Part-1.1

$$z = f(x, y) = ax + by + c$$

$$\frac{\partial z}{\partial x} = a + 0 + 0 = a$$

$$\frac{\partial z}{\partial y} = 0 + b + 0 = b$$

$$\nabla f(x, y) = \begin{bmatrix} a \\ b \end{bmatrix}$$

Part-1.2

$$z = f(X) = f(x_1, x_2, \cdots, x_N)$$

$$= \sum_{i=1}^{N} a_i (x_i - b_i) + s$$

$$= a_1 x_1 + a_2 x_2 + \cdots + a_N x_N + d$$

$$\frac{\partial z}{\partial x_1} = a_1 + 0 + \cdots + 0 = a_1$$

$$\frac{\partial z}{\partial x_2} = 0 + a_2 + \cdots + 0 = a_2$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$\frac{\partial z}{\partial x_N} = 0 + 0 + \cdots + a_N = a_N$$

$$\nabla f(x, y) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}$$

Part-1.3

$$Z = f(x, y) = A(x-x_0)^2 + B(y-y_0)^2 + C$$

$$Z = A(x^2 - 2x \cdot x_0 + x_0^2) + B(y-y_0)^2 + C$$

$$= A(x-x_0)^2 + B(y^2 - 2y \cdot y_0 + y_0^2) + C$$

$$\frac{\partial Z}{\partial x} = 2Ax - 2A \cdot x_0$$

$$= 2A(x - x_0)$$

$$\frac{\partial Z}{\partial y} = 2By - 2B \cdot y_0$$

$$= 2B(y - y_0)$$

$$\nabla f(x \cdot y) = \begin{bmatrix} 2A(x-x_0) \\ 2B(y-y_0) \end{bmatrix}$$

Part 1.4

$X^T = (3 \quad 1 \quad 4) \quad [1 \times 3]$

$Y^T = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} \quad [3 \times 1]$

$B^T = \begin{pmatrix} 5 & 2 & 4 \\ 3 & 5 & 1 \end{pmatrix} \quad [2 \times 3]$

$X \cdot X : (9 + 1 + 16) = 26$

$X \cdot Y^T : (6 + 5 + 4) = 15$

$X \times Y = \begin{pmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{pmatrix} \quad [3 \times 3]$

$Y \times X = (2 \times 3 + 1 \times 5 + 4 \times 1)$
$= (15) \quad [1 \times 1]$

$A \times X = \begin{pmatrix} 12+5+8 \\ 9+1+20 \\ 18+4+12 \end{pmatrix} = \begin{pmatrix} 25 \\ 30 \\ 34 \end{pmatrix} \quad [3 \times 1]$

$A \times B = \begin{pmatrix} 39, & 38 \\ 19, & 37 \\ 41, & 50 \end{pmatrix} \quad [3 \times 2]$

$B.\text{reshape}(1, 6) = (3 \quad 5 \quad 5 \quad 2 \quad 1 \quad 4) \quad [1 \times 6]$

Part-1 : LLS - Single Variable.

$$L(m, b) = \sum_{i=1}^{N} (y_i - M(\hat{x_i}, m, b))^2$$
$$= \sum_{i=1}^{N} (y_i - m\hat{x_i} - b)^2$$

$$\frac{\partial L}{\partial m} = 2 \cdot \sum_{i=1}^{N} (y_i - m\hat{x_i} - b) \cdot (-\hat{x_i})$$
$$= -2 \sum_{i=1}^{N} (y_i - m\hat{x_i} - b) \cdot \hat{x_i}$$
$$= -2 \sum_{i=1}^{N} (y_i \hat{x_i} - m\hat{x_i}^2 - b\hat{x_i})$$
$$= -2 \sum_{i=1}^{N} y_i \hat{x_i} + 2 \sum_{i=1}^{N} m\hat{x_i}^2 + 2 \sum_{i=1}^{N} b\hat{x_i}$$

$$\frac{\partial L}{\partial b} = 2 \sum_{i=1}^{N} (y_i - m\hat{x_i} - b) \cdot (-1)$$
let $0 = -2 \sum_{i=1}^{N} (y_i - m\hat{x_i} - b)$.
$$\Rightarrow \sum_{i=1}^{N} (y_i) = \sum_{i=1}^{N} m\hat{x_i} + n \cdot b$$
$$\cancel{n} \cdot \overline{y_i} = m \cdot \cancel{n} \cdot \overline{x_i} + \cancel{n} \cdot b$$
$$b = \overline{y_i} - m \cdot \overline{x_i}$$

Let $\frac{\partial L}{\partial m} = 0$, we have:

$$-\sum_{i=1}^{N} x_i y_i + m\sum_{i=1}^{N} x_i^2 + b\sum_{i=1}^{N} x_i = 0$$

$$\begin{cases} \sum_{i=1}^{N} x_i y_i = m\sum_{i=1}^{N} x_i^2 + b\sum_{i=1}^{N} x_i \\ b = \overline{y_i} - m\overline{x_i} \end{cases}$$

$$\Rightarrow \sum_{i=1}^{N} x_i y_i = m\sum_{i=1}^{N} x_i^2 + (\overline{y} - m\overline{x})\sum_{i=1}^{N} x_i$$

$$\sum_{i=1}^{N} x_i y_i = m\sum_{i=1}^{N} x_i^2 + \overline{y}\sum_{i=1}^{N} x_i - m\sum_{i=1}^{N} \overline{x}\cdot x_i$$

$$m\left(\sum_{i=1}^{N} x_i^2 - \sum_{i=1}^{N} \overline{x}\cdot x_i\right) = \sum_{i=1}^{N} x_i y_i - \overline{y}\sum_{i=1}^{N} x_i$$

$$\begin{cases} m = \dfrac{\sum_{i=1}^{N}(x_i y_i - x_i\overline{y})}{\sum_{i=1}^{N}(x_i^2 - x_i\overline{x})} \\ \overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i \ ; \ \overline{y} = \frac{1}{N}\sum_{i=1}^{N} y_i \end{cases}$$

$$\Rightarrow m = \frac{\sum_{i=1}^{N} x_i y_i - \frac{1}{N}\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} y_i}{\sum_{i=1}^{N} x_i^2 - \frac{1}{N}\cdot\sum_{i=1}^{N} x_i \cdot \sum_{i=1}^{N} x_i}$$

$$= \frac{\sum_{i=1}^{N} x_i y_i - n\cdot\overline{x}\cdot\overline{y}}{\sum_{i=1}^{N} x_i^2 - n\cdot\overline{x}^2}$$

$$= \frac{\sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{N} x_i^2 - \bar{x}\sum_{i=1}^{N} x_i}$$

$$= \frac{\operatorname{cov}(X\,Y)}{\sum_{i=1}^{N} x_i^2 - 2\bar{x}\sum_{i=1}^{N} x_i + \bar{x}\sum_{i=1}^{N} x_i}$$

$$= \frac{\operatorname{cov}(X, Y)}{\sum_{i=1}^{N} x_i^2 - 2\bar{x}\sum_{i=1}^{N} x_i + n \cdot \bar{x}^2}$$

$$= \frac{\operatorname{cov}(X, Y)}{\sum_{i=1}^{N} (x_i - \bar{x})^2}$$

$$= \frac{\operatorname{cov}(X, Y)}{\operatorname{Var}(X)}$$

$$\Rightarrow b = \bar{y} - \frac{\operatorname{cov}(X, Y)}{\operatorname{Var}(X)} \cdot \bar{X}$$

Multi-Variable

Let $X\beta = Y$

where

$$X = \begin{bmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 1 & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & x_{32} & \cdots & x_{mn} \end{bmatrix}, \quad \beta = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_n \end{bmatrix} = \vec{w}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$J(\beta) = \sum_{i=}^{m} \left| y_i - \sum_{j=1}^{n} x_{ij}\beta_j \right|^2 = \|y - x\beta^T\|^2$$

Let $\|y - x\beta^T\|^2 = 0$

$$\Rightarrow y = x\beta^T$$

$$X^T \cdot X \cdot \beta^T = X^T y$$

$$(X^T X)^{-1} X^T X \cdot \beta^T = (X^T X)^{-1} X^T y$$

$$\beta^T = (X^T X)^{-1} X^T y$$

$$\Rightarrow \vec{w} = (X^T X)^{-1} X^T y$$