**Probabilistic Forecast** Reconciliation **For Emergency Services Demand** 

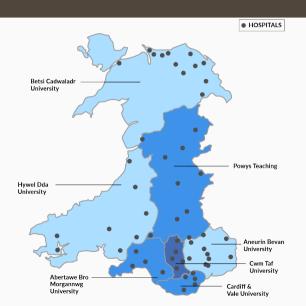
Rob J Hyndman Bahman Rostami-Tabar

MONASH University



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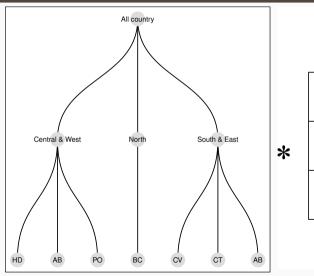
### **Wales Health Board Areas**

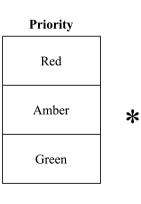


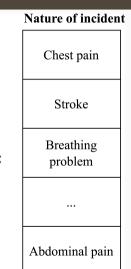
#### **Data**

- Daily number of attended incidents:
  - 1 October 2015 31 July 2019
- Disaggregated by:
  - control area
  - health board
  - priority
  - nature of incidents
- 2,142,000 rows observations from 1,530 time series.

### **Data structure**







### **Data structure**

Level	Number of series
All country	1
Control	3
Health board	7
Priority	3
Priority * Control	9
Priority * Health board	21
Nature of incident	35
Nature of incident * Control	105
Nature of incident * Health board	245
Priority * Nature of incident	104
Control * Priority * Nature of incident	306
Control * Health board * Priority * Nature of incident (Bottom level)	691
Total	1530

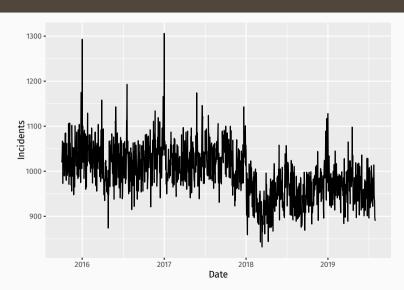
#### **Data**

```
# A tsibble: 2,142,000 x 6 [1D]
             region, category, nature, lhb [1,530]
# Kev:
   date
              region
                           category
                                        nature
                                                      lhb
                                                                   incident
                                                                      <dbl>
   <date>
              <chr*>
                           <chr*>
                                        <chr*>
                                                      <chr*>
 1 2015-10-01 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1020
 2 2015-10-02 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1021
 3 2015-10-03 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1025
 4 2015-10-04 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1043
 5 2015-10-05 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1067
                                                                       1063
 6 2015-10-06 <aggregated> <aggregated> <aggregated> <aggregated>
 7 2015-10-07 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                        973
 8 2015-10-08 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1057
 9 2015-10-09 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1026
10 2015-10-10 <aggregated> <aggregated> <aggregated> <aggregated>
                                                                       1063
# i 2,141,990 more rows
```

#### **Data**

```
# A tsibble: 2,142,000 x 6 [1D]
# Key:
           region, category, nature, lhb [1,530]
            region category nature lhb
  date
                                          incident
  <date> <chr*> <chr*> <chr*> <chr*>
                                                 <dbl>
1 2015-10-01 C
                  Amber ABDOMINAL HD
                                                    0
2 2015-10-01 C
                 Amber ABDOMINAL PO
                                                    0
3 2015-10-01 C Amber ABDOMINAL SB
                                                    0
4 2015-10-01 C Amber
                          ABDOMINAL <aggregated>
                                                    0
5 2015-10-01 C
                 Amber
                          ALLERGIES HD
                                                    0
6 2015-10-01 C
                 Amber ALLERGIES PO
7 2015-10-01 C
                 Amber
                          ALLERGIES SB
                                                    0
8 2015-10-01 C
                 Amber
                          ALLERGIES <aggregated>
9 2015-10-01 C
                 Amber
                          ANTMALBTT HD
10 2015-10-01 C
             Amber
                          ANIMALBIT PO
                                                     0
# i 2,141,990 more rows
```

# Aggregated daily incidents

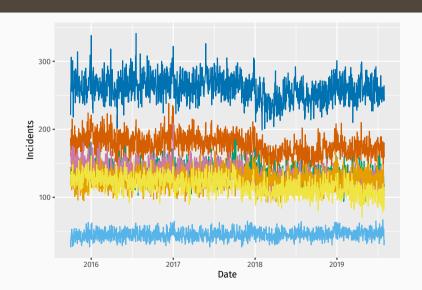


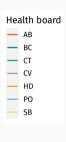
# Daily incidents by control area





# Data incidents by health board





# Data incidents by priority

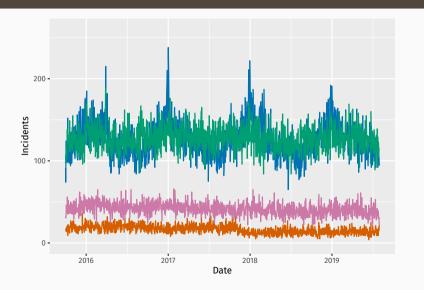




## Data incidents by nature of incident

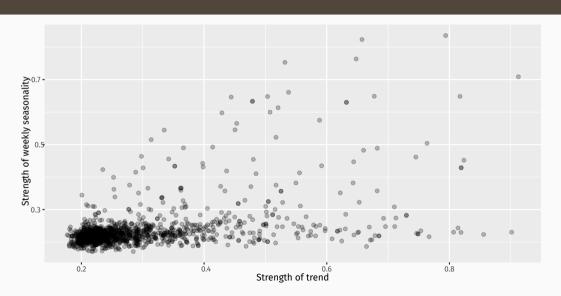


## Data incidents by nature of incident





## **Data features**



- **Naïve**: Empirical distribution of past daily attended incidents.
- **ETS**: Exponential Smoothing State Space models.
- **GLM**: Poission Regression with spline trend, day of the week, annual Fourier seasonality, public holidays, school holidays, Christmas Day, New Year's Day.
- **TSGLM**: Poisson Regression with same covariates plus three autoregressive terms.
- **Ensemble**: Mixture distribution of 1–4.

**Naïve**: Empirical distribution of past daily attended incidents.

$$y_{T+h|T} \sim \text{Empirical}(y_1, \dots, y_T)$$

**Naïve**: Empirical distribution of past daily attended incidents.

$$y_{T+h|T} \sim \text{Empirical}(y_1, \dots, y_T)$$

**ETS**: Exponential Smoothing State Space models.

$$y_{T+h|T} \sim \text{Normal}(\hat{y}_{T+h|T}, \hat{\sigma}_{T+h|T}^2)$$

**GLM**: Poission Regression

$$y_{T+h|T} \sim \text{Poisson}(\hat{y}_{T+h|T})$$

where

$$\hat{y}_{T+h|T} = \exp(\mathbf{x}'_{T+h}\beta)$$

and  $\mathbf{x}_{T+h}$  is a vector of covariates including

- spline trend
- day of the week
- annual Fourier seasonality

- public holidays
- school holidays
- Christmas Day
- New Year's Day

```
Estimate Std. Error z value Pr(>|z|)
(Intercept)
                6.998511
                           0.017412
                                     401.93
                                             < 2e-16 ***
Spline_1
                0.027859
                           0.004740
                                       5.88
                                             4.2e-09 ***
Spline 2
               -0.088244
                           0.006394
                                     -13.80 < 2e-16 ***
Spline 3
               -0.075036
                           0.004784
                                     -15.68 < 2e-16 ***
Spline 4
               -0.111854
                           0.010202
                                     -10.96
                                             < 2e-16 ***
Spline 5
                           0.004462
                                             < 2e-16 ***
               -0.043009
                                      -9.64
Monday
                0.019147
                           0.003174
                                      6.03
                                             1.6e-09 ***
Tuesday
               -0.016414
                           0.003180
                                      -5.16
                                             2.4e-07 ***
Wednesday
               -0.015479
                           0.003184
                                      -4.86
                                             1.2e-06 ***
Thursday
               -0.006804
                           0.003178
                                      -2.14
                                             0.03230 *
Friday
                0.012235
                           0.003156
                                       3.88
                                             0.00011 ***
Saturday
                0.005293
                           0.003165
                                       1.67
                                             0.09438 .
Fourier_S1_365
                0.005365
                           0.001294
                                       4.15
                                             3.4e-05 ***
Fourier_C1_365 0.008263
                           0.001263
                                       6.54
                                             6.1e-11 ***
Fourier S2 365
                0.004235
                           0.001271
                                       3.33
                                             0.00086 ***
Fourier_C2_365 -0.010510
                           0.001216
                                      -8.64
                                             < 2e-16 ***
Fourier_S3_365 -0.000556
                           0.001275
                                      -0.44
                                             0.66303
Fourier_C3_365
                0.002650
                           0.001243
                                             0.03294 *
                                       2.13
Public holidav
                0.033278
                           0.005697
                                       5.84
                                             5.2e-09 ***
School_holiday
                0.004857
                           0.002346
                                       2.07
                                             0.03843 *
Xmas
               -0.051902
                           0.016772
                                      -3.09
                                             0.00197 **
                0.120385
                           0.015573
                                       7.73
                                             1.1e-14 ***
New years day
```

#### Significance

\*\*\* p < 0.001

\*\* p < 0.01\* p < 0.05

p < 0.0

**TSGLM**: Poisson Regression

$$y_{T+h|T} \sim \mathsf{Poisson}(\hat{y}_{T+h|T})$$

where

$$\hat{y}_{T+h|T} = \exp\left(\mathbf{x}'_{T+h}\boldsymbol{\beta} + \sum_{k=1}^{3} \alpha_k \log(y_{T+h-k} + 1)\right)$$

and  $\mathbf{x}_{T+h}$  is a vector of covariates including

- spline trend
- day of the week
- annual Fourier seasonality

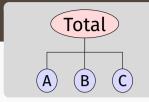
- public holidays
- school holidays
- Christmas Day
- New Year's Day

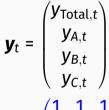
### **Notation**

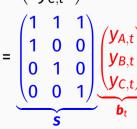
Every collection of time series with linear constraints can be written as

$$y_t = \mathbf{Sb_t}$$

- $\mathbf{y}_t$  = vector of all series at time t
- $y_{Total,t}$  = aggregate of all series at time t.
- $y_{X,t}$  = value of series X at time t.
- **b**<sub>t</sub> = vector of most disaggregated series at time t
- S = "summing matrix" containing the linear constraints.





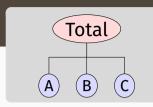


### **Notation**

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- Base forecasts:  $\hat{\mathbf{y}}_{T+h|T}$
- Reconciled forecasts:  $\tilde{y}_{T+h|T} = SG\hat{y}_{T+h|T}$ 
  - MinT:

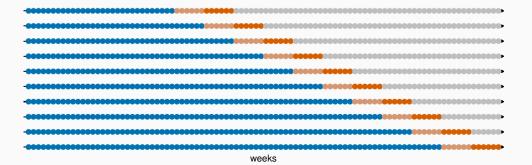
$$G = (S'W_h^{-1}S)^{-1}S'W_h^{-1}$$
  
where  $W_h$  is  
covariance matrix of  
base forecast errors.



## Nonparametric bootstrap reconciliation

- Fit model to all series and store the residuals as  $\varepsilon_t$ .
- These should be serially uncorrelated but cross-sectionally correlated.
- Draw iid samples from  $\varepsilon_1, \ldots, \varepsilon_T$  with replacement.
- Simulate future sample paths for model using the bootstrapped residuals.
- Reconcile each sample path using MinT.
- Combine the reconciled sample paths to form a mixture distribution at each forecast horizon.

- Ten-fold time series cross-validation
- Forecast horizon of 1–84 days
- Each training set contains an additional 42 days.
- Forecasts at 43–84 days correspond to planning horizon.



$$\mathsf{MASE} = \mathsf{mean}(|q_j|)$$

$$q_j = \frac{e_j}{\frac{1}{T - m} \sum_{t=m+1}^{T} |y_t - y_{t-m}|}$$

- $y_t$  = observation for period t
- $e_j$  = forecast error for forecast horizon j
- T = size of training set
- = m = 7

$$MSSE = mean(q_j^2)$$

$$q_j^2 = \frac{e_j^2}{\frac{1}{T-m} \sum_{t=m+1}^{T} (y_t - y_{t-m})^2}$$

- $y_t$  = observation for period t
- $e_j$  = forecast error for forecast horizon j
- T = size of training set
- = m = 7

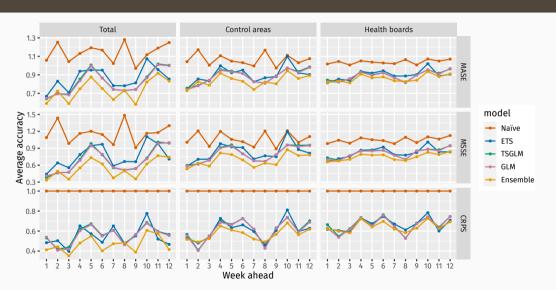


CRPS = mean
$$(p_j)$$

$$p_j = \int_{-\infty}^{\infty} \left( G_j(x) - F_j(x) \right)^2 dx,$$

- $G_i(x)$  = forecast distribution for forecast horizon j
- $F_j(x)$  = true distribution for same period

### **Forecast accuracy**



# Forecast accuracy: 43-84 days ahead

		MSSE			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	1.169	1.056	1.062	1.031
Base	ETS	0.979	0.875	0.816	0.975
Base	GLM	0.813	0.897	0.875	1.009
Base	TSGLM	0.822	0.901	0.875	1.050
Base	Ensemble	0.599	0.729	0.774	0.993
MinT	Naïve	1.168	1.057	1.062	2.095
MinT	ETS	0.785	0.852	0.845	0.994
MinT	GLM	0.720	0.827	0.837	1.803
MinT	TSGLM	0.722	0.833	0.839	1.851
MinT	Ensemble	0.560	0.706	0.765	1.557

# Forecast accuracy: 43-84 days ahead

		MASE			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	1.139	1.059	1.047	1.019
Base	ETS	0.963	0.930	0.899	1.038
Base	GLM	0.910	0.940	0.923	1.002
Base	TSGLM	0.911	0.939	0.924	1.005
Base	Ensemble	0.782	0.856	0.876	1.008
MinT	Naïve	1.138	1.059	1.047	2.651
MinT	ETS	0.877	0.916	0.915	1.289
MinT	GLM	0.848	0.901	0.902	2.493
MinT	TSGLM	0.852	0.903	0.903	2.513
MinT	Ensemble	0.753	0.844	0.872	2.260

# Forecast accuracy: 43-84 days ahead

		CRPS			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	30.387	10.882	5.500	0.302
Base	ETS	14.309	6.074	3.476	0.244
Base	GLM	15.396	6.253	3.576	0.244
Base	TSGLM	15.316	6.227	3.575	0.245
Base	Ensemble	12.978	5.727	3.430	0.243
MinT	Naïve	30.368	10.902	5.498	0.313
MinT	ETS	13.515	5.967	3.547	0.243
MinT	GLM	13.839	5.917	3.453	0.246
MinT	TSGLM	14.000	5.947	3.455	0.248
MinT	Ensemble	12.585	5.728	3.426	0.247

### **Conclusions**

- Ensemble mixture distributions give better forecasts than any component methods.
- Forecast reconciliation improves forecast accuracy, even when some component methods are quite poor.
- The ensemble without the Naïve method was worse.
- Forecast reconciliation allows coordinated planning and resource allocation.

### **More information**

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