



# Probabilistic forecasts for anomaly detection

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# Time series anomaly detection paradigms

- 1 **Identify anomalies within a time series in real time:**  
use one-step forecast distributions

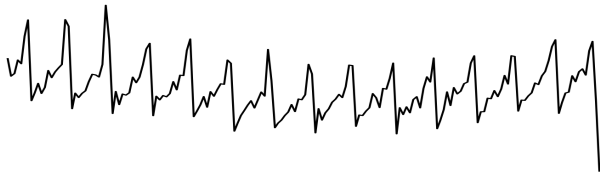


# Time series anomaly detection paradigms

## 1 Identify anomalies within a

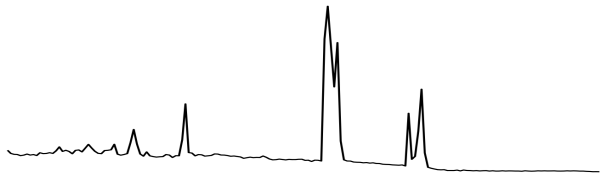
**time series in real time:**

use one-step forecast  
distributions



## 2 Identify anomalies within a time series in historical data:

use residual distributions  
from smoothing method



# Time series anomaly detection paradigms

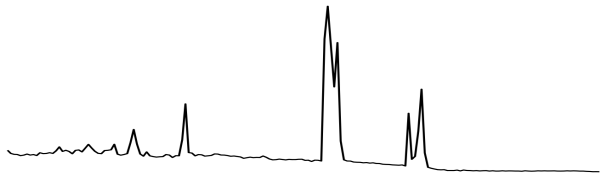
## 1 Identify anomalies within a time series in real time:

use one-step forecast distributions



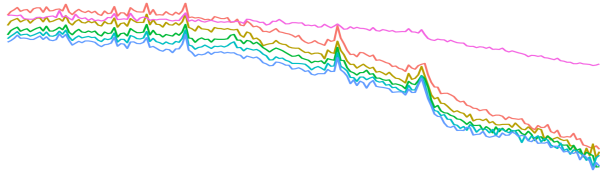
## 2 Identify anomalies within a time series in historical data:

use residual distributions from smoothing method



## 3 Identify an anomalous time series in a collection of time series:

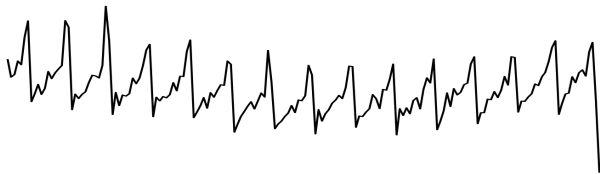
use feature-based approach



# Time series anomaly detection paradigms

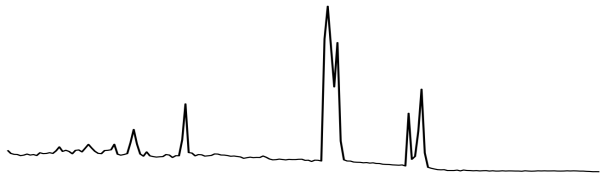
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use one-step forecast distributions



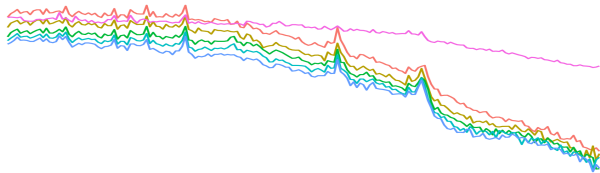
## 2 Identify anomalies within a time series in historical data:

use residual distributions from smoothing method



## 3 Identify an anomalous time series in a collection of time series:

use feature-based approach



# Australian PBS data

```
pbs
```

```
# A tsibble: 17,016 x 3 [1M]
```

```
# Key:      ATC2 [84]
```

```
  ATC2      Month Scripts  
  <chr>    <mth>    <dbl>
```

```
1 A01    1991 Jul     22.6
```

```
2 A01    1991 Aug     20.4
```

```
3 A01    1991 Sep     21.4
```

```
4 A01    1991 Oct     23.7
```

```
5 A01    1991 Nov     23.5
```

```
6 A01    1991 Dec     26.3
```

```
7 A01    1992 Jan     22.0
```

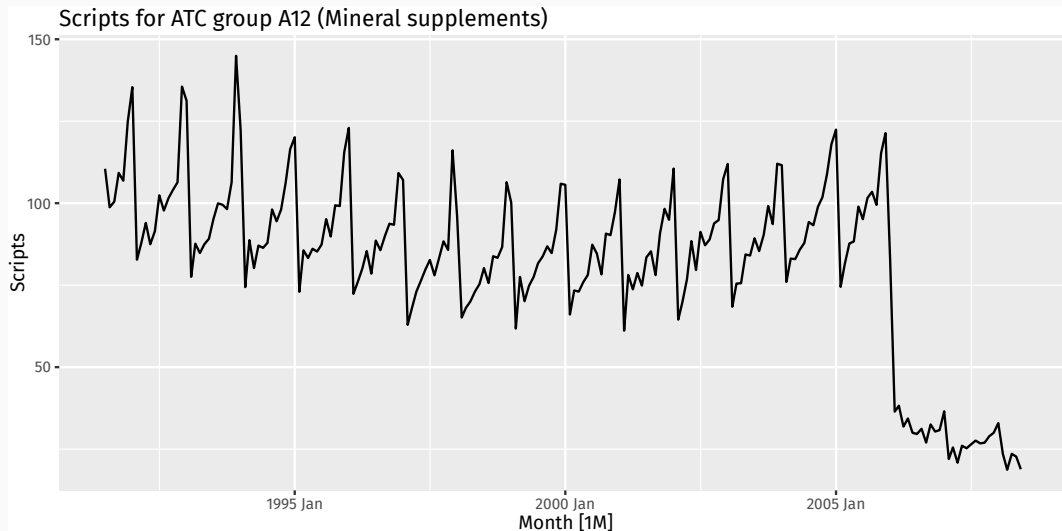
```
8 A01    1992 Feb     16.4
```

```
9 A01    1992 Mar     17.2
```

```
10 A01    1992 Apr     18.8
```

```
# i 17,006 more rows
```

# Australian PBS data

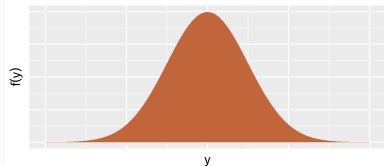


# Anomaly score distribution

## One-step forecast distribution: $N(\mu_t, \sigma^2)$

$$f(y_t | y_1, \dots, y_{t-1}) = \phi\left(\frac{y_t - \mu_t}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y_t - \mu_t)^2}{2\sigma^2}\right\}$$

One-step forecast density





# Anomaly score distribution

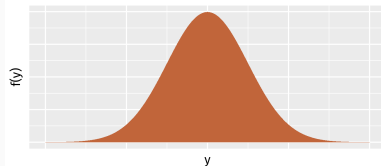
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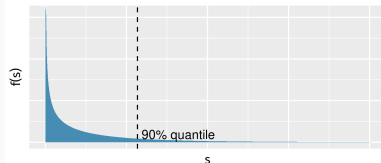
## Anomaly score distribution: $S \sim \frac{1}{2}\chi_1^2 + c$

$$s_t = -\log f(y_t | y_1, \dots, y_{t-1}) = \frac{1}{2} \left(\frac{y_t - \mu_t}{\sigma}\right)^2 + \frac{1}{2} \log(2\pi\sigma^2)$$

One-step forecast density



Anomaly score density



# Anomaly score distribution

## One-step forecast distribution: $N(\mu_t, \sigma^2)$

$$f(y_t | y_1, \dots, y_{t-1}) = \phi\left(\frac{y_t - \mu_t}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(y_t - \mu_t)^2}{2\sigma^2}\right\}$$

## Anomaly score distribution: $S \sim \frac{1}{2}\chi_1^2 + c$

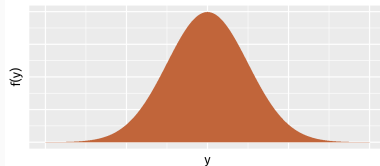
$$s_t = -\log f(y_t | y_1, \dots, y_{t-1}) = \frac{1}{2} \left(\frac{y_t - \mu_t}{\sigma}\right)^2 + \frac{1}{2} \log(2\pi\sigma^2)$$

## Extreme anomaly score distribution

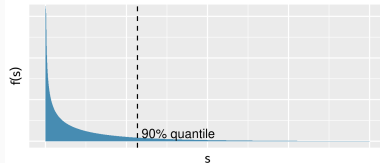
$$H(x) = P(S \leq u + x \mid S > u)$$

→ Generalized Pareto Distribution for almost all forecast distributions  $f$ .

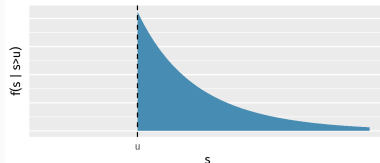
One-step forecast density



Anomaly score density



Anomaly score exceedance density



# Anomaly detection algorithm

For each  $t$ :

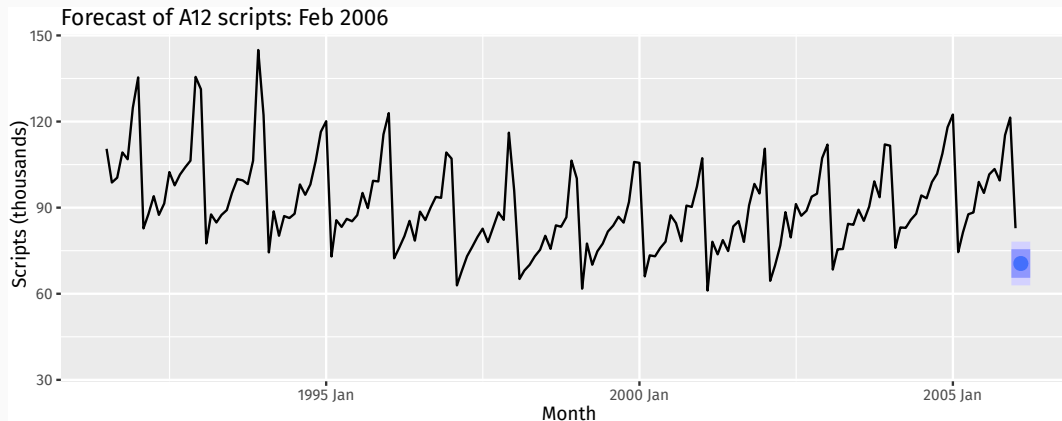
- Estimate one-step forecast density:  $f(y_t|y_1, \dots, y_{t-1})$ .
- Anomaly score:  $s_t = -\log \hat{f}(y_t|y_1, \dots, y_{t-1})$ .
- High anomaly score indicates potential anomaly.
- Fit a Generalized Pareto Distribution to the top 10% of anomaly scores seen so far.
- $y_t$  is anomaly if  $P(S > s_t) < 0.05$  under GPD.

# Example

```
a12 ← pbs ▷ filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))  
a12plus ← pbs ▷ filter(ATC2 == "A12", Month <= yearmonth("2006 Feb"))  
fc ← a12 ▷ model(ets = ETS(Scripts)) ▷ forecast(h = 1)
```

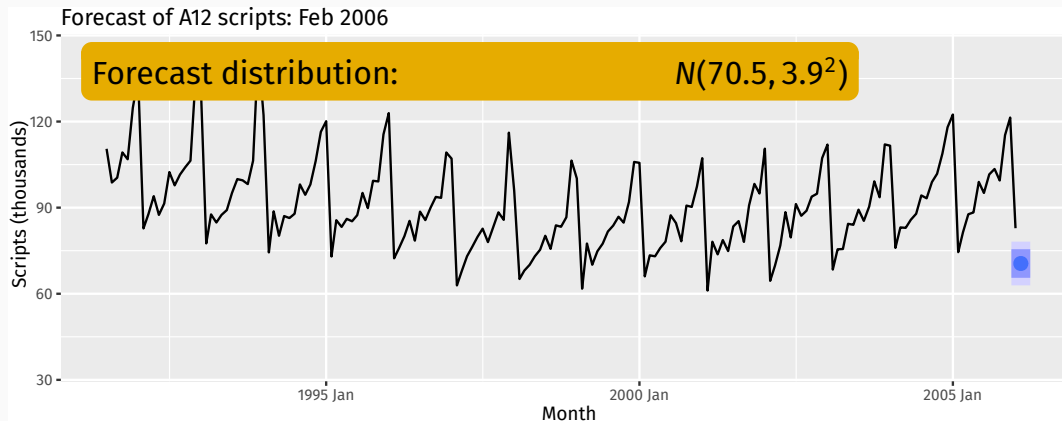
# Example

```
a12 <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))  
a12plus <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Feb"))  
fc <- a12 > model(ets = ETS(Scripts)) > forecast(h = 1)  
fc > autoplot(a12)
```



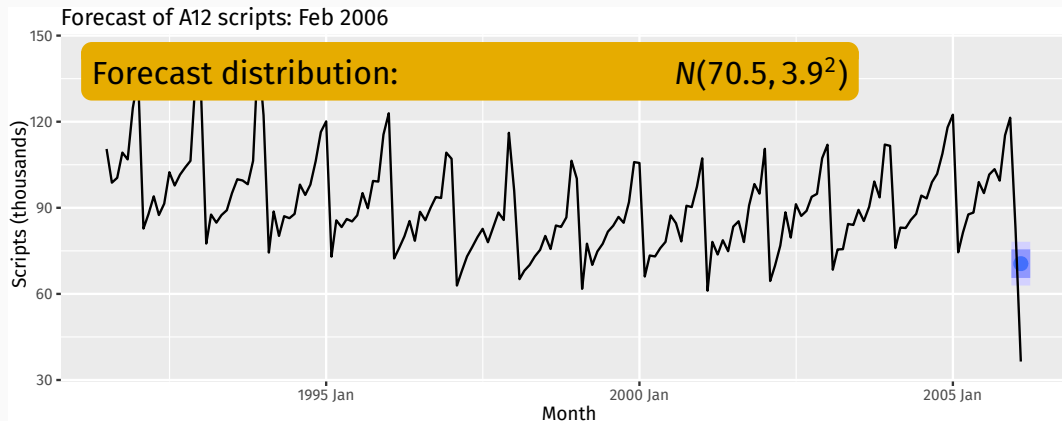
# Example

```
a12 <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))  
a12plus <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Feb"))  
fc <- a12 > model(ets = ETS(Scripts)) > forecast(h = 1)  
fc > autoplot(a12)
```



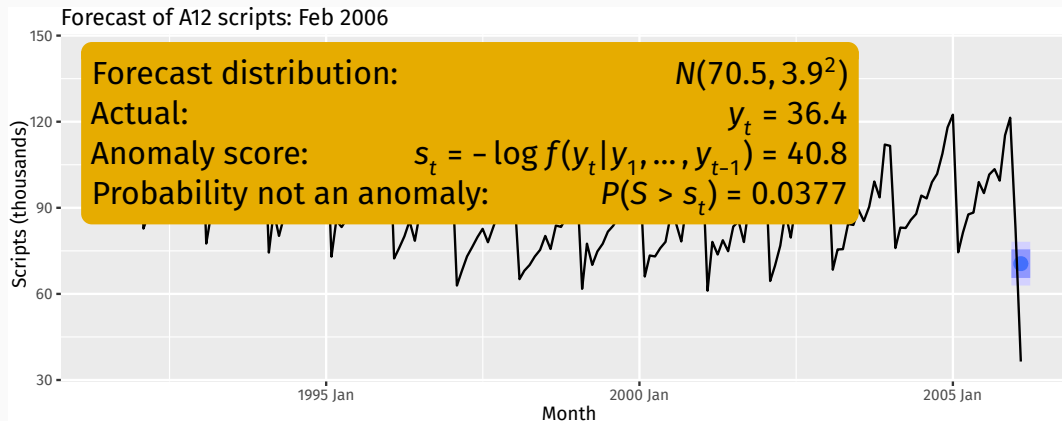
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a12 <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))  
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fc <- a12 > model(ets = ETS(Scripts)) > forecast(h = 1)  
fc > autoplot(a12plus)
```



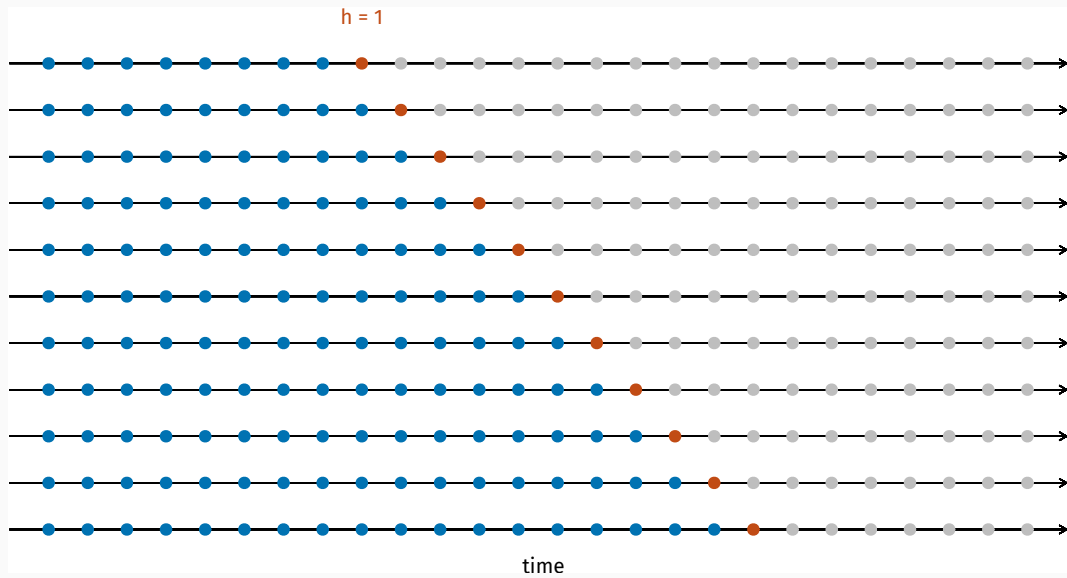
# Example

```
a12 <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))  
a12plus <- pbs > filter(ATC2 == "A12", Month <= yearmonth("2006 Feb"))  
fc <- a12 > model(ets = ETS(Scripts)) > forecast(h = 1)  
fc > autoplot(a12plus)
```





# Rolling origin forecasts



# Rolling origin forecasts

```
pbs_stretch ← stretch_tsibble(pbs, .step = 1, .init = 36)
```

```
# A tsibble: 1,684,884 x 4 [1M]
```

```
# Key:       .id, ATC2 [14,076]
```

	ATC2	Month	Scripts	.id
	<chr>	<mth>	<dbl>	<int>
1	A01	1991 Jul	22.6	1
2	A01	1991 Aug	20.4	1
3	A01	1991 Sep	21.4	1
4	A01	1991 Oct	23.7	1
5	A01	1991 Nov	23.5	1
6	A01	1991 Dec	26.3	1
7	A01	1992 Jan	22.0	1
8	A01	1992 Feb	16.4	1
9	A01	1992 Mar	17.2	1
10	A01	1992 Apr	18.8	1

```
# i 1,684,874 more rows
```

# Rolling origin forecasts

```
pbs_fit <- pbs_stretch > model(ets = ETS(Scripts))
```

```
# A mable: 14,076 x 3
# Key:      .id, ATC2 [14,076]
  .id ATC2      ets
  <int> <chr>    <model>
1     1 A01    <ETS(M,N,A)>
2     1 A02    <ETS(M,A,M)>
3     1 A03    <ETS(M,A,M)>
4     1 A04    <ETS(M,N,A)>
5     1 A05    <ETS(A,Ad,N)>
6     1 A06    <ETS(M,A,M)>
7     1 A07    <ETS(M,N,M)>
8     1 A09    <ETS(M,A,M)>
9     1 A10    <ETS(M,A,M)>
10    1 A11    <ETS(M,A,M)>
# i 14,066 more rows
```

# Rolling origin forecasts

```
pbs_fc ← forecast(pbs_fit, h = 1)
```

```
# A fable: 14,076 x 4 [1M]
```

```
# Key:      .id, ATC2 [14,076]
```

	.id	ATC2	Month	Scripts
	<int>	<chr>	<mth>	<dist>
1	1	A01	1994 Jul	N(23, 2.1)
2	1	A02	1994 Jul	N(590, 1054)
3	1	A03	1994 Jul	N(84, 19)
4	1	A04	1994 Jul	N(69, 15)
5	1	A05	2003 Jul	N(1.4, 0.014)
6	1	A06	1994 Jul	N(33, 4.2)
7	1	A07	1994 Jul	N(74, 17)
8	1	A09	1994 Jul	N(3.7, 0.029)
9	1	A10	1994 Jul	N(166, 54)
10	1	A11	1994 Jul	N(30, 3)

```
# i 14,066 more rows
```

# PBS anomalies

```
pbs_scores <- pbs_fc >
  left_join(pbs > rename(actual = Scripts), by = c("ATC2", "Month")) >
  group_by(.id) >
  mutate(
    s = -log_likelihood(Scripts, actual),
    prob = lookout(density_scores = s, threshold = 0.9)
  ) >
  ungroup()
```

# A tsibble: 14,076 x 7 [1M]

# Key: .id, ATC2 [14,076]

	.id	ATC2	Month	Scripts	actual	s	prob
	<int>	<chr>	<mth>	<dist>	<dbl>	<dbl>	<dbl>
1	1	A01	1994 Jul	N(23, 2.1)	20.9	2.46	1
2	1	A02	1994 Jul	N(590, 1054)	516.	6.97	0.575
3	1	A03	1994 Jul	N(84, 19)	80.5	2.75	1
4	1	A04	1994 Jul	N(69, 15)	66.1	2.62	1
5	1	A05	2003 Jul	N(1.4, 0.014)	1.47	-1.05	1
6	1	A06	1994 Jul	N(33, 4.2)	29.2	3.41	1

# PBS anomalies

```
pbs_scores > filter(prob < 0.05)
```

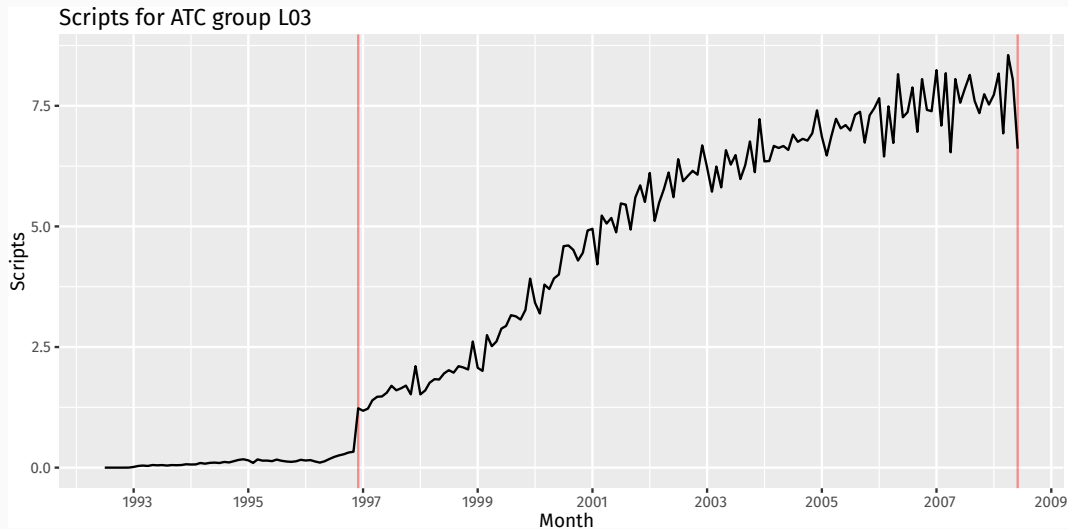
```
# A tsibble: 149 x 7 [1M]
```

```
# Key:           .id, ATC2 [149]
```

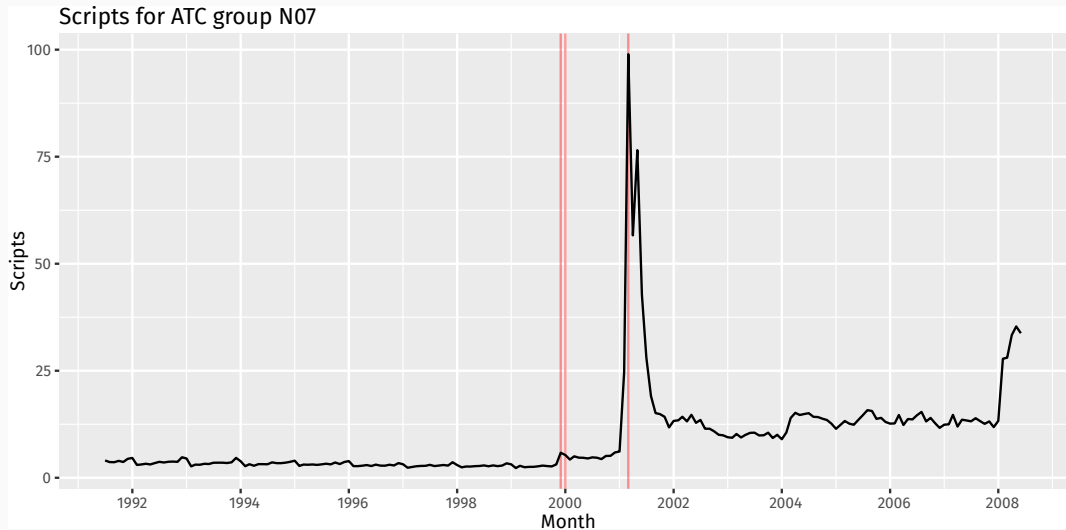
	.id	ATC2	Month	Scripts	actual	s	prob
	<int>	<chr>	<mth>	<dist>	<dbl>	<dbl>	<dbl>
1	1	S01	1994 Jul	N(403, 356)	347.	8.29	0.0249
2	2	J01	1994 Aug	N(1382, 5743)	1616.	10.0	0.0232
3	3	J01	1994 Sep	N(1558, 8941)	1552.	5.47	0.0330
4	4	J01	1994 Oct	N(1496, 7931)	1327.	7.21	0.0405
5	5	H01	1994 Nov	N(0.77, 0.0039)	1.05	7.89	0.0405
6	6	C02	1994 Dec	N(142, 39)	108.	17.1	0.00190
7	6	C07	1994 Dec	N(413, 324)	320.	17.0	0.0380
8	7	C07	1995 Jan	N(414, 541)	328.	10.9	0.0000572
9	8	R03	1995 Feb	N(416, 966)	507.	8.66	0.000770
10	9	C10	1995 Mar	N(205, 214)	260.	10.6	0.0261

```
# i 139 more rows
```

# PBS anomalies

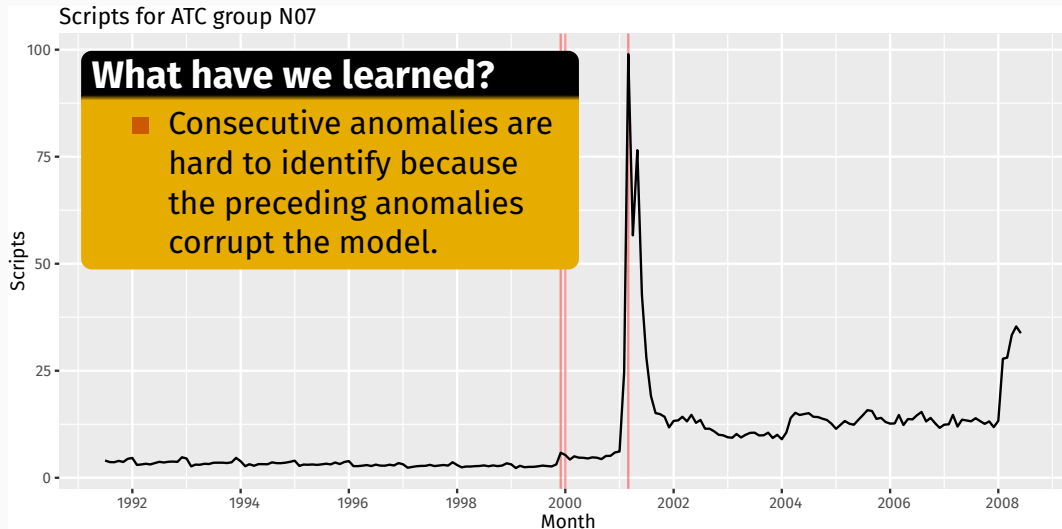


# PBS anomalies

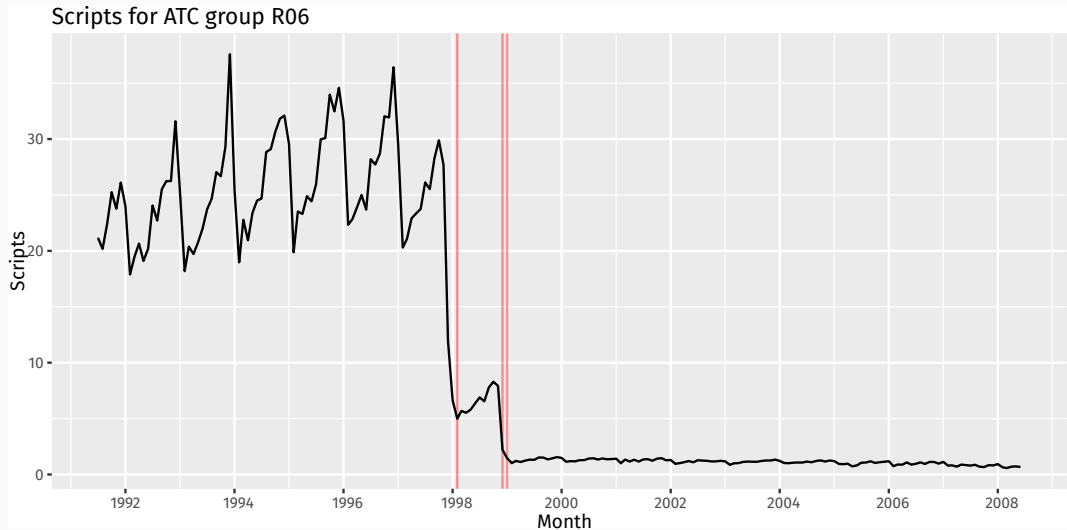




# PBS anomalies

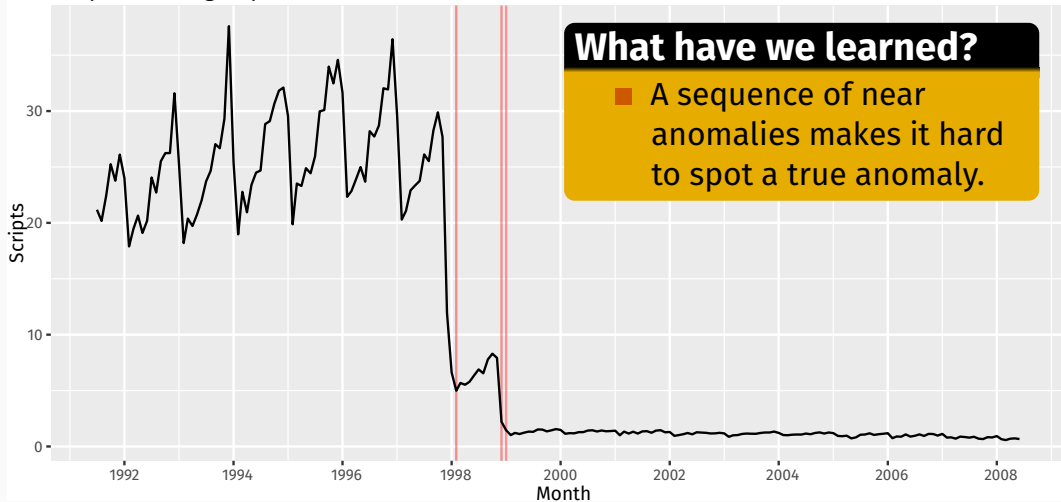


# PBS anomalies



# PBS anomalies

Scripts for ATC group R06



# Time series anomaly detection paradigms

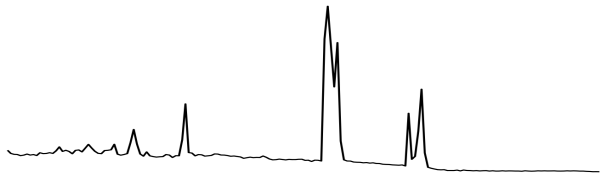
## 1 Identify anomalies within a time series in real time:

use one-step forecast distributions



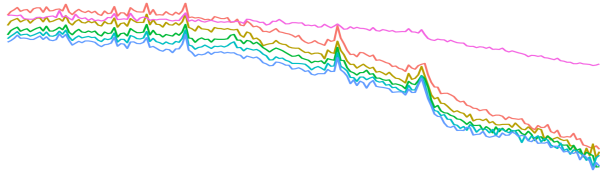
## 2 Identify anomalies within a time series in historical data:

use residual distributions from smoothing method



## 3 Identify an anomalous time series in a collection of time series:

use feature-based approach



# Time series anomaly detection paradigms

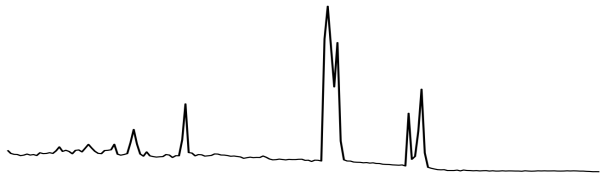
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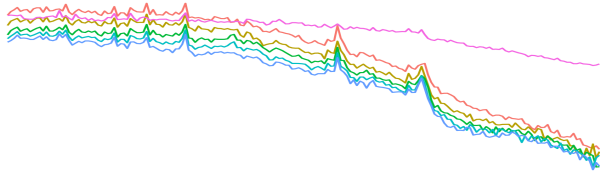
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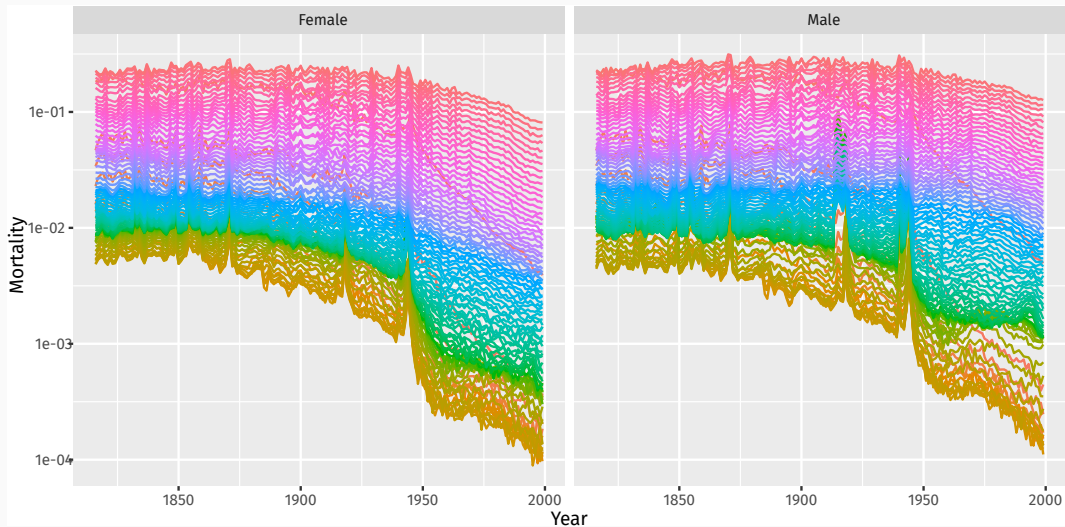


# Example: French mortality

```
fr_mortality
```

```
# A tsibble: 31,648 x 4 [1Y]
# Key:      Age, Sex [172]
   Year   Age Sex   Mortality
  <int> <int> <chr>    <dbl>
1  1816     0 Female    0.187
2  1817     0 Female    0.182
3  1818     0 Female    0.186
4  1819     0 Female    0.197
5  1820     0 Female    0.181
6  1821     0 Female    0.182
7  1822     0 Female    0.207
8  1823     0 Female    0.192
9  1824     0 Female    0.199
10 1825     0 Female    0.194
# i 31,638 more rows
```

# Example: French mortality



# Example: French mortality

```
fr_fit ← fr_mortality ▷  
  model(stl = STL(log(Mortality)))  
  
fr_sigma ← augment(fr_fit) ▷  
  group_by(Age, Sex) ▷  
  summarise(sigma = IQR(.innov)/1.349, .groups = "drop")  
  
fr_scores ← augment(fr_fit) ▷  
  left_join(fr_sigma) ▷  
  mutate(  
    s = -log(dnorm(.innov / sigma)),  
    prob = lookout(density_scores = s, threshold_probability = 0.9)  
  )
```



# Example: French mortality

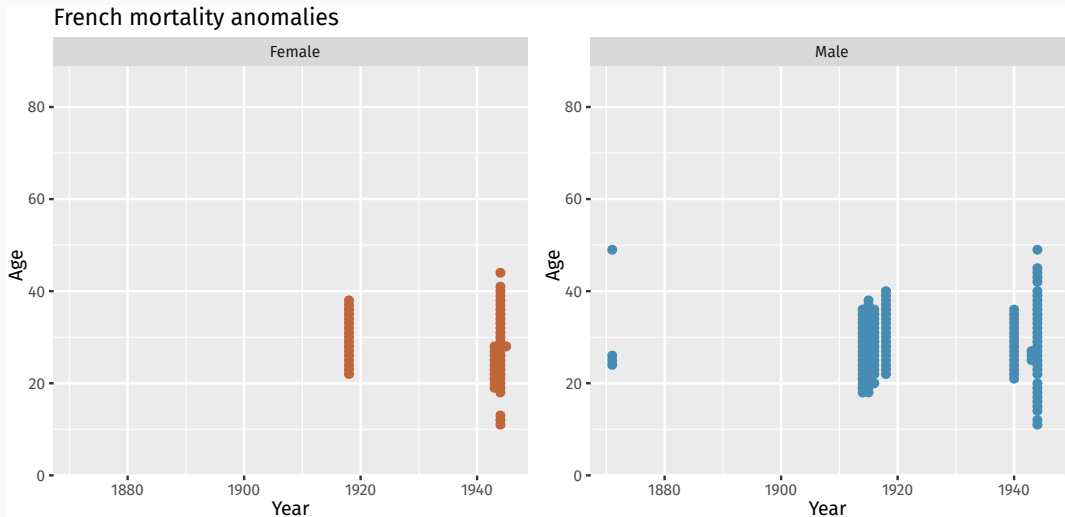
```
fr_scores ► arrange(prob)
```

```
# A tibble: 31,648 x 7
```

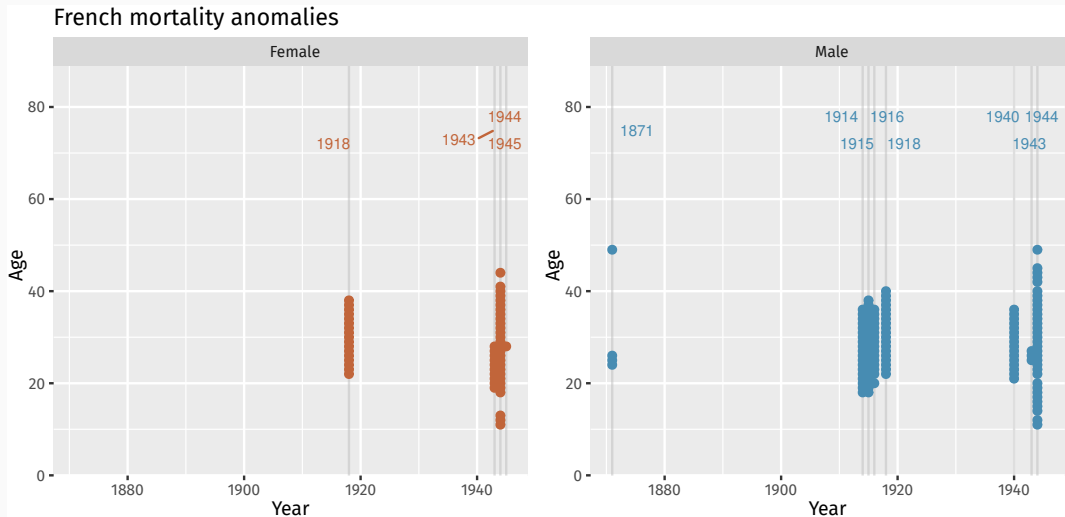
	Age	Sex	Year	Mortality	.innov	s	prob
	<int>	<chr>	<int>	<dbl>	<dbl>	<dbl>	<dbl>
1	28	Female	1944	0.0170	1.45	373.	0.00737
2	25	Female	1944	0.0191	1.59	331.	0.00831
3	26	Female	1944	0.0176	1.50	266.	0.0104
4	24	Female	1944	0.0150	1.40	259.	0.0106
5	27	Female	1944	0.0178	1.50	228.	0.0121
6	25	Male	1944	0.0432	1.89	170.	0.0163
7	18	Male	1914	0.0798	2.06	170.	0.0163
8	21	Female	1944	0.0120	1.29	168.	0.0165
9	27	Male	1944	0.0388	1.78	168.	0.0165
10	23	Female	1944	0.0134	1.29	167.	0.0166

```
# i 31,638 more rows
```

# Example: French mortality



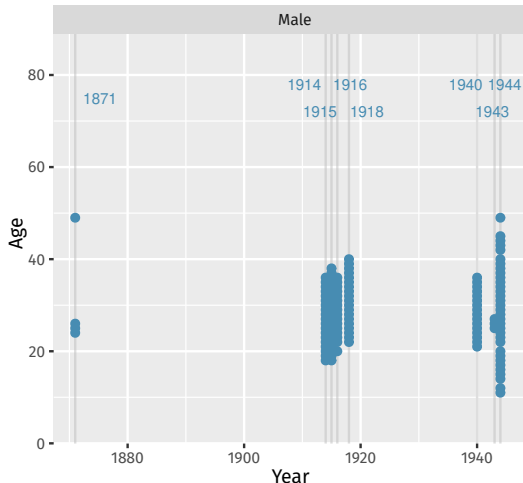
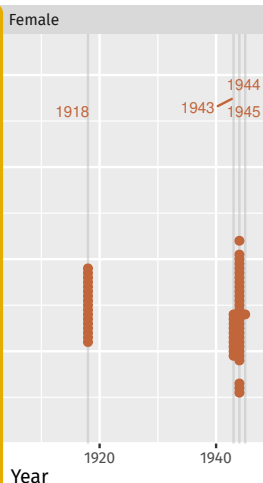
# Example: French mortality



# Example: French mortality

## French mortality anomalies

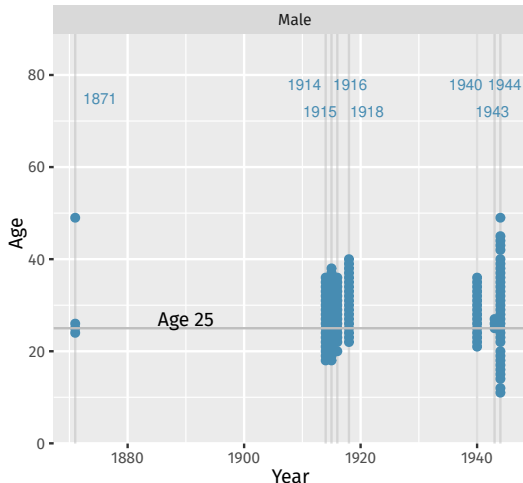
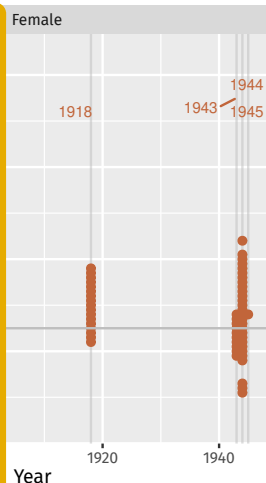
- 1870–1872: Franco-Prussian war and repression of the 'Commune de Paris'
- 1914–1918: World War I
- 1918: Spanish flu
- 1939–1945: World War II



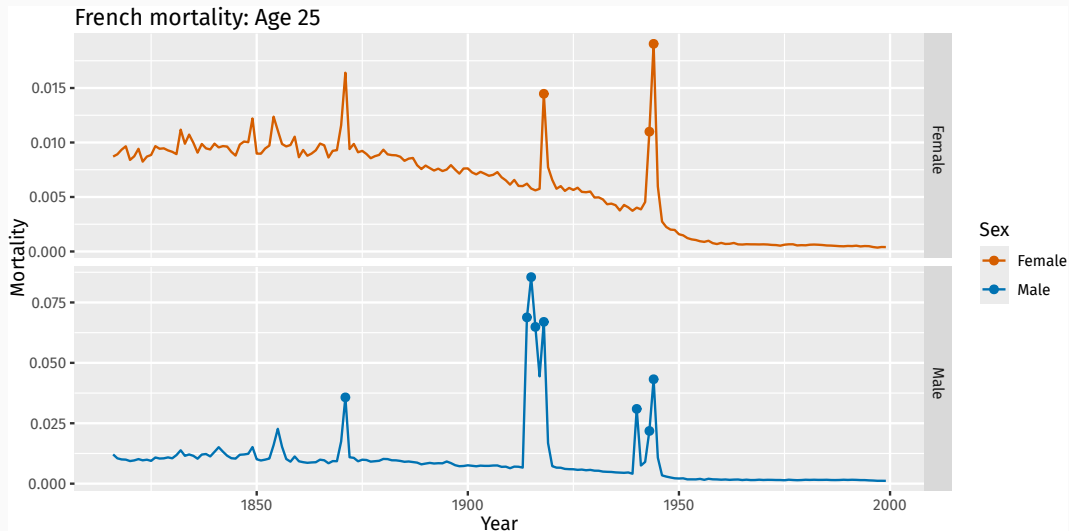
# Example: French mortality

## French mortality anomalies

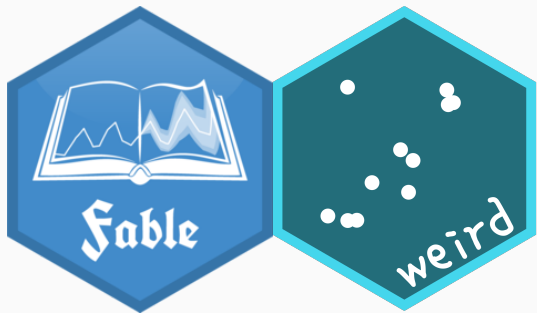
- 1870–1872: Franco-Prussian war and repression of the 'Commune de Paris'
- 1914–1918: World War I
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# Example: French mortality



# More information



# More information



**Slides:** [robjhyndman.com/isf2024](http://robjhyndman.com/isf2024)

**Incomplete book:** [OTexts.com/weird](http://OTexts.com/weird)



# More information



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✉ [rob.hyndman@monash.edu](mailto:rob.hyndman@monash.edu)