



PCA Example

Sample	Hours x_1	Score x_2
A	2	4
B	4	6
C	6	8

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

Compute the mean of each feature

Compute the mean of each column

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

$$\mu_1 = \frac{2 + 4 + 6}{3} = 4$$

$$\mu_2 = \frac{4 + 6 + 8}{3} = 6$$

$$\mu = [4 \quad 6]$$

Center the Data

$$A = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

$$\mu = [4 \quad 6]$$

Subtract the mean for each sample

$$A_{centered} = \begin{bmatrix} 2 - 4 & 4 - 6 \\ 4 - 4 & 6 - 6 \\ 6 - 4 & 8 - 6 \end{bmatrix}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

Compute the Covariance Matrix

$$\Sigma = \frac{1}{n-1} A_{centered}^T A_{centered}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \quad A_{centered}^T = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

Compute the transpose of $A_{centered}$

Compute the Covariance Matrix

$$\Sigma = \frac{1}{n-1} A_{centered}^T A_{centered}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \quad A_{centered}^T = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$

$$\Sigma = \frac{1}{n-1} A_{centered}^T A_{centered}$$

Use the formula by plugging
in the matrices

Compute the Covariance Matrix

$$\Sigma = \frac{1}{n-1} A_{centered}^T A_{centered}$$

$$A_{centered} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \quad A_{centered}^T = \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix}$$
$$\Sigma = \frac{1}{n-1} \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

Compute the Covariance Matrix

$$\Sigma = \frac{1}{n-1} \begin{bmatrix} -2 & 0 & 2 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

Multiply the two matrices

$$\Sigma = \frac{1}{3-1} \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Because we have three samples in our dataset, n will be 3

$$\Sigma = \frac{1}{2} \begin{bmatrix} 8 & 8 \\ 8 & 8 \end{bmatrix}$$

Now, divide the matrix by 2

$$\Sigma = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\Sigma = \begin{pmatrix} \text{cov}(X, X) & \text{cov}(X, Y) \\ \text{cov}(X, Y) & \text{cov}(Y, Y) \end{pmatrix}$$

$$\Sigma = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

- Variance of x_1 is 4
- Variance of x_2 is 4
- Covariance is 4

$$Av = \lambda v$$

Where:

A is a covariance matrix

v is the eigenvector

λ is the eigenvalue

$$Av = \lambda v$$

$$A v - \lambda v = 0$$

$$(A - \lambda)v = 0$$

$$(A - \lambda I)v = 0$$

①

$$\det(A - \lambda I) = 0$$

②

 I is an identity matrix $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Compute the determinant

$$\det(A - \lambda I) = 0 \quad 2$$

Plug in the identity matrix I
and covariance matrix A

$$\left| \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0$$

Perform scalar matrix multiplication

$$\left| \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = 0$$

Subtract matrices

$$\begin{vmatrix} 4 - \lambda & 4 \\ 4 & 4 - \lambda \end{vmatrix} = 0$$

Find the determinant of the matrix

$$(4 - \lambda)^2 - 16 = 0$$

Solve for the eigenvalues

$$(4 - \lambda)^2 - 16 = 0$$

Simplify

$$(4 - \lambda)(4 - \lambda) - 16 = 0$$

$$16 - 4\lambda - 4\lambda + \lambda^2 - 16 = 0$$

$$\lambda^2 - 8\lambda = 0$$

$$(\lambda - 0)(\lambda - 8) = 0$$

$$\lambda_1 = 8$$

$$\lambda_2 = 0$$

Find the eigenvectors

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_1 = 8$$

$$(A - \lambda I)v = 0$$

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Plug in the identity matrix I
and covariance matrix A

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Perform scalar matrix multiplication

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Replace lambda with the first eigenvalue

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} 8 & 0 \\ 0 & 8 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Subtract matrices

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Find the eigenvectors

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Multiply the two matrices

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} -4v_1 + 4v_2 = 0 \\ 4v_1 - 4v_2 = 0 \end{bmatrix} \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \begin{bmatrix} 4v_2 = 4v_1 \\ 4v_1 = 4v_2 \end{bmatrix} \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \begin{bmatrix} v_2 = v_1 \\ v_1 = v_2 \end{bmatrix} \begin{matrix} \curvearrowright \\ \curvearrowleft \end{matrix} \begin{bmatrix} v = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{bmatrix}$$

Find the eigenvectors

$$(A - \lambda I)v = 0$$

1

Plug in the identity matrix I
and covariance matrix A

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_2 = 0$$

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Perform scalar matrix multiplication

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Replace lambda with the first eigenvalue

$$\left(\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Subtract matrices

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Find the eigenvectors

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Multiply the two matrices

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$\lambda_2 = 0$$

$$\begin{bmatrix} 4v_1 + 4v_2 = 0 \\ 4v_1 + 4v_2 = 0 \end{bmatrix} \rightarrow \begin{bmatrix} 4v_2 = -4v_1 \\ 4v_1 = -4v_2 \end{bmatrix} \rightarrow \begin{bmatrix} v_2 = -v_1 \\ v_1 = -v_2 \end{bmatrix} \rightarrow v = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Check if correct

$$A v = \lambda v$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} v = \lambda v$$

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} v = 8v$$

$$\lambda_1 = 8$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 8 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 + 4 \\ 4 + 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$\begin{bmatrix} 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

Check if correct

$$A v = \lambda v$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} v = \lambda v$$

$$A = \begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} v = 0v$$

$$\lambda_2 = 0$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = 0 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -4 \\ 4 & -4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$$

Project the Data Along the Principal Component

$$\lambda_1 = 8$$

$$\lambda_2 = 0$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$v = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

We can see that our first principal component has a larger eigenvalue than the second principal component, we will use this because it has the highest maximum variance

Project the Data Along the Principal Component

$$u = \frac{v}{\|v\|}$$

$$\|v\| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\lambda_1 = 8$$

$$v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We need to normalize our eigenvectors so that they only represent directions not magnitude.

Project the Data Along the Principal Component

$$\text{new projection} = A_{\text{centered}} \times u$$

This is the formula to project our data along the axis of our principal component

$$A_{\text{centered}} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix}$$

$$u = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{new projection} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Project the Data Along the Principal Component

$$\text{new projection} = \begin{bmatrix} -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{bmatrix} \times \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{firstpoint} = \frac{-2 - 2}{\sqrt{2}} = \frac{-4}{\sqrt{2}} = -2\sqrt{2}$$

$$\text{secondpoint} = 0$$

$$\text{thirdpoint} = \frac{2 + 2}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

$$\text{new projection} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ 2\sqrt{2} \end{bmatrix}$$

Project the Data Along the Principal Component

$$\text{data in } 2D = \begin{bmatrix} 2 & 4 \\ 4 & 6 \\ 6 & 8 \end{bmatrix}$$

$$\text{new projection} = \begin{bmatrix} -2\sqrt{2} \\ 0 \\ 2\sqrt{2} \end{bmatrix}$$

This is the the 1D representation of our original data