LASTNOSTI MATRIČNEGA MNOŽENJA

① $O_{m\times n} = m\begin{array}{c} n \\ \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{bmatrix}\end{array}$ ničelna matrika

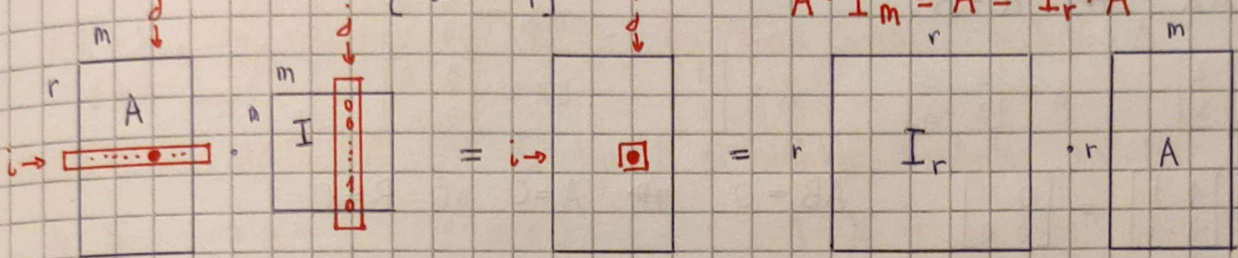
$A \in \mathbb{R}^{m\times n}$ $\quad A \cdot O_{n\times p} = O_{m\times p}$

$B \in \mathbb{R}^{n\times p}$ $\quad O \cdot B = O_{m\times p}$

② $A \in \mathbb{R}^{r\times m}$

$I_m = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \in \mathbb{R}^{m\times m}$ identična matrika (identiteta)

$A \cdot I_m = A = I_r \cdot A$



③ $A(BC) = (AB)C$ asociativnost *

(brez dokaza)

*(za ustrezno velike matrike)

④ $A(B+C) = AB + AC$ distributivnost *

$(A+B)C = AC + BC$

★ Dokaz prve distributivnosti: dovolj je pokazati, da za vsak element oz. par (i,j) velja $(A(B+C))_{i,j} = (AB+AC)_{i,j}$

$$(A(B+C))_{i,j} = \sum_{k=1}^{n} A_{i,k}(B+C)_{k,j} = \sum_{k=1}^{n} \underbrace{A_{i,k}}_{\in\mathbb{R}}(\underbrace{B_{k,j}}_{\in\mathbb{R}} + \underbrace{C_{k,j}}_{\in\mathbb{R}}) \overset{\text{distributivnost realnih števil}}{=}$$

$A \in \mathbb{R}^{m\times n}$; $B, C \in \mathbb{R}^{n\times p}$

$$= \sum_{k=1}^{n}(A_{i,k}B_{k,j} + A_{i,k}C_{k,j}) =$$

$$= \sum_{k=1}^{n} A_{i,k}B_{k,j} + \sum_{k=1}^{n} A_{i,k}C_{k,j} = (AB)_{i,j} + (AC)_{i,j} = (AB+AC)_{i,j} \quad \blacksquare$$

★ Dokaz druge distributivnosti: $A, B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times p}$

$$((A+B)C)_{i,j} = \sum_{k=1}^{n}(A+B)_{i,k}C_{k,j} = \sum_{k=1}^{n}(\underbrace{A_{i,k}}_{\in\mathbb{R}} + \underbrace{B_{i,k}}_{\in\mathbb{R}})\underbrace{C_{k,j}}_{\in\mathbb{R}} \overset{\text{distr. v } \mathbb{R}}{=} \sum_{k=1}^{n}(A_{i,k}C_{k,j} + B_{i,k}C_{k,j}) =$$

$$= \sum_{k=1}^{n} A_{i,k}C_{k,j} + \sum_{k=1}^{n} B_{i,k}C_{k,j} = (AC)_{i,j} + (BC)_{i,j} = (AC+BC)_{i,j} \quad \blacksquare$$

⑤ $A \in \mathbb{R}^{n\times n}$ (enako št. vrstic in stolpcev) → kvadratna matrika

$A^k = A \cdot A^{k-1}$ rekurzivna definicija k-te potence matrike A (kvadratne matrike!)

$A^0 = I_n$ $\quad (A^1 = A \cdot A^0 = A \cdot I_n = A)$

$A^2 = A \cdot A^1 = A \cdot A$ $\quad A^3 = A \cdot A^2 = A \cdot A \cdot A$ $\quad A^4 = \ldots$

Kaj pa NE velja?

- $AB \neq BA$

če $AB = BA$ velja, pravimo, da A in B **komutirata**

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 \\ 1 & 0 \end{bmatrix} \quad \left.\begin{aligned} AB &= \begin{bmatrix} 1\cdot4+2\cdot1 & 1\cdot0+2\cdot0 \\ 2\cdot4+1\cdot1 & 2\cdot0+1\cdot0 \end{bmatrix} = \begin{bmatrix} 6 & 0 \\ 5 & 0 \end{bmatrix} \\ BA &= \begin{bmatrix} 4\cdot1+0\cdot2 & 4\cdot2+0\cdot1 \\ 1\cdot1+0\cdot2 & 1\cdot2+0\cdot1 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} \end{aligned}\right\} AB \neq BA$$

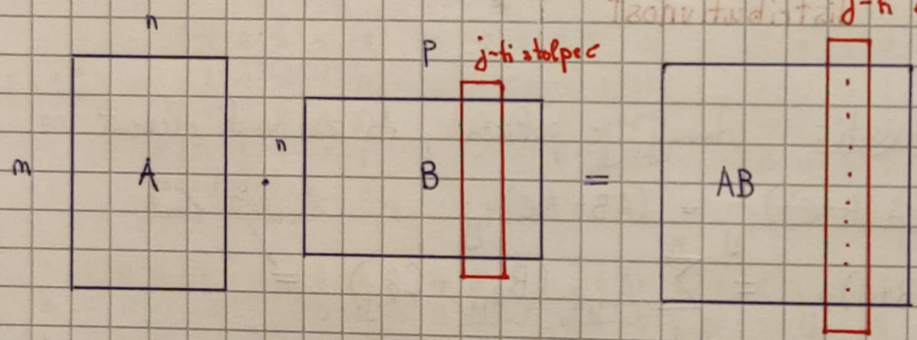
- $AX = BX \not\Rightarrow A = B$

$$A = \begin{bmatrix} -2 & 2 \\ -3 & 3 \end{bmatrix} \quad B = \begin{bmatrix} 2 & -2 \\ 3 & -3 \end{bmatrix} \quad X = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \left.\begin{aligned} AX &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ BX &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}\right\} \begin{gathered} AX = BX, \\ \text{ampak } A \neq B \end{gathered}$$

- $\underset{\neq 0}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \cdot \underset{\neq 0}{\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}} = \underset{= 0}{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$ $\quad AB = 0 \not\Rightarrow A = 0$ ali $B = 0$

BLOČNO MNOŽENJE MATRIK

$A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times p}$

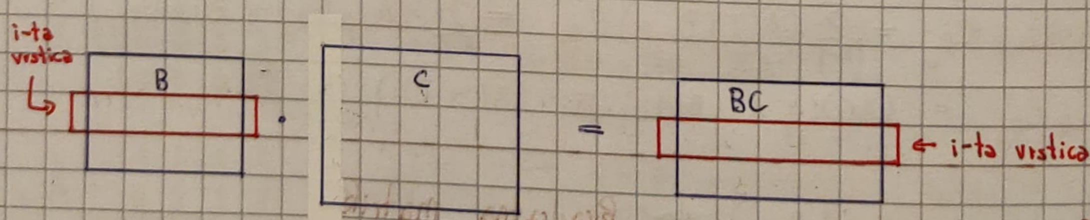


$B = \left[B^{(1)}\ B^{(2)} \ldots B^{(p)} \right]$ $\qquad AB = \left[AB^{(1)}\ AB^{(2)} \ldots AB^{(p)} \right]$

stolpci matrike B

$(AB)^{(j)} = A \cdot B^{(j)}$ ali pa $A \cdot B = \left[A\cdot B^{(1)}\ \ A\cdot B^{(2)}\ \ldots\ A\cdot B^{(p)} \right]$

$B \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times r}$



$$B = \begin{bmatrix} B_{(1)} \\ B_{(2)} \\ \vdots \\ B_{(n)} \end{bmatrix}$$ vrstice matrike B $\qquad (BC)_{(i)} = B_{(i)} \cdot C$ ali pa $BC = \begin{bmatrix} B_{(1)}C \\ B_{(2)}C \\ \vdots \\ B_{(n)}C \end{bmatrix}$

Uporaba v linearnem sistemu: $A\vec{x}=\vec{b}$ $\quad A\in\mathbb{R}^{m\times n},\ \vec{b}\in\mathbb{R}^{m},\ \vec{x}=\begin{bmatrix}x_1\\ \vdots\\ x_n\end{bmatrix}$

$$\left[A^{(1)}\dots A^{(n)}\right]\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}=\vec{b} \leftarrow \text{1 stolpec}$$

$$\vec{b}=\underset{\mathbb{R}^m}{A^{(1)}}\underset{\mathbb{R}}{x_1}+\underset{\mathbb{R}^m}{A^{(2)}}\underset{\mathbb{R}}{x_2}+\dots+\underset{\mathbb{R}^m}{A^{(n)}}\underset{\mathbb{R}}{x_n}$$

$$=x_1A^{(1)}+x_2A^{(2)}+\dots+x_nA^{(n)}$$ linearna kombinacija stolpcev matrike A

$A\vec{x}=\vec{b}$ ima rešitev $\Leftrightarrow$ $\vec{b}$ je lin. komb. stolpcev matrike A

INVERZNA MATRIKA

(obstaja le za nekatere kvadratne matrike)

v $\mathbb{R}$: $\quad a\cdot x=1 \quad /:a\neq 0$

$x=\frac{1}{a}=a^{-1}$ $\longrightarrow$ $A\cdot X=I=\begin{bmatrix}1 & & 0\\ & \ddots & \\ 0 & & 1\end{bmatrix}$

Matrika $A\in\mathbb{R}^{n\times n}$ je OBRNLJIVA, če obstaja $A^{-1}\in\mathbb{R}^{n\times n}$, da velja:

$$A\cdot A^{-1}=I_n=A^{-1}\cdot A$$

V nasprotnem primeru je A neobrnljiva. Če je A obrnljiva, matriko A^{-1} imenujemo INVERZ matrike A.

P $\quad A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ Ali je A obrnljiva?

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x & y\\ z & w\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}x & y\\ z & w\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}z & w\\ 0 & 0\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$0=1$ PROTISLOVJE $\rightarrow\leftarrow$

$A=\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ NI obrnljiva.

Če je $A \in \mathbb{R}^{n \times n}$ obrnljiva, obstaja taka X, da velja $AX = I$.

$$AX = \begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & & 1 \end{bmatrix} = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}$$

stolpci

samo ena enica

$$e_j = \begin{bmatrix} \\ 1 \\ \\ \end{bmatrix} \in j \in \mathbb{R}^n$$

$$\underbrace{AX^{(j)}} = I^{(j)} = e_j$$

za vsak stolpec $X^{(j)}$: $A \cdot X^{(j)} = e_j \xrightarrow{\text{REŠIMO}} [A \mid e_j] \xrightarrow{\text{G.E.}}$ rešitev $X^{(j)}$

dobimo en lin. sistem

za $j = 1, 2, \dots, n$

spreminjamo LE desno stran

I_n je A vedno isti

vedno isti A

$$\left[A \,\middle|\, e_1 e_2 \dots e_n\right] \xrightarrow{\text{G.E.}} \text{če } \left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \,\middle|\, B\right] : \quad I_n \cdot X = B \Rightarrow X = B$$

rešitev

hkrati rešimo n linearnih sistemov $AX^{(1)} = e_1, AX^{(2)} = e_2, \dots, AX^{(n)} = e_n$

$$\overbrace{\left[\underbrace{A}_{n} \,\middle|\, \underbrace{I}_{n}\right]}^{2n}$$

(n vrstic)

$A \cdot X = I$

Če je $\operatorname{rang} A = n$, potem matrika na desni strani zelo razširjene matrike po G.E. reši enačbo $A \cdot X = I_n$.

Denimo $\underbrace{AX = I_n}_{\substack{\text{desni inverz } A \\ (X)}}$ in $\underbrace{YA = I_n}_{\substack{\text{levi inverz } A \\ (Y)}}$. Izkaže se, da $X = Y$.

$$X = I_n \cdot X = (YA)X = Y(AX) = Y \cdot I_n = Y \quad \Rightarrow X = Y$$

V posebnem:

- če $AX = I_n \Rightarrow X = A^{-1}$
- če $AX = XA = I_n$ in $AY = YA = I_n \Rightarrow X = Y$

$\longrightarrow$ inverz matrike je en sam.

ALGORITEM ZA RAČUNANJE INVERZA MATRIKE $A \in \mathbb{R}^{n \times n}$

- Na zelo razširjeni matriki $[A \mid I] \in \mathbb{R}^{n \times (2n)}$ izvajamo G.E. po vrsticah
  - če $\operatorname{rang} A < n \Rightarrow A$ ni obrnljiva
  - če $\operatorname{rang} A = n$: $[A \mid I] \longrightarrow [I \mid A^{-1}]$

P Ali je $A=\begin{bmatrix}1&0&2\\0&2&-1\\2&-1&4\end{bmatrix}$ obrnljiva? Če da, izračunaj inverz.

$$\left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&2&-1&0&1&0\\2&-1&4&0&0&1\end{array}\right] \underset{III \leftarrow III-2I}{\sim} \left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&2&-1&0&1&0\\0&-1&0&-2&0&1\end{array}\right] \underset{II \leftrightarrow III\cdot(-1)}{\sim} \left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&0&2&0&-1\\0&2&-1&0&1&0\end{array}\right] \underset{III \leftarrow III-2II}{\sim} \left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&0&2&0&-1\\0&0&-1&-4&1&2\end{array}\right] \sim$$

$\operatorname{rang} A = 3 \Rightarrow A$ obrnljiva

$III \leftarrow -1\cdot III$

$$\sim \left[\begin{array}{ccc|ccc}1&0&2&1&0&0\\0&1&0&2&0&-1\\0&0&1&4&-1&-2\end{array}\right] \underset{I \leftarrow I-2III}{\sim} \left[\begin{array}{ccc|ccc}1&0&0&-7&2&4\\0&1&0&2&0&-1\\0&0&1&4&-1&-2\end{array}\right] \Rightarrow A^{-1}=\begin{bmatrix}-7&2&4\\2&0&-1\\4&-1&-2\end{bmatrix}$$

I A^{-1}

Kdaj je DIAGONALNA MATRIKA obrnljiva?

Matrika D je diagonalna, če so njeni edini neničelni elementi na diagonali.

$$D=\begin{bmatrix}\ddots & & 0\\ & \ddots & \\ 0 & & \ddots\end{bmatrix}$$

diagonala matrike

$$\left[\begin{array}{cccc|cccc}d_1 & & & 0 & 1 & & & 0\\ & d_2 & & & & \ddots & & \\ & & \ddots & & & & & \\ 0 & & & d_n & 0 & & & 1\end{array}\right] \xrightarrow{1:d_1,\ 1:d_2,\ \dots,\ 1:d_n} \left[\begin{array}{cccc|cccc}1 & & & 0 & \frac{1}{d_1} & & & 0\\ & 1 & & & & \frac{1}{d_2} & & \\ & & \ddots & & & & \ddots & \\ 0 & & & 1 & 0 & & & \frac{1}{d_n}\end{array}\right]$$

Če $d_1 \neq 0, d_2 \neq 0, \dots, d_n \neq 0$, potem je $\operatorname{rang} D = n$ in zato je D obrnljiva.

$$\begin{bmatrix}d_1 & & & 0\\ & d_2 & & \\ & & \ddots & \\ 0 & & & d_n\end{bmatrix}^{-1} = \begin{bmatrix}\frac{1}{d_1} & & & 0\\ & \frac{1}{d_2} & & \\ & & \ddots & \\ 0 & & & \frac{1}{d_n}\end{bmatrix},$$ če $d_i \neq 0$ za $i=1,\dots,n$

ZGORNJETRIKOTNA MATRIKA

Matrika je zgornjetrikotna, če so njeni elementi pod diagonalo enaki 0.

$$\begin{bmatrix}\ddots & * & * & *\\ & \ddots & * & *\\ & & \ddots & *\\ 0 & & & \ddots\end{bmatrix}$$

Zgornjetrikotna matrika ($Zg\Delta$) je obrnljiva, če so njeni diagonalni elementi (diagonali) neničelni.

$$\left[\begin{array}{cccc|ccc}\ddots & * & * & * & 1 & & 0\\ & \ddots & * & * & & \ddots & \\ & & \ddots & * & & & \\ 0 & & & \ddots & 0 & & 1\end{array}\right] \longrightarrow \left[\begin{array}{c|cccc} & \ddots & * & * & *\\ I & & \ddots & * & *\\ & & & \ddots & *\\ & & & & \ddots\end{array}\right]$$

GE izvajamo samo navzgor

↯ 2 Spodnje trikotna matrika

$(Sp\Delta)^T = Zg\Delta$

$$\left[\begin{array}{cccc|ccc}\ddots & & & 0 & 1 & & 0\\ * & \ddots & & & & \ddots & \\ * & * & \ddots & & & & \\ * & * & * & \ddots & 0 & & 1\end{array}\right] \longrightarrow \left[\begin{array}{c|cccc} & \ddots & & & \\ I & * & \ddots & & \\ & * & * & \ddots & \\ & * & * & * & \ddots\end{array}\right]$$

$$\begin{bmatrix}\ddots & & & 0\\ * & \ddots & & \\ * & * & \ddots & \\ * & * & * & \ddots\end{bmatrix}^T = \begin{bmatrix}\ddots & * & * & *\\ & \ddots & * & *\\ & & \ddots & *\\ 0 & & & \ddots\end{bmatrix}$$

GE izvajamo samo navzdol

UPORABA IN LASTNOSTI INVERZOV

① Če $A \in \mathbb{R}^{n \times n}$ obrnljiva ter $\vec{b} \in \mathbb{R}^n$:

$$A^{-1} \cdot \backslash \quad A\vec{x} = \vec{b}$$

množimo z leve!

$$A^{-1} A \vec{x} = A^{-1} \vec{b}$$

$$\underbrace{(A^{-1} A)}_{= I_n} \vec{x} = A^{-1} \vec{b} \implies \vec{x} = A^{-1} \vec{b}$$ edina rešitev tega lin. sistema

② A, B obrnljivi matriki, $A, B, C, X \in \mathbb{R}^{n \times n}$

Izrazimo X iz enakosti $AXB = C$

$$A^{-1} \cdot \backslash \quad AXB = C \quad / \cdot B^{-1}$$

$$\underbrace{A^{-1} A}_{I_n} \cdot X \cdot \underbrace{B B^{-1}}_{I_n} = A^{-1} C B^{-1}$$

$$I_n \cdot X \cdot I_n = X$$

$$X = A^{-1} C B^{-1}$$

③ Če je v sistemu $A\vec{x} = \vec{b}$ desna stran enaka $\vec{b} = \vec{0}$, potem tak sistem $A\vec{x} = \vec{0}$ imenujemo **homogeni sistem**.

→ homogeni sistem ima gotovo vsaj eno rešitev $\vec{x} = \vec{0}$ (trivialna rešitev)

- če A obrnljiva: $A^{-1} \cdot \backslash \quad A\vec{x} = \vec{0}$

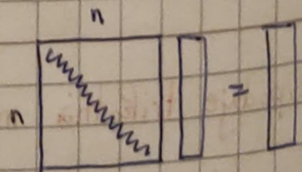
$$\vec{x} = A^{-1} \cdot \vec{0} = \vec{0}$$

$\Downarrow$

$\vec{0}$ je edina rešitev

- Obrat? Če $\vec{0}$ edina reš. $A\vec{x} = \vec{0}$, *

A mora biti taka, da ne dovoli neskončno rešitev (ne sme biti vrstice z samimi 0)



* potem je $\operatorname{rang} A = n \implies A$ je obrnljiva

Dokazali smo: $(A \in \mathbb{R}^{n \times n})$ A je obrnljiva $\iff \operatorname{rang} A = n \iff \vec{0}$ edina reš. homogenega sis. $A\vec{x} = \vec{0} \iff$ sistem $A\vec{x} = \vec{b}$ ima natanko eno rešitev (pri poljubnem $\vec{b} \in \mathbb{R}^n$)

④ $A \in \mathbb{R}^{n\times n}$ obrnljiva matrika $\quad A\cdot A^{-1} = I_n = A^{-1}\cdot A$

Kaj je $(A^{-1})^{-1}$? $\quad (A^{-1})^{-1} = A$

⑤ Če sta $A, B \in \mathbb{R}^{n\times n}$ obrnljivi, potem je $AB \in \mathbb{R}^{n\times n}$ obrnljiva.

$(AB)^{-1} = B^{-1}A^{-1}$

★ Dokaz: $(AB)(B^{-1}A^{-1}) = A\underbrace{(BB^{-1})}_{=I}A^{-1} = A I A^{-1} = \underbrace{AA^{-1}}_{=I} = I$

$\Rightarrow \quad (AB)^{-1} = B^{-1}A^{-1}$ ■

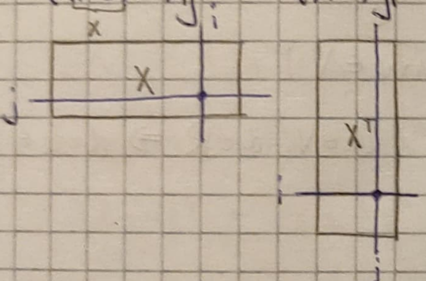
LASTNOSTI TRANSPONIRANE MATRIKE

① $(A^T)^T = A$ ② $(A+B)^T = A^T + B^T$ ③ $(\alpha A)^T = \alpha A^T$

④ $(AB)^T = B^T A^T$

★ $A = [A_{ij}]$, $B = [B_{ij}]$, $(AB) = [(AB)_{ij}]$

$((\underbrace{AB}_{X})^T)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk} B_{ki} = \sum_{k=1}^{n} (A^T)_{kj} (B^T)_{ik} =$



$= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij}$ ■

⑤ A obrnljiva $\Rightarrow$ A^T obrnljiva in $(A^T)^{-1} = (A^{-1})^T$

A obrnljiva $\Rightarrow A\cdot A^{-1} = I_n \quad /^T$

$(AA^{-1})^T = I_n^T$

po ④: $(A^{-1})^T \cdot A^T = I_n$ ■

$(A^T)^{-1}$ inverz A^T

$(A^{-1})^T = (A^T)^{-1}$

1

a) $A^{-1}\backslash\ AXB^{-1} = C \quad /\cdot B$

$A^{-1}AXB^{-1}B = A^{-1}CB$

$X = A^{-1}CB$ da

b)

c) NE, diag. elementi morajo biti neničelni.

d) NE; $A^{-1}\backslash\ A\vec{x} = \vec{b}$

$\underbrace{A^{-1}A}_{=I_n}\vec{x} = A^{-1}\vec{b}$

$\vec{x} = A^{-1}\vec{b}$ edina rešitev

e) DA; $\vec{b}$ je potem $\vec{0} \to \vec{0}$ je edina rešitev.

f) $AXA + A = 0$

$A^{-1}\backslash\ AXA = -A \quad /\cdot A^{-1}$

$X = A^{-1}(-A)\cdot A^{-1}$

$X = (-1)A^{-1}\cdot A\cdot A^{-1} = (-1)A^{-1}$

obrnljivo

verjetno DA