



PCA Example

Sample	Hours x_1	Score x_2
A	2	4
B	4	6
C	6	8

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

Compute the mean of each feature

Compute the mean of each column

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

$$\mu_1 = \frac{2 + 4 + 6}{3} = 4$$

$$\mu_2 = \frac{4 + 6 + 8}{3} = 6$$

$$\mu = [4 \quad 6]$$

Center the Data

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix} \quad \mu = [4 \quad 6]$$

Subtract the mean for each sample

$$A_{centered} = \begin{bmatrix} 2 - 4 & 4 - 6 \\ 4 - 4 & 6 - 6 \\ 6 - 4 & 8 - 6 \end{bmatrix}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$



Compute the Covariance Matrix

$$\Sigma = \frac{1}{n - 1} A_{centered}^T A_{centered}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \quad A_{centered}^T = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

Compute the transpose of $A_{centered}$

Compute the Covariance Matrix

$$\Sigma = \frac{1}{n - 1} A_{centered}^T A_{centered}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \quad A_{centered}^T = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\Sigma = \frac{1}{n - 1} A_{centered}^T A_{centered}$$

Use the formula by plugging
in the matrices

Compute the Covariance Matrix

$$\Sigma = \frac{1}{n - 1} A_{centered}^T A_{centered}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \quad A_{centered}^T = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\Sigma = \frac{1}{n - 1} \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$



Compute the Covariance Matrix

$$\Sigma = \frac{1}{n-1} \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

Multiply the two matrices

$$\Sigma = \frac{1}{3-1} \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Because we have three samples in our dataset, n will be 3

$$\Sigma = \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Now, divide the matrix by 2

$$\Sigma = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\Sigma = \begin{pmatrix} cov(X, X) & cov(X, Y) \\ cov(X, Y) & cov(Y, Y) \end{pmatrix}$$

$$\Sigma = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

- Variance of x_1 is 4
- Variance of x_2 is 4
- Covariance is 4



$$A\boldsymbol{v} = \lambda\boldsymbol{v}$$

Where:

A is a covariance matrix

\boldsymbol{v} is the eigenvector

λ is the eigenvalue

$$Av = \lambda v$$

$$A v - \lambda v = 0$$

$$(A - \lambda)v = 0$$

$$(A - \lambda I)v = 0 \quad 1$$

$$\det(A - \lambda I) = 0 \quad 2$$

I is an identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$



Compute the determinant

$$\det(A - \lambda I) = 0 \quad 2$$

Plug in the identity matrix I and covariance matrix A

$$\left| \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Perform scalar matrix multiplication

$$\left| \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

Subtract matrices

$$\left| \begin{array}{cc} 4 - \lambda & 4 \\ 4 & 4 - \lambda \end{array} \right| = 0$$

Find the determinant of the matrix

$$(4 - \lambda)^2 - 16 = 0$$



Solve for the eigenvalues

$$(4 - \lambda)^2 - 16 = 0$$

Simplify

$$(4 - \lambda)(4 - \lambda) - 16 = 0$$

$$16 - 4\lambda - 4\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 8\lambda = 0$$

$$(\lambda - 0)(\lambda - 8) = 0$$

$$\lambda_1 = 8$$

$$\lambda_2 = 0$$

Find the eigenvectors

$$(A - \lambda I)v = 0$$

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Plug in the identity matrix I and covariance matrix A

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Perform scalar matrix multiplication

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Replace lambda with the first eigenvalue

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Subtract matrices

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_1 = 8$$

Find the eigenvectors

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Multiply the two matrices

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} -4v_1 + 4v_2 = 0 \\ 4v_1 - 4v_2 = 0 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} 4v_2 = 4v_1 \\ 4v_1 = 4v_2 \end{bmatrix} \xrightarrow{\quad} \begin{bmatrix} v_2 = v_1 \\ v_1 = v_2 \end{bmatrix} \xrightarrow{\quad} v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Find the eigenvectors

$$(A - \lambda I)v = 0 \quad ①$$

Plug in the identity matrix I and covariance matrix A

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Perform scalar matrix multiplication

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Replace lambda with the first eigenvalue

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Subtract matrices

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_2 = 0$$

Find the eigenvectors

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Multiply the two matrices

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_2 = 0$$

$$\begin{bmatrix} 4v_1 + 4v_2 = 0 \\ 4v_1 + 4v_2 = 0 \end{bmatrix}$$

$$\begin{bmatrix} 4v_2 = -4v_1 \\ 4v_1 = -4v_2 \end{bmatrix}$$

$$\begin{bmatrix} v_2 = -v_1 \\ v_1 = -v_2 \end{bmatrix}$$

$$v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Check if correct

$$A \nu = \lambda \nu$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \nu = \lambda \nu \leftarrow$$

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \nu = 8\nu \leftarrow$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \leftarrow$$

$$\nu = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 + 4 \\ 4 + 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

Check if correct

$$A \mathbf{v} = \lambda \mathbf{v}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \mathbf{v} = \lambda \mathbf{v}$$

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \mathbf{v} = 0\mathbf{v}$$

$$\lambda_2 = 0$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Project the Data Along the Principal Component

$$\lambda_1 = 8$$

$$\lambda_2 = 0$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can see that our first principal component has a larger eigenvalue than the second principal component, we will use this because it has the highest maximum variance

Project the Data Along the Principal Component

$$u = \frac{v}{\|v\|}$$

$$\|v\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\lambda_1 = 8$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We need to normalize our eigenvectors so that they only represent directions not magnitude.



Project the Data Along the Principal Component

$$\text{new projection} = A_{centered} \times u$$

This is the formula to project our data along the axis of our principal component

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{new projection} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Project the Data Along the Principal Component

$$\text{new projection} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{firstpoint} = \frac{-2 - 2}{\sqrt{2}} = \frac{-4}{\sqrt{2}} = -2\sqrt{2}$$

$$\text{secondpoint} = 0$$

$$\text{thirdpoint} = \frac{2 + 2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\text{new projection} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ 2\sqrt{2} \end{bmatrix}$$



Project the Data Along the Principal Component

$$data \text{ in } 2D = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

$$new \text{ projection} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ 2\sqrt{2} \end{bmatrix}$$

This is the the 1D representation of our original data