BAZE VEKTORSKIH PROSTOROV V vektorski prostor

Vektorji $v_1, \dots, v_k \in V$ so **LINEARNO ODVISNI**, če lahko kakšnega izmed njih izrazimo kot lin. kombinacijo ostalih:

obstaja $j \in \{1, \dots, k\}$: $v_j = \alpha_1 v_1 + \dots + \alpha_{j-1} v_{j-1} + \alpha_{j+1} v_{j+1} + \alpha_k v_k$

Vektorji so **LINEARNO NEODVISNI**, če niso linearno odvisni.

Če $v_1, \dots, v_k$ lin. odvisni: $\alpha_1 v_1 + \dots + \alpha_{j-1} v_{j-1} + (-1) v_j + \alpha_{j+1} v_{j+1} + \dots + \alpha_k v_k = \vec{0}$

$\Rightarrow$ neka netrivialna lin. komb. vektorjev $v_1, \dots, v_k$ je enaka $\vec{0}$

niso vsi skalarji enaki 0

Lin. neodvisnost: edina lin. komb. $v_1, \dots, v_k$, ki je enaka $\vec{0}$ je tista z ničelnimi koef.

$\Rightarrow$ Če $\alpha_1 v_1 + \dots + \alpha_k v_k = \vec{0} \Rightarrow \alpha_1 = \dots = \alpha_k = 0$ LIN. NEODVISNI

$\boxed{P}$ Ali so $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ lin. neodvisne?

Če $\alpha \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, potem

$$\begin{bmatrix} \alpha+\beta & 2\alpha+\beta+\gamma \\ \beta+\gamma & 3\alpha+\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\left.\begin{array}{r} \alpha+\beta=0 \\ 3\alpha+\beta=0 \end{array}\right] -$$

$$\left.\begin{array}{l} 2\alpha = 0 \Rightarrow \alpha = 0 \\ \qquad\qquad \beta = 0 \end{array}\right\} \beta+\gamma=0 \Rightarrow \gamma=0$$

Odg.: DA, te matrike so lin. neodvisne.

$\boxed{P}$ Ali so $\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ lin. neodvisne?

$$\alpha \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} + \gamma \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} \alpha+\beta & 2\alpha+\beta+\gamma \\ -\beta+\gamma & 3\alpha+3\beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\gamma = \beta \Rightarrow \left.\begin{array}{r} 2\alpha+2\beta=0 \\ \alpha+\beta=0 \\ 3\alpha+3\beta=0 \end{array}\right\} \alpha = -\beta \text{ parameter}$$

$\alpha = -\beta, \beta, \gamma = \beta$

Odg.: NE, vsak α, β, γ, ki ustrezajo $\alpha = -\beta$, $\gamma = \beta$ predstavljajo lin. odvisno komb.

$\Rightarrow$ Niso lin. neodvisne.

Recimo $\alpha = -1$, $\beta = 1$, $\gamma = 1$

P $\vec{a} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \vec{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ Ali so $\vec{a}, \vec{b}, \vec{c}$ neodvisni?

$\cdot 2$

Opazimo, da $\vec{b} = 2\vec{a} = 2\vec{a} + 0 \cdot \vec{c}$
$\vec{a} = \frac{1}{2}\vec{b} + 0 \cdot \vec{c}$ } NE. $\vec{b}$ je lin. komb. $\vec{a}$ in $\vec{c}$.

Ampak: $\vec{c} = \alpha\vec{a} + \beta\vec{b} = \alpha\vec{a} + \beta \cdot 2\vec{a} = \vec{a}(\alpha + 2\beta) = (\alpha + 2\beta)\vec{a}$ $\rightarrow\leftarrow$

!! Odg: Niso lin. neodvisni – so lin. odvisni, ampak $\vec{c}$ ne moremo izraziti kot lin. komb. $\vec{a}, \vec{b}$.

N 1 $\vec{a}, \vec{b}, \vec{c}$ lin. neodvisni; pokaži, da so $\vec{a}+\vec{c}, \vec{b}+\vec{c}, \vec{c}$ lin. neodvisni

$\cancel{\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}$

$\alpha\vec{a} + \beta\vec{b} + \gamma\vec{c} = \vec{0} \Rightarrow \alpha = \beta = \gamma = 0$

$\alpha(\vec{a}+\vec{c}) + \beta(\vec{b}+\vec{c}) + \gamma\vec{c} = \vec{0}$

$\alpha\vec{a} + \alpha\vec{c} + \beta\vec{b} + \beta\vec{c} + \gamma\vec{c} = \vec{0}$

$\alpha\vec{c} + \beta\vec{c} = \vec{0} \Rightarrow 0 \cdot \vec{c} + 0 \cdot \vec{c} = \vec{0} \Rightarrow \vec{0} = \vec{0}$ ✓

Množica $B = \{b_1, \dots, b_k\} \subseteq V$ je **BAZA** vektorskega prostora V, če velja:

① $\mathcal{L}\{b_1, \dots, b_k\} = V$ vektorji v $b_1, \dots, b_k$ razpenjajo V

→ vsak vektor iz V je lin. komb. baznih vektorjev $\{b_1, \dots, b_k\}$

→ jih je dovolj veliko

② $b_1, \dots, b_k$ so lin. neodvisni

→ vektorjev ni preveč

P Določimo bazo

$$A = \left\{ \begin{bmatrix} x \\ y \\ -y \\ -x \end{bmatrix} ; x, y \in \mathbb{R} \right\} \subseteq \mathbb{R}^4$$

$$\begin{bmatrix} x \\ y \\ -y \\ -x \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \Rightarrow A = \mathcal{L}\left\{ \underset{\vec{a}}{\begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}}, \underset{\vec{b}}{\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}} \right\}$$

vsak vektor iz A je lin. komb. $\vec{a}$ in $\vec{b}$

Ali sta $\vec{a}, \vec{b}$ lin. neodvisna?

$\alpha\vec{a} + \beta\vec{b} = \vec{0}$

$$\begin{bmatrix} \alpha \\ \beta \\ -\beta \\ -\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \alpha = \beta = 0 \Rightarrow \vec{a} \text{ in } \vec{b} \text{ sta lin. neodvisna.}$$

$\Rightarrow A = \mathcal{L}\{\vec{a}, \vec{b}\}$ in $\{\vec{a}, \vec{b}\}$ lin. neodv.

$\{\vec{a}, \vec{b}\}$ je baza A. ← $\dim(A) = 2$

Lastnosti baz

① Vsak vektorski prostor ima neskončno baz.
Vse baze imajo enako število vektorjev.

To število imenujemo **DIMENZIJA PROSTORA** $\dim(V)$.

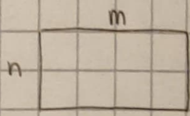
★ Denimo, da $\underbrace{\{b_1, \dots, b_n\}}_{B}$ in $\underbrace{\{c_1, \dots, c_m\}}_{C}$ bazi V, $m > n$.

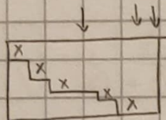
Ker je B baza

dovolj: $\mathcal{L}\{b_1, \dots, b_n\} = V$

$$\begin{aligned} c_1 &= \alpha_{11} b_1 + \alpha_{12} b_2 + \dots + \alpha_{1n} b_n \\ c_2 &= \alpha_{21} b_1 + \alpha_{22} b_2 + \dots + \alpha_{2n} b_n \\ &\vdots \\ c_m &= \alpha_{m1} b_1 + \alpha_{m2} b_2 + \dots + \alpha_{mn} b_n \end{aligned}$$

$$[c_1\ c_2 \dots c_m] = [b_1\ b_2 \dots b_n] \underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{21} & \dots & \alpha_{m1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{m2} \\ \vdots & \vdots & & \vdots \\ \alpha_{1n} & \alpha_{2n} & \dots & \alpha_{mn} \end{bmatrix}}_{A \in \mathbb{R}^{n \times m}}$$



$A \xrightarrow{G.E.}$  $\Rightarrow A\vec{x} = \vec{0}$ ima

rešitev $\vec{x}_0 = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} \neq \vec{0}$ (netrivialna reš.)

$[c_1\ c_2 \dots c_m] = [b_1\ b_2 \dots b_n] \cdot A \quad / \vec{x}_0 \quad (A\vec{x}_0 = \vec{0})$

$[c_1 \dots c_m] \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix} = \vec{0}$ $\qquad \beta_1 c_1 + \beta_2 c_2 + \dots + \beta_m c_m = \vec{0}$ (ker $\vec{c}_1, \dots, \vec{c}_m$ lin. neodvisni)

$c_1, \dots, c_m$ lin. neodvisni $\Rightarrow$ $\beta_1 = \beta_2 = \dots = \beta_m = 0$ → protislovje

▶ Torej: $m \le n$
Podobno pokažemo $n \le m$. $\Big\}$ $m = n$

② Dimenzija prostora $\dim(V)$ je
- največje št. lin. neodvisnih vektorjev
- najmanjše št. vektorjev, ki razpenjajo V

③ Če $\{b_1, \dots, b_k\}$ baza V, potem se vsak $v \in V$ izrazi na en sam (enoličen) način izrazi kot lin. komb. $v = \beta_1 b_1 + \beta_2 b_2 + \dots + \beta_k b_k$

★ Zakaj?
$$v = \beta_1 b_1 + \dots + \beta_k b_k = \gamma_1 b_1 + \dots + \gamma_k b_k$$
$$\beta_1 b_1 + \dots + \beta_k b_k - \gamma_1 b_1 - \dots - \gamma_k b_k = 0$$
$$(\beta_1 - \gamma_1) b_1 + \dots + (\beta_k - \gamma_k) b_k = 0$$

$b_1, \dots, b_k = 0$ lin. neodv.

$$\left.\begin{array}{lcl} \beta_1 - \gamma_1 = 0 & \Rightarrow & \beta_1 = \gamma_1 \\ \beta_2 - \gamma_2 = 0 & \Rightarrow & \beta_2 = \gamma_2 \\ & & \vdots \\ \beta_k - \gamma_k = 0 & \Rightarrow & \beta_k = \gamma_k \end{array}\right\}$$ en sam zapis kot lin. komb. $b_1, \dots, b_k$ ■

P Vektorski podprostori v $\mathbb{R}^3$

- $\dim U = 3 \Rightarrow$ 3 lin. neodv. vekt., njihova lin. ogrinjača mora biti $\mathbb{R}^3$
  $\Rightarrow U = \mathbb{R}^3$
- $\dim U = 2 \Rightarrow U$ je ravnina skozi koord. izhodišče
  $U = \mathcal{L}\{\vec{a}, \vec{b}\}$ $\vec{a}, \vec{b}$ lin. neodv.
- $\dim U = 1 \Rightarrow U$ je premica skozi koord. izhodišče
  $U = \mathcal{L}\{\vec{a}\}$
- $\dim U = 0 \Rightarrow U = \{\vec{0}\}$ trivialni vektorski prostor

P 1) V $\mathbb{R}^n$ označimo

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \vec{e_n} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

poljubni vektor v $\mathbb{R}^n$

$\{\vec{e_1}, \vec{e_2}, \dots, \vec{e_n}\}$ je baza $\mathbb{R}^n$:

- $\mathcal{L}\{\vec{e_1}, \dots, \vec{e_n}\} = \{\alpha_1 \vec{e_1} + \alpha_2 \vec{e_2} + \dots + \alpha_n \vec{e_n} \,;\, \alpha_i \in \mathbb{R}\} = \left\{ \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} ; \alpha_i \in \mathbb{R} \right\} = \mathbb{R}^n$
- $\vec{e_1}, \dots, \vec{e_n}$ so lin. neodvisni vektorji
  če $\alpha_1 \vec{e_1} + \dots + \alpha_n \vec{e_n} = \vec{0} \Rightarrow \alpha_1 = \dots = \alpha_n = 0$

$\Rightarrow \{\vec{e_1}, \dots, \vec{e_n}\}$ baza $\mathbb{R}^n$ STANDARDNA BAZA $\mathbb{R}^n$!!!

$\dim \mathbb{R}^n = n$

2) V $\mathbb{R}^{m \times n}$:

$$E_{ij} = \begin{array}{cc} & \\ i \rightarrow & \begin{bmatrix} & & \\ & & \\ & & 1 \\ & & \end{bmatrix} \\ & \uparrow \\ & j \end{array} \in \mathbb{R}^{m \times n}$$

STANDARDNA BAZA $\mathbb{R}^{m \times n}$

$\{E_{11}, E_{12}, \dots, E_{1n}, E_{21}, E_{22}, \dots, E_{2n}, \dots, E_{m1}, \dots, E_{mn}\}$ je baza $\mathbb{R}^{m \times n}$

$\dim \mathbb{R}^{m \times n} = mn$

- $\mathcal{L}\{E_{11}, E_{12}, \dots, E_{mn}\} = \{\alpha_{11} E_{11} + \alpha_{12} E_{12} + \dots + \alpha_{mn} E_{mn} \,;\, \alpha_i \in \mathbb{R}\} =$

$$= \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{m1} & \alpha_{m2} & \dots & \alpha_{mn} \end{bmatrix} ; \alpha_i \in \mathbb{R} \right\} = \mathbb{R}^{m \times n}$$

- če $\alpha_{11} E_{11} + \dots + \alpha_{mn} E_{mn} = 0 \Rightarrow$
  $\Rightarrow \alpha_{11} = \dots = \alpha_{mn} = 0$

TRIK: Pri G.E. se lin. ogrinjača ne spreminja:

① $\mathcal{L}\{u, v\} = \mathcal{L}\{v, u\}$ jasno

② $\mathcal{L}\{u, v_2, \dots, v_k\} = \mathcal{L}\{\alpha u, v_2, \dots, v_k\}$ jasno

③ $\mathcal{L}\{u, v, v_3, \dots, v_k\} = \mathcal{L}\{u, v + \alpha u, v_3, \dots, v_k\}$

$$\beta u + \gamma v + \dots = (\beta - \gamma\alpha)\cdot u + \gamma(v + \alpha u) + \dots$$

STOLPČNI PROSTOR matrike $A \in \mathbb{R}^{m\times n}$ je lin. ogrinjača njenih stolpcev.

Označimo ga z $C(A)$. Je vektorski podprostor v $\mathbb{R}^m$.

column space

$\boxed{P}$ a) Določimo $C(A)$ matrike
$$A = \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \end{bmatrix}$$

$$C(A) = \mathcal{L}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}2\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\} = \left\{\alpha\begin{bmatrix}1\\0\\1\end{bmatrix} + \beta\begin{bmatrix}1\\1\\0\end{bmatrix} + \gamma\begin{bmatrix}2\\1\\1\end{bmatrix} + \delta\begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$$

b) Določimo bazo $C(A)$ iz a).

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rang}(A) = 2$$

$II \leftarrow II - I$, $III \leftarrow III - 2I$; $III \leftarrow III - II$, $IV \leftarrow IV - II$

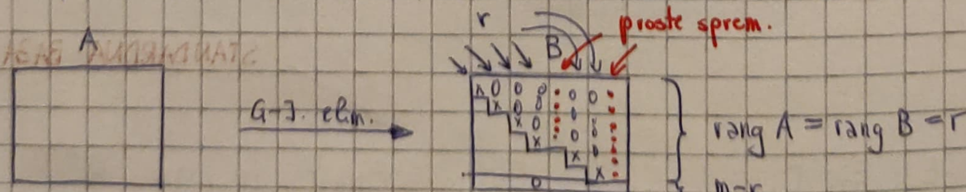
lin. neodvisni
prvega vektorja ne moremo izraziti z drugimi, drugega tudi ne itd.

$$\mathcal{L}\left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}$$

lin. neodvisna

$$\Rightarrow \text{baza } C(A) = \left\{\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\end{bmatrix}\right\}, \quad \dim C(A) = 2$$

$A \in \mathbb{R}^{m\times n}$, kakšni sta $\dim N(A)$ in $\dim C(A)$:



$A\vec{x} = \vec{0}$

$\dim N(A) = \dim N(B) = $ št. prostih sprem.

$\underset{r}{\text{rang} A} + \dim N(A) = n \leftarrow$ št. stolpcev matrike A

$\dim N(A) = n - r$

Rang matrike $A \in \mathbb{R}^{m \times n}$ je enak:

- št. neničelnih vrstic v vrstično stopničasti obliki matrike A
- št. pivotov v vrstično stopničasti obliki matrike A
- št. lin. neodv. vrstic matrike A
- št. lin. neodv. stolpcev matrike A
- dimenziji stolpčnega prostora $C(A)$ matrike A
- $\text{rang } A = n - \dim N(A)$ $\quad (\dim C(A) + \dim N(A) = n)$

$\boxed{\# 3}$ $\text{rang } A = \text{rang } A^T$ Zakaj? Po ^{T} vrstice "postanejo" stolpci, tj. $\text{rang } A$ ... št. lin. neodv. vrstic v A, $\Rightarrow$ $\text{rang } A^T$ ... št. lin. neodv. stolpcev v A^T

Naslednje trditve o matriki $A \in \mathbb{R}^{n \times n}$ so ekvivalentne:

- A je obrnljiva
- homogeni sistem enačb $A\vec{x} = \vec{0}$ ima le trivialno rešitev $\vec{x} = \vec{0}$
- sistem enačb $A\vec{x} = \vec{b}$ ima enolično rešitev za vsak $\vec{b} \in \mathbb{R}^n$
- reducirana vrstično stopničasta oblika matrike A je I
- $\text{rang } A = n$
- stolpci/vrstice matrike A so lin. neodvisni
- stolpci/vrstice matrike A razpenjajo $\mathbb{R}^n$
- stolpci/vrstice matrike A so baza $\mathbb{R}^n$
- $\dim N(A) = 0$
- $\dim C(A) = n$