For what follows, we're going to consider the set of real numbers to be the universe of discourse.

Convex Sets[1]

A **convex combination** is a linear combination of points where all coefficients are non-negative and sum to one.

Consider points (possibly vectors) $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. A general convex combination, which can be denoted $\mathbf{w}$, is

$$\mathbf{w} = k_1\mathbf{x} + k_2\mathbf{y} + k_3\mathbf{z}$$

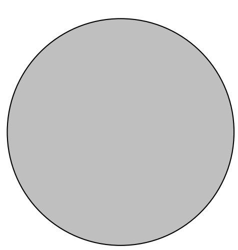
where $k_1 + k_2 + k_3 = 1$ and $k_i \geq 0, i = 1, 2, 3$.

The convex combination we are going to use most is:

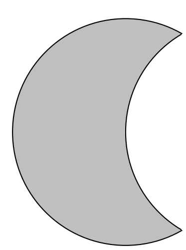
$$\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \qquad \alpha \in [0,1]$$

Think of it like a weighted average between two points (or vectors), where α determines the weight. The convex combinations made by all possible values of α will be a line between the two points.

$A \subseteq \mathbb{R}^n$ is a **convex set** iff $\alpha\mathbf{x} + (1-\alpha)\mathbf{y} \in A \quad \forall\ \mathbf{x}, \mathbf{y} \in A, \alpha \in [0,1]$



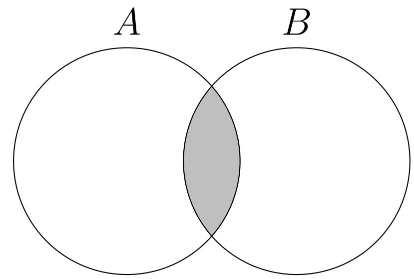
A Convex Set



A Non-Convex Set

[1] Prepared by Sarah Robinson

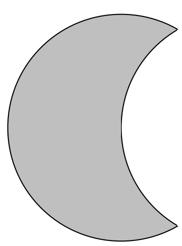
If A and B are both convex sets in $\mathbb{R}^n$, then $A \cap B$ is a convex set.



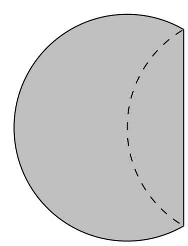
Intersection: $A \cap B$

Is $A \cup B$ a convex set?

The **convex hull** of set $B \subseteq \mathbb{R}^n$ is the smallest convex set containing B (the set of all convex combinations of points in B).



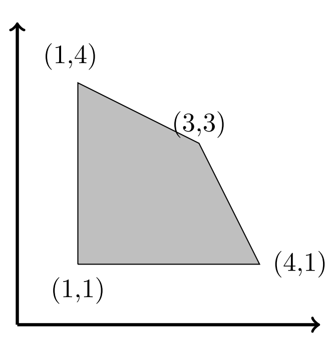
A Non-Convex Set



The Convex Hull

Example: A two-player prisoners' dilemma from game theory and the convex hull of the payoff profiles:

	C	D
C	(3, 3)	(1, 4)
D	(4, 1)	(1, 1)



Example: Consider set S:

$$S = \{x \mid x \in \mathbb{R} \land -1 \leq x \leq 1\}$$

Show that S is a convex set.

- $A \subseteq \mathbb{R}^n$ is a **convex set** iff $\alpha\mathbf{x}+(1-\alpha)\mathbf{y} \in A \quad \forall\, \mathbf{x}, \mathbf{y} \in A, \alpha \in [0,1]$

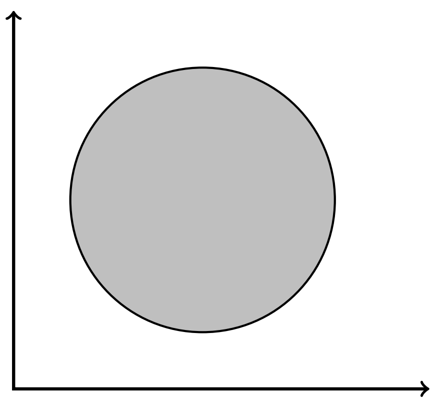
<u>To Show:</u>

<u>Proof:</u>

Closed Sets

A set $A \subseteq \mathbb{R}^n$ is **closed** iff for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in A$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in A$

- $\approx$ set A also includes its boundaries



A Closed Set in $\mathbb{R}^2$

A set is an **open set** if and only if its complement is a closed set.

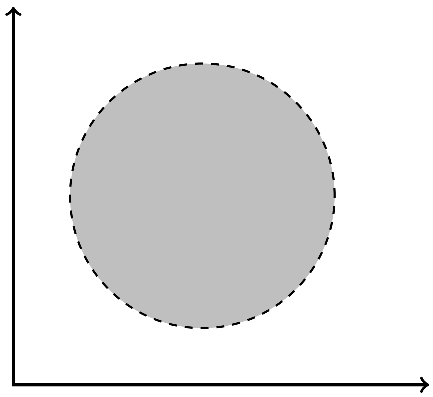
The following sets in $\mathbb{R}^n$ are open sets:

- The empty set $\emptyset$
- The entire space $\mathbb{R}^n$
- The union of any number of open sets
- The intersection of any finite number of open sets

The following sets in $\mathbb{R}^n$ are closed sets:

- The empty set $\emptyset$
- The entire space $\mathbb{R}^n$
- The union of any finite number of closed sets
- The intersection of any number of closed sets

We could also define open sets using the notion of an epsilon-neighborhood (a ball with radius ε). A set A is open if and only if for all $\mathbf{x} \in A$, there exists some $\varepsilon > 0$ such that the ε-ball centered at $\mathbf{x}$ is contained in A.



An Open Set in $\mathbb{R}^2$

For any point in an open set, we can always draw a tiny circle around the point that lies entirely within the set. I bring up this definition because ε-balls will come up in other contexts.

Example: Consider set S:

$$S = \{(x, y) \mid (x, y) \in \mathbb{R}^2 \wedge x^2 + y^2 \leq 1\}$$

Show that S is closed.

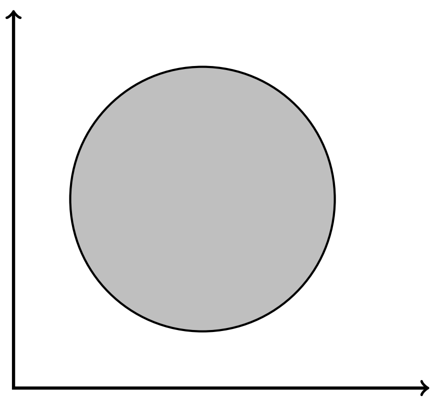
- $A \subseteq \mathbb{R}^n$ is **closed** iff for every sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ such that $\mathbf{x}_n \in A$ for all n and $\mathbf{x}_n \to \mathbf{x}$, it is also the case that $\mathbf{x} \in A$
- Theorem 1: If $a_n \to a$ and $b_n \to b$, then $a_n + b_n \to a + b$ and $a_n b_n \to ab$
- Theorem 2: If $a_n \to a$, then $a_n \leq b$ for all n implies $a \leq b$.

To Show:

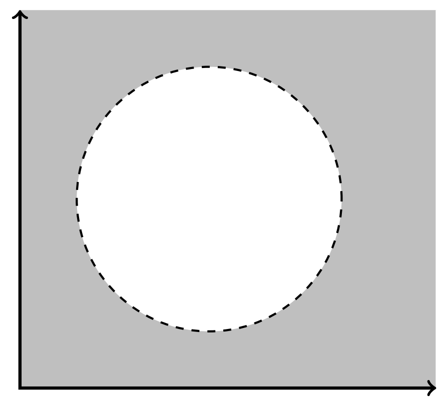
Proof:

Bounded Sets

A set $A \subseteq \mathbb{R}^n$ is **bounded** if and only if there exists an M and a point $\mathbf{c} \in \mathbb{R}^n$ such that the M-ball centered at $\mathbf{c}$ contains all of A.



A Bounded (Closed) Set



A Non-Bounded (Open) Set

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded:

- Pick a radius M
- Pick a center point $\mathbf{c}$
- Let $\mathbf{x} \in A$ – arbitrary $\mathbf{x}$
- Show that $\mathbf{x}$ is less than M distance away from $\mathbf{c}$

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded in the special case where the points furthest away from 0 are along the axes:

- Pick a radius M (use 0 as the center point)
- Let $\mathbf{x} \in A$ – arbitrary $\mathbf{x}$
- Show that $-M \leq x_i \leq M \quad \forall i = 1, \ldots, n$

A set $A \subseteq \mathbb{R}^n$ is **compact** if and only if it is closed and bounded.

Example: Consider set S:

$$S = \{(x, y) \mid (x, y) \in \mathbb{R}^2 \wedge x^2 + y^2 \leq 1\}$$

Show that S is bounded.

To prove a set in $A \subseteq \mathbb{R}^n$ is bounded in the special case where the points furthest away from 0 are along the axes:

- Pick a radius M (use 0 as the center point)
- Let $\mathbf{x} \in A$ – arbitrary $\mathbf{x}$
- Show that $-M \leq x_i \leq M \quad \forall i = 1, \ldots, n$

<u>To Show:</u> S is bounded

<u>Proof:</u>

$$\begin{aligned}
&\text{Let } M = 2 && \text{(by hypothesis)} \\
&\implies x^2 + y^2 \leq 1 && \text{(def. of } S\text{)} \\
&\implies (x^2 \leq 1) \wedge (y^2 \leq 1) && (x^2 \geq 0 \ \forall x) \\
&\implies (-1 \leq x \leq 1) \wedge (-1 \leq y \leq 1) && \text{(algebra)} \\
&\implies (-2 \leq x \leq 2) \wedge (-2 \leq y \leq 2) && \text{(algebra)} \\
&\implies (-M \leq x \leq M) \wedge (-M \leq y \leq M) && \text{(algebra)} \\
&\implies \text{S is bounded} && \text{(by def. of bounded)}
\end{aligned}$$

Continuity & Differentiability of Functions

Let f be a function with domain D and points (or vectors) $\mathbf{x}, \mathbf{y} \in D$.

f is **continuous** at $\mathbf{x}$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $||\mathbf{x} - \mathbf{y}|| < \delta \implies ||f(\mathbf{x}) - f(\mathbf{y})|| < \varepsilon$.

Recall that for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,

$$||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

f is **continuous** at $\mathbf{x}$ iff for every sequence $\mathbf{x}_n \in D$ such that $\mathbf{x}_n \to \mathbf{x}$, the sequence $f(\mathbf{x}_n) \to f(\mathbf{x})$.

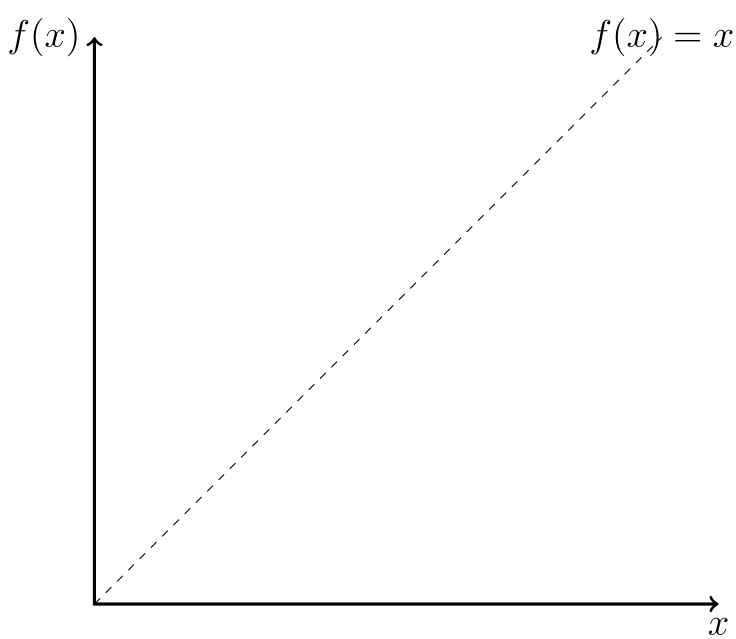
f is a continuous function if it continuous at every point in its domain.

Let $f : D \to \mathbb{R}$ be a continuous, real-valued function where D is non-empty, compact subset of $\mathbb{R}^n$. Then there exists a vector $\underline{\mathbf{x}} \in D$ and a vector $\overline{\mathbf{x}} \in D$ such that

$$\forall \mathbf{x} \in A, f(\underline{\mathbf{x}}) \leq f(\mathbf{x}) \leq f(\overline{\mathbf{x}})$$

That is, a continuous function $f(\mathbf{x})$ attains a maximum and a minimum on every compact set. (Weierstrass Extreme Value Theorem).

Let $D \subseteq \mathbb{R}^n$ be a non-empty compact, convex set. Let $f : D \to D$ be a continuous function. Then there exists at least one fixed point of f in D, that is, there exists $\mathbf{x}^* \in D$ such that $f(\mathbf{x}^*) = \mathbf{x}^*$. (Brouwer Fixed Point Theorem).



For $D = [0, 1]$ then any continuous $f : D \to D$ must cross the 45-degree line.

Let f be a function defined on an interval $(a, b) \subseteq \mathbb{R}$ and let $x \in (a, b)$. Then f is **differentiable** at x if and only if the limit of

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists and is finite.

If this is the case, then the limit is called the **derivative** of f at x and is denoted $f'(x)$ or $\frac{df(x)}{dx}$.

For a multivariate functions $f(\mathbf{x})$ where $\mathbf{x} \in \mathbb{R}^n$, the **partial derivative** of f with respect to x_i is given by:

$$f_i(\mathbf{x}) = \frac{\partial f}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1, \ldots, x_i, \ldots x_n)}{h}$$

$f'(x)$ is a function of x. Often, we want to discuss the value of the derivative at a particular point c:

$$f'(c) \qquad\qquad \left.\frac{df}{dx}\right|_c$$

Level Sets

We are focusing on real-valued functions:

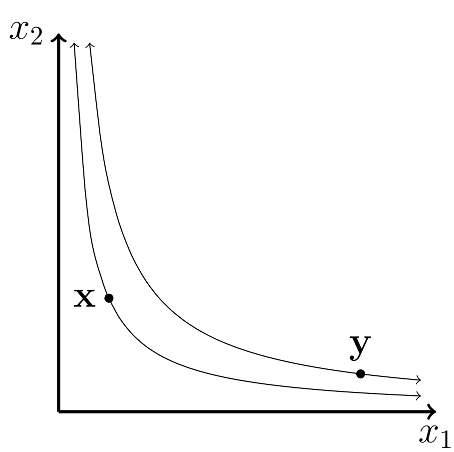
- $f : \mathbb{R} \to \mathbb{R}$ (univariate)
- $f : \mathbb{R}^n \to \mathbb{R}$ (multivariate)

Let f be a real valued function such that $f : D \to \mathbb{R}$ where $D \subseteq \mathbb{R}^n$. Then $L(\mathbf{x}_0)$ is a **level set** relative to $\mathbf{x}_0$ if and only if

$$L(\mathbf{x}_0) = \big\{\mathbf{x} \mid \mathbf{x} \in D \wedge f(\mathbf{x}) = f(\mathbf{x}_0)\big\}$$

Indifference curves are level sets. Consider the utility function:

$$u(x_1, x_2) = x_1^{1/2} x_2^{1/2}$$



- $\mathbf{x} = (1, 4)$ and $u(\mathbf{x}) = 2$
- $\mathbf{y} = \big(32, \frac{1}{2}\big)$ and $u(\mathbf{y}) = 4$

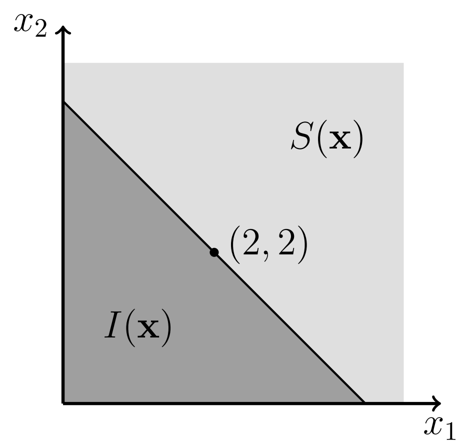
All the points on the curve running through $\mathbf{x}$ give a utility of 2, while all those on the curve running through $\mathbf{y}$ provide a utility of 4.

We can also define superior and inferior sets:

- $S(\mathbf{x}_0) = \{\mathbf{x} \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \geq f(\mathbf{x}_0)\}$ is the **superior set** relative to $\mathbf{x}_0$
- $I(\mathbf{x}_0) = \{\mathbf{x} \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \leq f(\mathbf{x}_0)\}$ is the **inferior set** relative to $\mathbf{x}_0$

If the weak inequalities are replaced with strict inequalities, then the sets are the **strictly superior set** and **strictly inferior set**, respectively.

Example: Consider the function $u(x_1, x_2) = x_1 + x_2$. The inferior and superior sets, relative to $\mathbf{x} = (2, 2)$ can be illustrated graphically as:



$\neq$ On and Above/Below

Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. The the set of points **on and below the graph** of f is defined as:

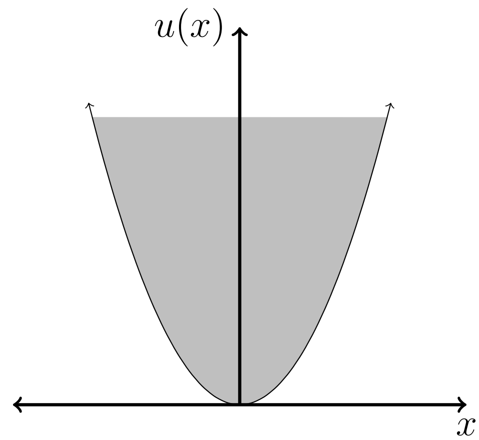
$$A = \{(\mathbf{x}, y) \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \geq y\}$$

Similarly, the set of points **on and above the graph** is defined as:

$$B = \{(\mathbf{x}, y) \mid \mathbf{x} \in D \wedge f(\mathbf{x}) \leq y\}$$

Note that superior/inferior sets are points in the domain, while points relative to graph are *ordered pairs*, $(n+1)$-tuples with elements from both the domain *and* the range.

Consider the set of points on and above the graph of the function $u(x) = x^2$.



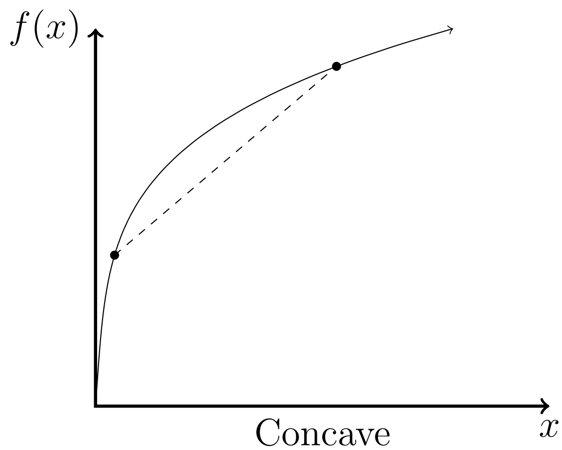
Concavity and Convexity

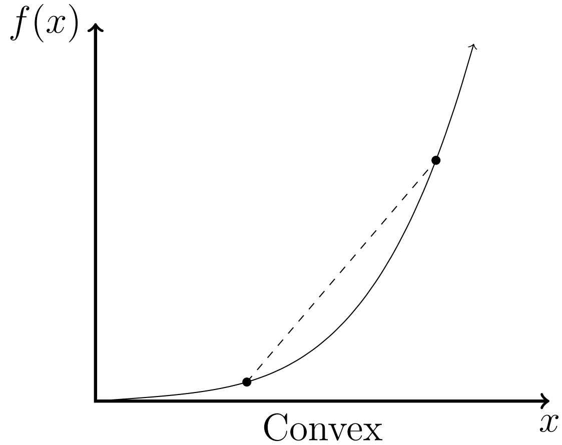
Let $f : D \to \mathbb{R}$ where D is a convex subset of $\mathbb{R}^n$. A function is **concave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \geq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

A function is **convex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \leq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$





Example: Consider $f : \mathbb{R} \to \mathbb{R}$ where $f(x) = |x|$. Prove that it is convex.

- A function is convex iff for all $\mathbf{x}_0, \mathbf{x}_1 \in D$ and $t \in [0, 1]$, $f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) \leq tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$
- The absolute value function $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$
- Theorem 1: $|ab| = |a||b|$
- Theorem 2: The triangle inequality, $|a+b| \leq |a| + |b|$

To show: $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$

Proof:

Let $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Then:

$$f \text{ is a concave function} \iff \text{ the set on and below } f \text{ is a convex set}$$

$$f \text{ is a convex function} \iff B \text{ the set on and above } f \text{ is a convex set}$$

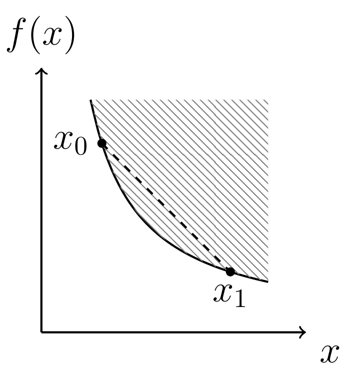
Let $f : D \to \mathbb{R}$ where D is a convex subset of $\mathbb{R}^n$. A function is **strictly concave** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0, 1)$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) > tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

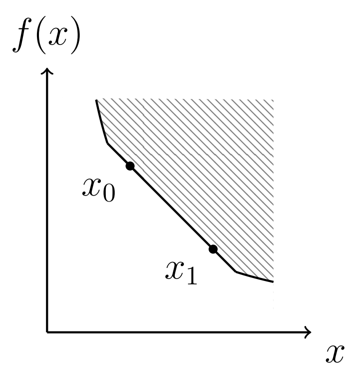
A function is **strictly convex** if and only if for all $\mathbf{x}_0, \mathbf{x}_1 \in D \ni \mathbf{x}_0 \neq \mathbf{x}_1$ and $t \in (0, 1)$:

$$f\Big(t\mathbf{x}_0 + (1-t)\mathbf{x}_1\Big) < tf(\mathbf{x}_0) + (1-t)f(\mathbf{x}_1)$$

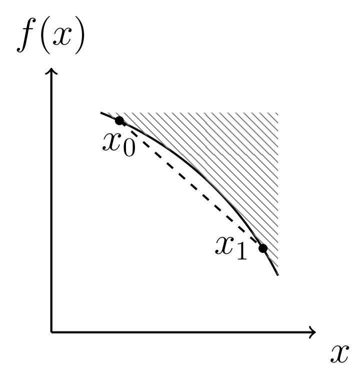
(We've changed the inequality, made sure the two points are distinct, and made t strictly between 0 and 1).



Strictly convex



Convex but not strictly



Strictly concave

Let D be a convex, non-degenerate interval on $\mathbb{R}$, such that on the interior of D, f is twice continuously differentiable. Then the following statements are equivalent:

1. f is concave
2. $f''(x) \leq 0$ for all non-endpoints $x \in D$.
3. For all $x_0 \in D$, $f(x) \leq f(x_0) + f'(x_0)(x - x_0)$
4. $f''(x) < 0$ for all non-endpoints $x \in D \implies f$ is strictly concave

The following statements are also equivalent:

1. f is convex
2. $f''(x) \geq 0$ for all non-endpoints $x \in D$.
3. For all $x_0 \in D$, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$
4. $f''(x) > 0$ for all non-endpoints $x \in D \implies f$ is strictly convex

This is all well and good for single-variable functions, but what about multivariable functions? We need to extend our concept of first and second derivatives, which lead us to the gradient and the Hessian matrix.

Let f be a twice continuously differentiable function, $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Then the **gradient** of f, denoted $\nabla f(\mathbf{x})$ is defined as the row vector of 1st-order partial derivatives:

$$\nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f(\mathbf{x})}{\partial x_1} & \dots & \frac{\partial f(\mathbf{x})}{\partial x_n} \end{bmatrix}$$

The **Hessian** of f, denoted H or $\mathbf{H}$, is the matrix of 2nd-order partial derivatives:

$$H = \begin{bmatrix} \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_1} & \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \dots \\ \frac{\partial^2 f(\mathbf{x})}{\partial x_1 \partial x_2} & \frac{\partial^2 f(\mathbf{x})}{\partial x_2 \partial x_2} & \dots \\ \vdots & & \ddots \end{bmatrix}$$

Let D be a convex subset of $\mathbb{R}^n$ with a non-empty interior on which f is twice continuously differentiable. Then

- H is negative semi-definite $\implies$ f is concave
- H is negative definite $\implies$ f is strictly concave
- H is positive semi-definite $\implies$ f is convex
- H is positive definite $\implies$ f is strictly convex

The critera from definiteness came up in linear algebra; recall our interpretation using second order total differentials for the intution.

Consider the function $f(x, y) = \ln(x) + \ln(y)$. We can establish that this function is strictly concave over it's domain $\mathbb{R}^2_{++}$ (note that this notation indicates we are only considering strictly positive values in $\mathbb{R}^2$:

$$f(x,y) = \ln(x) + \ln(y) \qquad \text{(the function)}$$

$$\nabla f(x,y) = \begin{bmatrix} \frac{1}{x} & \frac{1}{y} \end{bmatrix} \qquad \text{(the gradient)}$$

$$H = \begin{bmatrix} -\frac{1}{x^2} & 0 \\ 0 & -\frac{1}{y^2} \end{bmatrix} \qquad \text{(the Hessian)}$$

Recall first our notion of leading principle minors (note that these are determinant bars):

$$|H_1| = \left| -\frac{1}{x^2} \right| \qquad \text{(the first LPM)}$$

$$= -\frac{1}{x^2} < 0 \qquad \text{(simplifying)}$$

$$|H_2| = |H| \qquad \text{(the second LPM)}$$

$$= \frac{1}{x^2 y^2} > 0 \qquad \text{(simplifying)}$$

Thus, since our leading principle minors alternate in sign, beginning with a negative, the matrix is negative definite, implying our function is strictly concave.

Let f be a concave function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, concave function, $g : R \to \mathbb{R}$. Then the composite function defined as $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$ is a concave function.

This is a relatively straightforward theorem to prove:
<u>To show</u>: $(g \circ f)(t\mathbf{x} + (1-t)\mathbf{y}) \geq t(g \circ f)(\mathbf{x}) + (1-t)(g \circ f)(\mathbf{y})$
<u>Proof</u>

Let $\mathbf{x}, \mathbf{y} \in D$ and $t \in [0,1]$	(by hypothesis)
Consider $(g \circ f)(t\mathbf{x} + (1-t)\mathbf{y})$	(the composite)
$= g\big(f(t\mathbf{x} + (1-t)\mathbf{y})\big)$	(by def. of the composite)
$\geq g\big(tf(\mathbf{x}) + (1-t)f(\mathbf{y})\big)$	(by f concave and g increasing)
$\geq tg(f(\mathbf{x})) + (1-t)g(f(\mathbf{y}))$	(by g concave)
$= t(g \circ f)(\mathbf{x}) + (1-t)(g \circ f)(\mathbf{y})$	(by def. of the composite)

$\blacksquare$

Let f be a convex function such that $f : D \to R$, where $D \subseteq \mathbb{R}^n$ and $R \subseteq \mathbb{R}$. Let g be an increasing, convex function, $g : R \to \mathbb{R}$. Then the composite function defined as $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$ is a convex function.