Math Camp 2020 - Analysis (Reference)*

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1. **Topology**

   (a) Aside. We very briefly mentioned the concept of a field, i.e., a non-empty set equipped with addition and multiplication. With another assumption (that of completeness, which we won't discuss) lets us build all of calculus on the foundation of the real numbers. For what follows, we will consider the real numbers to be the universe of discourse.

   (b) Definition. A **convex combination** is a linear combination of points where all coefficients are non-negative and sum to one.

   (c) Example. Consider points (possibly vectors) $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. A general convex combination, which can be denoted $\mathbf{w}$, is

   $$\mathbf{w} = k_1\mathbf{x} + k_2\mathbf{y} + k_3\mathbf{z}$$

   where $k_1 + k_2 + k_3 = 1$ and $k_i \geq 0, i = 1, 2, 3$.

   (d) Definition. A set $A \subseteq \mathbb{R}^n$ is convex if and only if $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in A$ for all $\mathbf{x}, \mathbf{y} \in A$ and $\alpha \in [0, 1]$. In other words, a set is convex if whenever it contains two vectors, it contains the line connecting them as well.

   (e) Example.

   A Convex Set   A Non-Convex Set

   Of any two points we pick in the unit circle on the left, the straight line drawn between them will fall entirely within the circle. In the crescent shape on the right, there exist points in the set (e.g., the tips of the points) such that the line draw between them contains points *not in* in the set.

   (f) Example. Consider the points in the set A defined as

   $$A = \{x | x \in \mathbb{R} \wedge -1 \leq x \leq 1\}$$

   which is the close interval $[-1, 1]$. We can prove that this set is convex by using the definition. To show: $z = tx + (1-t)y \in A$

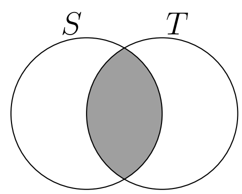
*These lecture notes are drawn principally from the mathematical appendices from *Microeconomic Theory*, by Andreu Mas-Colell, Michael D. Whinston, and Jerry R. Green, and *Advanced Microeconomic Theory*, by Geoffrey A. Jehle and Philip J. Reny. The material posted on this website is for personal use only and is not intended for reproduction, distribution, or citation. James Banovetz created the first edition of these awesome notes and graciously shared them.

Proof:

$$
\begin{aligned}
&\text{Let } x, y \in A \text{ and } z = tx + (1-t)y \text{ for } t \in [0,1] && \text{(by hypothesis)} \\
&\implies \big(-1 \leq x \leq 1\big) \wedge \big(-1 \leq y \leq 1\big) && \text{(by def. of } A\text{)} \\
&\implies \big(-t \leq tx \leq t\big) \wedge \big(-(1-t) \leq (1-t)y \leq 1-t\big) && \text{(multiplying by } t \text{ and } 1-t\text{)} \\
&\implies -1 \leq tx + (1-t)y \leq 1 && \text{(summing)} \\
&\implies -1 \leq z \leq 1 && \text{(by def. of } z\text{)} \\
&\implies z \in A && \text{(by def. of } A\text{)}
\end{aligned}
$$

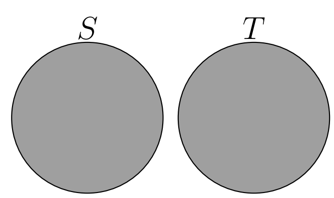
(g) Theorem (JR THM A1.1). Let S and T be convex sets in $\mathbb{R}^n$. Then $S \cap T$ is a convex set.

(h) Example. Consider the graphical representation of two sets:



Intersection: $S \cap T$

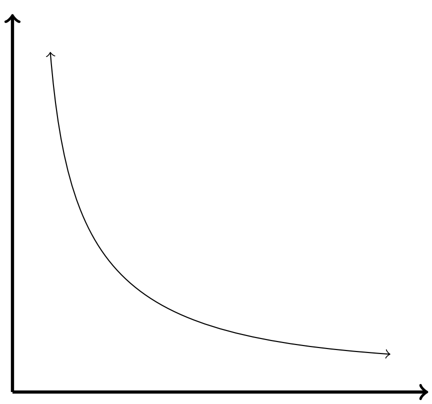
The intersection of the sets is clearly a convex set. While this does not suffice as a proof of the theorem, it should help solidfy the idea. Now consider a similar proposition: If S and T are convex sets, then $S \cup T$ is convex. This is NOT a true statement. Consider two disjoint sets:



$S \cup T$

Clearly, this is *not* a convex set.

(i) Aside. Recall our typical shape of indifference curves, e.g.,



An agent with this indifference curve is indifferent to everything on the curve, and strictly prefers everything above and to the right of the curve. Note that the "just as good as" set is thus a convex set! This ends up being a key assumption, that preferences are convex. When we make the assumption of convex preferences, moreover, we're invoking the set notion of convexity.

(j) Definition. Given a set $B \subseteq \mathbb{R}^n$, the **convex hull** of B, denoted $\text{Co}(B)$, is the smallest convex set containing B, that is, the intersection of all convex sets that contain B:

$$\text{Co}(B) = \left\{ \mathbf{y} \middle| \mathbf{y} = \sum_{j=1}^{J} \alpha_j \mathbf{x}_j, \left(\mathbf{x}_j \in B \; \forall \; j \right) \wedge \left(\alpha_j \geq 0 \; \forall \; j \right) \wedge \sum_{j=1}^{J} \alpha_j = 1 \right\}$$

Alternatively, the convex hull of B is the set of all convex combinations of points in B.

(k) Example. Consider one of the non-convex sets from a previous example:

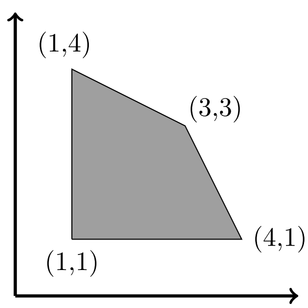
A Non-Convex Set   The Convex Hull

The set on the left is non-convex; the set on the right is convex (it's the set of all convex combinations of points in the first set). This is the smallest convext set containing everything in B.

(l) Example. Consider the following two-player prisoners' dilemma from game theory:

	C	D
C	(3, 3)	(1, 4)
D	(4, 1)	(1, 1)

The set of payoff profiles is $\{(1,1),\ (1,4),\ (4,1),\ (3,3)\}$. The convex hull of the payoff profiles can be described graphically:



(m) Definition. The **epsilon-neighborhood** (or ε-ball) with center $\mathbf{x}$ and radius ε is the subset of points in $\mathbb{R}^n$ defined as

$$B_\varepsilon(\mathbf{x}) = \{\mathbf{y} | \mathbf{y} \in \mathbb{R}^n \ni d(\mathbf{x}, \mathbf{y}) < \varepsilon\}$$

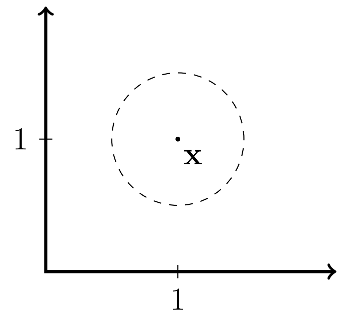
(n) Example. Let our universe of discourse be $\mathbb{R}$, equipped with the metric $d(x,y) = |x - y|$. Then the epsilon-neighborhood around 0 is given by:

$$B_\varepsilon(0) = \{y | y \in \mathbb{R} \ni -\varepsilon < y < \varepsilon\}$$

i.e., the open interval $(-\varepsilon, \varepsilon)$. We can think about higher dimensions as well. Consider $\mathbb{R}^2$ with the euclidean metric. Then the epsilon-neighborhood around the point $\mathbf{x} = (1,1)$ is given by:

$$B_\varepsilon(\mathbf{x}) = \{\mathbf{y} | \mathbf{y} \in \mathbb{R}^2 \ni ||\mathbf{y} - \mathbf{x}||_2 < \varepsilon\}$$

Graphically, this may be represented:



Where the radius of the circle is equal to ε.

(o) Definition. Let $S \subseteq \mathbb{R}^n$. S is an **open set** if and only if for all $\mathbf{x} \in A$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subseteq S$.

(p) Example. Consider the set S formed by the open interval (a, b), i.e., $S = \{x | a < x < b\}$. We can prove this is an open set.

- Def. of subset: $B_\varepsilon(x) \subseteq S$ if and only if $y \in B_\varepsilon(x) \implies y \in S$

To show: $y \in S$
Proof:

$$
\begin{array}{ll}
\text{Let } x \in S \text{ and } \varepsilon = \min\{b - x, x - a\} & \text{(by hypothesis)} \\
\text{Consider } B_\varepsilon(x) = (x - \varepsilon, x + \varepsilon) & \text{(defining an } \varepsilon\text{-ball)} \\
\text{Let } y \in (x - \varepsilon, x + \varepsilon) & \text{(by hypothesis)} \\
\implies x - \varepsilon < y < x + \varepsilon & \text{(by def. of an open interval)} \\
\implies x - (x - a) < y < x + (b - x) & \text{(by def. of } \varepsilon\text{)} \\
\implies a < y < b & \text{(simplifying)} \\
\implies y \in S & \text{(by def. of } S\text{)}
\end{array}
$$

■

The intuition behind this proof (and behind any open-set proof) is that for any point we pick in the set, we can always draw a tiny circle around the point that lies entirely within the set.

(q) Theorem (JR THM A1.2). The following sets in $\mathbb{R}^n$ are open sets:

i. The empty set $\emptyset$

ii. The entire space $\mathbb{R}^n$

iii. The union of any number of open sets

iv. The intersection of any finite number of open sets

(r) Definition. Let $S \subseteq \mathbb{R}^n$. S is a **closed set** if and only if its complement S^c is an open set.

(s) Example. Consider the closed interval $I = [a, b] \subseteq \mathbb{R}$. The complement is $I^c = (-\infty, a) \cup (b, \infty)$, which is the union of two open sets. Thus, I is closed.

(t) THM (JR THM A.1.4). The following sets in $\mathbb{R}^n$ are closed sets:

i. The empty set $\emptyset$

ii. The entire space $\mathbb{R}^n$

iii. The union of any finite collection of closed sets

iv. The intersection of any number of closed sets

(u) THM (JR THM A1.9) Let $D \subseteq \mathbb{R}^n$. D is closed if and only if for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in D$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in D$.

(v) Definition. A set $S \subseteq \mathbb{R}^n$ is **bounded** if and only if there exists an M and a point $\mathbf{x} \in \mathbb{R}^n$ such that $S \subseteq B_M(\mathbf{x})$. That is, there exists an M-ball that contains all of S.

(w) Theorem (JR THM A1.8). A set $S \subseteq \mathbb{R}^n$ is compact if and only if it is closed and bounded. This is known as the Heine-Borel Theorem.

(x) Aside. Note that compactness actually is a topological concept all on its own. The Heine-Borel Theorem establishes that that is equivalent to a set being closed and bounded. For our purposes, it is sufficient to work with the result from the preceding theorem.

(y) Example. We can prove that the set formed with the unit circle as its boundary, defined formally as

$$S = \{(x, y) | (x, y) \in \mathbb{R}^2 \land x^2 + y^2 \leq 1\}$$

is a compact set.

- Theorem (T1): If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$ and $a_n b_n \to ab$
- Theorem (T2): D is closed iff every convergent sequence of points in D has a limit in D
- Lemmna (L1): If $a_n \to a$, then $a_n \leq b$ for all n implies $a \leq b$.

To show: S is closed

Proof:

$$\begin{array}{ll}
\text{Let } (x_n, y_n) \in S \ \forall n \text{ such that } (x_n, y_n) \to (x, y) & \text{(by hypothesis)} \\
\implies x_n^2 + y_n^2 \leq 1 & \text{(by def. of } S\text{)} \\
\implies (x_n^2, y_n^2) \to (x^2, y^2) & \text{(by T1)} \\
\implies x_n^2 + y_n^2 \to x^2 + y^2 & \text{(by T1)} \\
\implies x^2 + y^2 \leq 1 & \text{(by L1)} \\
\implies (x, y) \in S & \text{(by def. of } S\text{)} \\
\implies S \text{ is closed} & \text{(by T2)}
\end{array}$$

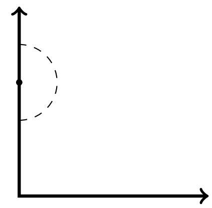
To show: S is bounded

Proof:

$$\begin{array}{ll}
\text{Let } (x, y) \in S & \text{(by hypothesis)} \\
\text{Let } M = 2 & \text{(by hypothesis)} \\
\implies x^2 + y^2 \leq 1 & \text{(def. of } S\text{)} \\
\implies (x^2 \leq 1) \land (y^2 \leq 1) & (x^2 \geq 0 \ \forall x) \\
\implies (-1 \leq x \leq 1) \land (-1 \leq y \leq 1) & \text{(algebra)} \\
\implies (-2 \leq x \leq 2) \land (-2 \leq y \leq 2) & \text{(algebra)} \\
\implies (-M \leq x \leq M) \land (-M \leq y \leq M) & \text{(algebra)} \\
\implies \text{S is bounded} & \text{(by def. of bounded)}
\end{array}$$

Thus, S is closed and bounded, implying that S is compact. ■

(z) Aside. Note that we can also define openess and closedness relative to other spaces. For example, $S \subseteq \mathbb{R}^n$ is open relative to $D \subseteq \mathbb{R}^n$ (the non-negative orthant) if and only if for every $\mathbf{x} \in S$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \cap D \subseteq S$. This comes in most handy for defining openess relative to the non-negative orthant, where we'd only consider the part of the ε-ball that fell within $\mathbb{R}_+$. Closedness relative to D is defined analoguosly, with $D - S$ in place of S^c.



2. **Continuity and Differentiability**

   (a) Definition. Let f be a function with domain $\mathcal{D}(f)$. The **limit** of f as $\mathbf{x}$ approaches $\mathbf{x}_0$ is $\mathbf{L}$ if and only if
   $$\forall\, \varepsilon > 0\; \exists\, \delta > 0 \ni d(\mathbf{x} - \mathbf{x}_0) < \delta \implies d(f(\mathbf{x}), \mathbf{L}) < \varepsilon$$
   In this case we write $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = \mathbf{L}$. A function is **continuous** at $\mathbf{x}_0 \in \mathcal{D}(f)$ if and only if
   $$\forall\, \varepsilon > 0\; \exists\, \delta > 0 \ni d(\mathbf{x} - \mathbf{x}_0) < \delta \implies d(f(\mathbf{x}), f(\mathbf{x}_0)) < \varepsilon$$
   f is a continuous function if it continuous at every point in its domain.

   (b) Theorem. Let $f : S \to \mathbb{R}$ be a continuous, real-valued function where S is non-empty, compact subset of $\mathbb{R}^n$. Then there exists a vector $\underline{\mathbf{x}} \in S$ and a vector $\overline{\mathbf{x}} \in S$ such that
   $$\forall \mathbf{x} \in S, f(\underline{\mathbf{x}}) \leq f(\mathbf{x}) \leq f(\overline{\mathbf{x}})$$
   That is, a continuous function $f(\mathbf{x})$ attains a maximum and a minimum on every compact set. This is known as the (Weierstrass) Extreme Value Theorem.

   (c) Theorem. Let $S \subseteq \mathbb{R}^n$ be a non-empty compact, convex set. Let $f : S \to S$ be a continuous function. Then there exists at least one fixed point of f in S, that is, there exists $\mathbf{x}^* \in S$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$. This is known as the (Brouwer) Fixed Point Theorem.

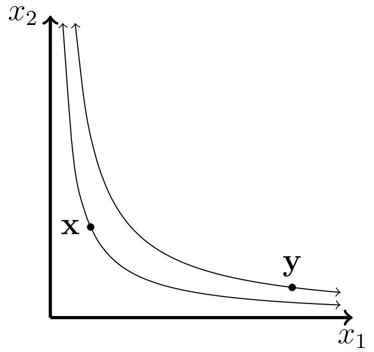
   (d) Aside. We won't spend too much time on these theorems, but you definitely need to be familiar with them moving forward. The first guarantees the existence of the maximum and the minimum of functions on compact sets–think about constrained optimization and (compact) budget sets, and it should be clear why this is important. Fixed points will come up in a major way during game theory (Nash Equilibria) and macro economics (convergence of dynamic programs).

   (e) Definition. Let f be a function defined on an interval $(a, b) \subseteq \mathbb{R}$ and let $c \in (a, b)$. Then f is **differentiable** at c if and only if the limit of
   $$\lim_{x\to c} \frac{f(x) - f(c)}{x - c}$$
   exists and is finite. If this is the case, then the limit is called the **derivative** of f and c and is denoted $f'(c)$ or $\frac{df(c)}{dx}$. For a multivariate functions $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$, the **partial derivative** of f with respect to x_i is given by:
   $$\frac{\partial f}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1, \ldots, x_i, \ldots x_n)}{h}$$
   and is sometimes denoted $f_i(\cdot)$

3. **Additional Sets**

   (a) Definition. Let f be a real valued function such that $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. Then $L(\mathbf{x}_0)$ is a **level set** relative to $\mathbf{x}_0$ if and only if
   $$L(\mathbf{x}_0) = \big\{\mathbf{x} | \mathbf{x} \in D \wedge f(\mathbf{x}) = f(\mathbf{x}_0)\big\}$$

(b) Example. This concept is intimately connected to our notion of indifference curves and isoquants. Indeed, both are simply level sets for the appropriate functions–utility and production, respectively. Consider the indifference curves for $u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$:



- $\mathbf{x} = (1, 4)$ and $u(\mathbf{x}) = 2$
- $\mathbf{y} = \left(32, \frac{1}{2}\right)$ and $u(\mathbf{y}) = 4$

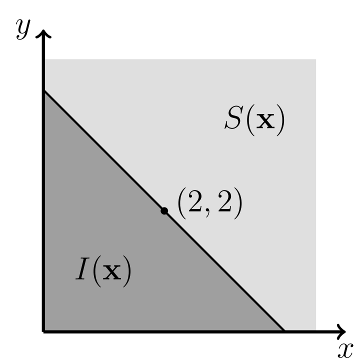
All the points on the curve running through $\mathbf{x}$ give a utility of 2, while all those on the curve running through $\mathbf{y}$ provide a utility of 4.

(c) Definition. Let be a real valued function such that $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. Then relative to a point $\mathbf{x}_0$:

- $S(\mathbf{x}_0) = \{\mathbf{x} | \mathbf{x} \in D \wedge f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$ is the **superior set** relative to $\mathbf{x}_0$
- $I(\mathbf{x}_0) = \{\mathbf{x} | \mathbf{x} \in D \wedge f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ is the **inferior set** relative to $\mathbf{x}_0$

If the weak inequalities are replaced with strict inequalities, then the sets are the **strictly superior set** and **strictly inferior set**, respectively.

(d) Example. Consider the function $u(x_1, x_2) = x_1 + x_2$. The inferior and superior sets, relative to $\mathbf{x} = (2, 2)$ can be illustrated graphically as:



(e) Aside. Note that these are the sets of of points in the domain that map to points greater/less than a point in the range. For univariate functions, e.g., $y = x^2$, the superior/inferior sets are points on the real-number line. For example, the superior set relative to $x = -1$ is $(-\infty, -1) \cup (1, \infty)$. Draw a picture of the graph to help solidify the idea! Please keep the idea of superior/inferior sets separate from the following idea:

(f) Definition. Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. The the set of points **on and below the graph** of f is defined as

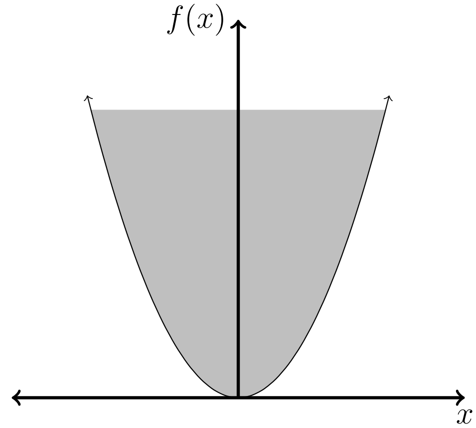
$$A = \{(\mathbf{x}, y) | \mathbf{x} \in D \wedge f(\mathbf{x}) \geq y\}$$

Similarly, the set of points **on and above the graph** is defined as

$$B = \{(\mathbf{x}, y) | \mathbf{x} \in D \wedge f(\mathbf{x}) \leq y\}$$

(g) Aside. Note that superior/inferior sets are points in the domain, while points relative to graph are *ordered pairs*, $(n + 1)$-tuples with elements from both the domain *and* the range.

(h) Example. Consider the set of points on and above the graph of the function $y = x^2$.



4. **Concavity and Convexity**

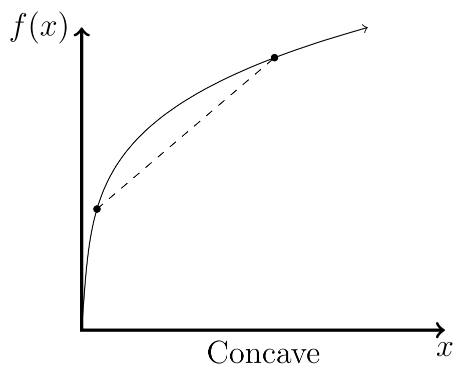
(a) Definition. Let $f : D \to \mathbb{R}$ where D is a convex subset of $\mathbb{R}^n$. A function is **concave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0,1]$:

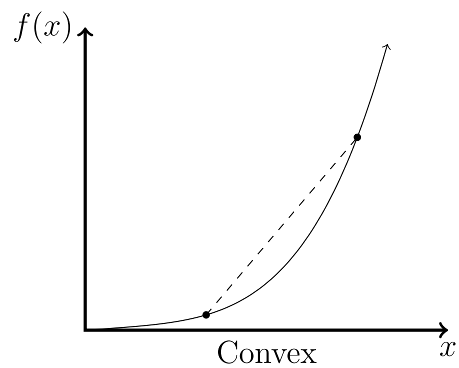
$$f\big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\big) \geq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

A function is **convex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0,1]$:

$$f\big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\big) \leq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

(b) Aside. We can think about this definition graphically using univariate functions:





In the graph on the left, the function lies above convex combination line, so the function is concave; on the right, the function lies below the convex combination line, so the function is convex. While drawing pictures helps solidify the intuition, we need to be well versed in the definition for proofs.

(c) Example. Consider the absolute value function, $f(x) = |x|$, with the domain being $\mathbb{R}$. We can prove this is convex using the definition (e.g., via convex combinations of points in the domain).

- The absolute value function $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Theorem (T1): $|ab| = |a||b|$
- Theorem (T2): The triangle inequality, $|a+b| \leq |a| + |b|$

To show: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$

Proof:

Let $x, y \in \mathbb{R} \wedge t \in [0,1]$	(by hypothesis)
$f(tx + (1-t)y) = \lvert tx + (1-t)y \rvert$	(by def. of $f(x)$)
$\leq \lvert tx \rvert + \lvert (1-t)y \rvert$	(by T2)
$= t\lvert x \rvert + (1-t)\lvert y \rvert$	(by $t \geq 0$)
$= tf(x) + (1-t)f(y)$	(by def. of $f(x)$)

■

(d) Theorem (JR THM A1.13 and A1.17). Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let A be the set of points on and below the graph of f. Then

$$f \text{ is a concave function} \iff A \text{ is a convex set}$$

Similarly, let B be the set of points on and above the graph of f. Then

$$f \text{ is a convex function} \iff B \text{ is a convex set}$$

(e) Aside. We probably wont use this theorem too frequently, but it is another tool to help with concavity/convexity proofs. It may occasionally be easier to prove that the set of points above/below a graph is a convex set that a proof via the definition. Further, the theorem may help visualize what we mean by concavity and convexity.

(f) Definition. Let $f : D \to \mathbb{R}$ where D is a convex subset of $\mathbb{R}^n$. A function is **strictly concave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0,1)$:

$$f\big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\big) > tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

A function is **strictly convex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0,1)$:

$$f\big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\big) < tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

(g) Aside. Note that several things changed between our definitions. First, we must be sure to state that we aren't consider to equivalent points. Second, t must be strictly between zero and one (so all of our weight can't be on a single point). Finally, the inequality becomes strict. Note that by this definition, the absolute value function *is not* strictly convex. Rather than employing definitions and graphs/sets, more frequently we will use calculus criteria to establish concavity/convexity.

(h) Theorem. Let D be a convex, non-degenerate interval on $\mathbb{R}$, such that on the interior of D, f is twice continuously differentiable. Then the following statements are equivalent:

i. f is concave

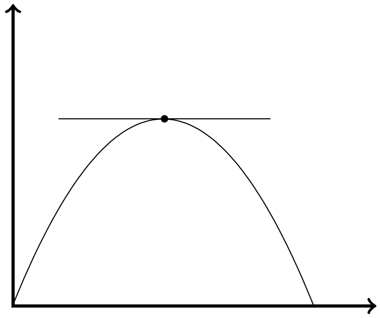
ii. $f''(x) \leq 0$ for all non-enpoints $x \in D$.

iii. For all $x_0 \in D$, $f(x) \leq f(x_0) + f'(x_0)(x - x_0)$

Further, we can relate strict concavity to the second derivative:

iv. If $f''(x) < 0$ for all non-endpoints $x \in D$, then f is strictly concave

(i) Aside. The first two points are what we will use most frequently and are probably the critera for which you are most familiar. The third critera give us another way to picture concavity–the tangent line at a point must lie above the graph:



Finally, note that the the third point is a *conditional*, NOT a bidcondional. A counter-example is $y = -x^4$. This is a strictly concave function, but its derivative at 0 is 0.

(j) Theorem. Let D be a convex, non-degenerate interval on $\mathbb{R}$, such that on the interior of D, f is twice continuously differentiable. Then the following statements are equivalent:

i. f is convex

ii. $f''(x) \geq 0$ for all non-enpoints $x \in D$.

iii. For all $x_0 \in D$, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$

Further, we can relate strict convexity to the second derivative:

iv. If $f''(x) > 0$ for all non-endpoints $x \in D$, then f is strictly convex

(k) Aside. This is all well and good for single-variable functions, but what about multivariable functions? We need to extend our concept of first and second derivatives, which lead us to the gradient and the Hessian matrix.

(l) Definition. Let f be a twice continuously differentiable function, $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Then the **gradient** of f, denoted $\nabla f(\mathbf{x})$ is defined as the row vector of 1st-order partial derivatives:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

The **Hessian** of f, denoted H or $\mathbf{H}$, is the matrix of 2nd-order partial derivatives:

$$H = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \dots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_2} & \dots \\ \vdots & & \ddots \end{bmatrix}$$

(m) Theorem. Let D be a convex subset of $\mathbb{R}^n$ with a non-empty interior on which f is twice continuously differentiable. Then

- H is negative semi-definite $\implies$ f is concave
- H is negative definite $\implies$ f is strictly concave
- H is positive semi-definite $\implies$ f is convex
- H is positive definite $\implies$ f is strictly convex

(n) Aside. The critera from definiteness came up in linear algebra; recall our interpretation using second order total differentials for the intution.

(o) Example. Consider the function $f(x, y) = \ln(x) + \ln(y)$. We can establish that this function is strictly concave over it's domain $\mathbb{R}^2_{++}$ (note that this notation indicates we are only considering strictly positive values in $\mathbb{R}^2$:

$$f(x,y) = \ln(x) + \ln(y) \qquad \text{(the function)}$$
$$\nabla f(x,y) = \begin{bmatrix} \frac{1}{x} & \frac{1}{y} \end{bmatrix} \qquad \text{(the gradient)}$$
$$H = \begin{bmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix} \qquad \text{(the Hessian)}$$

Recall first our notion of leading principle minors (note that these are determinant bars):

$$\begin{aligned} |H_1| &= \left| -\frac{1}{x^2} \right| && \text{(the first LPM)} \\ &= -\frac{1}{x^2} < 0 && \text{(simplifying)} \\ |H_2| &= |H| && \text{(the second LPM)} \\ &= \frac{1}{x^2 y^2} > 0 && \text{(simplifying)} \end{aligned}$$

Thus, since our leading principle minors alternate in sign, beginning with a negative, the matrix is negative definite, implying our function is strictly concave.

(p) Theorem. Let f be a concave function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, concave function, $g : R \to \mathbb{R}$. Then the composite function defined as $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$ is a concave function.

(q) Example. This is a relatively straightforward theorem to prove:
To show: $(g \circ f)(t\mathbf{x} + (1-t)\mathbf{y}) \geq t(g \circ f)(\mathbf{x}) + (1-t)(g \circ f)(\mathbf{y})$
Proof

$$\begin{aligned} &\text{Let } \mathbf{x}, \mathbf{y} \in D \text{ and } t \in [0,1] && \text{(by hypothesis)} \\ &\text{Consider } (g \circ f)(t\mathbf{x} + (1-t)\mathbf{y}) && \text{(the composite)} \\ &= g\big(f(t\mathbf{x} + (1-t)\mathbf{y})\big) && \text{(by def. of the composite)} \\ &\geq g\big(tf(\mathbf{x}) + (1-t)f(\mathbf{y})\big) && \text{(by } f \text{ concave and } g \text{ increasing)} \\ &\geq tg(f(\mathbf{x})) + (1-t)g(f(\mathbf{y})) && \text{(by } g \text{ concave)} \\ &= t(g \circ f)(\mathbf{x}) + (1-t)(g \circ f)(\mathbf{y}) && \text{(by def. of the composite)} \end{aligned}$$

$\blacksquare$

(r) Theorem. Let f be a convex function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, convex function, $g : R \to \mathbb{R}$. Then the composite function defined as $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$ is a convex function.

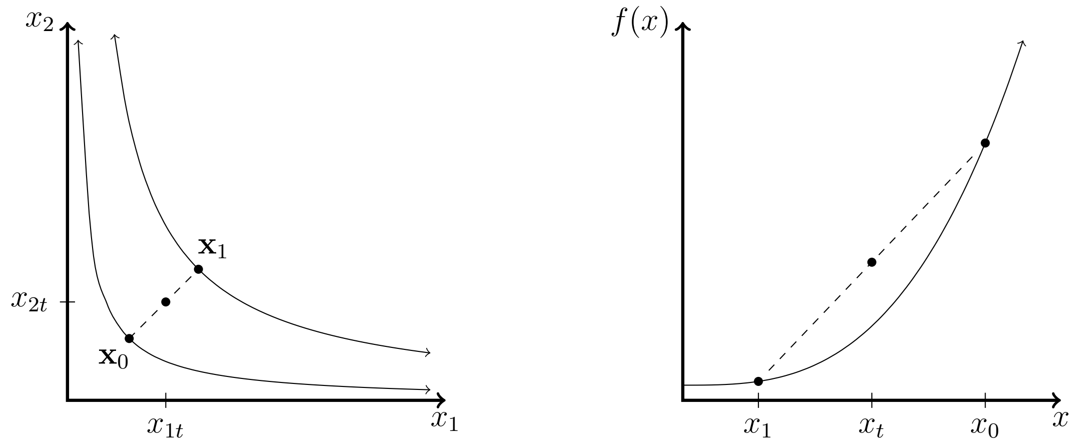
5. **Quasiconcavity and Quasiconvexity**

(a) Aside. Concavity and convexity are very helpful concepts when we're dealing with unconstrained optimization. As economists, however, we deal with constrained optimization more frequently. Under constrained optimization, we can relax our assumptions about concavity and convexity, employing a weaker notion.

(b) Definition. Let D be a convex subset of $\mathbb{R}^n$. Then the function $f : D \to \mathbb{R}$ is **quasiconcave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0,1]$,

$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \geq \min\big\{f(\mathbf{x}_0), f(\mathbf{x}_1)\big\}$$

(c) Example. Consider the level sets from a function with domain $\mathbb{R}^2_+$ and the graph of a function with domain $\mathbb{R}$:



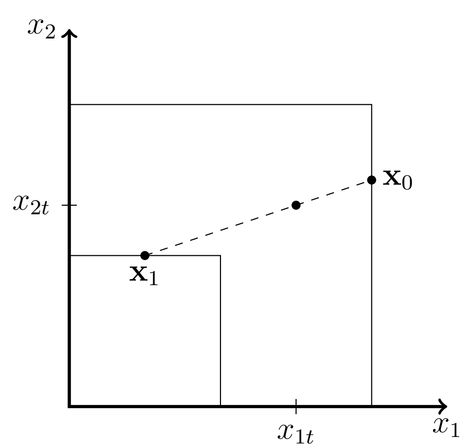
On the left graph, we pick two points, $\mathbf{x}_0$ and $\mathbf{x}_1$, and plot the corresponding level sets. Any point we pick on the convex combination line between $\mathbf{x}_0$ and $\mathbf{x}_1$ (denoted here by the ordered pair (x_{1t}, x_{2t})) will lie on a level set *at least as high* as the one $\mathbf{x}_0$ is on. On the right graph, we picked two points x_0 and x_1 and evaluate the function at each. Any point we pick on the convex combination line (denoted here as x_t) will correspond to a function value *at least as high* as the function evaluated at x_1. Both of these functions are quasiconcave.

(d) Definition. Let D be a convex subset of $\mathbb{R}^n$. Then the function $f : D \to \mathbb{R}$ is **quasiconvex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$,

$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) \leq \max\{f(\mathbf{x}_0), f(\mathbf{x}_1)\}$$

(e) Aside. Consider the graphs above. The level sets on the left *do not* represent a quasiconvex function–think about picking a point on the right-most part of the lower level set (the one associated with $\mathbf{x}_0$), and the upper-most part on the upper level set (the one associated with $\mathbf{x}_1$). The tangent line would cut above the upper level set, violatin the definition. The function on the right graph, however, *is* quasiconvex, as every point on the tangent line corresponds to a functional value less than $f(x_0)$.

(f) Example. Consider the level sets for a function with domain $\mathbb{R}^2_+$ (note that the function increases as we move right and up):



In this graph, we pick two points, $\mathbf{x}_0$ and $\mathbf{x}_1$, and plot the corresponding level sets. any point we pick on the convex combination line (once again denoted (x_{1t}, x_{2t})) will lie on a level set *no higher than* the one associated with $\mathbf{x}_0$.

(g) Example. We can show that the function $f(x, y) = \min\{x, y\}$, defined on $\mathbb{R}^2$, is quasiconcave but *not* quasiconvex using the applicable definitions.

- Lemma 1 (L1): $tx + (1-t)y \geq \min\{x, y\}\ \forall t \in [0, 1]$
- Lemma 2 (L2): $\min\big\{\min\{w, x\}, \min\{y, z\}\big\} = \min\{w, x, y, z\}$

To show: $f(x_t, y_t) \geq \min\{f(x_1, y_1), f(x_2, y_2)\}$
Proof:

Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2_+$ and $t \in [0, 1]$ (by hypothesis)

Let $x_t = tx_1 + (1-t)x_2$ and $y_t = ty_1 + (1-t)y_2$ (by hypothesis)

Consider $f(x_t, y_t) = \min\big\{tx_1 + (1-t)x_2, ty_1 + (1-t)y_2\big\}$ (the function)

$\geq \min\Big\{\min\{x_1, x_2\}, \min\{y_1, y_2\}\Big\}$ (by L1)

$= \min\big\{x_1, x_2, y_1, y_2\big\}$ (by L2)

$= \min\Big\{\min\{x_1, y_1\}, \min\{x_2, y_2\}\Big\}$ (by L1)

$= \min\big\{f(x_1, y_1), f(x_2, y_2)\big\}$ (by def. of f)

■

Thus, $f(x_t, y_t) \geq \min\{f(x_1, y_1), f(x_2, y_2)\}$, satisfying the definition of quasiconcavity. To prove that the function is *not* quasiconvex, it suffices to provide a counter example:

Consider $(x_1, y_1) = (1, 9)$ and $(x_2, y_2) = (9, 1)$ and $t = 0.5$ (picking values)

$f(x_1, y_1) = \min\{1, 9\} = 1$ (by def. of f)

$f(x_2, y_2) = \min\{9, 1\} = 1$ (by def. of f)

$(x_t, y_t) = (5, 5)$ (the convex combo.)

$f(x_t, y_t) = \min\{5, 5\} = 5$ (by def. of f)

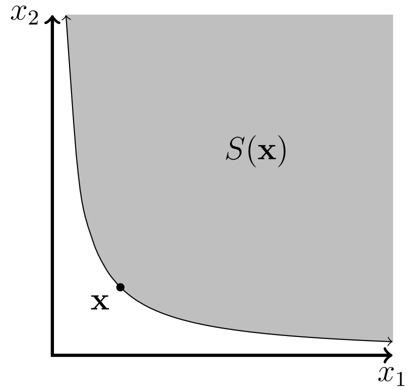
Thus, $f(x_t, y_t) \not\leq \max\{f(x_1, y_1), f(x_2, y_2)\}$ ■

Remember, to prove a proposition is false, we simply need one instance where the definition is violated.

(h) Aside. The definition is a bit unwieldy, but luckily, we have another way to think about quasiconcavity and quasiconvexity. Recall out pictures of our level sets above. They are shaped very particularly (convex-to or convex-away from the origin). This leads to the next characterization of quasi-concavity/convexity:

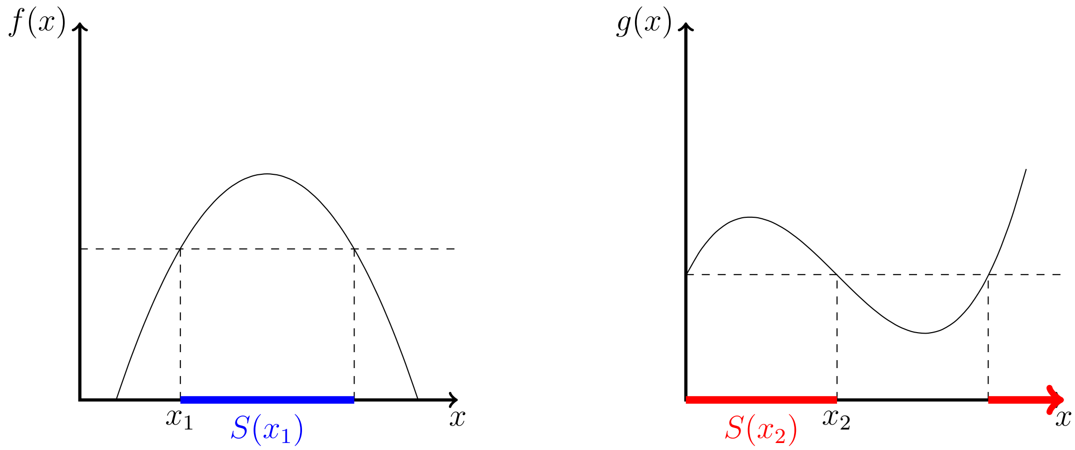
(i) Theorem (JR THM A1.14). A function $f : D \to \mathbb{R}$ is quasiconcave iff its superior set $S(\mathbf{x})$ is a convex set for all $\mathbf{x} \in D$.

(j) Example. Consider again one of our level sets with our typical "indifference curve" look, where the shaded area is the superior set:



If this function's superior sets are all convex (as this one is), then the function is quasiconcave. Note that this level set looks very similar to an indifference curve–this is no coincidence. In fact, our assumption that "indifference curves are convex towards the origin" has a technical mathematical definition as well: the utility function is quasiconcave. Similarly, a picture like this corresponds to the assumption of convex preferences, discussed earlier.

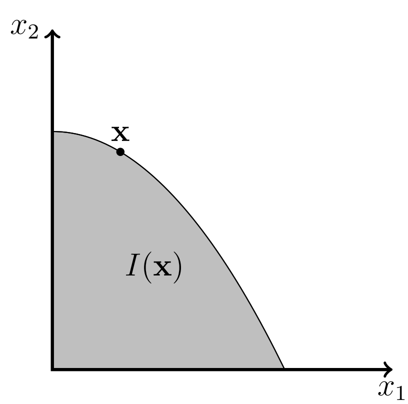
(k) Example. Consider two univariate functions:



On the left graph, the set of points in the domain that produce a function value at least as high as $f(x_1)$ is convex (given by the shaded blue area). On the right graph, the set of points in the domain that produce a function value at least as high as $g(x_2)$ is *not* convex (the shaded red areas). Thus, the graph on the left is quasiconcave, but the one on the right is not.

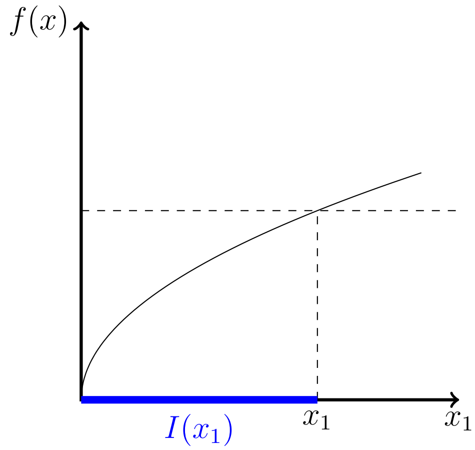
(l) Theorem (JR THM A1.18). A function $f : D \to \mathbb{R}$ is quasiconcave iff its inferior set $I(\mathbf{x})$ is a convex set for all $\mathbf{x} \in D$.

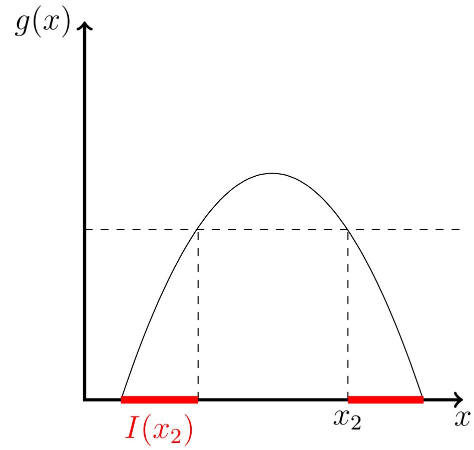
(m) Example. Consider level sets of a function that are concave towards the origin:



This inferior set is a convex set; if all inferior sets are convex, then the function is quasiconvex. Note that we don't use quasiconvexity nearly as much as quasiconcavity; indeed, indifference curves that look this way would be highly irregular.

(n) Example. Consider two univariate functions:





On the left graph, the inferior set $I(x_1)$ is a convex set (the shaded blue area), so the function is quasiconvex. On the right, the inferior set $I(x_2)$ is not a convex set (the shaded red areas), so the function *is not* quasiconvex.

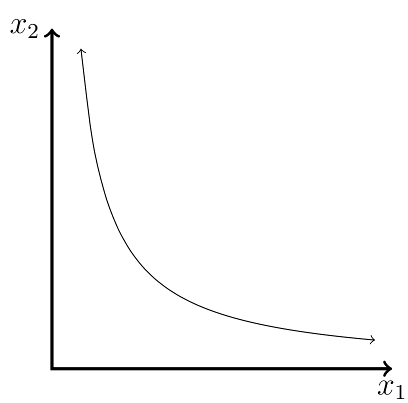
(o) Definition. Let D be a convex subset of $\mathbb{R}^n$. Then the function $f : D \to \mathbb{R}$ is **strictly quasiconcave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0, 1)$,

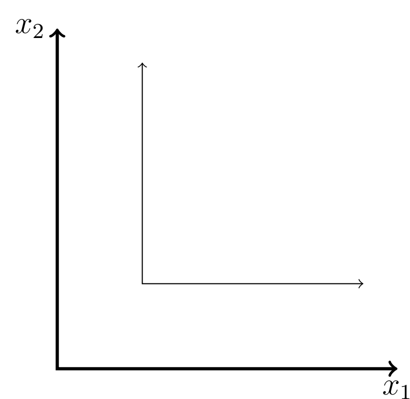
$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) > \min\left\{f(\mathbf{x}_0), f(\mathbf{x}_1)\right\}$$

The function $f : D \to \mathbb{R}$ is **strictly quasiconvex** if and only if

$$f(t\mathbf{x}_0 + (1-t)\mathbf{x}_1) < \max\left\{f(\mathbf{x}_0), f(\mathbf{x}_0)\right\}$$

(p) Example. Consider level sets from two different functions, one that is strictly quasiconcave, and one that is not:





On the left, pick any two points on the level set–any convex combination that is *strictly* between the two points will fall on a level set higher than first. On right right, if we pick two points on the level set, *it is* necessary that the convex combination will be on a higher level set (e.g., pick two points on the horizontal portion of the level set).

(q) Definition. Let f be a twice continuously differentiable function, $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Then the **bordered Hessian**, denoted $\bar{H}$, is the matrix of 2nd-order partial derivatives bordered by the first derivatives, i.e.,

$$\bar{H} = \begin{bmatrix} 0 & f_1(\mathbf{x}) & f_2(\mathbf{x}) & \dots \\ f_1(\mathbf{x}) & f_{11}(\mathbf{x}) & f_{12}(\mathbf{x}) & \\ f_1(\mathbf{x}) & f_{12}(\mathbf{x}) & f_{22}(\mathbf{x}) & \\ \vdots & & & \ddots \end{bmatrix}$$

where $f_i(\mathbf{x}) = \frac{\partial f(\mathbf{x})}{\partial x_i}$ and $f_{ij} = \frac{\partial^2 f(\mathbf{x})}{\partial \mathbf{x}_i \partial \mathbf{x}_j}$.

(r) Theorem (SB THM 21.19 and CW 12.26). Let f be a twice continuously differentiable function on a convex domain $D \subseteq R_n$. Consider the associated bordered Hessian:

- f is quasiconcave if the second to nth order leading principle minors alternate in sign, beginning with a negative, for all $\mathbf{x} \in D$.

$$|\bar{H}_1 = 0|, \quad |\bar{H}_2| \leq 0, \quad |\bar{H}_3| \geq 0, \quad |\bar{H}_4| \leq 0, \dots$$

- f is quasiconvex if the second to nth order leading principle minors are negative for all $\mathbf{x} \in D$.

$$|\bar{H}_1 = 0|, \quad |\bar{H}_2| \leq 0, \quad |\bar{H}_3| \leq 0, \quad |\bar{H}_4| \leq 0, \dots$$

If $D \subseteq \mathbb{R}^n_+$ (i.e., the domain is the non-negative orthant), then

- f is strictly quasiconcave if the 2nd to nth order leading principle minors alternate in sign with strict inequalities, beginning with a negative, for all $\mathbf{x} \in D$:

$$|\bar{H}_1 = 0|, \quad |\bar{H}_2| < 0, \quad |\bar{H}_3| > 0, \quad |\bar{H}_4| < 0, \dots$$

- f is strictly quasiconvex if the 2nd to nth order leading principle minors are strictly negative, beginning with a $\mathbf{x} \in D$:

$$|\bar{H}_1 = 0|, \quad |\bar{H}_2| < 0, \quad |\bar{H}_3| < 0, \quad |\bar{H}_4| < 0, \dots$$

(s) Example. Consider a function $f : \mathbb{R}^2_{++} \to \mathbb{R}$ where $f(x, y) = \ln(x) + \ln(y)$. We can show that this function is strictly quasiconcave.

$$\bar{H} = \begin{bmatrix} 0 & \frac{1}{x} & \frac{1}{y} \\ \frac{1}{x} & -\frac{1}{x^2} & 0 \\ \frac{1}{y} & 0 & -\frac{1}{y^2} \end{bmatrix} \qquad \text{(the borderd Hessian)}$$

$$|\bar{H}_1| = 0 \qquad \text{(the first LPM)}$$

$$|\bar{H}_2| = \begin{vmatrix} 0 & \frac{1}{x} \\ \frac{1}{x} & 0 \end{vmatrix} \qquad \text{(the second LPM)}$$

$$|\bar{H}_2| = -\frac{1}{x^2} < 0 \qquad \text{(evaluating)}$$

$$|\bar{H}_3| = \begin{vmatrix} 0 & \frac{1}{x} & \frac{1}{y} \\ \frac{1}{x} & -\frac{1}{x^2} & 0 \\ \frac{1}{y} & 0 & -\frac{1}{y^2} \end{vmatrix} \qquad \text{(the third LPM)}$$

$$= -\left(\frac{1}{x}\right) \begin{vmatrix} \frac{1}{x} & 0 \\ \frac{1}{y} & -\frac{1}{y^2} \end{vmatrix} + \left(\frac{1}{y}\right) \begin{vmatrix} \frac{1}{x} & -\frac{1}{x^2} \\ \frac{1}{y} & 0 \end{vmatrix} \qquad \text{(evaluating the determinant)}$$

$$|\bar{H}_3| = \frac{2}{x^2 y^2} > 0 \qquad \text{(simplifying)}$$

Thus, for all values in our domain, the signs of the leading principle minors go: 0, strictly negative, strictly positive. Thus, the function is strictly quasiconcave.

(t) Theorem. Let $f : D \to \mathbb{R}$ be a concave function, where $D \subseteq \mathbb{R}^n$ is convex and $R \subseteq \mathbb{R}$, and let $g : R \to \mathbb{R}$ be an increasing function. Then

i. f is a quasiconcave function

ii. $g \circ f$ is a quasiconcave function

That is, concavity implies quasiconcavity, and a monotonic transformation of a concave function is quasiconcave.

(u) Aside. It is also the case that convexity implies quasiconvexity and a monotonic transformation of a convex function is quasiconvex, but we tend to focus on concavity and quasiconcavity in the first-year micro sequence.

6. **Homogeneity and Homotheticity**

(a) Definition. A real valued function $f(\mathbf{x})$ is **homogeneous** of degree k if and only if

$$f(t\mathbf{x}) = t^k f(\mathbf{x})$$

(b) Example. We can show that the functions $f(x_1, x_2) = x_1^3 x_2 + 4x_1^2 x_2^2$ is homogeneous of degree 4 (HOD 4) by applying the definition:

$$\begin{aligned}
f(tx_1, tx_2) &= (tx_1)^3(tx_2) + 4(tx_1)^2(tx_2)^2 && \text{(evaluating at } t\mathbf{x}\text{)} \\
&= t^4(x_1^3 x_2) + t^4(4x_1^2 x_2^2) && \text{(by distributivity)} \\
&= t^4(x_1^3 x_2 + 4x_1^2 x_2^2) && \text{(factoring)} \\
&= t^3 f(x_1, x_2) && \text{(by def. of } f\text{)}
\end{aligned}$$

The t in front is raised to the 4th power, so the function is HOD 4.

(c) Example. A function that is HOD 1 is sometimes called "linearly homogeneous," and in the context of production represents a constant-returns-to-scale function. For example, consider a Cobb-Douglas production function:

$$F(K, L) = AK^\alpha L^{1-\alpha}$$

where $K \geq 0$ and $L \geq 0$. Again, we can show that it is HOD 1 by applying the definition:

$$\begin{aligned}
F(tK, tL) &= A(tK)^\alpha (tL)^{1-\alpha} && \text{(scaling inputs by } t\text{)} \\
&= At^\alpha K^\alpha t^{1-\alpha} L^{1-\alpha} && \text{(distributing the exponent)} \\
&= tAK^\alpha L^{1-\alpha} && \text{(rearranging)} \\
&= tF(K, L) && \text{(by def. of } F\text{)}
\end{aligned}$$

The interpretation here is that if we double inputs, we also double the output; hence "constant returns to scale." This will be a common assumption in a lot of macroeconomic theory for the first year.

(d) Aside. Note that in the context of production, the degree of homogeneity determines returns to scale. That is, if the production function is HOD > 1, then it is increasing returns to scale; HOD < 1, it is decreasing returns to scale.

(e) Example. If we have a demand function $x^*(\mathbf{p}, M)$, where $\mathbf{p}$ is a vector of prices and M income, our theory will dictate that

$$x^*(t\mathbf{p}, tM) = x^*(\mathbf{p}, M)$$

In other words, demand is HOD 0 in prices and income. This should be fairly intuitive–if I instaneously added a zero to every dollar bill in circulation and a zero to every price, consumers' behavior shouldn't change.

(f) Theorem (JR THM A2.6). If $f(\mathbf{x})$ is homogeneous of degree k, then the first-order partial derivatives are homogeneous of degree $k-1$.

Proof:

$$\begin{aligned}
&\text{Let } f : D \to \mathbb{R} \text{ where } D \subseteq \mathbb{R}^n \text{ and } f \text{ is HOD } k && \text{(by hypothesis)}\\
\implies\ & f(t\mathbf{x}) = t^k f(\mathbf{x}) && \text{(by def. of HOD } k\text{)}\\
\implies\ & \frac{\partial f(t\mathbf{x})}{\partial x_i} t = t^k \frac{\partial f(\mathbf{x})}{\partial x_i} && \text{(differentiating w.r.t. } x_i\text{)}\\
\implies\ & \frac{\partial f(t\mathbf{x})}{\partial x_i} = t^{k-1} \frac{\partial f(\mathbf{x})}{\partial x_i} && \text{(dividing by } t\text{)}
\end{aligned}$$

(g) Theorem (JR THM A2.7). $f : D \to \mathbb{R}$ is differentiable, then $f(\mathbf{x})$ and homogeneous of degree k if and only if

$$kf(\mathbf{x}) = \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{\partial x_i} x_i$$

Proof:

$$\begin{aligned}
&\text{Let } f : D \to \mathbb{R} \text{ where } D \subseteq \mathbb{R}^n \text{ and } f \text{ is HOD } k && \text{(by hypothesis)}\\
\implies\ & f(t\mathbf{x}) = t^k f(\mathbf{x}) && \text{(by def. of HOD } k\text{)}\\
\implies\ & \sum_{i=1}^{n} \frac{\partial f(t\mathbf{x})}{x_i} x_i = kt^{k-1} f(\mathbf{x}) && \text{(differentiating w.r.t. } t\text{)}\\
\implies\ & \sum_{i=1}^{n} \frac{\partial f(\mathbf{x})}{x_i} x_i = kf(\mathbf{x}) && \text{(evaluating at } t = 1\text{)}
\end{aligned}$$

(h) Aside. This theorem ends up being very useful in macroeconomics during the first year. In particular, if we have a CRS production function such as $F(K, L) = AK^{\alpha}L^{1-\alpha}$, then we have the relationship:

$$K \cdot F_K(\cdot) + L \cdot F_L(\cdot) = F(K, L)$$

(i) Definition. Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. $f(\mathbf{x})$ is homothetic if and only if there exists a function $g : D \to R$ that is HOD k and a strictly increasing function $h : R \to \mathbb{R}$ such that $f(\mathbf{x}) = (h \circ g)(\mathbf{x})$. Note that this implies all homogeneous functions are also homothetic.

(j) Example. Consider the function $f : \mathbb{R}^n_{++} \to \mathbb{R}$ defined as

$$f(\mathbf{x}) = \sum_{i=1}^{n} \alpha_i \ln(x_i)$$

where $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i > 0$ for all i. We can show that this is a homothetic function.

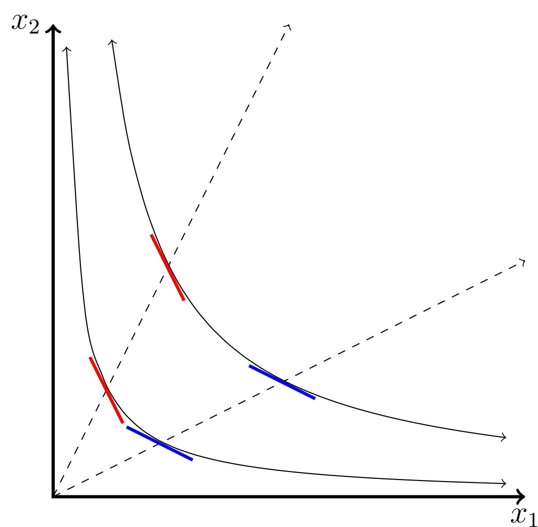
$$\begin{aligned}
\mathbf{e}^{f(\mathbf{x})} &= \exp\left\{\sum_{i=1}^{n} \alpha_i \ln(x_i)\right\} && \text{(an exponential transformation)} \\
&= \prod_{i=1}^{n} \exp\left\{\alpha_i \ln(x_i)\right\} && \text{(rearranging)} \\
&= \prod_{i=1}^{n} x_i^{\alpha_i} && \text{(simplifying)}
\end{aligned}$$

Now, we have our function $g(\mathbf{x})$ that is HOD 1. The function that reverses the exponential transformation is the natural log function, $\ln(y)$. Thus,

$$\sum_{i=1}^{n} \alpha_i \ln(x_i) = \ln\left(\prod_{i=1}^{n} x_i^{\alpha_i}\right)$$

so the the function $f(\mathbf{x})$ is homothetic.

(k) Aside. Homogeneity and homotheticity are related in much that same way as concavity and quasiconcavity, in that the homogeneity is a cardinal concept, while homotheticity is an ordinal one. Graphically, homothetic functions look like:



Along rays from the origin, the function's level sets have the same slope–in the indifference curve context, this means that the marginal rate of substitution is constant along the rays. The difference between homogeneity and homotheticity (on the picture) is how the level sets are denoted. If we had a CRS function (HOD 1), if we move twice the distance from the origin, the level set is twice as high–a cardinal notion. Homotheticity preserves the shape, but does not tell us anything about the height of the function relative to the distance between level sets.