Math Camp 2020 - Relations (Reference)*

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1. **Relations**

   (a) Definition. Let A and B be sets. R is a **relation** from A to B if and only if R is a subset of $A \times B$. If $(a, b) \in R$, we write aRb and say a is R-related to b.

   (b) Examples. Consider the coordinate system $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. We can think of the less-than operator as a relation $<$. Namely, the set of all (x, y) such that $x < y$ is a subset of all $(x, y) \in \mathbb{R}^2$

   (c) Example. A very common relation we use in economics is the "preference relation." Given two elements a and b, we read $a \succsim b$ as "a is preferred to b." Much of choice theory can be built using this relation, including $\sim$, the "indifference" relation, and $\succ$, the "strictly preferred to" relation.

   (d) Definition. The **domain** of a relation R from A to B is the set

   $$\mathcal{D}(R) = \{x \in A | \exists\, y \in B \ni xRy\}$$

   The **range** of the relation R is the set

   $$\mathcal{R}(R) = \{y \in B | \exists\, x \in A \ni xRy\}$$

   (e) Example. Consider the relation R defined to be the set

   $$R = \big\{(-1, 5), (2, 4), (2, 1), (4, 2)\big\}$$

   Then the domain and range of R are, respectively,

   $$\mathcal{D}(R) = \{-1, 2, 4\} \quad \text{and} \quad \mathcal{R}(R) = \{1, 2, 4, 5\}$$

   (f) Definition. If R is a relation from A to B, then the **inverse** of R is the relation

   $$R^{-1} = \{(y, x) | (x, y) \in R\}$$

   (g) Aside. Note that the inverse of a relation is simply the set with the ordered pairs reversed. This is distinct from the inverse of a function, which only exists in certain circumstances. The inverse of a relation, however, lets us think about inverses more generally.

*These lecture notes are drawn principally from *A Transition to Advanced Mathematics*, 7th ed., by Douglas Smith, Maurice Eggen, and Richard St. Andre. The material posted on this website is for personal use only and is not intended for reproduction, distribution, or citation. James Banovetz created the first edition of these awesome notes and graciously shared them.

(h) Definition. Let A be a set and R be a relation on A. Then

- R is **reflexive** if and only if for all $x \in A$,xRx
- R is **complete** if and only if for all $x, y \in A$,$xRy \vee yRx$
- R is **transitive** if and only if for all $x, y, z \in A$, $xRy \wedge yRz \implies xRz$
- R is **symmetric** if and only if for all $x, y \in A$, $xRy \implies yRx$
- R us **antisymmetric** if and only if for all $x, y \in A$, $xRy \wedge yRx \implies x = y$
- These are *potential* properties that a relation might satisfy.

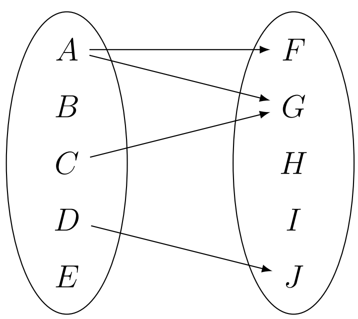
2. **Functions**

(a) Definition. A **function** from A to B is a relation f from A to B such that

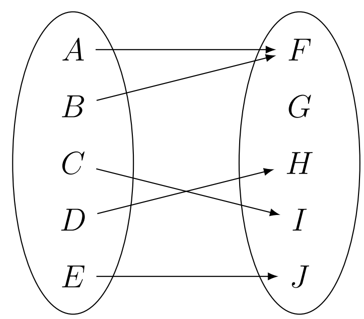
- the domain of f is A
- if $(x, y) \in f$ and $(x, z) \in f$, then $y = z$.

We write this using the notation $f : A \implies B$, which reads "f maps A to B." The set B is called the **codomain** of f.

(b) Example. In each of the graphs, let $\{A, B, C, D, E\}$ be domains, and $\{F, G, H, I, J\}$ be codomains:



Not a function!



Function!

The figure on the left does not represent a function for two reasons. First, A maps to more than one element, violating the second point in the definition. Second, not everything in the domain is mapped to something in the codomain. The right figure does represent a function, as everything in the domain is mapped to only one element in the codomain. Note that the range of a function does not need to be the same as the codomain.

(c) Definition. Let $f : A \to B$. $y = f(x)$ when $(x, y) \in f$, where y **is the value of** f **at** x (or the image of f at x).

(d) Definition. Let f be a function. f is a **real valued** function if and only if $\mathbb{R}$ is the codomain. f is a **vector valued** function if and only if the codomain is $\mathbb{R}^n$ for $n > 1$.

(e) Definition. A function $f : A \to B$ is **onto** if the range of f is the codomain, i.e.,

$$\mathcal{R}(f) = B$$

Alternatively, we say that f is a **surjection**.

(f) Example. We can prove that the function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x + 2$ is onto. Essentially, we need to prove that for all y in the codomain, there exists an x in the domain such that $f(x) = y$.

$$\forall y \in \mathbb{R}_c \; \exists \, x \in \mathbb{R}_d \ni f(x) = y$$

- For universal qualifier proofs, we need to pick a y, show it works
- Closure of the reals

To show: $f(x) = y$.
Proof:

$$
\begin{aligned}
&\text{Let } y \in \mathbb{R} && \text{(by hypothesis)}\\
&\implies y - 2 \in \mathbb{R} && \text{(by closure)}\\
&\text{Let } x = y - 2 && \text{(by hypothesis)}\\
&x \in \mathbb{R} && \text{(a tautology)}\\
&\implies f(x) = (y-2) + 2 && \text{(by def. of } x \text{ and } f)\\
&\implies f(x) = y && \text{(by associativity)}
\end{aligned}
$$

■

Note that per our format for doing "for all" proofs, we chose an arbitrary value y and show it worked in general. Thus, it must work for any and all values we could specify for y.

(g) Definition. A function $f : A \to B$ is **one-to-one** or $1-1$ if and only if whenever $f(x) = f(y)$, then $x = y$. Alternatively, we say that f is an injection.

(h) Example. Consider the function $f : D \to \mathbb{R}$ defined as $f(x) = x^2$. f is a one-to-one function if the domain does not "cross-zero," i.e., so long as the domain contains only non-negative or (exclusive!) non-positive values. For example, it is $1-1$ if $D = [-1, 0]$ of $D = [0, 1]$, but it is *not* $1-1$ if $D = [-1, 1]$.

(i) Definition. A function $f : A \to B$ is a **one-to-one correspondence**, or a **bijection**, if and only if f is one-to-one and onto.

(j) Theorem. Let f be a function $f : A \to B$. If f is a bijection, then there exists an **inverse function** $f^{-1} : B \to A$ such that for $x \in A$, $f^{-1}(f(x)) = x$.

(k) Example. Consider one again our function $f(x) = x^2$. While this will always have a inverse (as defined for relations), this may or may not have an inverse function depending on the domain and codomain. Consider $f : D \to \mathbb{R}_+$ where $D = [0, \infty)$. Then the inverse function $f^{-1} : \mathbb{R}_+ \to D$ is $\sqrt{x}$.

(l) Aside. While we have until now been considering mapping individual elements, we can also consider mappings of collections of points to sets.

(m) Definitions. Let $f : A \to B$ and let $X \subset A$ and $Y \subset B$. The **image set** of X is

$$f(X) = \{y \in B | y \in f(x) \text{ for some } x \in X\}$$

and the **inverse image** is

$$f^{-1}(Y) = \{x \in A | f(x) \in Y\}$$

(n) Example. We can prove that the following two sets are equal: $f^{-1}(B \cap C) = f^{-1}(B) \cap f^{-1}(C)$, again using a biconditional proof.

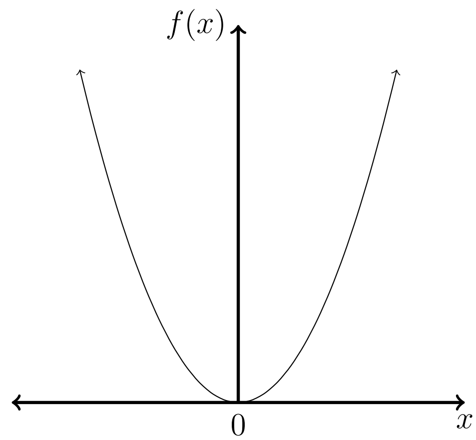
- Definition of the inverse image
- Definition of set intersection
- Definition of a subset

To show: $x \in f^{-1}(B) \cap f^{-1}(C)$.
Proof:

$$\begin{aligned}
&\text{Let } x \in f^{-1}(B \cap C) && \text{(by hypothesis)}\\
&\iff f(x) \in B \cap C && \text{(def. of the inverse image)}\\
&\iff f(x) \in B \wedge f(x) \in C && \text{(by def. of } \cap)\\
&\iff x \in f^{-1}(B) \wedge x \in f^{-1}(C) && \text{(by def. of the inverse image)}\\
&\iff x \in f^{-1}(B) \cap x \in f^{-1}(C) && \text{(by def. of } \cap)
\end{aligned}$$

(o) Definition. Let f be a real-valued function whose domain includes interval I. Then f is **increasing** on I if and only if for all $x, y \in I$, if $x < y$ then $f(x) < f(y)$. Similarly, f is **decreasing** if $x < y \implies f(x) > f(y)$.

(p) Example. Consider the function $f : I \to \mathbb{R}$, where $f(x) = x^2$. f is an increasing function if the interval I is defined to include only non-negative values, e.g., $I = [0, \infty)$; it might be decreasing, e.g., $I = (-\infty, 0]$; or it might be neither if $I = (-\infty, \infty)$.



(q) Aside. We've been using greather-than and less-than (and their weak analogues) symbols for numbers in $\mathbb{R}$, but we can extend these notions to $\mathbb{R}^n$. First, if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, then $\mathbf{x} \geq \mathbf{y}$ if every element in $\mathbf{x}$ is at least as large as every element in $\mathbf{y}$, or $x_i \geq y_i$ for all $i = 1, \ldots, n$. Similarly, we use the notation $\mathbf{x} >> \mathbf{y}$ if $x_i > y_i$ for all $i = 1, \ldots, n$

(r) Definition. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$. Then f is

- **increasing** if and only if $f(\mathbf{x}_0) \geq f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \geq \mathbf{x}_1$.
- **strictly increasing** if and only if $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 >> \mathbf{x}_1$.
- **strongly increasing** if and only if $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 \geq \mathbf{x}_1$

(s) Definition. Let $f : D \to \mathbb{R}$ where $D \subset \mathbb{R}^n$. Then f is

- **decreasing** if and only if $f(\mathbf{x}_0) \geq f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \leq \mathbf{x}_1$.
- **strictly decreasing** if and only if $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 << \mathbf{x}_1$.
- **strongly decreasing** if and only if $f(\mathbf{x}_0) > f(\mathbf{x}_1)$ whenever $\mathbf{x}_0 \neq \mathbf{x}_1$ and $\mathbf{x}_0 \leq \mathbf{x}_1$

3. **Sequences**

(a) Definition. A function x with domain $\mathbb{N}$ is called an **infinite sequence**. The image of n is written x_n and is called the nth term of the sequence.

(b) Example. Consider the function $x : \mathbb{N} \to \mathbb{R}$ defined as

$$x_n = \frac{1}{n+1}$$

The first term of this sequence is 1/2; the 100th term is 1/101. Note that we also frequently use set notation $\{x_n\}_{n=1}^{\infty}$

(c) Aside. Sequences end up being extremely useful tools in the first year sequences. For example, concepts like closure and continuity can be defined in terms of sequences; further, there are concepts in game theory, like trembling hand equilibria, sequential equilibria, minimax regret, and others.

(d) Definition. For a sequence x of real numbers and a real number L, x has a **limit** L (or x **converges** to L) if and only if for all $\varepsilon > 0$, there exists a natural number N such that if $n > N$, then $|x_n - L| < \varepsilon$. This may be represented

$$\lim_{n\to\infty} x_n = L \quad \text{or} \quad x_n \to L$$

If no number L exists, then x does not converge (or the limit does not exist).

(e) Aside. While we will be considering this definition using absolute values, note that this is not necessarily required. We're thinking in terms of the real number line and our most common metric for $\mathbb{R}$ is $d(x, y) = |x - y|$. For a more general definition, however, we could use generic notation $d(x, y)$ in place of the absolute value.

(f) Example. We can show that the sequence x with the nth term

$$x_n = \frac{(-1)^n}{n+1}$$

converges to zero. In order to show this, we need to fix a general ε, then find a n "large enough" to satisfy our definition. Note that the end goal is to show $|x_n - 0| < \varepsilon$, so we typically do some scratch work to find an n that works:

$$|x_n - 0| = |x_n| = \left|\frac{(-1)^n}{n+1}\right| = \frac{1}{n+1} < \varepsilon \implies n > \frac{1}{\varepsilon} - 1$$

Now that we've "picked a n," we can prove that the sequence converges.
To show: $|x_n - 0| < \varepsilon$.
Proof:

$$\begin{aligned}
&\text{Let } \varepsilon > 0 && \text{(by hypothesis)}\\
&\text{Let } N > \frac{1}{\varepsilon} - 1 && \text{(by hypothesis)}\\
&\text{Let } n > N && \text{(by hypothesis)}\\
&|x_n - 0| = \left|\frac{(-1)^n}{n+1} - 0\right| && \text{(by def. of } x_n\text{)}\\
&= \frac{1}{n+1} && \text{(simplifying)}\\
&< \frac{1}{\left(\frac{1}{\varepsilon} - 1\right) + 1} && \left(n > \frac{1}{\varepsilon} - 1\right)\\
&= \varepsilon && \text{(simplifying)}
\end{aligned}$$

■

Basically, we're picking a very small ε, arbitrarily close to zero, and then picking a very large N, which depends on the very small ε. If we pick a large enough value, then our sequence is closer than ε to the limit 0.

(g) Example. Consider the following sequences:
- $\{1, -1, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{4}, \dots\}$
- $y_n = (-1)^n$
- $\{a_n\}_{n=1}^{\infty} = \{1, 0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, \dots\}$
- $x_n = \dfrac{3n^2}{n^2+1}$

The first converges to zero–it oscillates in every smaller degree, getting arbitrarily close to zero. The second does not converge, as it flips from 1 to -1. The third also does not coverge, as no matter how large a N we pick, there will always be a 1. Finally, the third sequence converges to 3.

(h) Theorem (SES THM 4.6.1). If a sequence x converges, then its limit is unique.

(i) Theorem (SES THM 4.6.2). Let a, b and c be sequences of real numbers such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. If $a_n \to L$ and $c_n \to L$, then $b_n \to L$. This is known as the sandwich theorem.

(j) Theorem. Let a and b be sequences of real numbers such that $a_n \to a$ and $b_n \to b$ Then
  i. $a_n + b_n \to a + b$
  ii. $ka_n \to ka$ for any real number k
  iii. $a_n b_n \to ab$
  iv. If $b \neq 0$ and $b_n \neq 0$ for any n, then $a_n/b_n \to a/b$

(k) Example. We can prove the first of these fairly easily.
- $a_n \to a \iff \forall\, \varepsilon > 0 \exists N \ni n > N \implies |a_n - a| < \varepsilon$
- Triangle inequality: $|x + y| \leq |x| + |y|$

To show: $|a_n + b_n - (a + b)| < \varepsilon$
Proof:
Let $\varepsilon > 0$ (by hypothesis)
Let $a_n \to a \wedge b_n \to b$ (by hypothesis)
$\implies \left(\exists N_a \ni n_a > N_a \Rightarrow |a_{n_a} - a| < \varepsilon/2\right) \wedge \left(\exists N_b \ni n_b > N_b \Rightarrow |b_{n_b} - b| < \varepsilon/2\right)$ (by def. of convergence)
Let $n > \max\{N_a, N_b\}$ (by hypothesis)
$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)|$ (a tautology)
$\leq |a_n - a| + |b_n - b|$ (by the triangle ineq.)
$< \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2}$ (by our assumption)
$= \varepsilon$ ■

(l) Aside. This is probably more involved than you will need to get with sequential proofs, or sequences generally, but it is important to have a working knowledge of ε proofs. There are many results in economics that depend on similar ideas; further, later on we'll get to more general ε-concepts that are extensions of this line of reasoning. For the first year, however, we'll frequently work with things like $\varepsilon_n \to 0$, without explicitly defining a sequence. The concept of "trembling-hand perfection" in game theory, for example, relies on the notion of convergence.

4. **Series**

(a) Definition. Given a sequence $\{a_n\}$, an **infinite series** is an infinite sum of the elements in the sequence, denoted $\sum_{i=1}^{\infty} a_i$. The sequence $\{S_n\}$ is the **sequence of partial sums**, defined as

$$S_n = \sum_{i=1}^{n} a_i$$

(b) Definition. The series $\sum_{i=1}^{\infty} a_i$ converges to L if the sequence of partial sums converges to L:

$$S_n \to L$$

(c) Aside. Infinite series are extremely important in your first year classes. In particular, the geometric series and the Taylor series are used in a wide array of homework problems, theorems, proofs, etc. While there is a whole field of study dedicated to series, we will make a quick point about the geometric series now, and talk about the Taylor series in a few classes.

(d) Definition. A **geometric series** is a series of the form

$$a + ar + ar^2 + ar^3 + \dots$$

(e) Theorem Let $a + ar + ar^2 + \dots$ be a geometric series. If $a \neq 0$ and $|r| < 1$, then the series converges to $\frac{a}{1-r}$. Proof:

$$\begin{aligned} S_n &= a + ar + ar^2 + \dots + ar^{n-1} && \text{(the partial sum)} \\ rS_n &= ar + ar^2 + \dots + ar^n && \text{(multiplying by } r\text{)} \\ S_n - rS_n &= a - ar^n && \text{(substracting)} \\ S_n &= \frac{a - ar^n}{1 - r} && \text{(solving for } S_n\text{)} \end{aligned}$$

At this point, notice that $\lim r^n = 0$ and $n \to \infty$, since $|r| < 1$. Thus:

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$