Probabilistic forecasts for anomaly detection

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Time series anomaly detection paradigms

1 **Identify anomalies within a time series in real time**: use one-step forecast distributions

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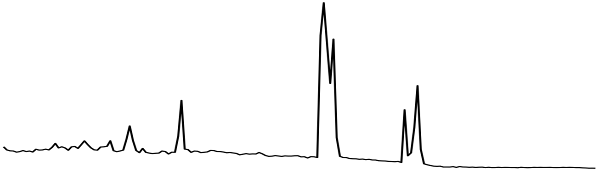
2. **Identify anomalies within a time series in historical data:** use residual distributions from smoothing method



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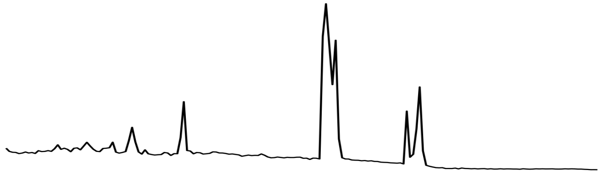

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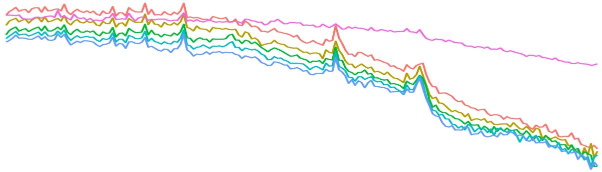

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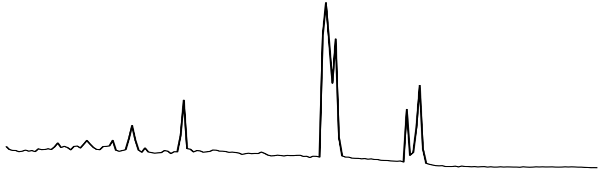

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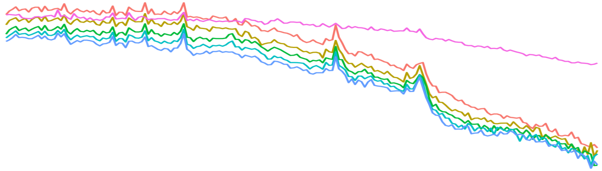
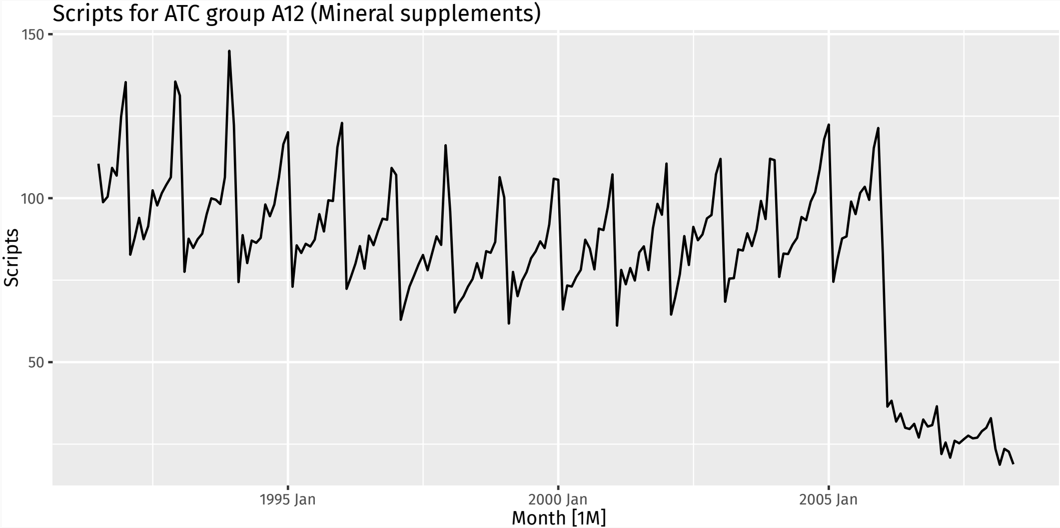


Australian PBS data

```
pbs
```

```
# A tsibble: 17,016 x 3 [1M]
# Key:       ATC2 [84]
   ATC2     Month Scripts
   <chr>    <mth>   <dbl>
 1 A01   1991 Jul    22.6
 2 A01   1991 Aug    20.4
 3 A01   1991 Sep    21.4
 4 A01   1991 Oct    23.7
 5 A01   1991 Nov    23.5
 6 A01   1991 Dec    26.3
 7 A01   1992 Jan    22.0
 8 A01   1992 Feb    16.4
 9 A01   1992 Mar    17.2
10 A01   1992 Apr    18.8
# i 17,006 more rows
```

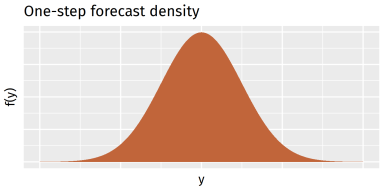
Australian PBS data



Anomaly score distribution

One-step forecast distribution: $N(\mu_t, \sigma^2)$

$$f(y_t|y_1,\dots,y_{t-1}) = \phi\left(\frac{y_t - \mu_t}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}}\exp\left\{\frac{(y_t-\mu_t)^2}{2\sigma^2}\right\}$$



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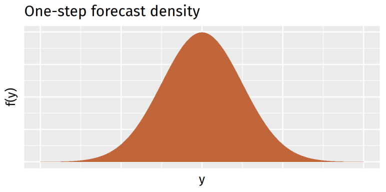
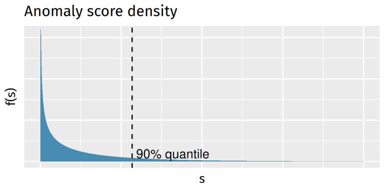
Anomaly score distribution: $S \sim \frac{1}{2}\chi^2_1 + c$

$$s_t = -\log f(y_t|y_1,\dots,y_{t-1}) = \frac{1}{2}\left(\frac{y_t - \mu_t}{2\sigma}\right)^2 + \frac{1}{2}\log(2\pi\sigma^2)$$





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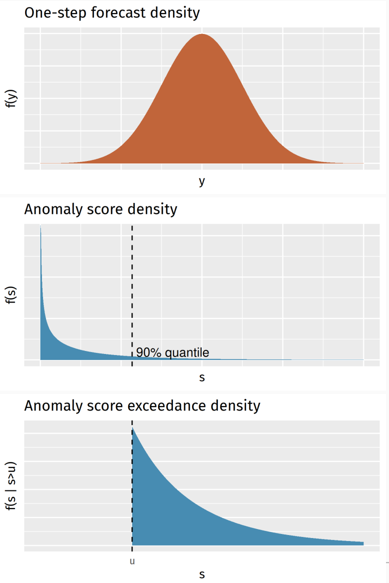
Extreme anomaly score distribution

$$H(x) = P(S \le u + x \mid S > u)$$

→ Generalized Pareto Distribution for almost all forecast distributions f.



Anomaly detection algorithm

For each t:

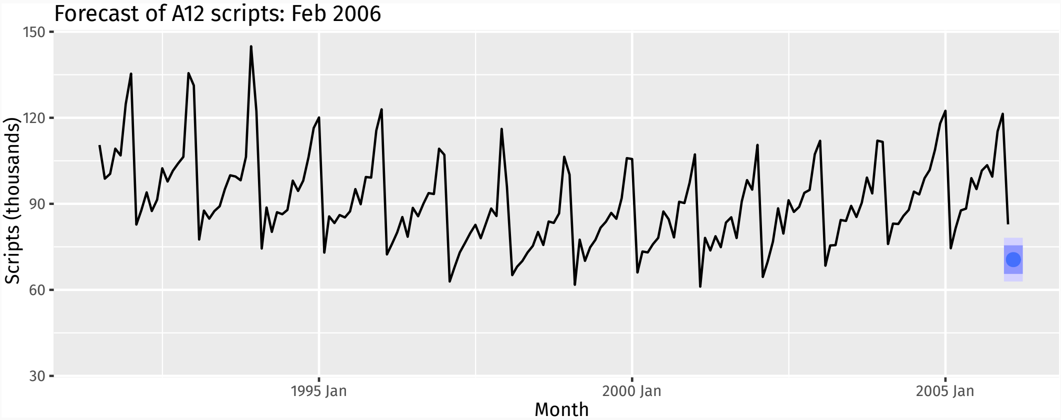
- Estimate one-step forecast density: $f(y_t|y_1, \dots, y_{t-1})$.
- Anomaly score: $s_t = -\log \hat{f}(y_t|y_1, \dots, y_{t-1})$.
- High anomaly score indicates potential anomaly.
- Fit a Generalized Pareto Distribution to the top 10% of anomaly scores seen so far.
- y_t is anomaly if $P(S > s_t) < 0.05$ under GPD.

Example

```
a12 <- pbs |> filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))
a12plus <- pbs |> filter(ATC2 == "A12", Month <= yearmonth("2006 Feb"))
fc <- a12 |> model(ets = ETS(Scripts)) |> forecast(h = 1)
```

Example

```
a12 ← pbs ▷ filter(ATC2 == "A12", Month <= yearmonth("2006 Jan"))
a12plus ← pbs ▷ filter(ATC2 == "A12", Month <= yearmonth("2006 Feb"))
fc ← a12 ▷ model(ets = ETS(Scripts)) ▷ forecast(h = 1)
fc ▷ autoplot(a12)
```



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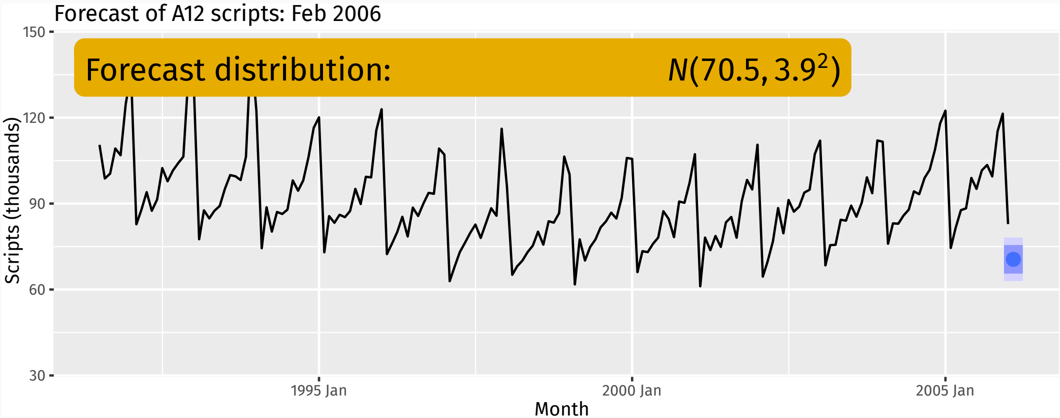

Forecast distribution: $N(70.5, 3.9^2)$

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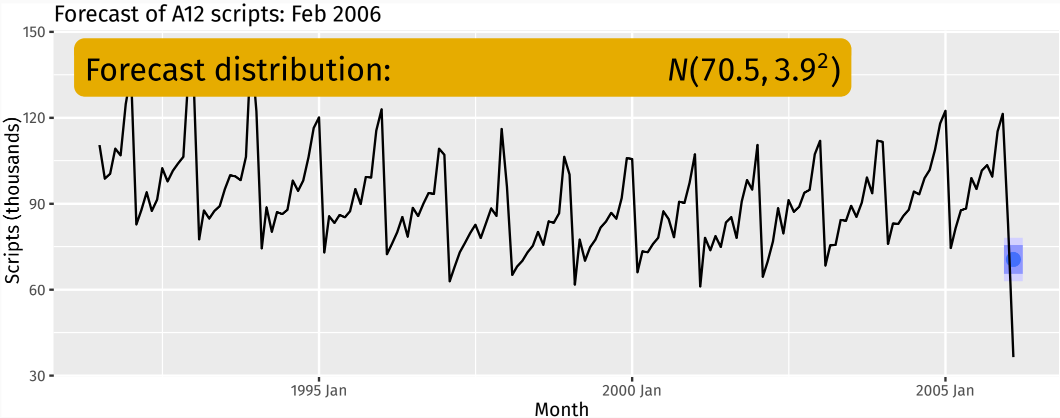
fc ▷ autoplot(a12plus)
```



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```
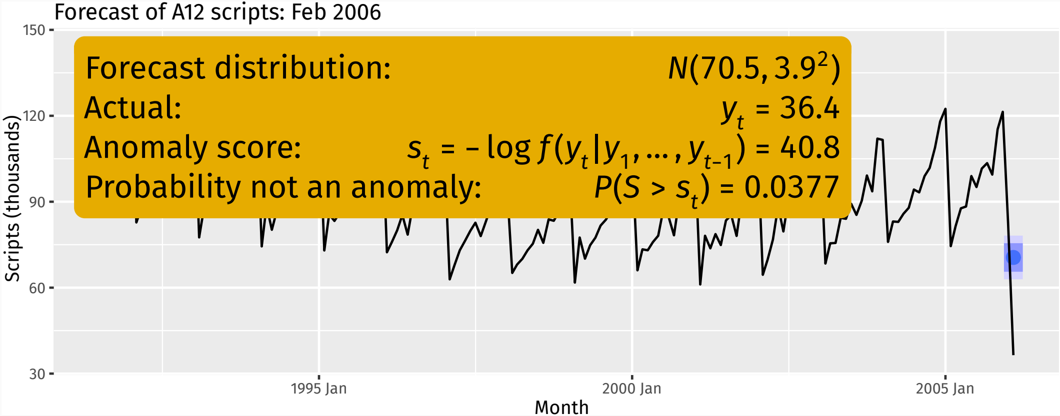



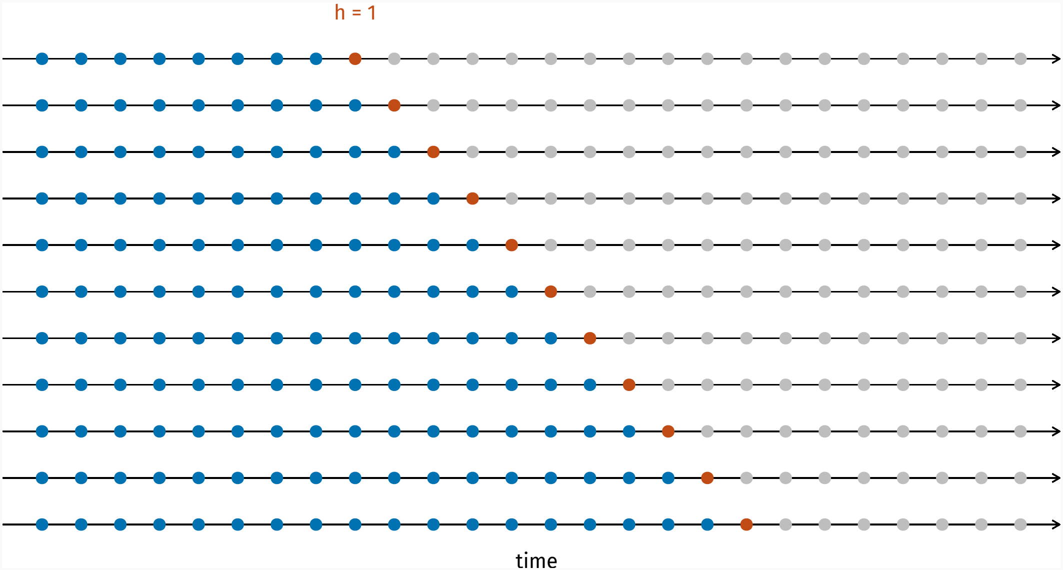
Forecast distribution: $N(70.5, 3.9^2)$

Actual: $y_t = 36.4$

Anomaly score: $s_t = -\log f(y_t | y_1, \dots, y_{t-1}) = 40.8$

Probability not an anomaly: $P(S > s_t) = 0.0377$

Rolling origin forecasts



Rolling origin forecasts

```r
pbs_stretch <- stretch_tsibble(pbs, .step = 1, .init = 36)
```

```
# A tsibble: 1,684,884 x 4 [1M]
# Key:       .id, ATC2 [14,076]
   ATC2      Month Scripts   .id
   <chr>     <mth>   <dbl> <int>
 1 A01    1991 Jul    22.6     1
 2 A01    1991 Aug    20.4     1
 3 A01    1991 Sep    21.4     1
 4 A01    1991 Oct    23.7     1
 5 A01    1991 Nov    23.5     1
 6 A01    1991 Dec    26.3     1
 7 A01    1992 Jan    22.0     1
 8 A01    1992 Feb    16.4     1
 9 A01    1992 Mar    17.2     1
10 A01    1992 Apr    18.8     1
# i 1,684,874 more rows
```

Rolling origin forecasts

```
pbs_fit <- pbs_stretch |> model(ets = ETS(Scripts))
```

```
# A mable: 14,076 x 3
# Key:     .id, ATC2 [14,076]
     .id ATC2           ets
   <int> <chr>      <model>
 1     1 A01    <ETS(M,N,A)>
 2     1 A02    <ETS(M,A,M)>
 3     1 A03    <ETS(M,A,M)>
 4     1 A04    <ETS(M,N,A)>
 5     1 A05   <ETS(A,Ad,N)>
 6     1 A06    <ETS(M,A,M)>
 7     1 A07    <ETS(M,N,M)>
 8     1 A09    <ETS(M,A,M)>
 9     1 A10    <ETS(M,A,M)>
10     1 A11    <ETS(M,A,M)>
# i 14,066 more rows
```

Rolling origin forecasts

```r
pbs_fc <- forecast(pbs_fit, h = 1)
```

```
# A fable: 14,076 x 4 [1M]
# Key:     .id, ATC2 [14,076]
     .id ATC2     Month        Scripts
   <int> <chr>    <mth>         <dist>
 1     1 A01   1994 Jul    N(23, 2.1)
 2     1 A02   1994 Jul  N(590, 1054)
 3     1 A03   1994 Jul     N(84, 19)
 4     1 A04   1994 Jul     N(69, 15)
 5     1 A05   2003 Jul N(1.4, 0.014)
 6     1 A06   1994 Jul    N(33, 4.2)
 7     1 A07   1994 Jul     N(74, 17)
 8     1 A09   1994 Jul N(3.7, 0.029)
 9     1 A10   1994 Jul    N(166, 54)
10     1 A11   1994 Jul      N(30, 3)
# i 14,066 more rows
```

PBS anomalies

```r
pbs_scores <- pbs_fc |>
  left_join(pbs |> rename(actual = Scripts), by = c("ATC2", "Month")) |>
  group_by(.id) |>
  mutate(
    s = -log_likelihood(Scripts, actual),
    prob = lookout(density_scores = s, threshold = 0.9)
  ) |>
  ungroup()
```

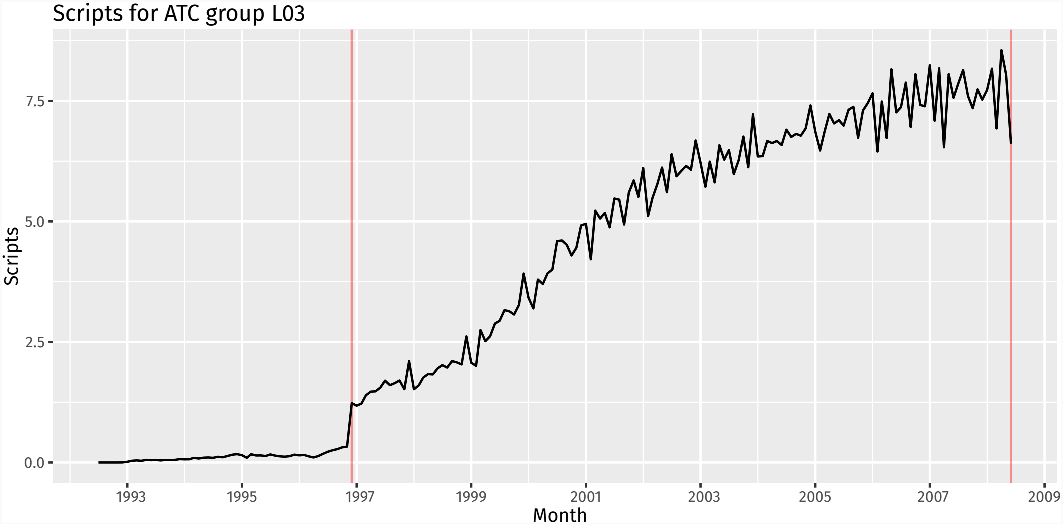
```
# A tsibble: 14,076 x 7 [1M]
# Key:       .id, ATC2 [14,076]
     .id ATC2     Month       Scripts actual     s  prob
   <int> <chr>    <mth>        <dist>  <dbl> <dbl> <dbl>
 1     1 A01   1994 Jul    N(23, 2.1)  20.9   2.46 1
 2     1 A02   1994 Jul  N(590, 1054) 516.    6.97 0.575
 3     1 A03   1994 Jul     N(84, 19)  80.5   2.75 1
 4     1 A04   1994 Jul     N(69, 15)  66.1   2.62 1
 5     1 A05   2003 Jul N(1.4, 0.014)   1.47 -1.05 1
 6     1 A06   1994 Jul    N(33, 4.2)  29.2   3.41 1
```

PBS anomalies

```r
pbs_scores |> filter(prob < 0.05)
```

```
# A tsibble: 149 x 7 [1M]
# Key:       .id, ATC2 [149]
     .id ATC2     Month          Scripts  actual     s       prob
   <int> <chr>    <mth>           <dist>   <dbl> <dbl>      <dbl>
 1     1 S01   1994 Jul     N(403, 356)   347.    8.29 0.0249
 2     2 J01   1994 Aug   N(1382, 5743) 1616.   10.0  0.0232
 3     3 J01   1994 Sep   N(1558, 8941) 1552.    5.47 0.0330
 4     4 J01   1994 Oct   N(1496, 7931) 1327.    7.21 0.0405
 5     5 H01   1994 Nov N(0.77, 0.0039)    1.05  7.89 0.0405
 6     6 C02   1994 Dec      N(142, 39)   108.   17.1  0.00190
 7     6 C07   1994 Dec     N(413, 324)   320.   17.0  0.0380
 8     7 C07   1995 Jan     N(414, 541)   328.   10.9  0.0000572
 9     8 R03   1995 Feb     N(416, 966)   507.    8.66 0.000770
10     9 C10   1995 Mar     N(205, 214)   260.   10.6  0.0261
# i 139 more rows
```

PBS anomalies



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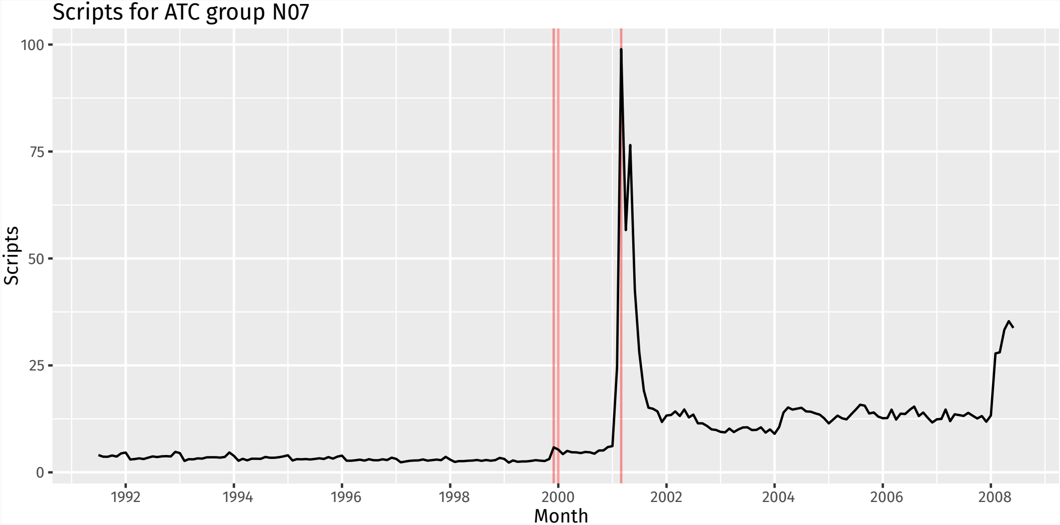

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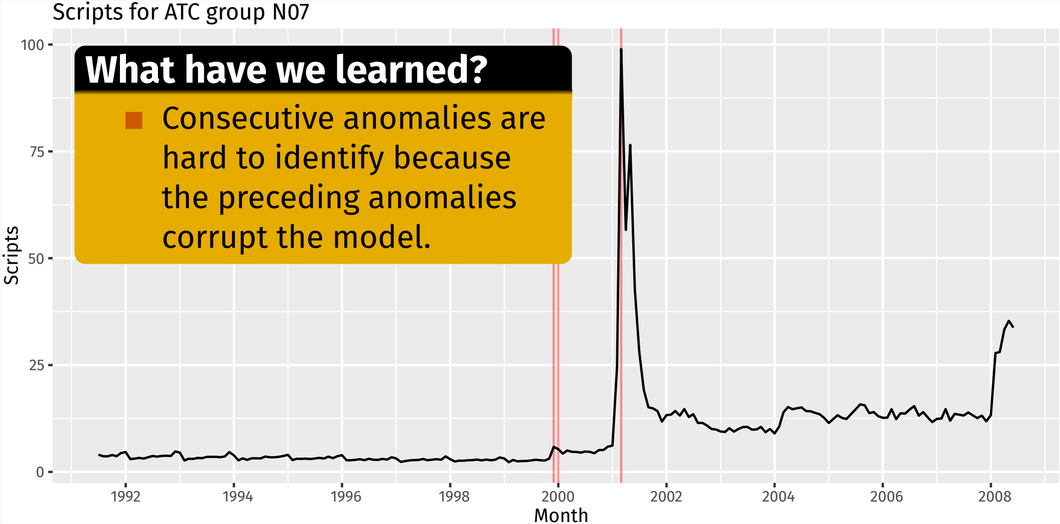
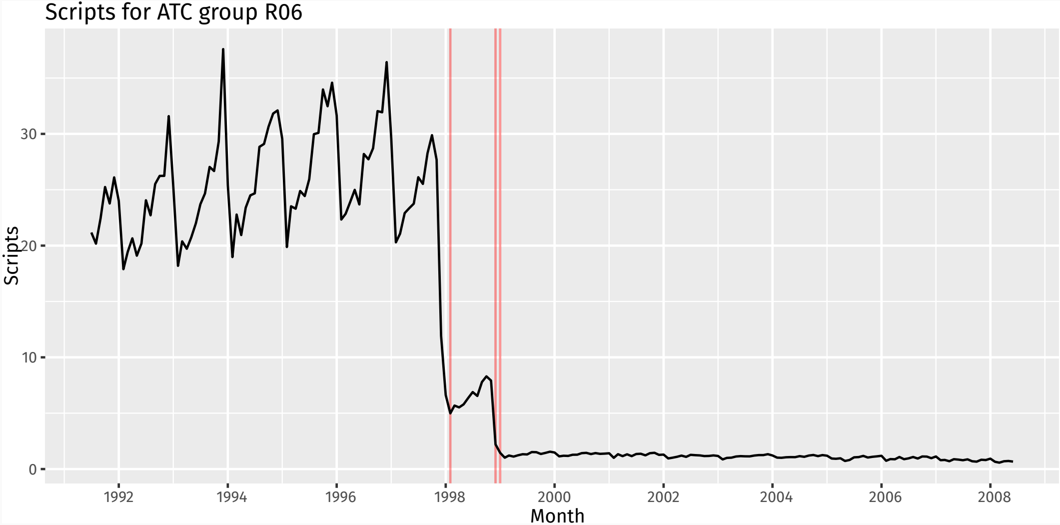


What have we learned?

- Consecutive anomalies are hard to identify because the preceding anomalies corrupt the model.

PBS anomalies



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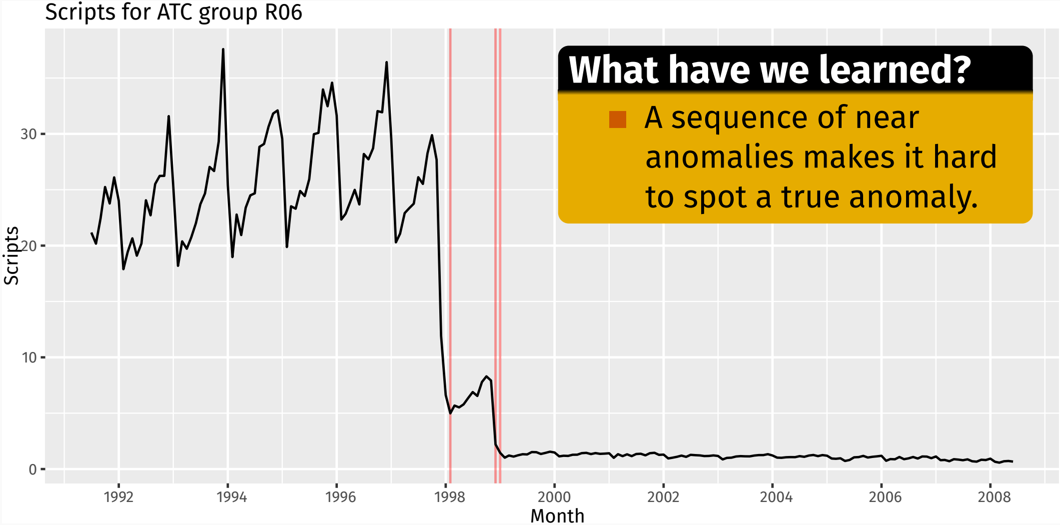

What have we learned?

- A sequence of near anomalies makes it hard to spot a true anomaly.

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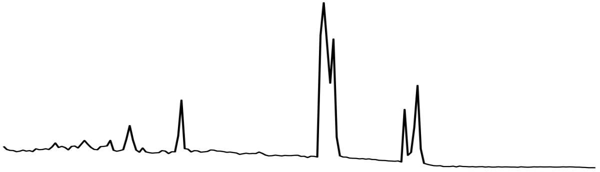

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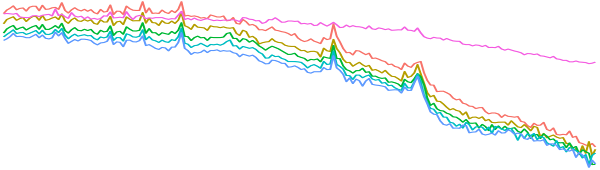

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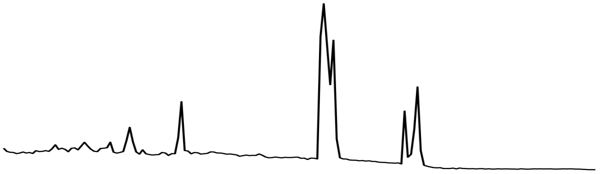

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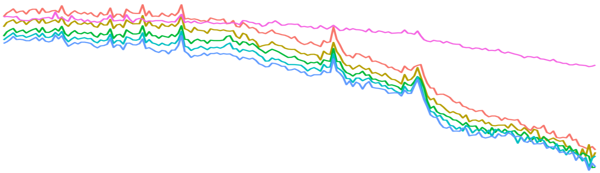
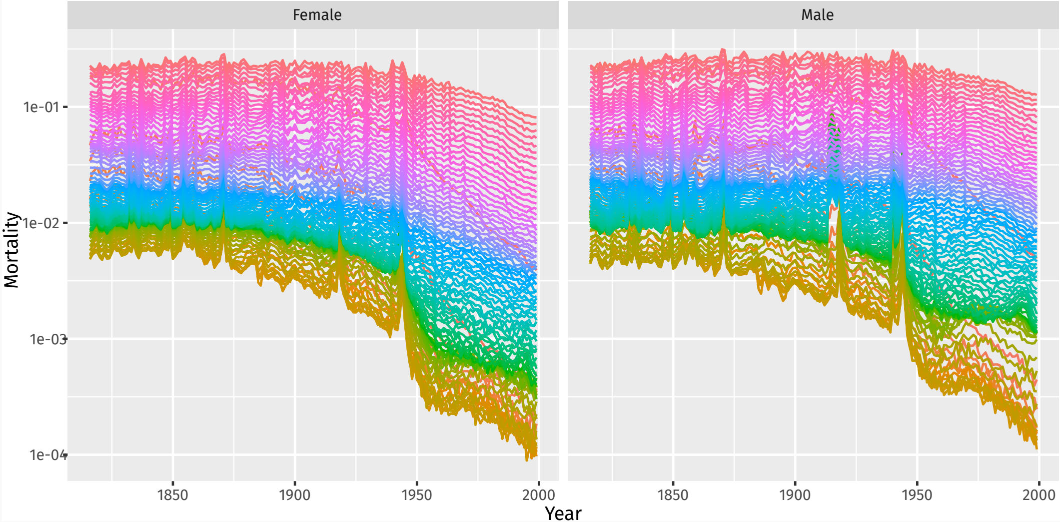


Example: French mortality

```r
fr_mortality
```

```
# A tsibble: 31,648 x 4 [1Y]
# Key:       Age, Sex [172]
    Year   Age Sex    Mortality
   <int> <int> <chr>      <dbl>
 1  1816     0 Female     0.187
 2  1817     0 Female     0.182
 3  1818     0 Female     0.186
 4  1819     0 Female     0.197
 5  1820     0 Female     0.181
 6  1821     0 Female     0.182
 7  1822     0 Female     0.207
 8  1823     0 Female     0.192
 9  1824     0 Female     0.199
10  1825     0 Female     0.194
# i 31,638 more rows
```

Example: French mortality



Example: French mortality

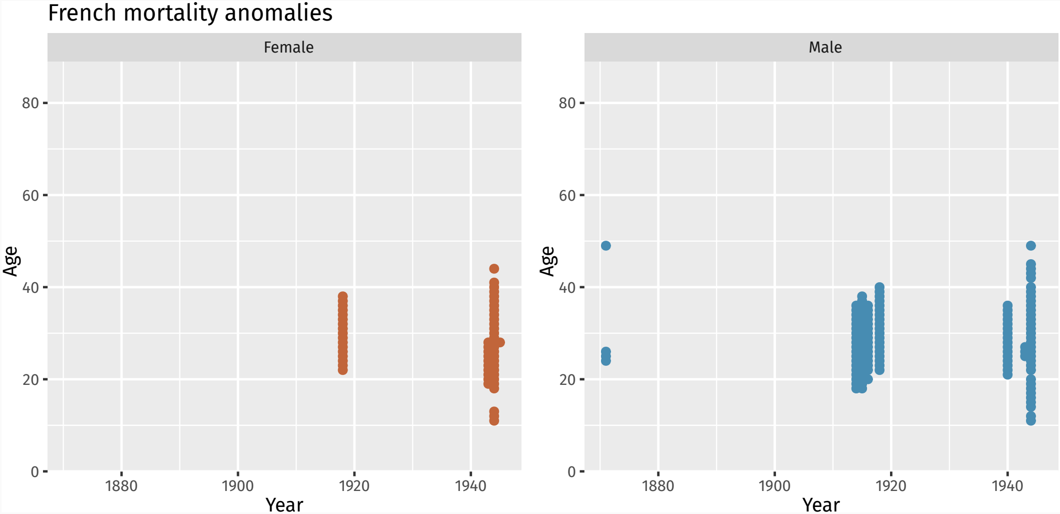
```r
fr_fit ← fr_mortality ▷
  model(stl = STL(log(Mortality)))

fr_sigma ← augment(fr_fit) ▷
  group_by(Age, Sex) ▷
  summarise(sigma = IQR(.innov)/1.349, .groups = "drop")

fr_scores ← augment(fr_fit) ▷
  left_join(fr_sigma) ▷
  mutate(
    s = -log(dnorm(.innov / sigma)),
    prob = lookout(density_scores = s, threshold_probability = 0.9)
  )
```

Example: French mortality

```r
fr_scores |> arrange(prob)
```

```
# A tibble: 31,648 x 7
     Age Sex     Year Mortality .innov     s    prob
   <int> <chr>  <int>     <dbl>  <dbl> <dbl>   <dbl>
 1    28 Female  1944    0.0170   1.45  373. 0.00737
 2    25 Female  1944    0.0191   1.59  331. 0.00831
 3    26 Female  1944    0.0176   1.50  266. 0.0104
 4    24 Female  1944    0.0150   1.40  259. 0.0106
 5    27 Female  1944    0.0178   1.50  228. 0.0121
 6    25 Male    1944    0.0432   1.89  170. 0.0163
 7    18 Male    1914    0.0798   2.06  170. 0.0163
 8    21 Female  1944    0.0120   1.29  168. 0.0165
 9    27 Male    1944    0.0388   1.78  168. 0.0165
10    23 Female  1944    0.0134   1.29  167. 0.0166
# i 31,638 more rows
```

Example: French mortality



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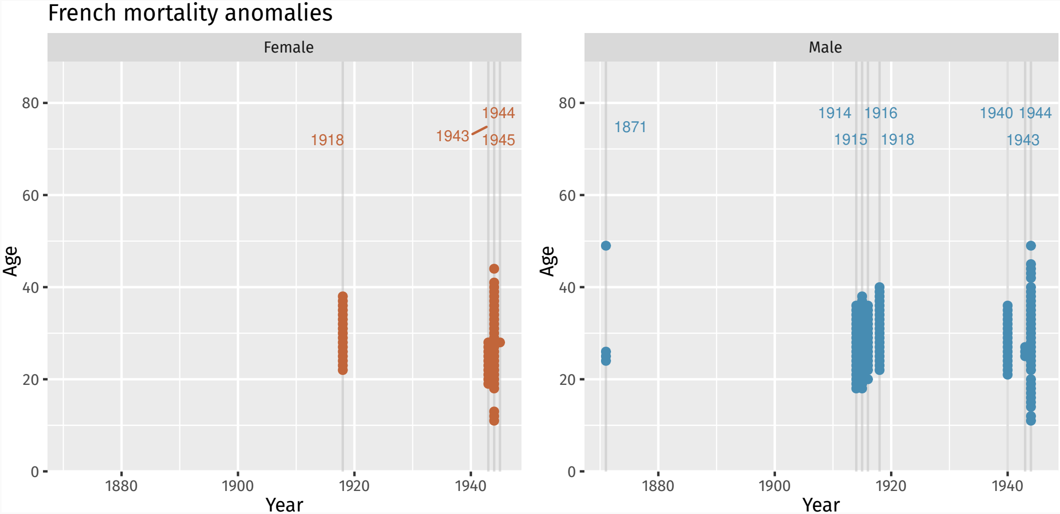
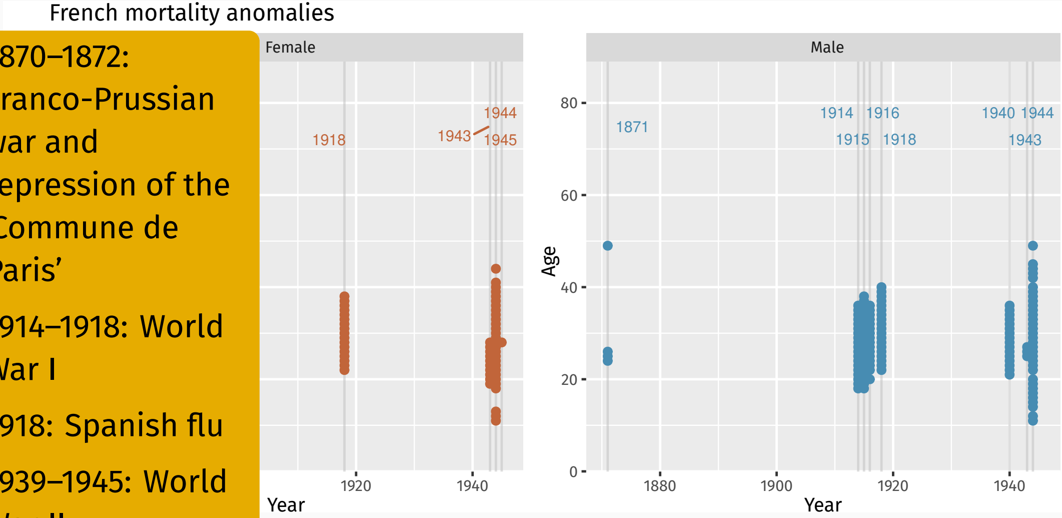


Example: French mortality

- 1870–1872: Franco-Prussian war and repression of the ‘Commune de Paris’
- 1914–1918: World War I
- 1918: Spanish flu
- 1939–1945: World War II



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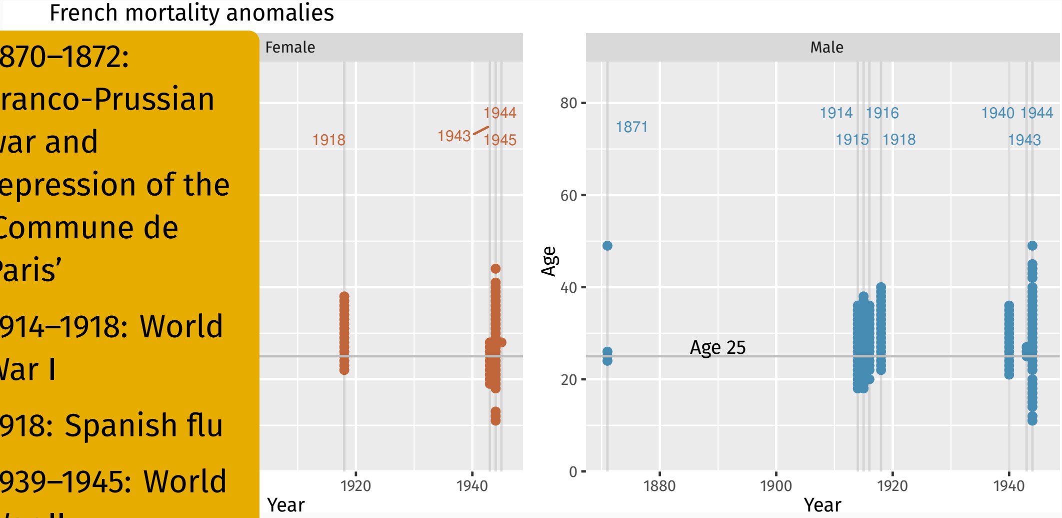

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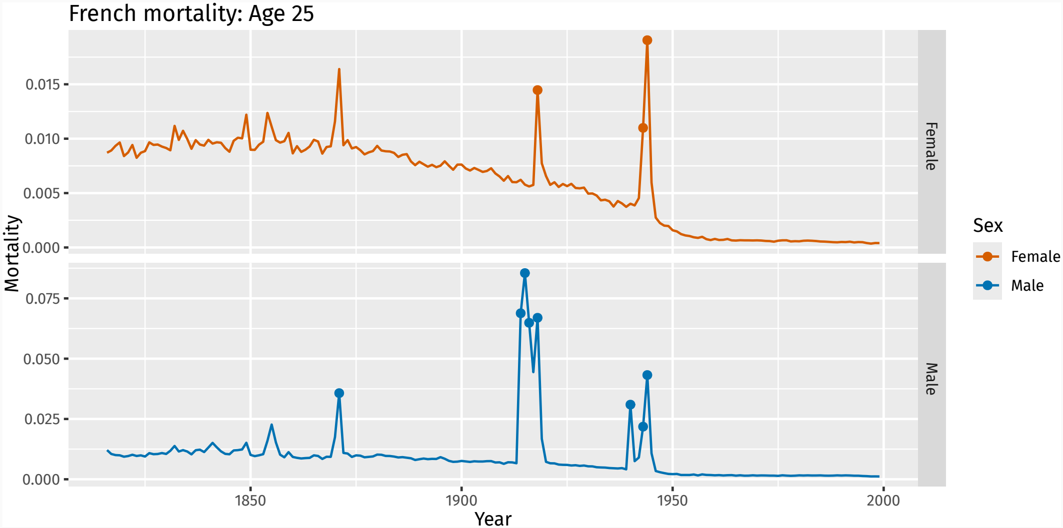

More information

- **Slides:** robjhyndman.com/isf2024
- **Incomplete book:** OTexts.com/weird