varepsilon} - 1$$

<u>To show</u>: $|x_n - 0| < \varepsilon$.
<u>Proof</u>:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 && \text{(by hypothesis)} \\
&\text{Let } N > \frac{1}{\varepsilon} - 1 && \text{(by hypothesis)} \\
&\text{Let } n > N && \text{(by hypothesis)} \\
&|x_n - 0| = \left|\frac{(-1)^n}{n+1} - 0\right| && \text{(by def. of } x_n\text{)} \\
&= \frac{1}{n+1} && \text{(simplifying)} \\
&< \frac{1}{\left(\frac{1}{\varepsilon} - 1\right) + 1} && \left(n > \frac{1}{\varepsilon} - 1\right) \\
&= \varepsilon && \text{(simplifying)}
\end{aligned}$$

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During the first year, going to focus on:

- A generic sequence that converges to zero $\varepsilon_n \to 0$
- Convergence in probability $\xrightarrow{p}$
- Convergence in distribution $\xrightarrow{d}$

Let a and b be sequences of real numbers such that $a_n \to a$ and $b_n \to b$:

- $a_n + b_n \to a + b$
- $ka_n \to ka$ for any real number k
- $a_n b_n \to ab$
- If $b \neq 0$ and $b_n \neq 0$ for any n, then $a_n/b_n \to a/b$

Let a, b and c be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$. (Sandwich Theorem)

Series

Given an infinite sequence $\{a_n\}$, an **infinite series** is the sum of the elements in the sequence.

$$a_1 + a_2 + a_3 + \cdots = \sum_{i=1}^{\infty} a_i$$

The sequence $\{S_n\}$ is the **sequence of partial sums**:

$$S_n = \sum_{i=1}^{n} a_i$$

Example: Geometric series

$$\begin{aligned}
a_n &= \delta^{n-1} a \\
S_n &= \sum_{i=1}^{n} a_i \\
&= \sum_{i=1}^{n} \delta^{i-1} a \\
&= a + \delta a + \delta^2 a + \delta^3 a + \cdots + \delta^{n-1} a \\
S_1 &= a \\
S_2 &= a + \delta a \\
S_3 &= a + \delta a + \delta^2 a
\end{aligned}$$

The series $\sum_{i=1}^{\infty} a_i$ converges to L if the sequence S_n to L:

$$S_n \to L$$

$$L = \sum_{i=1}^{\infty} a_i$$

If $a \neq 0$ and $|\delta| < 1$, then a geometric converges to $\frac{a}{1-\delta}$

$$\begin{aligned}
S_n &= a + \delta a + \delta^2 a + \cdots + \delta^{n-1} a \\
\delta S_n &= \delta a + \delta^2 a + \delta^3 a + \cdots + \delta^n a \\
S_n - \delta S_n &= a - \delta^n a \\
S_n &= \frac{a - \delta^n a}{1-\delta} \\
\lim_{n\to\infty} S_n &= \frac{a}{1-\delta}
\end{aligned}$$

$$\begin{aligned}
\sum_{i=1}^{\infty} \delta^{i-1} a &= \frac{a}{1-\delta} \\
\sum_{i=0}^{\infty} \delta^{i} a &= \frac{a}{1-\delta}
\end{aligned}$$

Example: Taylor Series Expansion

$$\sum_{i=0}^{\infty} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

$$= f(a) + \frac{f'(a)}{1}(x-a) + \frac{f''(a)}{1 \cdot 2}(x-a)^2 + \frac{f'''(a)}{1 \cdot 2 \cdot 3}(x-a)^3 + \dots$$

$$= f(x)$$

We can approximate an $n-$times differentiable function around a point using a n^{th} order Taylor series expansion. The higher n, the better our approximation.

$$f(x) \approx \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$$

Example:
Use a Taylor series expansion at $a = 8$ to approximate $f(x) = \sqrt[3]{x}$ at $x = 10$

$$f(a) = \sqrt[3]{8} = 2$$

$$\frac{f'(a)}{1}(x-a) = \frac{1}{3a^{2/3}}(x-a) = \frac{1}{12}(2) = 0.166667$$

$$\frac{f''(a)}{1 \cdot 2}(x-a)^2 = \frac{1}{2}\,\frac{1}{3}\,\frac{-2}{3}\,\frac{1}{a^{5/3}}(x-a) = \frac{-1}{288}(2)^2 = -0.013889$$

$$f(x) \approx 2.152778$$

$$f(x) = 2.154434$$

Recap of Functions

Let A and B be sets. Function f maps A to B such that, for every element $a \in A$, there is exactly one mapped element $b \in B$.

- $f : \mathbb{R} \to \mathbb{R}$
- $f : \mathbb{R}^n \to \mathbb{R}$
- $f : \mathbb{R} \to \mathbb{R}^m$
- $f : \mathbb{R}^n \to \mathbb{R}^m$
- $f : 2^A \to 2^B$

For all of these, f is mapping an element to an element (e.g., a vector $\mathbf{x} \in \mathbb{R}^n$ to a vector $\mathbf{y} \in \mathbb{R}^m$).

We can think of f as a set of ordered pairs.

$f : A \to B$	Function f maps domain A to codomain B
$f \subseteq A \times B$	f is a subset of the cross product of A and B
$(a, b) \in f$	f maps element a to element b

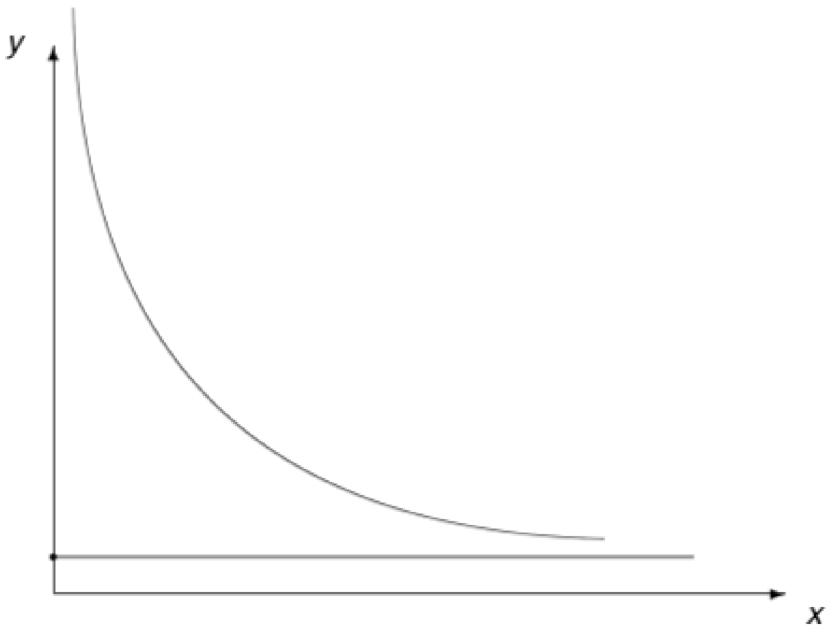
Correspondences

Correspondence ϕ maps elements $a \in A$ to sets $\phi(a) \subseteq B$.

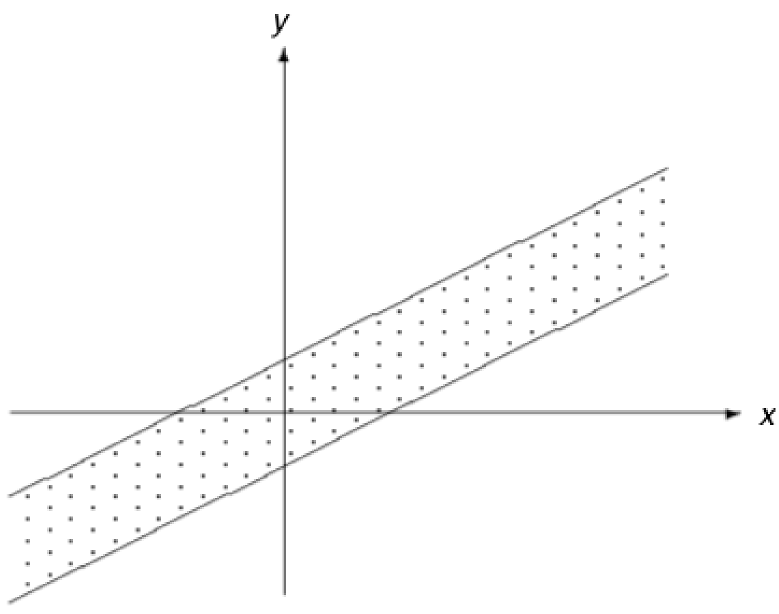
$\phi : A \rightrightarrows B$	Correspondence ϕ maps domain A to codomain B
$\phi \subseteq A \times B$	ϕ is a subset of the cross product of A and B
$(a, b) \in \phi$	ϕ maps element a to element b
$(a, c) \in \phi$	ϕ maps element a to element c
$b, c \in \phi(a)$	ϕ maps element a to elements b and c (b and c are elements in the set $\phi(a)$)

$\phi(x) = \{y \in \mathbb{R} \mid y^2 = x\} \qquad y \in \phi(x)$

A and B could be sets of real numbers or sets of vectors or sets of sets (as with functions). For now lets consider only sets of real numbers.



$\phi : \mathbb{R}^+ \rightrightarrows \mathbb{R}^+$



$\phi : \mathbb{R} \rightrightarrows \mathbb{R}$

Preference Relation

Functions and correspondences are both special cases of relations.

Let A and B be sets. A relation is a subset of $A \times B$.

Example: The operator $\geq$ is a relation. It's a subset of $\mathbb{R}^2$ with elements (x, y) such that $x \geq y$.

A very important relation is the **preference relation** $\succsim$ on a choice set X.

$\succsim = \{(x, y) \mid$ "x is (weakly) preferred to y" $\forall x, y \in X\}$

$\succsim \subseteq \; X \times X$

Both of these mean "x is (weakly) preferred to y":

$$x \succsim y \qquad\qquad (x, y) \in \succsim$$

$$X \times X$$

	apple pie	*brownies*	*cookies*
apple pie			
brownies			
cookies		(b, c)	

Below are some *potential* properties the preference relation (or relations more generally) might satisfy.

Let X be a set and $\succsim$ be a relation on X:

$\succsim$ is **reflexive** if and only if $x \succsim x \ \ \forall x \in X$

$\succsim$ is **complete** if and only if $x \succsim y \ \lor \ y \succsim x \ \ \forall x, y \in X$

$\succsim$ is **transitive** if and only if $x \succsim y \ \land \ y \succsim z \implies x \succsim z \ \ \forall x, y, z \in X$

$\succsim$ is **antisymmetric** if and only if $x \succsim y \ \land \ y \succsim x \implies x = y \ \ \forall x, y \in X$

Choice Function (loosely speaking)

The input to the function is a menu (the dessert menu I get), and the output is a choice (the dessert item I order).

The **choice function** is going to map (sets of sets) to (sets)

- Domain: set of potential menus ($\approx$ the power set)
- Element in the domain: a potential menu $\{apple\ pie, brownies\}$
- Codomain: set of potential options $\{apple\ pie, brownies, cookies, \ldots\}$
- Element in the codomain: the chosen option $brownies$

The **choice correspondence** is going to map (sets of sets) to (sets of sets)

- The domain is the same
- Element in the codomain: sets I randomly choose among $\{brownies, cookies\}$
- Domain: set of those sets