

# Time Series Analysis & Forecasting Using R

[bit.ly/fable2023](https://bit.ly/fable2023)

## 8. ARIMA models



# Outline

- 1 ARIMA models
- 2 Lab Session 16
- 3 Seasonal ARIMA models
- 4 Lab Session 17
- 5 Forecast ensembles

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# ARIMA models

- AR:** autoregressive (lagged observations as inputs)
- I:** integrated (differencing to make series stationary)
- MA:** moving average (lagged errors as inputs)

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An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

# Stationarity

## Definition

If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

# Stationarity

## Definition

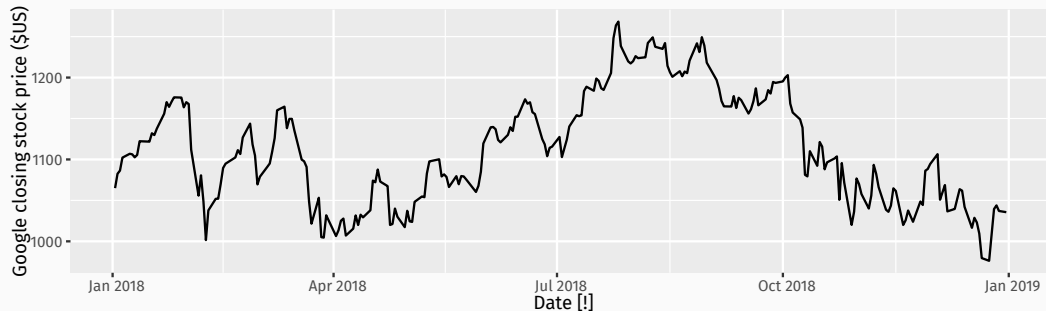
If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

# Stationary?

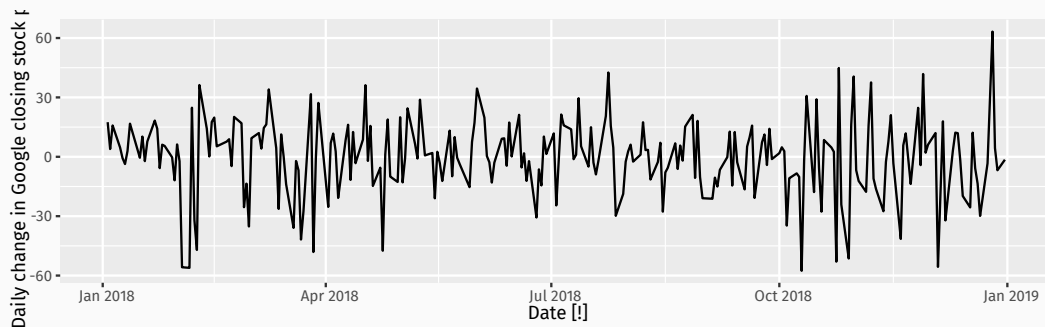
```
gafa_stock |>  
  filter(Symbol == "GOOG", year(Date) == 2018) |>  
  autoplot(Close) +  
  labs(y = "Google closing stock price ($US)")
```





# Stationary?

```
gafa_stock |>  
  filter(Symbol == "GOOG", year(Date) == 2018) |>  
  autoplot(difference(Close)) +  
  labs(y = "Daily change in Google closing stock price")
```



# Differencing

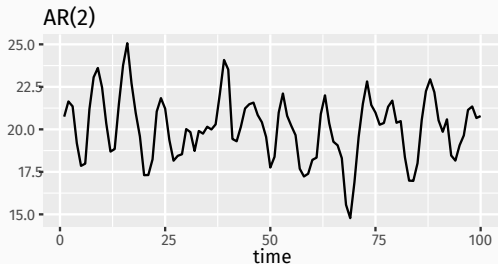
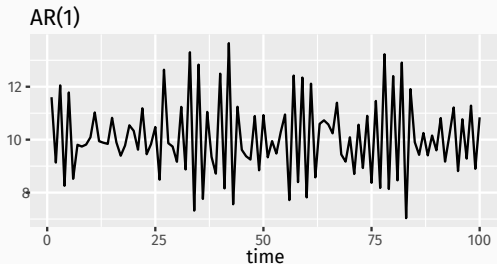
- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series.
- Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time.
- In practice, it is almost never necessary to go beyond second-order differences.

# Autoregressive models

## Autoregressive (AR) models:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where  $\varepsilon_t$  is white noise. A multiple regression with **lagged values** of  $y_t$  as predictors.



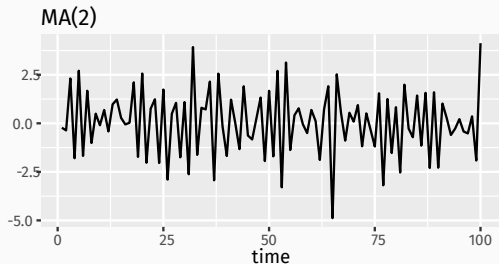
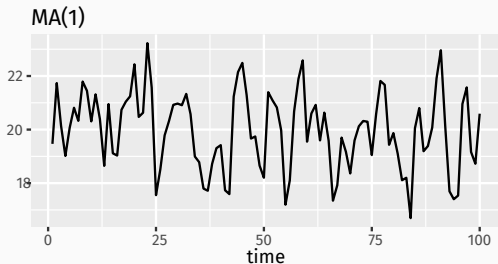
— Cyclic behaviour is possible when  $p \geq 2$

# Moving Average (MA) models

## Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \dots + \theta_q \varepsilon_{t-q},$$

where  $\varepsilon_t$  is white noise. A multiple regression with **lagged errors** as predictors. *Don't confuse with moving average smoothing!*



# ARIMA models

## Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

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- Predictors include both **lagged values of  $y_t$  and lagged errors.**

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## Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing.**
- $d$ -differenced series follows an ARMA model.
- Need to choose  $p$ ,  $d$ ,  $q$  and whether or not to include  $c$ .

# ARIMA models

## ARIMA( $p, d, q$ ) model

AR:  $p$  = order of the autoregressive part

I:  $d$  = degree of first differencing involved

MA:  $q$  = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR( $p$ ): ARIMA( $p,0,0$ )
- MA( $q$ ): ARIMA(0,0, $q$ )



# Example: National populations

```
fit <- global_economy |>  
  model(arima = ARIMA(Population))  
fit
```

```
# A mable: 263 x 2
```

```
# Key:      Country [263]
```

Country	arima
<fct>	<model>
1 Afghanistan	<ARIMA(4,2,1)>
2 Albania	<ARIMA(0,2,2)>
3 Algeria	<ARIMA(2,2,2)>
4 American Samoa	<ARIMA(2,2,2)>
5 Andorra	<ARIMA(2,1,2) w/ drift>
6 Angola	<ARIMA(4,2,1)>
7 Antigua and Barbuda	<ARIMA(2,1,2) w/ drift>
8 Arab World	<ARIMA(0,2,1)>
9 Argentina	<ARIMA(2,2,2)>

# Example: National populations

```
fit |>  
  filter(Country == "Australia") |>  
  report()
```

Series: Population

Model: ARIMA(0,2,1)

Coefficients:

ma1

-0.661

s.e. 0.107

sigma^2 estimated as 4.063e+09: log likelihood=-699

AIC=1401 AICc=1402 BIC=1405

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Series: Population

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Coefficients:

ma1  
-0.661  
s.e. 0.107

$$y_t = 2y_{t-1} - y_{t-2} - 0.7\varepsilon_{t-1} + \varepsilon_t$$
$$\varepsilon_t \sim \text{NID}(0, 4 \times 10^9)$$

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# Understanding ARIMA models

- If  $c = 0$  and  $d = 0$ , the long-term forecasts will go to zero.
- If  $c = 0$  and  $d = 1$ , the long-term forecasts will go to a non-zero constant.
- If  $c = 0$  and  $d = 2$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 0$ , the long-term forecasts will go to the mean of the data.
- If  $c \neq 0$  and  $d = 1$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 2$ , the long-term forecasts will follow a

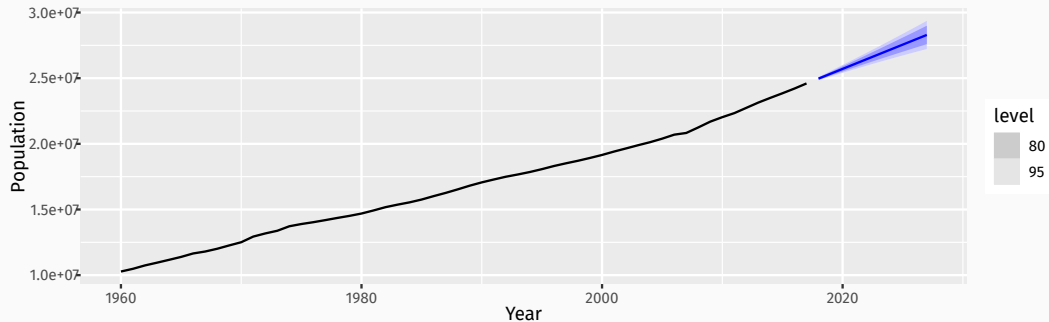
# Understanding ARIMA models

## Forecast variance and $d$

- The higher the value of  $d$ , the more rapidly the prediction intervals increase in size.
- For  $d = 0$ , the long-term forecast standard deviation will go to the standard deviation of the historical data.

# Example: National populations

```
fit |>  
  forecast(h = 10) |>  
  filter(Country == "Australia") |>  
  autoplot(global_economy)
```



# How does ARIMA() work?

## Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences  $d$  via KPSS test.
- Select  $p, q$  and inclusion of  $c$  by minimising AICc.
- Use stepwise search to traverse model space.

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$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[ 1 + \frac{(p + q + k + 2)}{T - p - q - k - 2} \right].$$

where  $L$  is the maximised likelihood fitted to the *differenced* data,  $k = 1$  if  $c \neq 0$  and  $k = 0$  otherwise.



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where  $L$  is the maximised likelihood fitted to the *differenced* data,  $k = 1$  if  $c \neq 0$  and  $k = 0$  otherwise.

Note: Can't compare AICc for different values of  $d$ .

# How does ARIMA() work?

**Step1:** Select current model (with smallest AICc) from:

ARIMA(2,  $d$ , 2)

ARIMA(0,  $d$ , 0)

ARIMA(1,  $d$ , 0)

ARIMA(0,  $d$ , 1)

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**Step 2:** Consider variations of current model:

- vary one of  $p$ ,  $q$ , from current model by  $\pm 1$ ;
- $p$ ,  $q$  both vary from current model by  $\pm 1$ ;
- Include/exclude  $c$  from current model.

Model with lowest AICc becomes current model.

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- $p$ ,  $q$  both vary from current model by  $\pm 1$ ;
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**Repeat Step 2 until no lower AICc can be found.**

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## Lab Session 16

For the United States GDP data (from `global_economy`):

- Fit a suitable ARIMA model for the logged data.
- Produce forecasts of your fitted model. Do the forecasts look reasonable?

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# Seasonal ARIMA models

ARIMA	$(p, d, q)$	$(P, D, Q)_m$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

- $m$  = number of observations per year.
- $d$  first differences,  $D$  seasonal differences
- $p$  AR lags,  $q$  MA lags
- $P$  seasonal AR lags,  $Q$  seasonal MA lags

Seasonal and non-seasonal terms combine multiplicatively

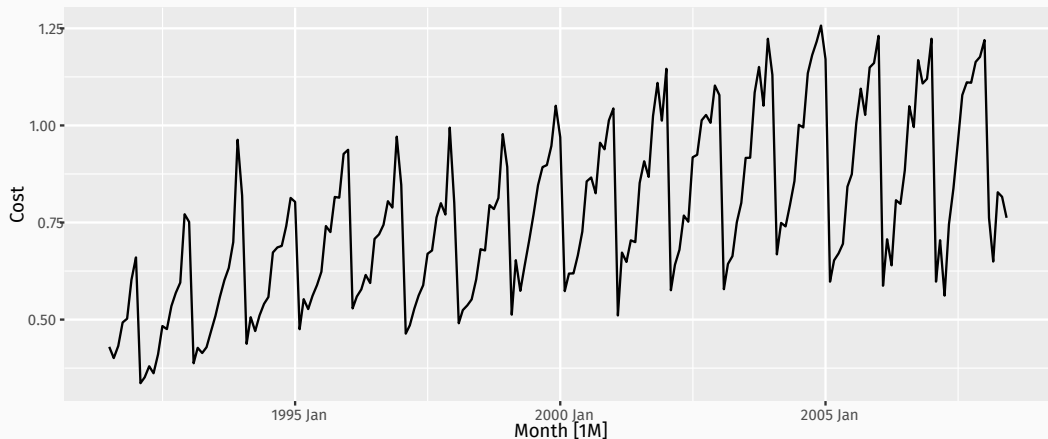


# Corticosteroid drug sales

```
h02 <- PBS |>  
  filter(ATC2 == "H02") |>  
  summarise(Cost = sum(Cost) / 1e6)
```

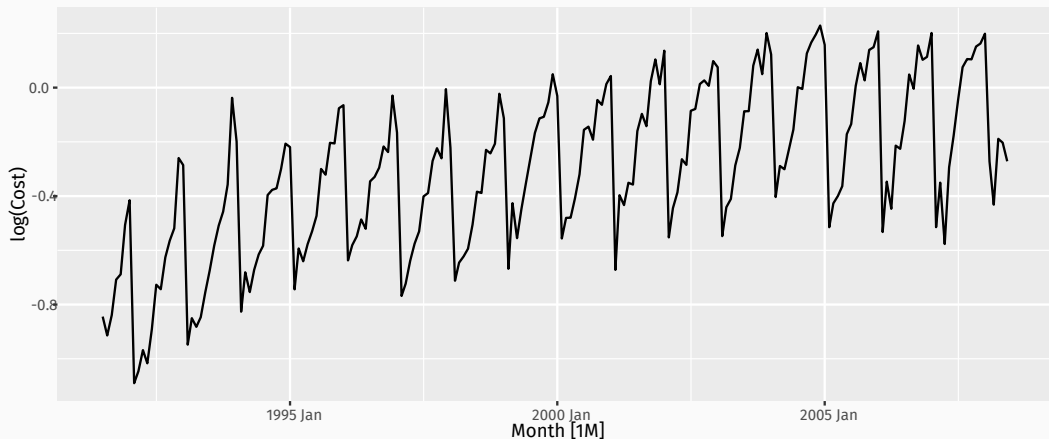
# Corticosteroid drug sales

```
h02 |> autoplot(  
  Cost  
)
```



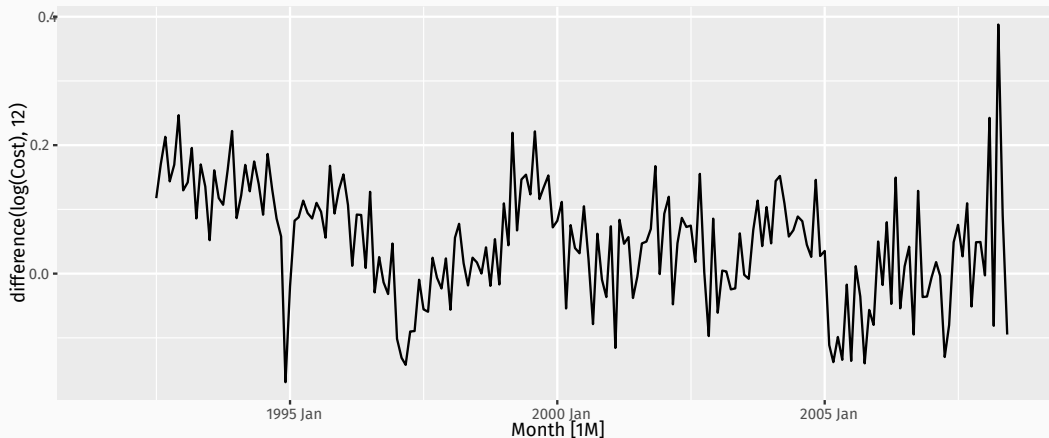
# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost)  
)
```



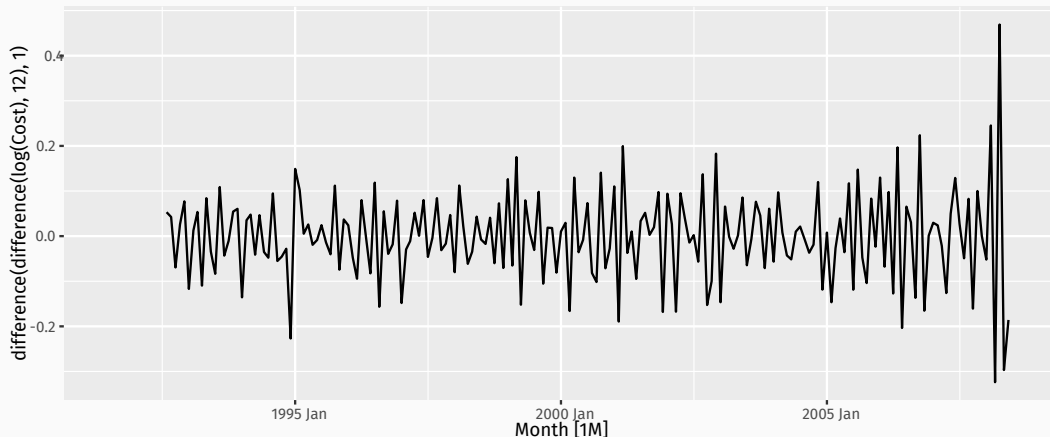
# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost) |> difference(12)  
)
```



# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost) |> difference(12) |> difference(1)  
)
```



# Example: US electricity production

```
h02 |>  
  model(arima = ARIMA(log(Cost))) |>  
  report()
```

Series: Cost

Model: ARIMA(2,1,0)(0,1,1)[12]

Transformation: log(Cost)

Coefficients:

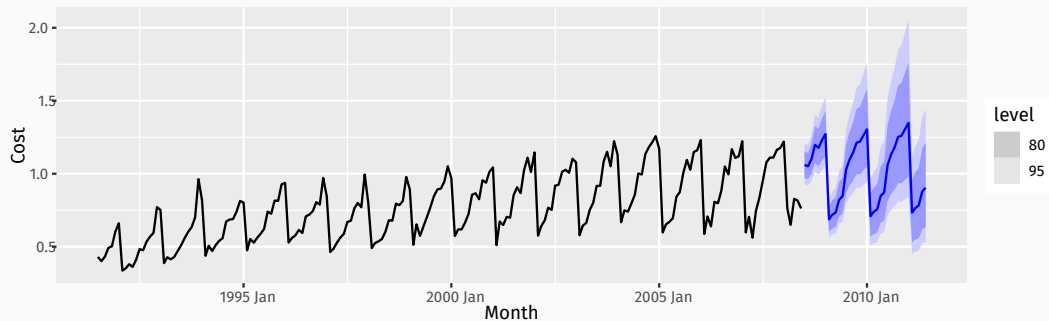
	ar1	ar2	sma1
	-0.8491	-0.4207	-0.6401
s.e.	0.0712	0.0714	0.0694

sigma^2 estimated as 0.004387: log likelihood=245

AIC=-483 AICc=-483 BIC=-470

# Example: US electricity production

```
h02 |>  
  model(arima = ARIMA(log(Cost))) |>  
  forecast(h = "3 years") |>  
  autoplot(h02)
```



# Corticosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE,
    approximation = FALSE,
    order_constraint = p + q + P + Q <= 9
  ))
report(fit)
```

Series: Cost

Model: ARIMA(4,1,1)(2,1,2)[12]

Transformation: log(Cost)

Coefficients:

	ar1	ar2	ar3	ar4	ma1	sar1	sar2	sma1	sma2
	-0.0425	0.210	0.202	-0.227	-0.742	0.621	-0.383	-1.202	0.496
s.e.	0.2167	0.181	0.114	0.081	0.207	0.242	0.118	0.249	0.213

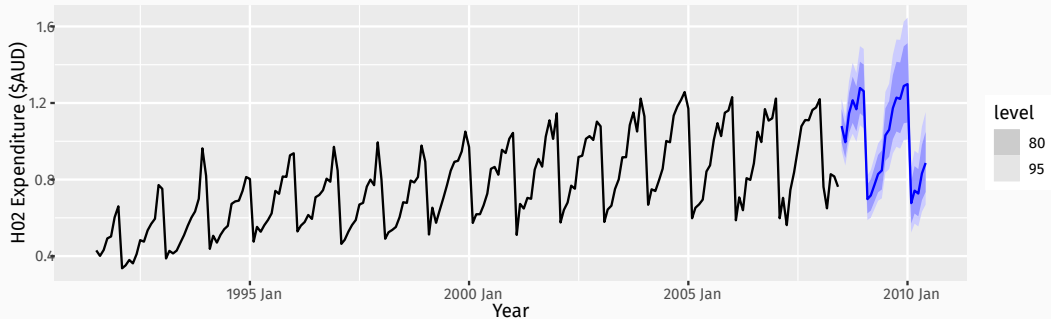
sigma^2 estimated as 0.004049: log likelihood=254

AIC=-489 AICc=-487 BIC=-456



# Corticosteroid drug sales

```
fit |>  
  forecast() |>  
  autoplot(h02) +  
  labs(y = "H02 Expenditure ($AUD)", x = "Year")
```



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## Lab Session 17

For the Australian tourism data (from `tourism`):

- Fit a suitable ARIMA model for all data.
- Produce forecasts of your fitted models.
- Check the forecasts for the “Snowy Mountains” and “Melbourne” regions. Do they look reasonable?

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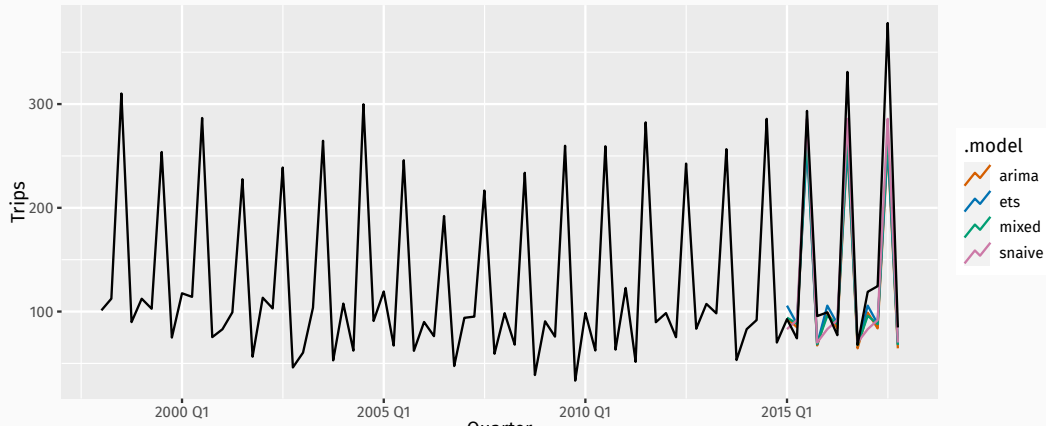
# Forecast ensembles

```
train <- tourism |>
  filter(year(Quarter) <= 2014)
fit <- train |>
  model(
    ets = ETS(Trips),
    arima = ARIMA(Trips),
    snaive = SNAIVE(Trips)
  ) |>
  mutate(mixed = (ets + arima + snaive) / 3)
```

- Ensemble forecast `mixed` is a simple average of the three fitted models.
- `forecast()` will produce distributional forecasts taking into account the correlations between the forecast errors of the component models.

# Forecast ensembles

```
fc <- fit |> forecast(h = "3 years")  
fc |>  
  filter(Region == "Snowy Mountains", Purpose == "Holiday") |>  
  autoplot(tourism, level = NULL)
```



# Forecast ensembles

```
accuracy(fc, tourism) |>  
  group_by(.model) |>  
  summarise(  
    RMSE = mean(RMSE),  
    MAE = mean(MAE),  
    MASE = mean(MASE)  
  ) |>  
  arrange(RMSE)
```

```
# A tibble: 4 x 4  
  .model  RMSE    MAE  MASE  
  <chr>   <dbl> <dbl> <dbl>  
1 mixed   19.8  16.0  0.997  
2 ets     20.2  16.4  1.00  
3 snaive  21.5  17.3  1.17  
4 arima   21.9  17.8  1.06
```

# Forecast ensembles

**Can we do better than equal weights?**



# Forecast ensembles

## Can we do better than equal weights?

- Hard to find weights that improve forecast accuracy.
- Known as the “forecast combination puzzle”.
- Solution: FFORMA

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## FFORMA (Feature-based FOfRecast Model Averaging)

- Vector of time series features used to predict best weights.
- A modification of xgboost is used.
- Method came 2nd in the 2018 M4 international forecasting competition.
- Main author: Pablo Montero-Manso (Monash U)
- Not (yet) available for fable.