

Seasonal functional autoregressive models

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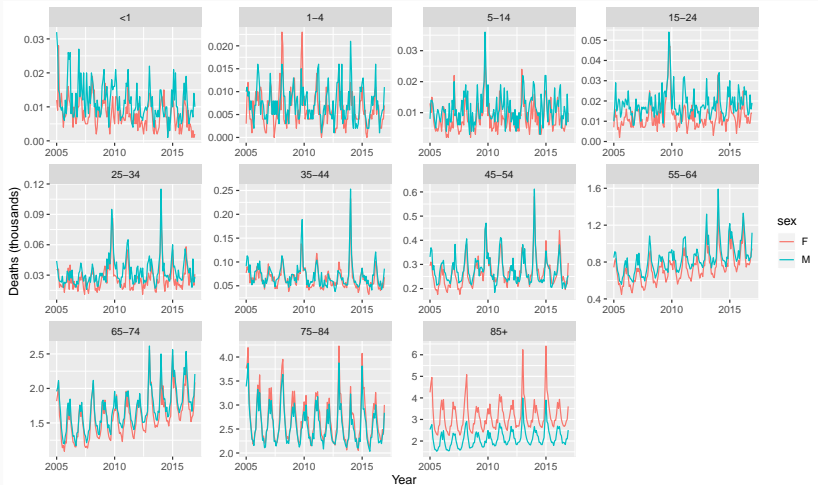
Outline

- 1 Motivation
- 2 SFAR(1)_S processes
- 3 Estimation
- 4 Forecasting
- 5 SFAR(P)_S processes

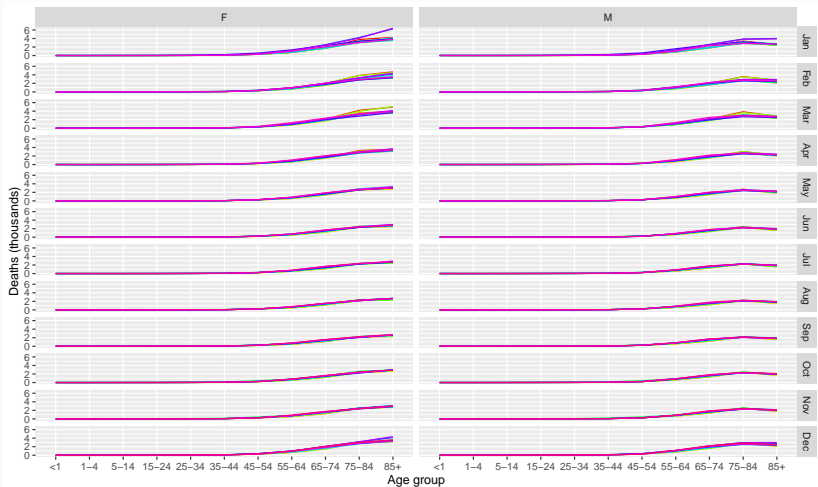
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Flu/respiratory mortality (US)



Flu/respiratory mortality (US)



Examples

Notation

$X_t(u)$ where $t = 1, \dots, T$ indexes regularly spaced time and u is a continuous variable in \mathbb{R} or \mathbb{R}^2

- 1 $X_t(u)$ = mortality rate for people aged u in month t .
- 2 $X_t(u)$ = vegetation index at location u in month t , measured by average satellite observations.

Sometimes u may denote a second time variable.

- 3 $X_t(u)$ = pollution level observed every 30 minutes. u denotes time-of-day, t denotes day.

Seasonality

Notation

$X_t(u)$ where $t = 1, \dots, T$ indexes regularly spaced time and u is a continuous variable in \mathbb{R} or \mathbb{R}^2

Seasonality occurs when $X_t(u)$ is influenced by seasonal factors (e.g., the quarter of the year, the month, the day of the week, etc.).

A (possibly de-trended) series is seasonal of period S if

$$E[X_t(u)] = E[X_{t+S}(u)].$$

Functional autoregression

FAR(p) processes – introduced by Bosq (2000)

$$X_t = \phi_1(X_{t-1}) + \cdots + \phi_p(X_{t-p}) + \varepsilon_t,$$

- $\{\varepsilon_t\}$ is a functional H -white noise process
- ϕ_j are Hilbert-Schmidt integral operators in $\mathcal{L}(H)$
- $\phi_p \neq 0$
- Some stationarity conditions on ϕ_1, \dots, ϕ_p .

$H =$ separable real Hilbert space of square integrable functions.

$\mathcal{L}(H) =$ space of continuous linear operators from H to H .

Seasonal univariate autoregression

AR(p) processes

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + \cdots + \phi_p Y_{t-p} + \varepsilon_t.$$

- $\{\varepsilon_t\}$ is a real white noise process.
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Seasonal univariate autoregression

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SAR(P)_S processes

$$Y_t = \Phi_1 Y_{t-S} + \Phi_2 Y_{t-2S} + \cdots + \Phi_P Y_{t-PS} + \varepsilon_t.$$

- $\{\varepsilon_t\}$ is a real white noise process.
- $\Phi_P \neq 0$
- Some stationarity conditions on Φ_1, \dots, Φ_P

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SFAR(1)_S processes

$$X_t = \Phi(X_{t-s}) + \varepsilon_t,$$

- $\{\varepsilon_t\}$ is a functional H -white noise process
- Φ is Hilbert-Schmidt integral operator in $\mathcal{L}(H)$
- $\Phi \neq 0$

SFAR(1)_S processes

$$X_t = \Phi(X_{t-S}) + \varepsilon_t,$$

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Stationarity of a SFAR(1)_S process

If there exists an integer $j_0 \geq 1$ such that $\|\Phi^{j_0}\|_{\mathcal{L}} < 1$,
then $X_t = \Phi(X_{t-S}) + \varepsilon_t$,

has a unique stationary solution given by

$$X_t = \sum_{j=0}^{\infty} \Phi^j(\varepsilon_{t-jS}),$$

where the series converges in L_H^2 with probability 1.

SFAR(1)_S processes

$$X_t = \Phi(X_{t-s}) + \varepsilon_t,$$

- $\{\varepsilon_t\}$ is a functional H -white noise process
- Φ is Hilbert-Schmidt integral operator in $\mathcal{L}(H)$
- $\Phi \neq 0$

Link to SAR(1)_S processes

Let

$$\Phi = \sum_{j=1} \alpha_j e_j \otimes e_j$$

be a symmetric compact operator on H .

Then, X_t is a SFAR(1)_S process if and only if $\langle X_t, e_k \rangle$ is a SAR(1)_S process.

SFAR(1)_S processes

Let $\mathbf{Y}_t = (X_t, \dots, X_{t-S+1})'$ and $\boldsymbol{\varepsilon}_t = (\varepsilon_t, 0, \dots, 0)'$.

Define operator ρ on H^S :

$$\rho = \begin{bmatrix} 0 & 0 & \dots & 0 & \Phi \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{bmatrix}, \quad \text{where } I \text{ is identity operator.}$$

Lemma

If \mathbf{X} is SFAR(1)_S process, associated with (ε, Φ) , then \mathbf{Y} is FAR(1) with values in product Hilbert space H^S associated with (ε, ρ) , i.e., $\mathbf{Y}_n = \rho \mathbf{Y}_{n-1} + \boldsymbol{\varepsilon}_n$.

SFAR(1)_S processes

Theorem

Let $\mathcal{L}(H^S)$ be space of bounded linear operators on H^S equipped with norm $\|\cdot\|_{\mathcal{L}^S}$. If

$$\|\rho^{j_0}\|_{\mathcal{L}^S} < 1, \quad \text{for some } j_0 \geq 1,$$

then SFAR(1)_S has unique stationary solution given by

$$X_t = \sum_{j=0}^{\infty} (\pi \rho)(\varepsilon_{t-j}),$$

where the series converges in $L^2_{H^S}$ with probability 1 and π is the projector of H^S onto H , defined as $\pi(x_1, \dots, x_S) = x_1, \quad (x_1, \dots, x_S) \in H^S$.

Limit Theorems for SFAR(1)_S processes

Theorem: Law of large numbers for X

If X is a standard SFAR(1)_S then, as $T \rightarrow \infty$,

$$\frac{T^{0.25}}{(\log T)^\beta} \frac{(X_1 + X_2 + \dots + X_T)}{T} \rightarrow 0, \quad \text{for } \beta > 0.5.$$

Central Limit Theorem

Let X be a standard SFAR(1)_S associated with a strong white noise ε and such that $I - \Phi$ is invertible. Then

$$\frac{(X_1 + X_2 + \dots + X_T)}{\sqrt{T}} \rightarrow \mathcal{N}(0, \Gamma),$$

where $\Gamma = (I - \Phi)^{-1} C_\varepsilon (I - \Phi^*)^{-1}$.

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Method of Moments

Covariance operator

$$C_k^X = E(X_t \otimes X_{t-k})$$

$$\hat{C}_k^X = \frac{1}{T} \sum_{t=k+1}^T X_t \otimes X_{t-k}$$

Eigendecomposition

For any $x \in H$,

$$\Phi(X) = \sum_{j=1}^{\infty} \langle x, \nu_j \rangle \Phi(\nu_j) = \sum_{j=1}^{\infty} \frac{C_S^X(\nu_j)}{\lambda_j} \langle x, \nu_j \rangle$$

■ Estimate λ_j and ν_j from \hat{C}_0^X .

■ $\hat{\Phi}(X) = \sum_{j=1}^J \frac{\hat{C}_S^X(\hat{\nu}_j)}{\hat{\lambda}_j} \langle x, \hat{\nu}_j \rangle$

Yule-Walker estimates

For $\text{SFAR}(P)_S$ processes,

$$C_0 = \Phi C_{-S} + C_\varepsilon$$

$$C_h = \Phi C_{h-S}, \quad h \geq 1.$$

- Replace C_j with empirical counterpart to estimate Φ and C_ε .

Unconditional Least Squares

Let $X_{tk} = \langle X_n, \nu_k \rangle$ be projection of t th observation onto k th largest FPC.

$$X_{nk} = \sum_{j=1}^p \Phi_{kj} X_{n-S,j} + \delta_{nk}, \quad k = 1, \dots, p$$

where $\Phi_{kj} = \langle \Phi(\nu_j), \nu_k \rangle$. Note that the δ_{nk} are not iid.

- Replace ν_k by $\hat{\nu}_k$ to get \hat{X}_{tk}

Unconditional Least Squares

Set $\mathbf{X}_n = (\hat{X}_{n1}, \dots, \hat{X}_{np})'$, $\boldsymbol{\delta}_n = (\delta_{n1}, \dots, \delta_{np})'$,

$\boldsymbol{\Phi} = (\Phi_{11}, \dots, \Phi_{1p}, \Phi_{21}, \dots, \Phi_{2p}, \dots, \Phi_{p1}, \dots, \Phi_{pp})'$.

$$\mathbf{Z}_n = \begin{bmatrix} \mathbf{X}'_n & \mathbf{0}'_p & \dots & \mathbf{0}'_p \\ \mathbf{0}'_p & \mathbf{X}'_n & \dots & \mathbf{0}'_p \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{0}'_p & \mathbf{0}'_p & \dots & \mathbf{X}'_n \end{bmatrix}, \quad \mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \\ \vdots \\ \mathbf{X}_N \end{bmatrix}, \quad \boldsymbol{\delta} = \begin{bmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{bmatrix}, \quad \mathbf{Z} = \begin{bmatrix} \mathbf{Z}_{1-S} \\ \mathbf{Z}_{2-S} \\ \vdots \\ \mathbf{Z}_{N-S} \end{bmatrix}$$

$$\hat{\boldsymbol{\Phi}} = (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'\mathbf{X}$$

The Kargin-Onatski Method

Find A , approximating Φ , minimizing $E\|X_t - A(X_{t-s})\|^2$.

Let $\hat{C}_\alpha = \hat{C}_0 + \alpha I$, $\alpha > 0$

Let $\{v_{\alpha,i}\}$ be eigenfunctions of $\hat{C}_\alpha^{-1/2} \hat{C}'_S \hat{C}_S \hat{C}_\alpha^{-1/2}$,
corresponding to eigenvalues $\{\hat{u}_{\alpha,i}\}$, $\hat{u}_{\alpha,j} > \hat{u}_{\alpha,j+1}$.

$$\hat{\Phi}_{\alpha,k_n} = \sum_{i=1}^{k_n} \hat{C}_\alpha^{-1/2} v_{\alpha,i} \otimes \hat{C}_S \hat{C}_\alpha^{-1/2} v_{\alpha,i}.$$

$\hat{\Phi}_{\alpha,k_n}$ is a consistent estimator of Φ if $\{k_n\}$ is sequence of positive integers such that $Kn^{-1/4} \leq k_n \leq n$, for some $K > 0$ and $\alpha \sim n^{-1/6}$.

Simulations

Let $\{X_t\}$ follow a SFAR(1) $_S$ model,

$$X_t(u) = \Phi X_{t-S}(u) + \varepsilon_t(u), \quad t = 1, \dots, T,$$

where Φ is an integral operator with *parabolic* kernel

$$k_\Phi(u, v) = \gamma_0 \left(2 - (2u - 1)^2 - (2v - 1)^2 \right),$$

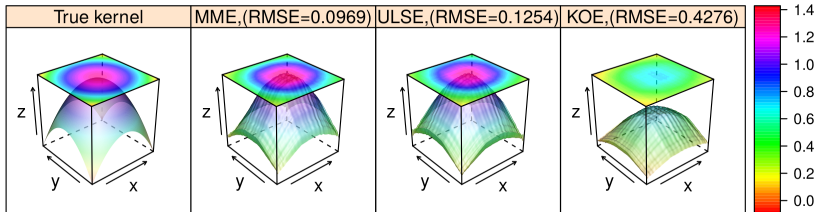
and γ_0 is such that $\|\Phi\|_S^2 = \int_0^1 \int_0^1 |k_\Phi(u, v)|^2 du dv = 0.9$.

- White noise terms $\varepsilon_t(u)$ are independent standard BM on $[0, 1]$ with variance 0.05.
- $B = 1000$ trajectories simulated
- Φ estimated using MME, ULSE and KOE.

- $$\text{RMSE} = \sqrt{\frac{1}{B} \sum_{i=1}^B \|\hat{\Phi}_i - \Phi\|_S^2}$$

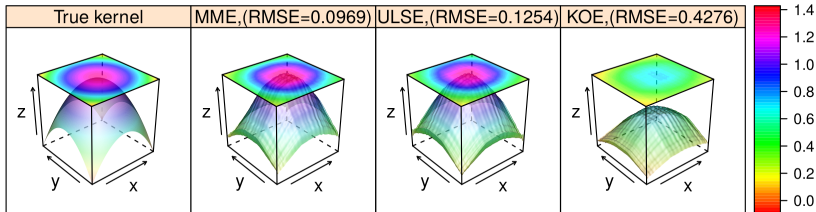
Simulations

$T = 200$, $\|\Phi\|_{\mathcal{S}} = 0.9$ and $k_n = 1$.

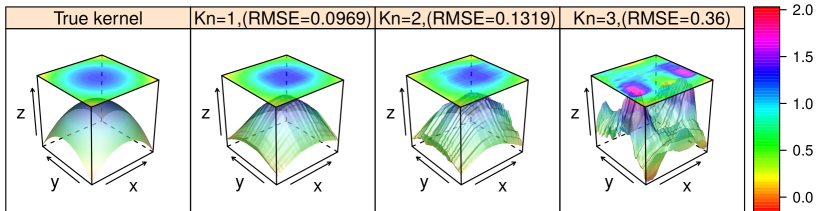


Simulations

$T = 200$, $\|\Phi\|_{\mathcal{S}} = 0.9$ and $k_n = 1$.



$T = 200$, $\|\Phi\|_{\mathcal{S}} = 0.9$ and $k_n = 1, 2, 3$. MME only



Simulations: RMSE

n	k_n	$\ \Phi\ _{\mathcal{S}} = 0.1$			$\ \Phi\ _{\mathcal{S}} = 0.5$			$\ \Phi\ _{\mathcal{S}} = 0.9$		
		MME	ULSE	KOE	MME	ULSE	KOE	MME	ULSE	KOE
50	1	0.1750	0.1645	0.0951	0.2403	0.2838	0.3716	0.1986	0.2323	0.5096
	2	0.5484	0.5189	0.0959	0.4931	0.7000	0.3720	0.4387	1.0381	0.5099
	3	1.0239	0.9657	0.0961	0.9988	1.1478	0.3721	1.0435	1.7282	0.5099
	4	1.5573	1.4934	0.0962	1.5340	1.6513	0.3721	1.6382	2.4725	0.5099
100	1	0.1222	0.1183	0.0861	0.2050	0.2579	0.3539	0.1387	0.1709	0.4134
	2	0.3662	0.3598	0.0866	0.3325	0.6087	0.3541	0.2743	0.9728	0.4136
	3	0.6830	0.6798	0.0868	0.6661	0.8723	0.3541	0.6694	1.3925	0.4136
	4	1.0645	1.0243	0.0868	1.0245	1.1973	0.3542	1.0193	1.9377	0.4136
150	1	0.1033	0.1027	0.0825	0.1946	0.2505	0.3460	0.1205	0.1517	0.3735
	2	0.2903	0.2900	0.0830	0.2666	0.5704	0.3462	0.2149	0.9478	0.3737
	3	0.5533	0.5449	0.0831	0.5387	0.7601	0.3462	0.5272	1.2493	0.3736
	4	0.8560	0.8237	0.0831	0.8256	1.0040	0.3462	0.8106	1.6683	0.3736
200	1	0.0917	0.0935	0.0798	0.1879	0.2457	0.3393	0.1114	0.1419	0.3496
	2	0.2490	0.2610	0.0803	0.2285	0.5568	0.3394	0.1818	0.9411	0.3497
	3	0.4790	0.4745	0.0804	0.4684	0.7047	0.3394	0.4542	1.1896	0.3497
	4	0.7438	0.7127	0.0804	0.7199	0.9042	0.3394	0.7040	1.5134	0.3497

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Forecasting

Let $\mathbf{X}_T = (X_1, X_2, \dots, X_T)'$.

Let G be closure of $\{\ell_0 \mathbf{X}_T; \ell_0 \in \mathcal{L}(H^T, H)\}$.

Best linear h -step predictor of X_{T+h} is projection of X_{T+h} on G , i.e., $\hat{X}_{T+h} = P_G X_{T+h}$.

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Proposition (based on Bosq 2014)

For $h \in \mathbb{N}$ the following statements are equivalent:

- 1 There exists $\ell_0 \in \mathcal{L}(H^T, H)$ such that $\mathbf{C}_{\mathbf{X}_T, X_{T+h}} = \ell_0 \mathbf{C}_{\mathbf{X}_T}$.
- 2 $P_G X_{T+h} = \ell_0 \mathbf{X}_T$ for some $\ell_0 \in \mathcal{L}(H^T, H)$.

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- 1 There exists $\ell_0 \in \mathcal{L}(H^T, H)$ such that $C_{\mathbf{X}_T, X_{T+h}} = \ell_0 C_{\mathbf{X}_T}$.
- 2 $P_G X_{T+h} = \ell_0 \mathbf{X}_T$ for some $\ell_0 \in \mathcal{L}(H^T, H)$.

How to find $\ell_0 \in \mathcal{L}(H^T, H)$ such that $C_{\mathbf{X}_T, X_{T+h}} = \ell_0 C_{\mathbf{X}_T}$?

Forecasting

Forecast horizon $h = aS + c$, $a \geq 0$ and $0 \leq c < S$.

$$C_{X_n, X_{n+h}}(\mathbf{x}) = E \left(\langle \mathbf{X}_n, \mathbf{x} \rangle_{H^n} X_{n+h} \right) = \Phi_{n-S+c}^{a+1} C_{X_n}(\mathbf{x}),$$

where Φ_j^i is an n -vector of zeros with Φ^i in j th position.

$$\hat{X}_{n+h} = P_G X_{n+h} = \Phi_{n-S+c}^{a+1} \mathbf{X}_n = \Phi^{a+1} X_{n-S+c}$$

Based on KOE, 1-step ahead predictor of X_{n+1} is:

$$\hat{X}_{n+1} = \sum_{i=1}^{k_n} \langle X_{n-S+1}, \hat{z}_{\alpha,i} \rangle \hat{C}_S(\hat{z}_{\alpha,i}),$$

where
$$\hat{z}_{\alpha,i} = \sum_{j=1}^q \hat{u}_j^{-1/2} \langle v_{\alpha,i}, \hat{v}_j \rangle \hat{v}_j + \alpha v_{\alpha,i}.$$

Select q by cumulative variance method and set $k_n = q$.

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SFAR(P) $_S$ processes

Definition

A sequence $\{X_t; t \in \mathbb{Z}\}$ of functional random variables is said to be a seasonal functional autoregressive process of order P with seasonality S if

$$X_t - \mu = \Phi_1 (X_{t-S} - \mu) + \cdots + \Phi_P (X_{t-PS} - \mu) + \varepsilon_t,$$

where $\{\varepsilon_t, t \in \mathbb{Z}\}$ is H -white noise, $\mu \in H$, and $\Phi_1, \dots, \Phi_P \in \mathcal{L}(H)$, with $\Phi_P \neq 0$.

SFAR(P) _{S} processes

Let $Y_t = (X_t, X_{t-S}, \dots, X_{t-PS+S})'$, $\varepsilon'_t = (\varepsilon_t, 0, \dots, 0)'$,
where 0 appears $P - 1$ times, and

$$\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \dots & \Phi_P \\ I & 0 & \dots & 0 \\ 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \quad (1)$$

where I and 0 denote identity and zero operator on H .

Lemma

If X is a SFAR(P) _{S} associated with associated with
($\varepsilon, \phi_1, \dots, \phi_P$), then Y is a SFAR(1) _{S} with values in the
product Hilbert space H^P associated with $(\varepsilon, \phi_1, \dots, \phi_P)$

SFAR(P)_S processes

Theorem

Let X_n be a SFAR(P)_S zero-mean process associated with $(\varepsilon, \phi_1, \phi_2, \dots, \phi_P)$. Suppose that there exist $\nu \in H$ and $\alpha_1, \dots, \alpha_P \in \mathbb{R}$, $\alpha_P \neq 0$, such that $\phi_j(\nu) = \alpha_j \nu_j$, $j = 1, \dots, P$ and $E \langle \varepsilon_0, \nu \rangle^2 > 0$. Then, $(\langle X_n, \nu \rangle, n \in \mathbb{Z})$ is a SAR(P) process, i.e.,

$$\langle X_n, \nu \rangle = \sum_{j=1}^P \alpha_j \langle X_{n-jS}, \nu \rangle + \langle \varepsilon_n, \nu \rangle, \quad n \in \mathbb{Z}. \quad (2)$$

SFAR(P)_S processes

Theorem

If X is a standard SFAR(P)_S process, then

$$C_h = \sum_{j=1}^P \phi_j C_{h-jS}, \quad h = 1, 2, \dots, \quad (3)$$

$$C_0 = \sum_{j=1}^P \phi_j C_{jS} + C_\varepsilon, \quad (4)$$

where C_ε is the covariance operator of the innovation process ε .