Do road-traffic injuries show a power-law scaling relationship?

August 27, 2019

1 Introduction

The term "safety in numbers" reflects the observation that a change in the number of road users is not met by a linear change in the number of injuries caused to or by the road-user group. The number of injuries, I, is generally formulated as follows, in terms of a base rate, α, the number of road users of one type (say, motorists), M, the number of road users of the second type (say, cyclists), C, and "safety in numbers" exponents for each, β_1 and β_2 (Elvik and Bjørnskau, 2017):

$$I = \alpha M^{\beta_1} C^{\beta_2}. \tag{1}$$

For completeness, one might include other covariates, e.g.

$$I = \alpha M^{\beta_1} C^{\beta_2} \exp\left(\sum_{i=3}^{n} \beta_i X_i\right), \tag{2}$$

but these are not central to the present discussion, which focuses around the variables in Equation 1.

Beyond observational inference, such analyses are proposed to inform public-health forecasting (Schepers and Heinen, 2013). This includes assessment of likely health benefits following policy change, as well as forecasting healthcare needs given the expected change to transport-related behaviour, e.g. the increase in motor vehicle ownership expected in cities in fast-growing economies in the coming years. We have previously used this formulation to make predictions of numbers of road injuries for specific cities (Accra, Sao Paulo), as well as in the developing generic software ITHIM-R. (ITHIM: integrated transport and health impact model.)

In this work, we question the validity of the assumptions implicit in the model as well as its applications. Our intention here is to examine (1) the interpretation of (and language around) Equation 1, (2) its proposed application to health-impact modelling, and (3) what "safety in numbers" is and how it might be measured. We might consider two types of study: (a) small scale, and (b) inter city. In this work we focus on the latter, but make reference to the former, as it provides some insights, it might be used for prediction, and in the end we would like to have a single comprehensive framework. The picture of questions is shown in Table 1.

Using a theoretical approach, we derive the result that coefficients $\beta_1 = \beta_2 = 0.5$ learnt from multiple settings correspond to linear scaling across time and space. Put another way, these coefficients permit "tiling" of a small space to create a larger space whose properties are the same. We confirm this result through simulation, which shows also that the coefficients corresponding to variation in density given fixed space are $\beta_1 = \beta_2 = 1$. These provide null hypotheses for studies assessing the impact of road-user number on collision risk.

We highlight the necessity of inclusion of density for the proper specification of a null hypothesis. For a study that considers multiple settings of different sizes but constant density, an appropriate null hypothesis is $\beta_1 = \beta_2 = 0.5$. For a study that considers multiple settings of different densities but

Table 1: Questions & answers for different study types

Question	*Study level* (a) Small scale	(b) Inter city
(1) What does it mean?	Safety in numbers? Linearity?	$\beta_1 + \beta_2 = 1$ means linearity (Section 3.1)
(2) How do we use it to predict?	Multiply then sum (Section 5.1)	Account for density (Section 5.4)
(3) How do we study it?	...As before? (Section 6.1)	Account for density (Section 6.2)

constant size, an appropriate null hypothesis is $\beta_1 = \beta_2 = 1$. For a study that considers multiple settings of varies sizes and densities, an appropriate null hypothesis would need to be derived from theoretical considerations or simulations.

2 Background

2.1 Equation 1 in the literature

"Safety in numbers" exponents are estimated in analyses of road-traffic injuries through fitting a regression model such as Equation 1 or Equation 2 to data. At minimum, these data consist of counts of road injuries, and of two road user types, often cars and pedestrians or cyclists. The exponents β_1 and β_2 are estimated, and reported with confidence intervals and p-values corresponding to the probability of observing the data under the null hypothesis $\beta_1 = 0$, $\beta_2 = 0$.

The units of measurement are not a primary concern in studies reporting safety-in-numbers effects. On the contrary, what is identified is apparently a unit-independent, scale-invariant effect that maps cumulative road usage into injury counts. For example, Miranda-Moreno et al. (2011); Geyer et al. (2006); Garder et al. (1998); Schepers et al. (2011); Nordback et al. (2014) and Leden (2002) count vehicles (usually in terms of average daily number of vehicles per unit space, though sometimes the time unit is annual or hourly). In contrast, Prato et al. (2016) and Schepers and Heinen (2013) work in terms of km. Injuries are counted as the total in one or more years. The areas considered in the studies range in size from intersections (Nordback et al., 2014) to municipalities (Schepers and Heinen, 2013) and "local authorities" (Aldred et al., 2017).

From this we see that time and space are not explicitly included in published models. Instead, it is implicit that the numbers of injuries are linear in time and space. We return to this issue later, where we see that this specification results in a base rate (α in Equation 1) that depends on e.g. the number of years of study (Section 4.1).

2.2 Illustration with data for England

To give some context, I will use data pertaining to road-traffic injuries recorded in England in the years 2005 to 2015. From among these data I isolate all injuries to cyclists that occurred in events involving at least one car. Alongside these data, I use Road Traffic Statistics estimates of distance travelled by cars and bikes in these areas[1], as well as census estimates for population numbers in the year 2011.

We begin with an illustration of how injuries to cyclists (involving cars) scales with the total distance travelled by cyclists (Figure 1). In this Figure, we show side by side how the scaling changes as we include only more severe cases. We repeat the analysis with London boroughs only in Figure 2.

We look also at many possible relationships between single predictors and injury counts (Figure 3). Note the similar trend as the less serious casualties are excluded (KSI: killed or seriously injured). These values are summarised in Table 2, where we include also the result of a regression of the form of Equation 1. Note that the sum aligns with the other rows.

Table 2: Coefficients

Predictor	All injuries	KSI	Fatalities
Total travel (β)	0.45	0.6	0.85
Population (β)	0.74	0.88	0.99
Bike (β)	0.68	0.77	0.86
Car (β)	0.44	0.59	0.85
Bike+Car ($\beta_1 + \beta_2$)	0.66	0.78	0.93

[1] We use data covering all road types. Meagre attempts to remove motorway travel for cars have not changed the general picture that emerges.

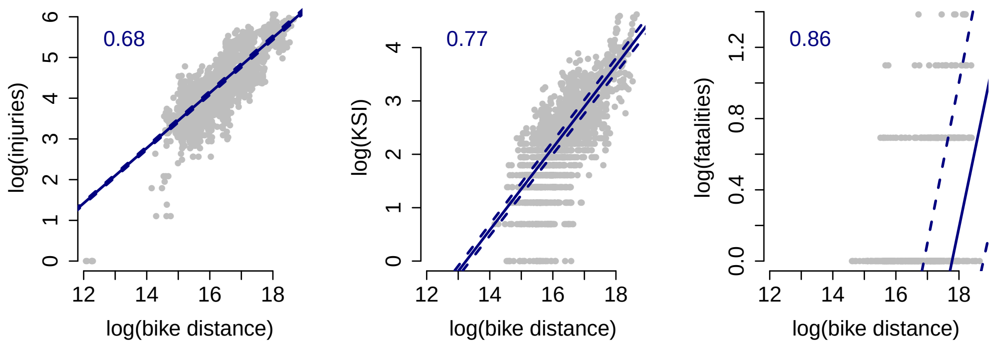

Figure 1: Fit of equation $I = \alpha C^{\beta}$ to data for local areas in England. There are 148 areas including counties and London boroughs, and 11 years (grey points). I is (left) total incidents involving a car in which a cyclist was injured; (middle) the subset of these that were KSI; and (right) the subset of these that were fatal. C counts the total distance travelled in the area. In blue is the line of best fit, with the gradient of the line (the value for β) in the top-left corner.

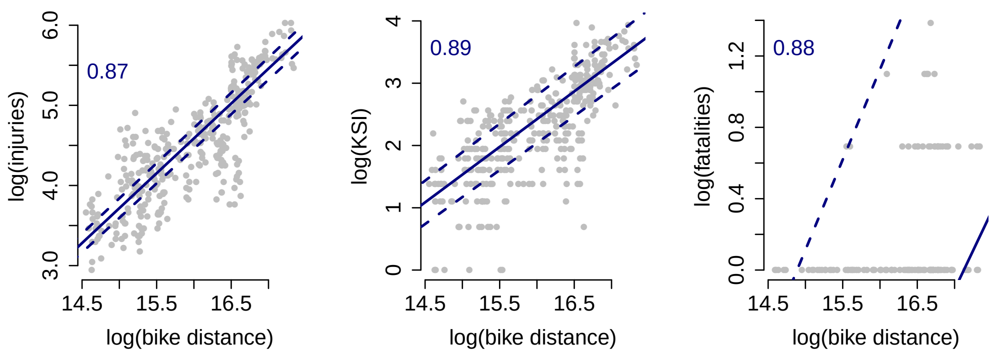

Figure 2: As in Figure 1, with only the London boroughs.

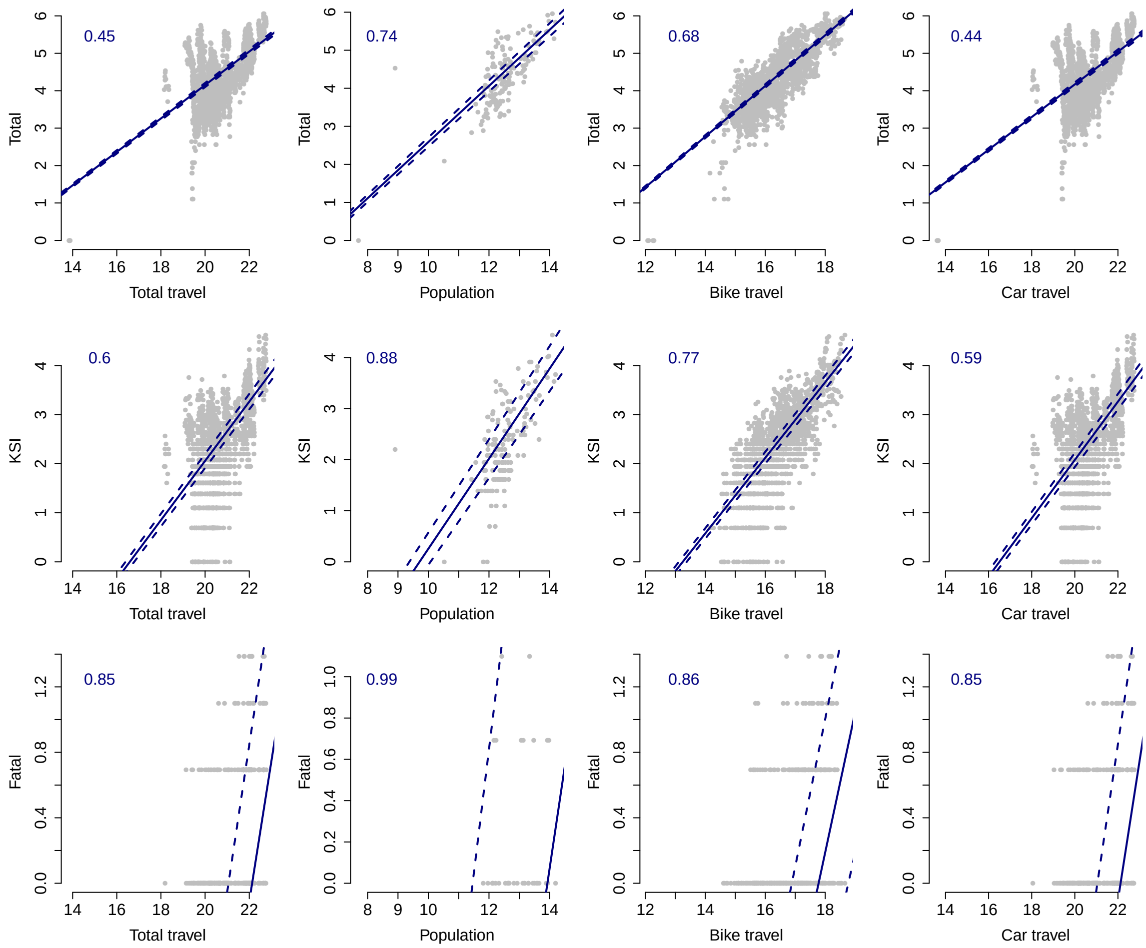

Figure 3: As in Figure 1, showing different relationships and the β coefficients they generate.

3 Examination of Equation 1

In general, it is posited that $\beta_i < 1$ for an equation of the form of Equation 1 implies safety in numbers for mode i (Elvik and Bjørnskau, 2017). However, given the construction of Equation 1, we ought to expect $\beta_i < 1$ for all i for any study counting motorists, cyclists and injuries across scales, because the mode counts C and M are functions of scale. This has implications both for our interpretation and for our application of such a model.

3.1 Interpretation of Equation 1

To illustrate, consider again the example of the English counties, excluding London boroughs, and cyclist KSI counts. Applying Equation 1, we find coefficients $\beta_1 = 0.15$ and $\beta_2 = 0.66$, and we might interpret this as safety in numbers.

However, Figure 4 shows that there is an approximately linear relationship between population (N) and cyclist KSI, between population and car distance, and between population and cyclist distance. If we consider population to be akin to a "latent variable" that explains the relationships between the other three variables, and apply the equation $I = \alpha N^{\beta}$, we find $\beta = 0.96$ – i.e., injuries are close to linear in population.

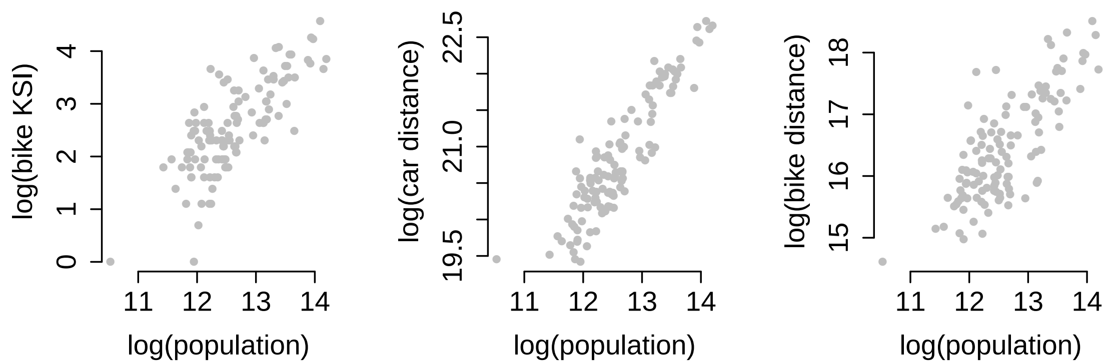

Figure 4: Linear relationships between population (N) and cyclist KSI, between population and car distance, and between population and cyclist distance in English counties.

The act of decomposing the population into two component parts splits the predictive coefficient. We would observe a three-way split were we to consider three modes. We would see the same phenomenon were we to apply this equation to, say, second-hand car sales, where I is the number of sales, and we divide the population N into M vendors and C prospective purchasers. We would see the same in HIV infection rates, if we were to count the number of new infections I and divide the population N into the susceptible group, C, and the infectious group, M (Bettencourt et al., 2007). We wouldn't describe either of these cases as examples of "safety in numbers".

3.1.1 Methodology applied to infectious disease

Let's look more closely at the infectious disease analogy, to see what are the implications of the regression method. Let's start with hypothetical, perfect "data" that illustrate the relationship $I = \alpha C^{\beta_1} M^{\beta_2}$, yielding $\beta_1 = \beta_2 = 0.5$, exemplary safety in numbers (Table 3).

Now let's relabel the columns and consider instead the number of new infections in a year (I), the number of infectious people (C) and the number of susceptible people (M), Table 4.

Table 3: A simple example of idealised data representing safety in numbers.

Injuries I	Motorists M	Cyclists C
1	100	10
2	200	20
3	300	30
4	400	40

Table 4: A simple example of idealised data representing safety in numbers applied to infectious disease, where we count the number of new infections per year as a function of the number of susceptible and the number of infectious people.

New infections I	Susceptible M	Infectious C
1	100	10
2	200	20
3	300	30
4	400	40

So we see that for infectious diseases we also have safety in numbers. In fact, the rate of a disease (per capita) is often studied as a function of the total population, N, rather than the infectious and susceptible populations. We see from Table 5 that infections are linear in population ($I = \alpha' N$), even though there is safety in numbers for both infectious and susceptible parties.

Table 5: A simple example of idealised data representing safety in numbers applied to infectious disease, where we count the number of new infections per year as a function of total population.

New infections I	Susceptible M	Infectious C	Population N
1	100	10	110
2	200	20	220
3	300	30	330
4	400	40	440

For HIV infections, a relationship of $I \sim \alpha' N^{1.2}$ has been observed (Bettencourt et al., 2007). Given that $I = \alpha' N$ corresponds to $I = \alpha C^{0.5} M^{0.5}$, how would we write $I = \alpha C^{\beta_1} M^{\beta_2}$ for $I \sim \alpha' N^{1.2}$? Perhaps $I = \alpha C^{0.6} M^{0.6}$? Although this is pure speculation, it seems reasonable as a first guess. In any event, we would not expect either β_1 or β_2 to exceed 1. The resulting inference in the safety-in-numbers framework is then that the biggest susceptible populations M are the safest. However, the biggest susceptible populations are in the locations with the biggest populations, which have, according to the observation $I \sim \alpha' N^{1.2}$, the highest rates of new infections per year per capita.

3.2 Application of Equation 1

We ought also to consider the consequences of colinearity in using our models to make predictions. It is well known that the validity of predictive models worsens as one departs from the training space; all the more so with colinear variables, whose predictive performance is poor even within the training space (Kiers and Smilde, 2007). In addition, in many scenarios we are likely to consider in health-impact modelling, we are particularly interested in mode shifts (Schepers and Heinen, 2013): that is, we are considering transitions that break the linearity present in the construction of the model, increasing one mode and decreasing another, rather than increasing or decreasing both together (Figure 5). We return to these dynamics with a particular focus on density in a simulation study in Section 5.4.

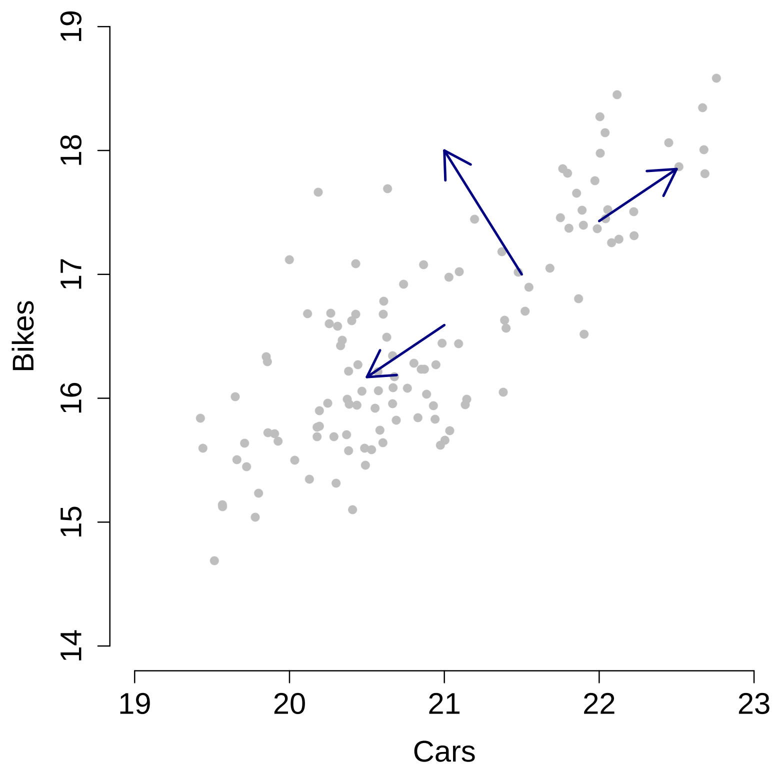

Figure 5: Where we have two colinear predictors, it seems reasonable to make predictions in spaces that align with their colinearity. However, does it make sense to make predictions in spaces orthogonal to the colinearity?

4 Scalability of Equation 1

Equation 1 does not scale (unless $\beta_1 + \beta_2 = 1$). In part, this is due to working in numbers rather than density: the equation does not distinguish between an increase in number due to extended measurement and an increase in number due to there being more road users. The consequence is that each parametrised model is particular to its own setting, which makes the universality of finding a "safety in numbers" effect suspicious. An illustration and an example are given in Sections 4.1 and 4.2. In fact, the absence of accounting for scale might be what gives rise to the "safety in numbers" observation in the first place: see Section 4.3 for illustration and explanation.

4.1 Illustration 1

We simulate a study of observations made over different numbers of years. Starting with Equation 1, we set $\alpha = \exp(-6.879)$, $\beta_1 = 0.591$, and $\beta_2 = 0.32$, and simulate data for a single study. To simulate many studies that vary by duration, we simply multiply the data by the size of the study. We learn the parameters through a regression model corresponding to Equation 1.

Figure 6 shows that, as the size of the study increases, the base rate, α, also increases, even though the data are exactly the same. This identifies the trade-off between the parameters β_i and α: if β_1 and β_2 are fixed, our base rate α will increase as the size of the study increases. My intuition is that, in this trade-off, it is the rate α that should stay constant, while the exponents might vary with study size (Figure 7).

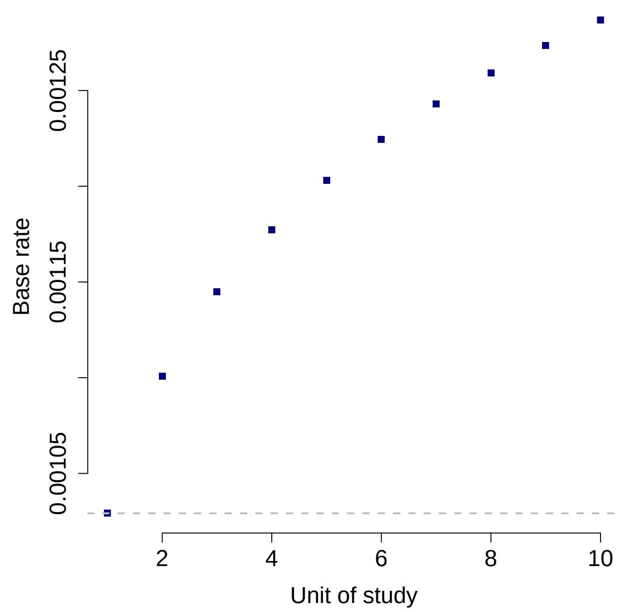

Figure 6: Base rate (α) as the size of a study increases, and safety-in-numbers exponents are fixed. The grey dashed line shows the true base rate.

4.2 Example: England injury data

Our example uses the county-level injury data for England in the years 2005 to 2015. Counties are grouped into nine regions. We apply Equation 1 (a) to county-level data within regions and make predictions for each region, and (b) to county-level data for the whole country and make predictions for the country.

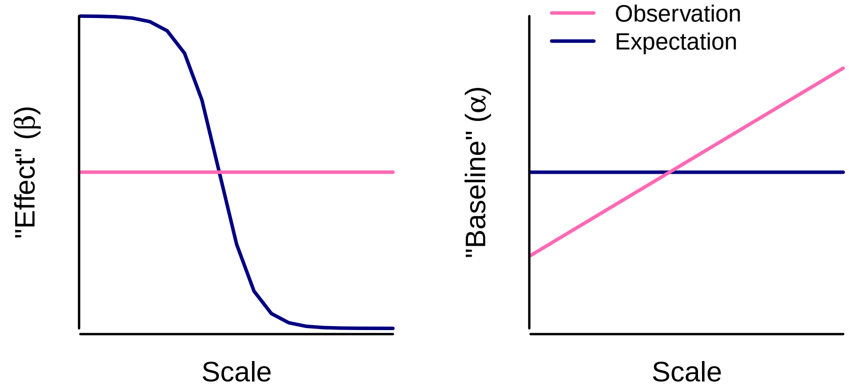

Figure 7: Safety-in-numbers effect (β) and base rate (α) as the size of a study increases. Studies observe the effect to stay more or less constant across scales, while the base rate increases (pink). I would expect instead the effect to diminish across scales and the base rate to stay constant (navy blue).

Figure 8 shows the coefficients we infer for cyclist KSI counts resulting from collisions with other cyclists, motorcyclists, cars, vans, buses and heavy-goods vehicles. Empty circles are region estimates based on the counties they contain. In orange are the country estimates using all counties. In turquoise are the country estimates using the nine regions. Note the trend of clustering towards the line $\beta_1+\beta_2 = 1$.

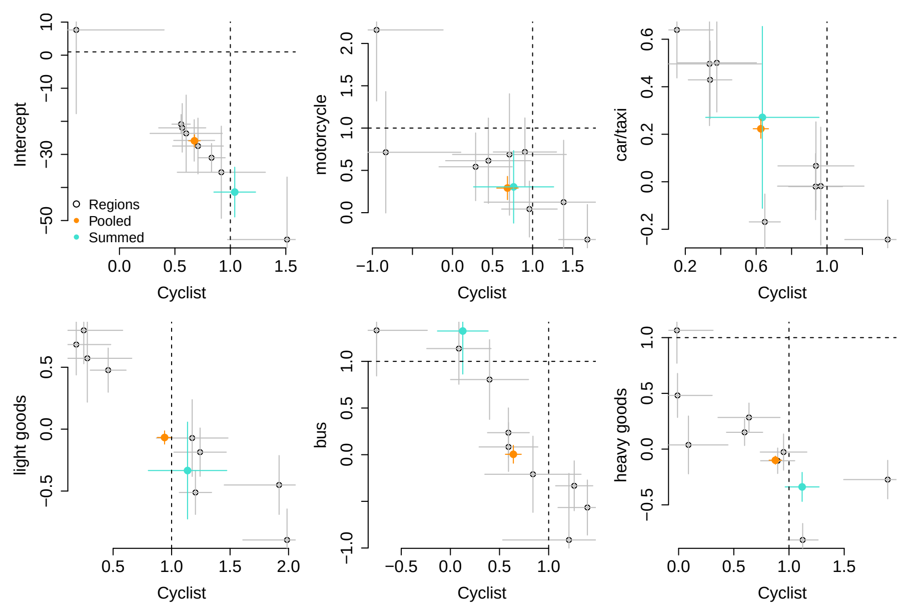

Figure 8: Coefficients we infer for cyclist KSI counts resulting from collisions with other cyclists, motorcyclists, cars, vans, buses and heavy-goods vehicles. Empty circles are region estimates based on the counties they contain. In orange are the country estimates using all counties. In turquoise are the country estimates using the nine regions.

We now use the coefficients to make predictions at higher scales (Figure 9). There is a bias in underestimating mixed-mode injuries and overestimating same-mode injuries. In particular, we observe ~16,000 injuries to cyclists caused by cars in the data, but predict only ~8,000.

The reason for the discrepancy is the deceptive universality of the inferred values for coefficients

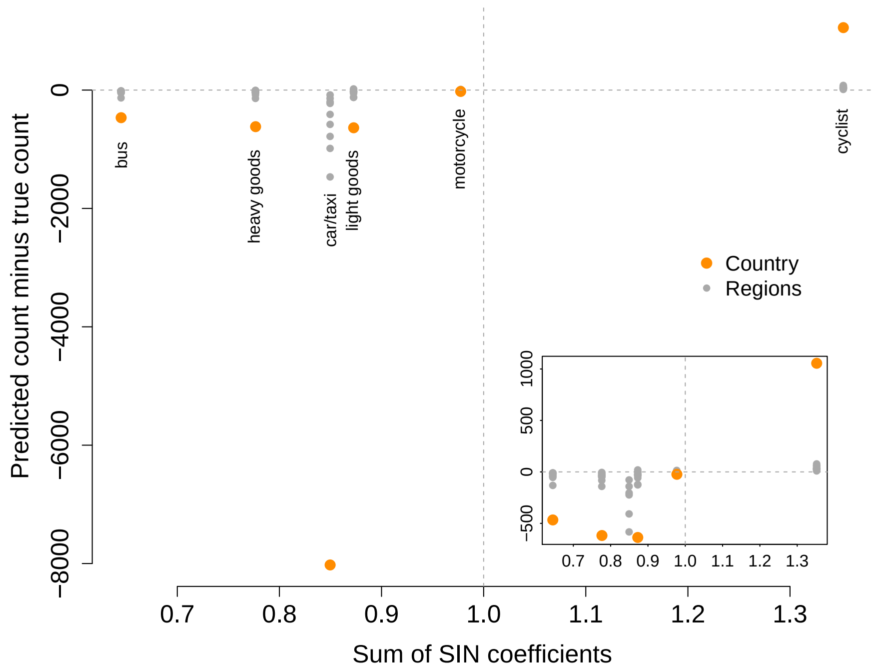

Figure 9: Predictions we make for cyclist KSI counts relative to observed counts, resulting from collisions with other cyclists, motorcyclists, cars, vans, buses and heavy-goods vehicles.

β_1 and β_2. While the persistence of observed values for β_1 and β_2 has been proposed as evidence that "safety in numbers" is a phenomenon, it is, to me, suggestive that it isn't. If the observed effect is independent of scale, i.e. it does not diminish as the scale increases, then we can rule out a "local protective effect" that might be conferred between proximal cyclists. If we extend the scale to include arbitrarily large areas in space and time, we can rule out policy and infrastructure effects. Then the "truth" that is conveyed by consistent discovery of values is not a feature of safety but rather a feature of the data as described by the model.

4.3 Illustration 2: a micro study

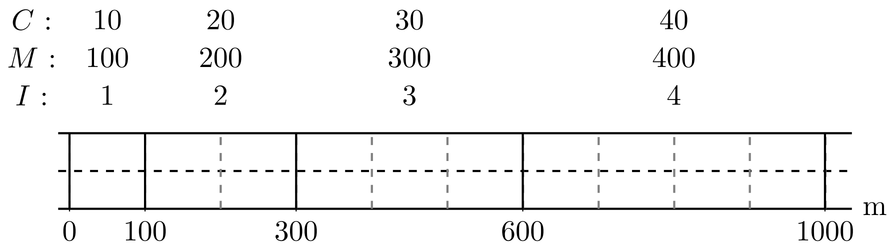

Figure 10: A 1 km stretch of road, divided into four unequal parts. Each 100 m stretch is the same: it counts ten cyclists (C), 100 motorists (M), and one injury (I).

Figure 10 shows a 1 km stretch of road, divided into four unequal parts. Each 100 m stretch is the same: it counts ten cyclists (C), 100 motorists (M), and one injury (I). The unequal segmentation of the road results in a "safety in numbers" effect as observed in regression studies: we have an exact fit to the equation

$$I = \alpha\sqrt{CM}$$

where $\alpha = 0.032$. However, the actual risk is uniform across the stretch of road. Thus, we can induce and infer an observable "safety in numbers" effect just through observing areas of different sizes. This demonstrates that, to understand injury risk as a function of road use, we must consider densities rather than numbers. It also highlights a concern about the use of the words "safety in numbers", if we are able to produce this effect with no actual gradation in safety.

More formally, we start with a single observation $I = \alpha M^{\beta_1} C^{\beta_2}$, and make a second observation identical to the first, yielding $2I = \alpha(2M)^{\beta_1}(2C)^{\beta_2}$. We can solve these equations together to learn about the β values where we have assumed linearity:

$$\frac{2I}{I} = \frac{\alpha(2M)^{\beta_1}(2C)^{\beta_2}}{\alpha M^{\beta_1} C^{\beta_2}}; \tag{3}$$

$$2 = \frac{2^{\beta_1} M^{\beta_1} 2^{\beta_2} C^{\beta_2}}{M^{\beta_1} C^{\beta_2}}; \tag{4}$$

$$= \frac{2^{\beta_1} 2^{\beta_2}}{1}; \tag{5}$$

$$= 2^{\beta_1 + \beta_2} \tag{6}$$

So $\beta_1 + \beta_2 = 1$ when injuries are linear across a study.

5 Application of learnt coefficients in new models

We are particularly interested in applying learning from regression modelling to making predictions. First, we explore the possibility of applying coefficients learnt at smaller scales to city-level prediction models. We describe a general method and discuss issues remaining to be resolved.

5.1 Scaling

How can we use the results of studies such as those documented in Elvik and Bjørnskau (2017) in a meaningful, appropriate way in city-level prediction models? We can define an area of effect from the studies, and extrapolate to the area for which we are predicting. This is in order to be consistent with the assumptions of the studies whose results we apply. The studies implicitly define a sphere of influence in that mode users confer protection to others within their area and not those in other areas. Therefore, in applying these models, it is consistent to employ a similar or the same area as the sphere of influence, and then scale up. Note that to use non-linear coefficients in sizes different from that of the study area defined is a violation of the assumptions inherent in the construction of the regression model.

We define the area of our unit of analysis as A, and our area of application as nA (e.g. n years). We write the total number of motorists as M_n and the total number of cyclists C_n. We approximate the numbers of motorists and cyclists in each unit A as M_n/n and C_n/n respectively. We calculate the number of injuries, $I(n)$, as the "sum" over n identical units:

$$I(n) = \alpha n \left(\frac{M_n}{n}\right)^{\beta_1} \left(\frac{C_n}{n}\right)^{\beta_2}. \tag{7}$$

Equivalently, we can calculate updated exponents as follows:

$$\beta_1' = \frac{(0.5 - \beta_1)\log(n) + \beta_1 \log(M_n)}{\log(M_n)} = \frac{(0.5 - \beta_1)\log(n)}{\log(M_n)} + \beta_1 \tag{8}$$

Then we would have, as in Equation 1,

$$I(n) = \alpha M_n^{\beta_1'} C_n^{\beta_2'}. \tag{9}$$

This correction fixes the error of Figure 9; see Figure 11.

While this allows us to predict injuries for a whole area through consideration of small subunits, it still needs to be extended to model mode shifts and density changes in that larger region. In Section 5.4 we will join this up with the bigger picture.

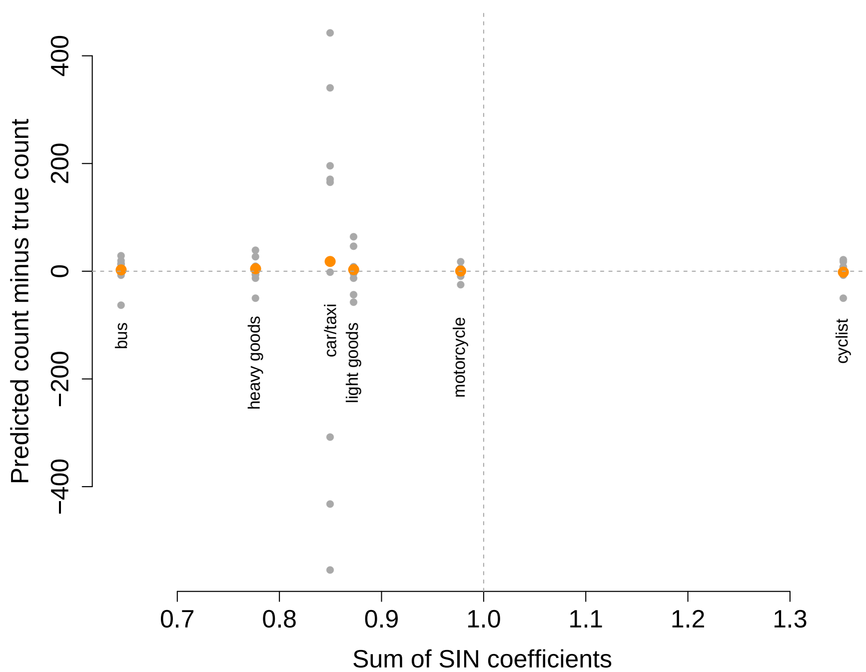

Figure 11: Predictions we make for cyclist KSI counts relative to observed counts, resulting from collisions with other cyclists, motorcyclists, cars, vans, buses and heavy-goods vehicles, as in Figure 9, but with corrected coefficients using Equation 8.

5.2 A remaining problem

However, this model is not robust to arbitrary redefinition of modes, e.g. we fit different models if we consider that both "red buses" and "blue buses" are the same "mode", vs. considering that they are different modes.

Let W be the number of red buses, Y the number of blue buses, and $Z = W + Y$ the total number of buses. For collisions with cycles, using equations of the form of Equation 1, we have

$$\lambda_W = \alpha_W W^{\beta_W} C^{\beta_2}, \quad (10)$$
$$\lambda_Y = \alpha_Y Y^{\beta_Y} C^{\beta_2}, \quad (11)$$
$$\lambda_Z = \alpha_Z Z^{\beta_Z} C^{\beta_2}. \quad (12)$$

Let's assume there are three times the number of red buses as blue buses, assume they have the same rate of causing injury to cyclists, and use some simple numbers to illustrate, given in Table 6.

Table 6: Idealised bus and cyclist injury data.

Observation	C	Buses (Z)	I_Z	Red (W)	I_W	Blue (Y)	I_Y
1	10	80	4	60	3	20	1
2	20	160	8	120	6	40	2
3	30	240	12	180	9	60	3
4	40	320	16	240	12	80	4

As with the illustration in Section 4.3, we can fit these models simply, finding in each case that $\beta_i = 0.5 \, \forall i$. The intercepts, i.e. the base rates, are:

$$\alpha_W = 0.12, \quad (13)$$
$$\alpha_Y = 0.07, \quad (14)$$
$$\alpha_Z = 0.14. \quad (15)$$

It appears that both bus categories are "safer" when considered alone, with blue buses being safer than red buses, despite all bus categorisations posing the same actual danger.

We arrive at the same conclusion by defining $\exists \alpha : \alpha = \alpha_W = \alpha_Y = \alpha_Z$, reflecting our definition that all bus mode categorisations pose the same risk, and testing whether $\lambda_Z = \lambda_W + \lambda_Y$:

$$\lambda_W + \lambda_Y = \alpha_W W^{\beta_W} C^{\beta_2} + \alpha_Y Y^{\beta_Y} C^{\beta_2}, \quad (16)$$
$$= (\alpha_W W^{\beta_W} + \alpha_Y Y^{\beta_Y}) C^{\beta_2}, \quad (17)$$
$$= (\alpha_W W^{0.5} + \alpha_Y Y^{0.5}) C^{0.5}, \quad (18)$$
$$\overset{\exists \alpha}{=} \alpha (W^{0.5} + Y^{0.5}) C^{0.5}, \quad (19)$$
$$\neq \alpha (W + Y)^{0.5} C^{0.5} = \lambda_Z. \quad (20)$$

If using this implementation for health-impact modelling, it is necessary to provide a solution to this curious problem, which suggests that we could improve road safety by painting buses. (The real analogue of this example is the combination and disaggregation of modes in injury modelling. Buses are variously combined with trucks or minibuses; cars and taxis can be one category or two; scooters, motorbikes and three-wheelers can also be considered together or independently.)

5.3 How many modes?

Our predictions are highly sensitive to the number of modes we include in the model. Just as we can arbitrarily redefine buses as "red buses" and "blue buses", we can arbitrarily assume that an injury rate depends on one mode or two. This has profound consequences for our prediction methodology.

To simplify exposition, we assume that injuries modelled as a function of one mode are linear in that mode, and injuries modelled as a function f of two modes are linear in the product of those two modes. More explicitly,

$$I = f(C) = \alpha_c C, \tag{21}$$

$$I = f(M, C) = \alpha_{mc} M^{0.5} C^{0.5}, \tag{22}$$

where we use subscripts c and mc to distinguish the two models. Note that these models are both consistent with the toy example in Table 3.

The problem arises in our predictive model when we predict injuries based on one mode alone, and keep another constant. Suppose we predict the number of injuries to cyclists due to collisions with trucks. We do not change truck distance in our scenario, so a model of the type of Equation 21 seems appropriate, as we are predicting the number of cyclist injuries as a function of cyclists, not of cyclists and trucks. However, they produce very different predictions for the case of doubling cyclists: $2I$ injuries predicted from Equation 21, vs. $\sqrt{2}I$ injuries predicted from Equation 22.

What are the consequences of this insight for predictive models where one mode changes not at all, or only a little? Does this bring us back to the problem discussed in Section 4? If yes, how ought we rewrite e.g. Equation 8 to account for the varying dependence of the function on the mode arguments? We explore some of these questions in the next section (Section 5.4).

5.4 Scaling city-level models

Here we look explicitly at city-level models, that is, studies that consider multiple cities (rather than small localities such as junctions) in order to understand how safety scales as a function of road use. As at other scales, coefficients $\beta_1, \beta_2 < 1$ are understood to correspond to safety in numbers and, again, coefficients learnt in such studies are suggested as parameters to be used in predictions of injury numbers in mode-shift scenarios.

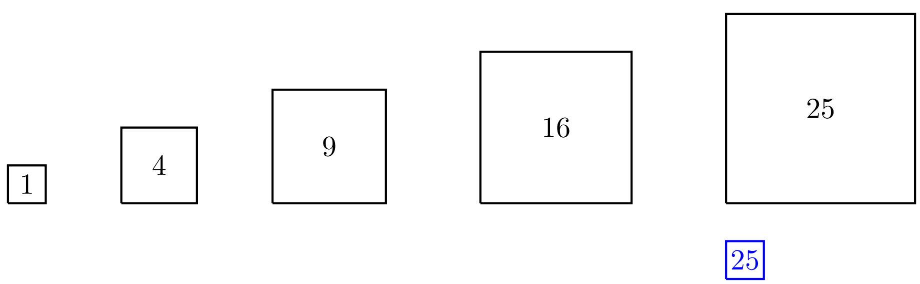

Figure 12: Schematic representation of an inter-city study, demonstrating scaling across cities. If we fit a model using these cities, and then make a prediction for "City 1" with 25 times the travel, then the best estimate we make will correspond to the observation for "City 25" (black). However, when modelling mode-shift scenarios, what we actually want to predict is represented in blue: the total travel of "City 25", taking place in a city of the size of "City 1".

Figure 12 demonstrates the fundamental problem in using cities that vary across scales to parametrise models intended for use in predicting for a city that will change in number but not scale, i.e. it will change in density. In the remainder of the section we develop the ideas summarised in Figure 12 through a simulation study.

5.4.1 Simulation model

We create a simulator with three variables: the number of cyclists, the number of motorists, and the dimension of the space, which we equate to the size of the city. Each cyclist and each motorist is a five-pixel by five-pixel square. At each time step each body moves one body-width. We simulate 500 time steps. Each body undertakes a biased random walk: that is, with probability 5/6 it continues in the same direction, and with probability 1/6 it chooses a direction randomly from among the directions available to it (including the same direction). Moves that would take a body out of the frame wrap around to the other side of the frame. The frame size is some multiple of the step size that we vary as an input parameter.

We count the number of collisions between cyclists and motorists, which is defined as any overlap in pixels between a cyclist and a motorist. In the style of PacMan, upon collision the cyclist disappears and reappears randomly in the next time step. The cyclist disappears immediately, so a collision involving two motorists and one cyclist will be counted as one event.

5.4.2 Simulated study

In order to emulate inter-city regression studies, we simulate 50 times frames of different sizes and constant density. We define our vector of sizes to be the integers $x = \{5, ..., 14\}$. Then the numbers of cyclists and motorists are Poisson-distributed random variables with mean x^2, and the dimensions of the frames are $20\times5x$ pixels by $20\times5x$ pixels. There are 500 simulations in total: ten sizes with fifty repetitions of each. The results of the simulations are summarised in Figure 13. We fit a Poisson regression model to the resulting number of collisions to learn the parameters β_1 and β_2, finding $\beta_1 \approx \beta_2 \approx 0.5$ as expected. We then use (a) the model to make new predictions and (b) the simulator to test them.

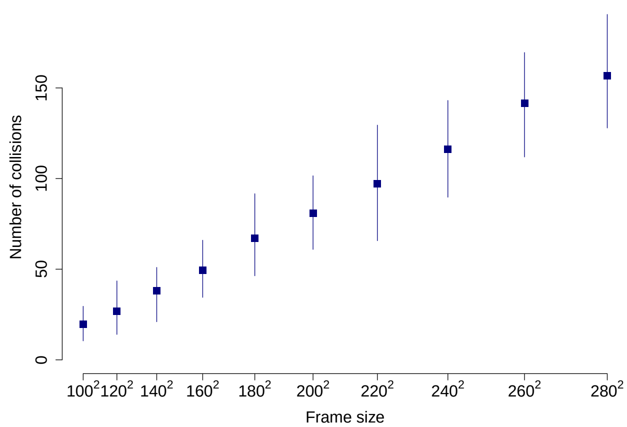

Figure 13: Simulation study. The number of collisions is a function of frame size, number of cyclists and number of motorists. Each bar shows 90% of the range of 50 simulations. In every simulation the density of cyclists and the density of motorists is the same.

5.4.3 Comparison of model predictions to simulations

First, we consider every frame size with 100 motorists and 100 cyclists. This was one of our inputs into the regression model data: frame size 200^2. Because our model is a function of road-user numbers alone, we make the same prediction for every frame size: 80 collisions. In Figure 14 we plot this

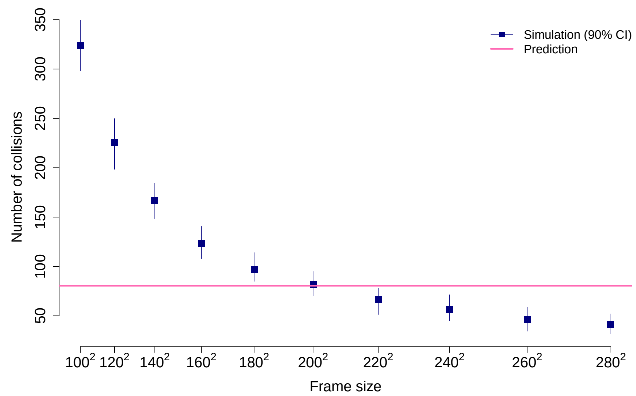

Figure 14: Simulated vs. predicted results for 100 motorists and 100 cyclists with varying frame sizes. The predicted value is 80 for all frame sizes.

against the simulated number over 50 repetitions for each frame size. Note that only for the frame size corresponding to mode numbers of 100 does the 90% confidence range overlap the predicted value.

Second, using the same set up, we consider that the number of motorists is constant at 100 in the corresponding frame size of 200^2, and we vary the number of cyclists. Again, we predict the number of collisions using our model, which this time varies with cyclist number. Note in Figure 15 that the prediction aligns with the simulation when the cyclist value corresponds to the frame size and density of the predictive model. When this number is exceeded, the estimate is too low, and at a lower density, the prediction is too high. We should keep this picture in mind when we consider e.g. cyclist injuries when cyclist numbers are changing but the other mode, such as truck, is not.

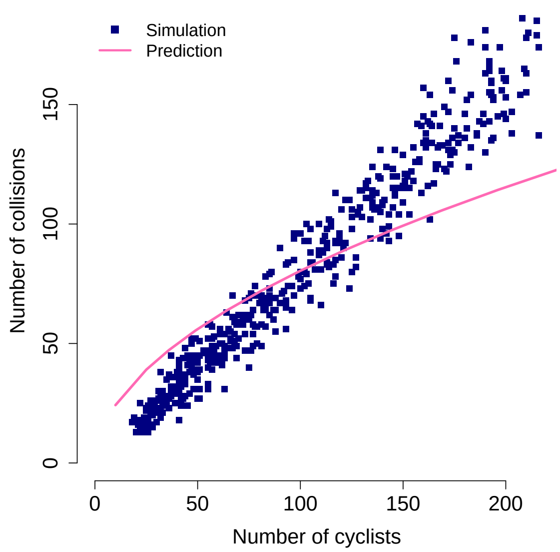

Figure 15: We fix motorists at 100 and frame size at 200^2, and vary the number of cyclists as Poisson random variables with means from 25 to 196. We simulate and predict using the model the number of collisions. This represents the type of error we might make if we predict for one mode that varies when another stays constant.

Third, we consider that the number of cyclists is constant at 150 in a frame size of 200^2, and we vary the number of motorists. Again, we predict the number of collisions using our model, which this time varies with motorist number. Note in Figure 16 that the prediction aligns with the simulation when the motorist value corresponds to the difference between the motorist value corresponding to the frame size and density of the predictive model, and the change in cyclist number from its corresponding value. (Precisely: the errors cancel out when the number of new motorists = the number of old motorists $\times$ the number of old cyclists / the number of new cyclists = $100 \times 100/150 = 66.7$.) When this number is exceeded, the estimate is too low, and at a lower density, the prediction is too high. We should keep this picture in mind when we consider e.g. that we have six modes each with 100 road users, and we predict for a scenario in which cyclists increase to 150 and all other modes decrease to 90.

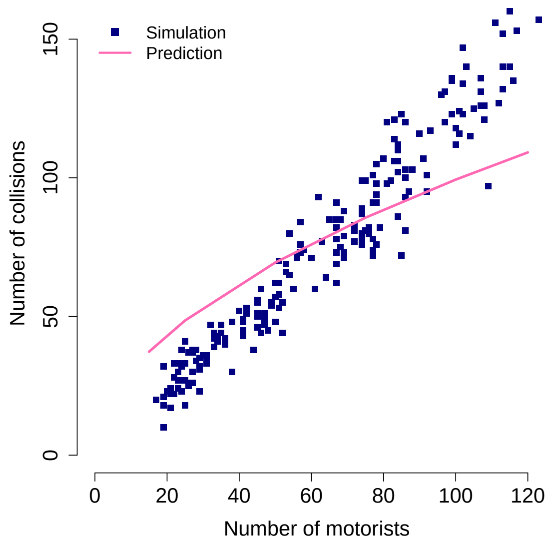

Figure 16: We fix cyclists at 150 and frame size at 200^2, and vary the number of motorists as Poisson random variables with means from 25 to 100. We simulate and predict using the model the number of collisions. This represents the type of error we might make if we predict for one mode that varies as a result of reallocation of other mode types. Note the deviation between model and prediction, which is minimised close to the point of constant density, i.e. that the increase in cyclists is close to the decrease in motorists.

5.4.4 Simulated safety in numbers

We repeat the simulation study, this time imposing a safety-in-numbers effect. We specify that the probability of collision is represented by some number $p = p(C)$, which is a function of the number of cyclists C. We recreate Figure 13 with $p = C^{-0.25}$. In terms of "safety in numbers", this corresponds to a raw exponent of 0.75. The probability p doesn't depend on M, so its exponent is 1.

We see the effect of cyclist number on collision number in Figure 17. Note the non-linear gradient, compared to Figure 13. For these data, we infer for a model of the form of Equation 1 scaling exponents $\beta_1 + \beta_2 \approx 0.75$, which is equal to the sum of the raw exponents minus 1.

Next, we predict the number of collisions for this model with doubled numbers of cyclists and motorists and varying frame sizes. The results are shown in Figure 18 along with the prediction from the the scaling exponents inferred from Figure 17.

Concatenating data in Figures 17 and 18, we fit a model with scaling exponents $\beta_1 + \beta_2 \approx 1$, recovering the tiling parameters. When we control for frame size (which eliminates confounding), we isolate the density effect: we will refer to these density exponents as δ_1 and δ_2, analogous to β_1 and β_2. We find that $\delta_1 + \delta_2 \approx 1.78$, close to the original exponents.

We repeat the whole process for $p = C^{-0.5}$, i.e. exponents 0.5 and 1. Analogously for Figure 17 we find $\beta_1 + \beta_2 \approx 0.5$, the sum of the original exponents minus one. Again we simulate data for doubled

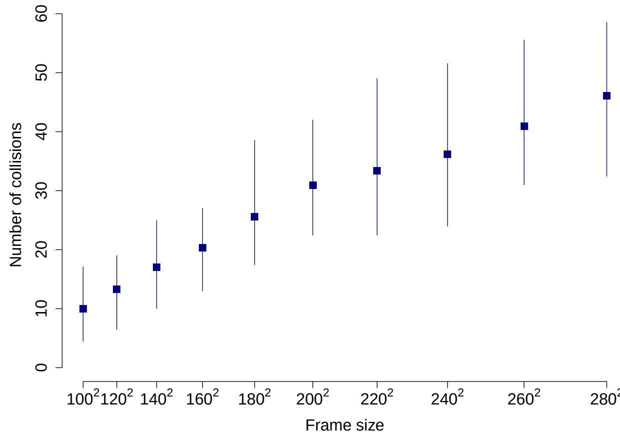

Figure 17: Simulation study. The number of collisions is a function of frame size, number of cyclists and number of motorists. Each bar shows 90% of the range of 50 simulations. In every simulation the density of cyclists and the density of motorists is the same. Collisions are a function of the number of cyclists and occur with probability $p = C^{-0.25}$.

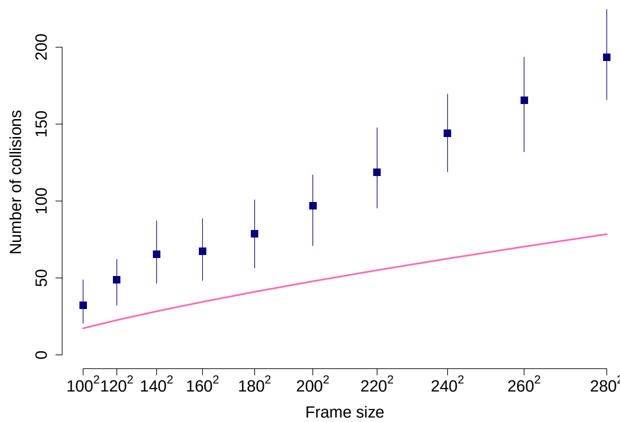

Figure 18: As in Figure 17, with double the number of motorists and cyclists. The prediction generated from data in Figure 17 is shown in pink.

cyclists and motorists which, concatenated to the original data, yield scaling exponents $\beta_1 + \beta_2 \approx 1$ and density exponents $\delta_1 + \delta_2 \approx 1.5$ when size is accounted for.

We are trying to build up a picture of how Equation 1 behaves across scales and densities. It seems that the predictive equation we are aiming for requires density exponents; that studies likely return scaling exponents; and that, for two modes, the sum of the density exponents is one more than the sum of the scaling exponents: $\delta_1 + \delta_2 \approx \beta_1 + \beta_2 + 1$.

5.4.5 Discussion

We do not model in order to suggest that we have understood and can recapitulate the mechanism. Rather, we use a simple simulation model in order to test simple assumptions and infer simple principles, for example the relationship between size and density. If we can make a general statement about what happens in this simple simulation, in which we have full control of the workings, we have the opportunity to illuminate possible factors influencing observational inferences such as those in safety-in-numbers studies.

There are some key differences between our simulation study and real data: we have no spatial or temporal factors, and no relationship between e.g. density and speed. One respect in which our simulation differs from the England study of Section 4.2 is that we consider here constant density across scales, whereas the regions in England decline in density as size increases. In contrast, in general, it is posited that, globally, as city size increases, so does density (see schematic of the model space, Figure 19). However, we could tailor our simulation to match in some way a phenomenon we are interested to capture.[2]

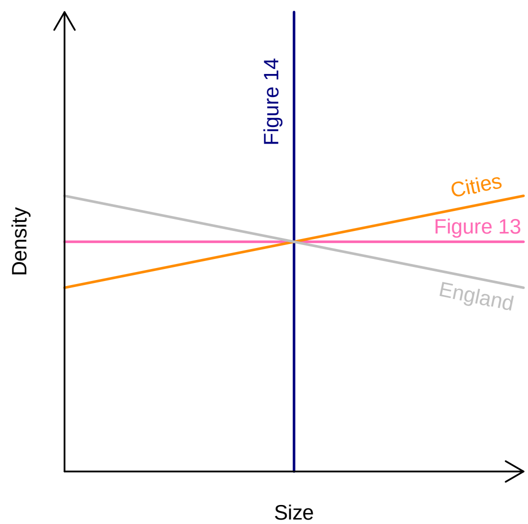

Figure 19: A depiction of the model space covered by the simulation model. We mark out in pink and navy blue a line of constant density and a line of constant size through the model space, depicted in Figures 13 and 14, respectively. The study of areas in England might be represented in this model as occupying the space of increasing size and decreasing density (grey), and expect that a study of many cities will occupy the space of increasing size and increasing density (orange). (NB: these are schematic.)

The point here is not how much our simulation set up resemble what one imagines happens in cities. The point is that we can contrive data from which we can learn coefficients that fit the general trend $\beta_1 \approx \beta_2 \approx 0.5$, and we can test this model against data simulated from the same original source. This means we can test the range of applicability of the model.

5.4.6 Conclusion

The conclusion of this test is that a predictive model that does not take into account factors of scale fails to predict outside its training space. Further, we can identify the direction of the bias: uncaptured increases in density lead to underprediction of collisions, and uncaptured decreases in density lead to overprediction. This calls into question the assumption that density-independent regression models can be used to predict the number of collisions, or injuries, that will occur in mode-shift scenarios.

A heuristic for correction is offered: for our simulation, scaling exponents of 0.5 corresponded to density exponents of 1, with two modes simulated. A starting point for application of inter-city studies is then the translation of a scaling exponent of β_i to a density exponent of $\delta_i = \beta_i + 0.5$, where the number of modes we model is two. This would correct the errors seen in Figures 14, 15 and 16.

[2] For example, we could introduce density-dependent speeds. Concretely, in a fixed space, doubling cyclists and doubling motorists increases density by a factor of four. If we then divide every mode speed by that value - four - we undo exactly the effect of the increased density.

Recall that in Section 5.1, we found that, for two modes (and an estimated n units of space) $\lim_{n\to\infty} \beta_i = 0.5$, which corresponds to the size scaling we have just explored. Then our full pipeline, from small-scale study to exponents for injuries as a function of distance, where density is taken into account, would be:

1. Identify small-scale exponents
2. Calculate city-level scaling exponents using Equation 8
3. Use city-level scaling exponents to calculate city-level density exponents by adding a total of 1 to the parameters, e.g. 0.5 to each.

Then our null hypotheses are, for models of the form of Equation 1, that $\beta_1 = \beta_2 = 0.5$ for constant density and varying scale, and $\delta_1 = \delta_2 = 1$ for constant scale and varying density. These models are mutually consistent, and they are complementary presentations of the simulation model.

We can formally write this as:

$$I_n = \frac{\alpha}{n}(nM)^{\delta_1}(nC)^{\delta_2} \tag{23}$$

where n is the number of spatial units. For the baseline, we would use $n = 1$. Then, in a scenario, we can distinguish between a doubling of density,

$$\begin{aligned} I_n^{\text{(double density)}} &= \frac{\alpha}{n}(2nM)^{\delta_1}(2nC)^{\delta_2} && (24)\\ &= 2^{\delta_1+\delta_2}\frac{\alpha}{n}(nM)^{\delta_1}(nC)^{\delta_2} && (25)\\ &= 2^{\delta_1+\delta_2} I_n, && (26) \end{aligned}$$

and a doubling of size:

$$\begin{aligned} I_n^{\text{(double size)}} &= \frac{\alpha}{2n}(2nM)^{\delta_1}(2nC)^{\delta_2} && (27)\\ &= 2^{\delta_1+\delta_2-1}\frac{\alpha}{n}(nM)^{\delta_1}(nC)^{\delta_2} && (28)\\ &= 2^{\delta_1+\delta_2-1} I_n. && (29) \end{aligned}$$

With this formulation we can also derive the relationship with β_1 and β_2, assuming observations for a size of 1 and a size of n. From Equation 23 we have

$$\begin{aligned} \frac{I_n}{I_1} &= \frac{\frac{\alpha}{n}(nM)^{\delta_1}(nC)^{\delta_2}}{\alpha M^{\delta_1} C^{\delta_2}} && (30)\\ &= n^{\delta_1+\delta_2-1}. && (31) \end{aligned}$$

We can derive the equivalent relation using Equation 1:

$$\begin{aligned} \frac{I_n}{I_1} &= \frac{\alpha(nM)^{\beta_1}(nC)^{\beta_2}}{\alpha M^{\delta_1} C^{\delta_2}} && (32)\\ &= n^{\beta_1+\beta_2}. && (33) \end{aligned}$$

Thus we see that for these two models to be consistent, we require $\beta_1 + \beta_2 + 1 = \delta_1 + \delta_2$.

We propose the model of Equation 23 as a model that is consistent with our observations and our insights. There will be other models that also fulfill those criteria. There will be other models that contain ours within them as a subset or a special case. An example would be an extension that takes account also of speed. Whatever model we use, however, ought to be consistent with the observations made so far, and this is one such model.

In terms of application, we don't know how to add 1 to β_1 and β_2. We could simply define $\delta_i = \beta_i + 0.5$, which seems a reasonable starting point. However, inference of β_1 and β_2 is likely confounded by space and/or time, so we might instead choose $\delta_1 = \delta_2 = (\beta_1 + \beta_2 + 1)/2$. This is an equitable solution. We might like to test it, particularly for cases where one mode vastly outnumbers the other.

Finally, the problem identified in Section 5.3 remains for $\delta_1 \neq 1$ where δ_1 is applied to a mode that is arbitrarily redefined. A quick fix would be to define $\delta_1 = 1$ for this mode, or group of modes, and $\delta_2 = \beta_1 + \beta_2$. Of course this won't help when both modes are arbitrarily defined.

6 Investigating safety in numbers

What does this mean for safety in numbers? There exist a few lines of enquiry we could follow, but I suspect that none will lead to a formulation as appealing or convenient as Equation 1. A regression in the form of Equation 1 will only be relevant for the setting in which it was fit, and any attempt to relate parameters across settings should include consideration of both the scales of the studies and the other parameters inferred (e.g. the intercept α).

6.1 Recommendations for studies

Studies investigating group-mode-use effects will need to take account of density in order to correctly specify the null hypothesis to test (Figure 19). In assessing goodness of fit, testing should be done on per person quantities, rather than at the level of the population (Shalizi, 2011). The implications of the inference should be tested (e.g. when we split or combine modes (see Section 5.2); when we break linearity; when we try to apply it beyond the scope of its inference (e.g. Section 4.2)).

We could aim to engage a wider audience and share data in order to find alternative ways to understand and formulate the problem. This includes statistical modellers who have previously considered quantities claimed to exhibit power-law scaling properties (Leitão et al., 2016), and mechanistic epidemic modellers such as those who consider discretisation of space (and time) in a "contact matrix" that describes interactions across partitions (Birrell et al., 2011).

6.2 Power-law scaling relationships

The relationship described by Equation 1 closely resembles the family of power-law scaling models in e.g. Bettencourt et al. (2007). We could introduce this field of study to that family, describing safety in numbers as another relationship showing a power-law scaling relationship, meriting further investigation, for which solutions are occasionally proposed (Sim et al., 2015; Shalizi, 2011). This would be a very different way of approaching the problem, though there are some insights that might be useful in and of themselves.

Following the publication of Bettencourt et al. (2007) and other, similar work, some useful suggestions were made for studies such as these. Particularly relevant for our application is the requirement to fit models to per capita data (see Figure 20), if that is the quantity of interest to us (Shalizi, 2011). In the same work and in others (Leitão et al., 2016) we can find methods for investigating these types of data – perhaps there exists a statistical model already that would work for us. Also presented are various methods for robust specification and testing of null hypotheses, and discussion around competing models and in which circumstances one might conclude that "scaling is nonlinear" (Leitão et al., 2016).

Here are some questions, specifically with that field in mind:

(1) Is there a power-law scaling occurring in road injuries? Considering numbers alone (if we want to consider numbers at all), it seems to depend on exactly what we measure and exactly how we model it, e.g. injuries vs. KSI vs. fatalities (see e.g. Table 2). What does that mean for our interpretation? Study of no-other-vehicle casualties might be relevant here (Figure 1 vs. Figure 21).

(2) Within a power-law scaling model, can we elicit the contributions of different modes? Could we apply a latent variable model, and learn the effects of fluctuations of mode types on the outcome?

(3) Can we identify local (protective) effects?

(4) If yes, can we design a mechanism that links local effects to the power-law scaling model, i.e. explains the deviance of $\beta_1 + \beta_2$ from 1 as a function of system size?

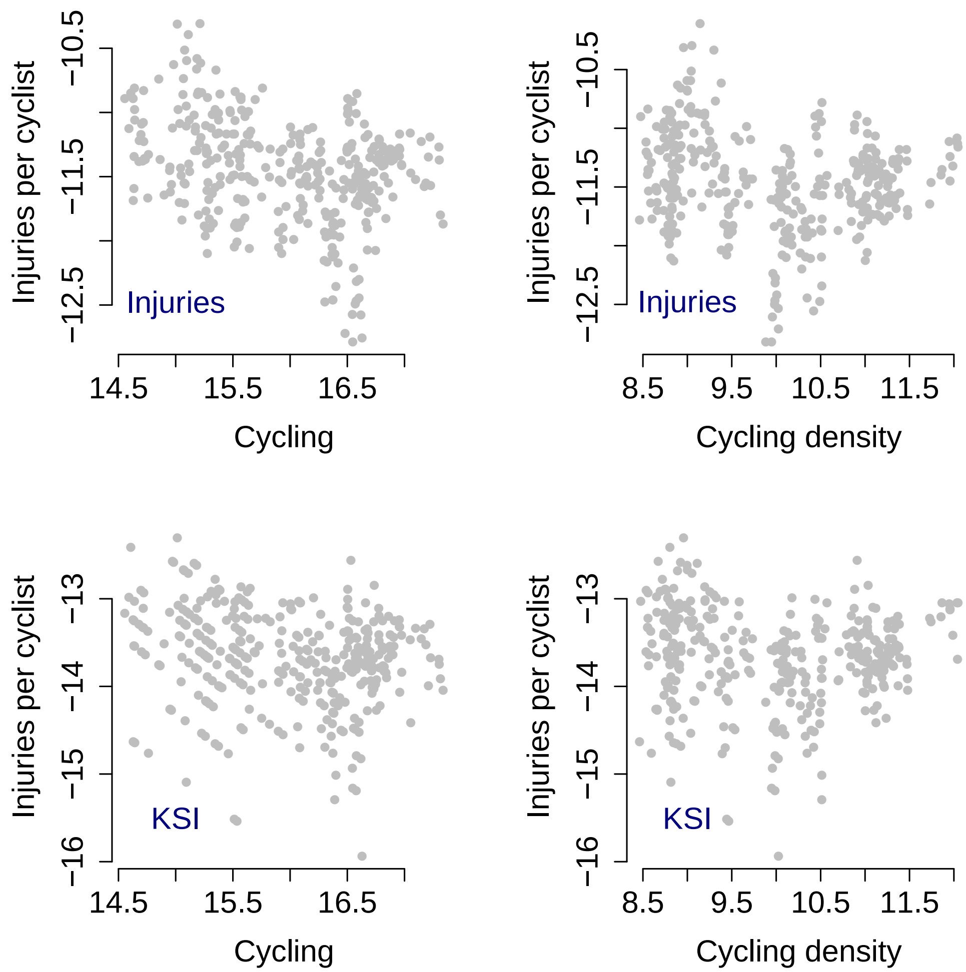

Figure 20: London injuries per cyclist distance, as a function of cyclist distance (left) and cyclist density (right). Cyclist density is calculated as total distance travelled divided by total distance of A, B and minor roads.

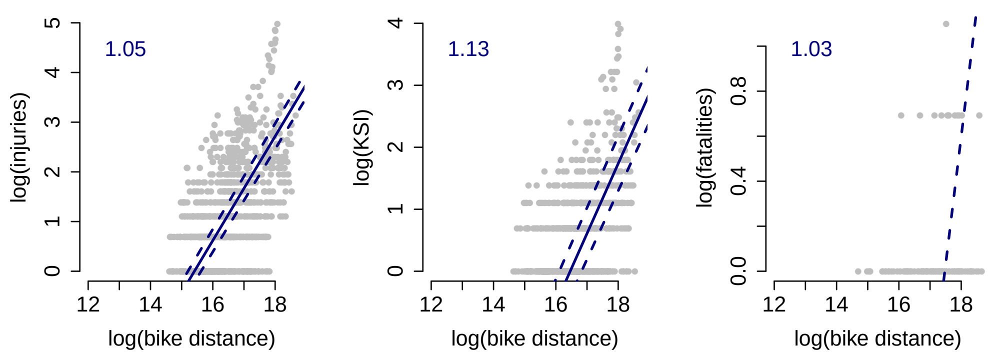

Figure 21: As in Figure 1, counting only injuries that occur in incidents involving no other vehicle (NOV).

7 Conclusions

In conclusion, for studies that scale across sizes, $\beta_1 + \beta_2 = 1$ represents linearity in numbers. That the same (or similar) coefficients are observed across scales is perhaps the biggest indication that whatever is being captured is not an effect of cyclists conferring protection to other nearby cyclists. Further, we don't know how the predictions from models built from colinear variables might behave when we depart from the training space.

While it might exist, "safety in numbers" is not what is being measured in studies that include multiple sites that differ in size. It would be more useful for predictive models, and perhaps more meaningful, to aim to identify "safety in density". This will have particular challenges: first, quantifying density. This will include consideration of time and space in at least one dimension; road width and infrastructure might also characterise density in some way. Following this, it would be instructive to explore the effects of heterogeneity in density, and to introduce another dimension to the model: speed.

Complementing data-driven analyses, simulation models can be used to develop relational models. These can test implications of comprehensive mechanistic models of injuries as a function of space and its occupancy. For now, simulation and theory provide us with a number of null hypotheses, which (a) give us an objective to test and reject, and (b) provide a justified basis for prediction. These hypotheses, in reference to Equation 1, are:

(1) We estimate the city-scaling exponent from small-scale exponents by scaling β coefficients in such a way that we approximate the process of "multiplying then summing" rather than "summing then multiplying" (Section 5.1).

(2) City-level size-scaling exponents $\beta_1 = \beta_2 = 0.5$ correspond to linearity (Section 4.3).

(3) City-level size-scaling exponents $\beta_1 = \beta_2 = 0.5$ correspond to city-level density-scaling exponents $\delta_1 = \delta_2 = 1$, where mode speeds are assumed independent of density.

(4) To predict the consequence of a change in density using city-level size-scaling exponent β, we can use density exponents $\delta_1 + \delta_2 = 1 + \beta_1 + \beta_2$, where there are two modes involved (Section 5.4.4).

A Worked example: implementation in ITHIM-R

We consider the setting of Accra, for which we have a list of recorded fatalities over multiple years. Each record contains the following information: the year, the mode of the casualty, the mode of the other party, the age of the casualty, and the gender of the casualty. In addition, we have a travel survey, from which we learn total travel by each mode (and by demographic group, which we omit for now, for simplicity).

A.1 Constructing the model

We fit the observed data (the number of injuries, I) to an equation of the form

$$I \sim \text{Poisson}(\lambda), \tag{34}$$

$$\lambda = \alpha M^{\beta_1} C^{\beta_2} \exp\left(\sum_{i=3}^{n} X_i \beta_i\right) \tag{35}$$

with α a fixed intercept, C and M the distances travelled by cyclists and cars, respectively, based on the travel survey, and X the model matrix built from all the covariates (here, we consider only the two modes; gender and age of the casualty are omitted for simplicity). We do not use the "year" covariate but instead suppose that we have multiple observations for a single "year" (i.e. we reuse the distance data). Finally, the coefficients to fit using `glm` are α and β_i for $i \geq 2$, and we supply β_1 and β_2 as fixed parameters so that $M^{\beta_1} C^{\beta_2}$ is our offset.

Note that there are many combinations of modes, so this model is linked via the model matrix X to the number of pedestrian casualties in collisions with buses, etc. The contingency table of injury counts between all mode pairings forms the "who hit whom" matrix for the city.

A.2 Making predictions

We use the same model equation to make predictions in hypothesised scenarios. The prediction equation requires us to specify the distances travelled in the scenario: call them $\hat{M}$ and $\hat{C}$. Then we predict the expected number of injuries in the scenario, $\hat{I}$, as:

$$\hat{I} = \alpha \hat{M}^{\beta_1} \hat{C}^{\beta_2} \exp\left(\sum_{i=3}^{n} X_i \beta_i\right). \tag{36}$$

To aid interpretation, we can consider the ratio of the expected injuries in the scenario to the expected injuries in the baseline:

$$\frac{\hat{I}}{\mathbb{E}(I)} = \frac{\hat{M}^{\beta_1} \hat{C}^{\beta_2}}{M^{\beta_1} C^{\beta_2}} \tag{37}$$

$$= \left(\frac{\hat{M}}{M}\right)^{\beta_1} \left(\frac{\hat{C}}{C}\right)^{\beta_2}. \tag{38}$$

Then we can immediately read out, for example, that if M does not change ($\hat{M} = M$) then the fold change in injuries is equal to the fold change in cycling raised to the power β_2: if $\beta_2 = 1$, then if cycling increases 25 times, so does the injury count. If $\beta_2 = 0.5$, then if cycling increases 25 times, the injury count increases five times.

A.3 β_1 and β_2 parameters

The question we need to answer is, given that we are using this model, what values should we choose for β_1 and β_2? This choice will impact on the other parameters to fit (α and β_i for $i \geq 2$) and, crucially, on the number of injuries we predict in scenarios.

Recall that there are multiple casualty modes and multiple "other party" modes, including NOV (no other vehicle). Another question we need to answer is how these values should differ for different modes, in particular (a) where a mode's distance is not changing at all (or even very little) in scenarios, (b) where a mode is a combination of multiple modes, and (c) where there is no other mode.

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