# Probabilistic forecasts for anomaly detection

Rob J Hyndman

3 July 2024

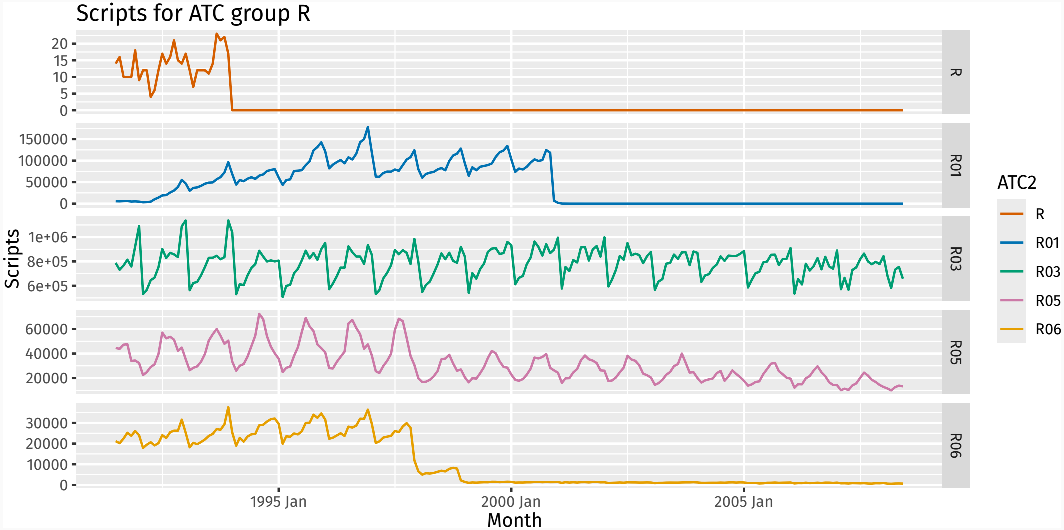
MONASH University

# Australian PBS data

```r
pbs
```

```
# A tsibble: 17,016 x 3 [1M]
# Key:       ATC2 [84]
   ATC2     Month Scripts
   <chr>    <mth>   <dbl>
 1 A01   1991 Jul   22615
 2 A01   1991 Aug   20443
 3 A01   1991 Sep   21389
 4 A01   1991 Oct   23746
 5 A01   1991 Nov   23477
 6 A01   1991 Dec   26316
 7 A01   1992 Jan   22041
 8 A01   1992 Feb   16393
 9 A01   1992 Mar   17207
10 A01   1992 Apr   18847
# i 17,006 more rows
```

# Australian PBS data

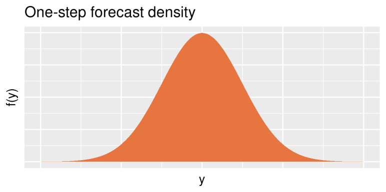


# Main idea

- Estimate one-step forecast densities: $f(y|y_1,\dots,y_{t-1})$.
- Anomaly score: $s_t = -\hat{f}(y_t|y_1,\dots,y_{t-1})$.
- High anomaly scores indicate potential anomalies.
- Fit a Generalized Pareto Distribution to the top 5% of anomaly scores.
- Estimate the probability of each observation being an anomaly.

# Anomaly score distribution

Suppose one-step forecasts are $N(\mu_t, \sigma^2)$.

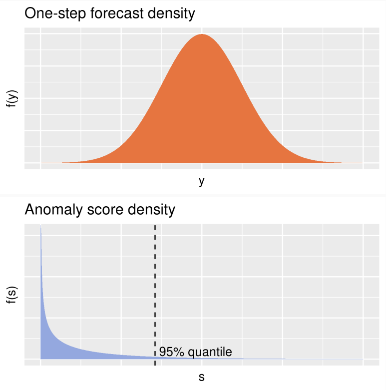


# Anomaly score distribution

Suppose one-step forecasts are $N(\mu_t, \sigma^2)$.

Then $s_t = -\log \phi\left(\frac{y_t - \mu_t}{\sigma}\right) = \frac{1}{2}\left(\frac{y_t - \mu_t}{\sigma}\right)^2 + \frac{1}{2}\log(2\pi\sigma^2)$

So anomaly scores have distribution: $S \sim \frac{1}{2}\chi^2_1 + c$



# Anomaly score distribution

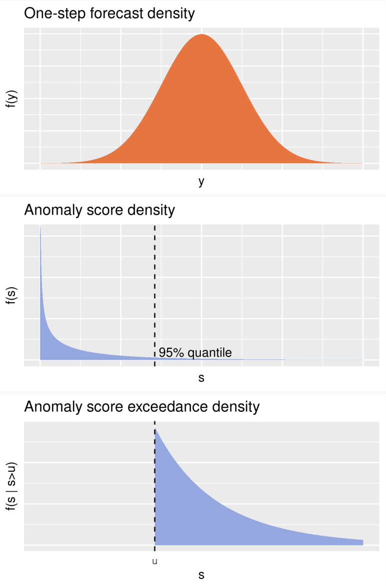
Suppose one-step forecasts are $N(\mu_t, \sigma^2)$.

Then $s_t = -\log \phi\left(\frac{y_t - \mu_t}{\sigma}\right) = \frac{1}{2}\left(\frac{y_t - \mu_t}{\sigma}\right)^2 + \frac{1}{2}\log(2\pi\sigma^2)$

So anomaly scores have distribution: $S \sim \frac{1}{2}\chi^2_1 + c$

Conditional probability distribution of scores above threshold $u$ is Generalized Pareto:

$$H(x) = P(S \leq u + x \mid S > u) = 1 - (1 + \xi x/v)^{-1/\xi}$$

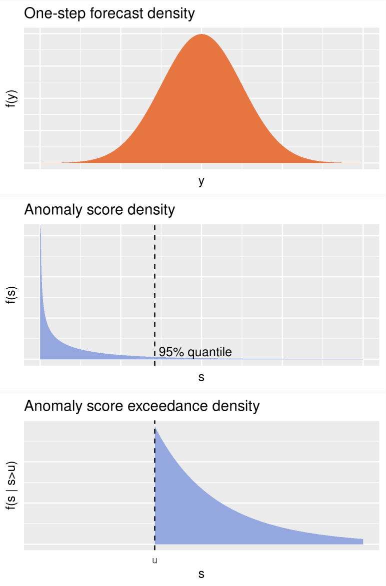


# Anomaly score distribution

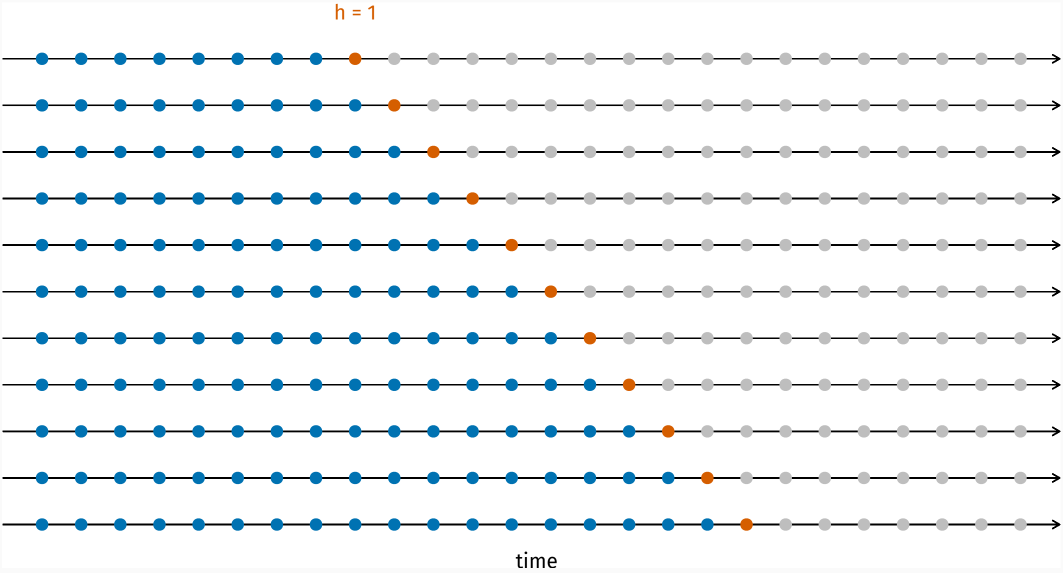
Extreme value theory shows that the Generalized Pareto distribution is a good approximation to the distribution of the largest anomaly scores for almost all possible forecast distributions.

Conditional probability distribution of scores above threshold $u$ is Generalized Pareto:

$$H(x) = P(S \leq u + x \mid S > u) = 1 - (1 + \xi x/v)^{-1/\xi}$$



# Rolling origin forecasts



# Rolling origin forecasts

```r
pbs_stretch <- stretch_tsibble(pbs, .step = 1, .init = 36)
```

```
# A tsibble: 1,684,884 x 4 [1M]
# Key:       .id, ATC2 [14,076]
   ATC2     Month Scripts   .id
   <chr>    <mth>   <dbl> <int>
 1 A01   1991 Jul   22615     1
 2 A01   1991 Aug   20443     1
 3 A01   1991 Sep   21389     1
 4 A01   1991 Oct   23746     1
 5 A01   1991 Nov   23477     1
 6 A01   1991 Dec   26316     1
 7 A01   1992 Jan   22041     1
 8 A01   1992 Feb   16393     1
 9 A01   1992 Mar   17207     1
10 A01   1992 Apr   18847     1
# i 1,684,874 more rows
```

# Rolling origin forecasts

```
pbs_fit <- pbs_stretch |> model(ets = ETS(Scripts))
```

```
# A mable: 14,076 x 3
# Key:     .id, ATC2 [14,076]
      .id ATC2           ets
    <int> <chr>      <model>
 1      1 A01    <ETS(M,N,A)>
 2      1 A02    <ETS(M,A,M)>
 3      1 A03    <ETS(M,A,M)>
 4      1 A04    <ETS(M,N,M)>
 5      1 A05   <ETS(A,Ad,N)>
 6      1 A06    <ETS(M,N,M)>
 7      1 A07    <ETS(M,A,M)>
 8      1 A09    <ETS(M,A,M)>
 9      1 A10    <ETS(M,A,M)>
10      1 A11    <ETS(M,A,M)>
# i 14,066 more rows
```

# Rolling origin forecasts

```
pbs_fc <- forecast(pbs_fit, h = 1)
```

```
# A fable: 14,076 x 4 [1M]
# Key:     .id, ATC2 [14,076]
     .id ATC2     Month             Scripts
   <int> <chr>    <mth>              <dist>
 1     1 A01   1994 Jul  N(22722, 2441206)
 2     1 A02   1994 Jul N(588422, 1.1e+09)
 3     1 A03   1994 Jul  N(84529, 1.9e+07)
 4     1 A04   1994 Jul  N(70220, 1.5e+07)
 5     1 A05   2003 Jul     N(1372, 13768)
 6     1 A06   1994 Jul  N(30624, 5537439)
 7     1 A07   1994 Jul  N(78305, 1.5e+07)
 8     1 A09   1994 Jul     N(3658, 28241)
 9     1 A10   1994 Jul N(166969, 5.4e+07)
10     1 A11   1994 Jul  N(30575, 2947627)
# i 14,066 more rows
```

# Finding anomalies

```r
pbs_scores <- pbs_fc |>
  left_join(pbs |> rename(actual = Scripts), by = c("ATC2", "Month")) |>
  mutate(
    s = -log_likelihood(Scripts, actual), # Density scores
    prob = lookout(density_scores = s)     # Probability not an anomaly
  )
```

```
# A fable: 14,076 x 7 [1M]
# Key:     .id, ATC2 [14,076]
    .id ATC2     Month            Scripts actual     s  prob
  <int> <chr>    <mth>             <dist>  <dbl> <dbl> <dbl>
1     1 A01   1994 Jul  N(22722, 2441206)  20854  8.99 1
2     1 A02   1994 Jul N(588422, 1.1e+09) 516122 13.8  0.583
3     1 A03   1994 Jul  N(84529, 1.9e+07)  80471  9.73 1
4     1 A04   1994 Jul  N(70220, 1.5e+07)  66125  9.74 1
5     1 A05   2003 Jul      N(1372, 13768)   1468  6.02 1
6     1 A06   1994 Jul  N(30624, 5537439)  29194  8.87 1
```

# Finding anomalies

```r
pbs_scores |> filter(prob < 0.05)
```

```
# A fable: 36 x 7 [1M]
# Key:      .id, ATC2 [36]
     .id ATC2      Month              Scripts actual     s    prob
   <int> <chr>     <mth>               <dist>  <dbl> <dbl>   <dbl>
 1    11 P03    1995 May     N(2191, 40081)   3827  39.6 0.0475
 2    18 A11    1995 Dec N(45094, 5793676)  25108  43.2 0.0422
 3    18 C05    1995 Dec   N(33126, 5e+06)   2456 102.  0.0149
 4    18 D02    1995 Dec   N(43704, 6e+06)  10005 103.  0.0149
 5    18 D08    1995 Dec   N(5537, 117972)   1404  79.2 0.0199
 6    18 G04    1995 Dec N(56160, 8452311)   9666 137.  0.0109
 7    18 L03    1996 Dec        N(329, 536)   1231 763.  0.00185
 8    20 D02    1996 Feb N(-7835, 1208830)   5059  76.7 0.0207
 9    21 C05    1996 Mar         N(-16, 13)     40 120.  0.0125
10    21 J06    1996 Mar        N(2.1, 1.4)     14  53.0 0.0323
# i 26 more rows
```

# Online anomaly detection

- Demonstrate using weird package with (a) univariate models for tourism data; and (b) univariate models for age-specific time series from French mortality.