

Forecast reconciliation

4. Probabilistic forecast reconciliation

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Outline

1 Example: Australian electricity generation

Notation reminder

- Data: $\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$ where \mathbf{S} is a summing matrix and \mathbf{b}_t is a vector of disaggregated time series
- Base forecasts: $\hat{\mathbf{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$
- MinT: $\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}$ where \mathbf{W}_h is covariance matrix of base forecast errors.

Probabilistic forecasts

- Gaussian
- Non-parametric
- Count

Probabilistic forecast reconciliation

Key papers

- Ben Taieb, Taylor, Hyndman (*ICML*, 2017)
- Jeon, Panagiotelis, Petropoulos (*EJOR*, 2019)
- Ben Taieb, Taylor, Hyndman (*JASA*, 2020)
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020). robjhyndman.com/publications/coherentprob/

Probabilistic forecast reconciliation

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- The reconciled multivariate density must lie on the coherent subspace.
- The univariate density at each node is a convolution of the densities of its children.

Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_b(\mathbf{b}) = |\mathbf{G}^*| \int \hat{f}(\mathbf{G}^- \mathbf{b} + \mathbf{G}_\perp \mathbf{a}) d\mathbf{a}$$

- \hat{f} is density of incoherent base probabilistic forecast
- \mathbf{G}^- is $n \times m$ generalised inverse of \mathbf{G} st $\mathbf{G}\mathbf{G}^- = \mathbf{I}$
- \mathbf{G}_\perp is $n \times (n - m)$ orthogonal complement to \mathbf{G} st $\mathbf{G}\mathbf{G}_\perp = \mathbf{0}$
- $\mathbf{G}^* = (\mathbf{G}^- : \mathbf{G}_\perp)$, and \mathbf{b} and \mathbf{a} are obtained via

the change of variables $\mathbf{y} = \mathbf{G}^* \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_y(\mathbf{y}) = |\mathbf{S}^*| \tilde{f}_b(\mathbf{S}^- \mathbf{y}) \mathbb{1}\{\mathbf{y} \in \mathfrak{s}\}$$

- $\mathbf{S}^* = (\mathbf{S}^{-'} \ \mathbf{S}_{\perp})'$
- \mathbf{S}^- is $m \times n$ generalised inverse of \mathbf{S} such that $\mathbf{S}^- \mathbf{S} = \mathbf{I}$,
- \mathbf{S}_{\perp} is $n \times (n - m)$ orthogonal complement to \mathbf{S} such that

Gaussian reconciliation

If the incoherent base forecasts are $N(\hat{\mu}, \hat{\Sigma})$, then the reconciled density is $N(\mathbf{S}\mathbf{G}\hat{\mu}, \mathbf{S}\mathbf{G}\hat{\Sigma}\mathbf{G}'\mathbf{S}')$.

Bootstrap reconciliation

Reconciling sample paths from incoherent distributions works.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule	Coherent v Incoherent	Coherent v Coherent
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Log Score Not proper

- Ordering preserved if compared using bottom-level only

Energy Score Proper

- Full hierarchy should be used.
- Rankings may

Score optimal reconciliation

Algorithm proposed by Panagiotelis et al (2020) for optimizing \mathbf{G} using stochastic gradient descent to optimize Energy Score.

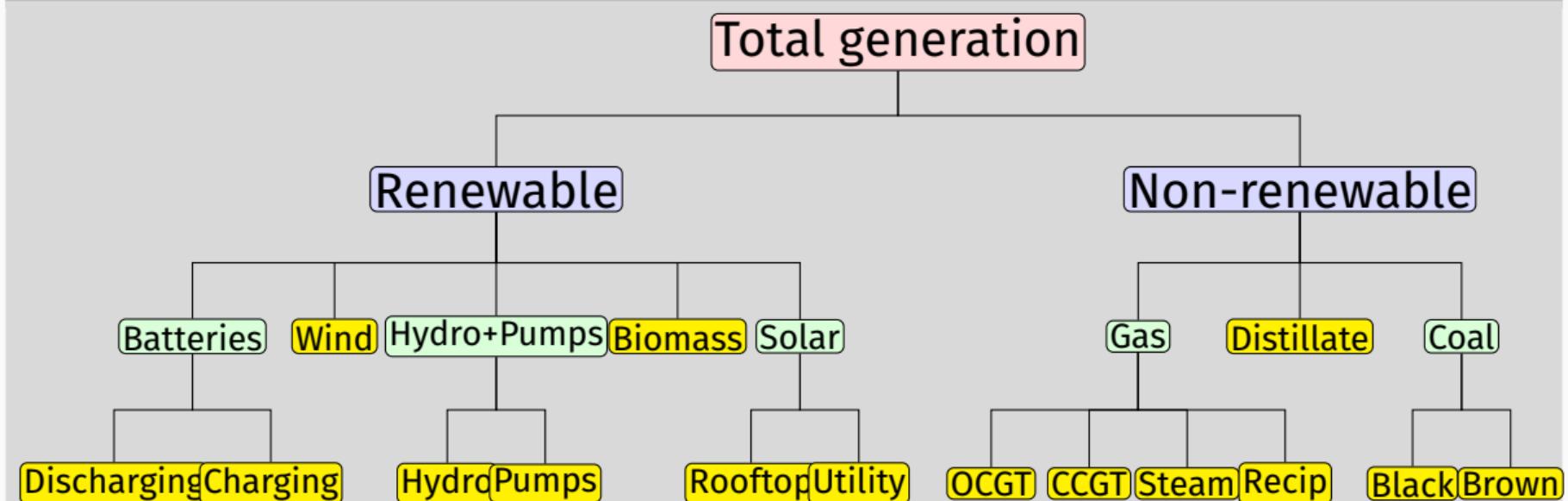
- 1 Compute base forecasts over a test set.
- 2 Compute OLS reconciliation: $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
- 3 Iteratively update \mathbf{G} using SGD with Adam method and ES objective over a test set

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Example: Australian electricity generation

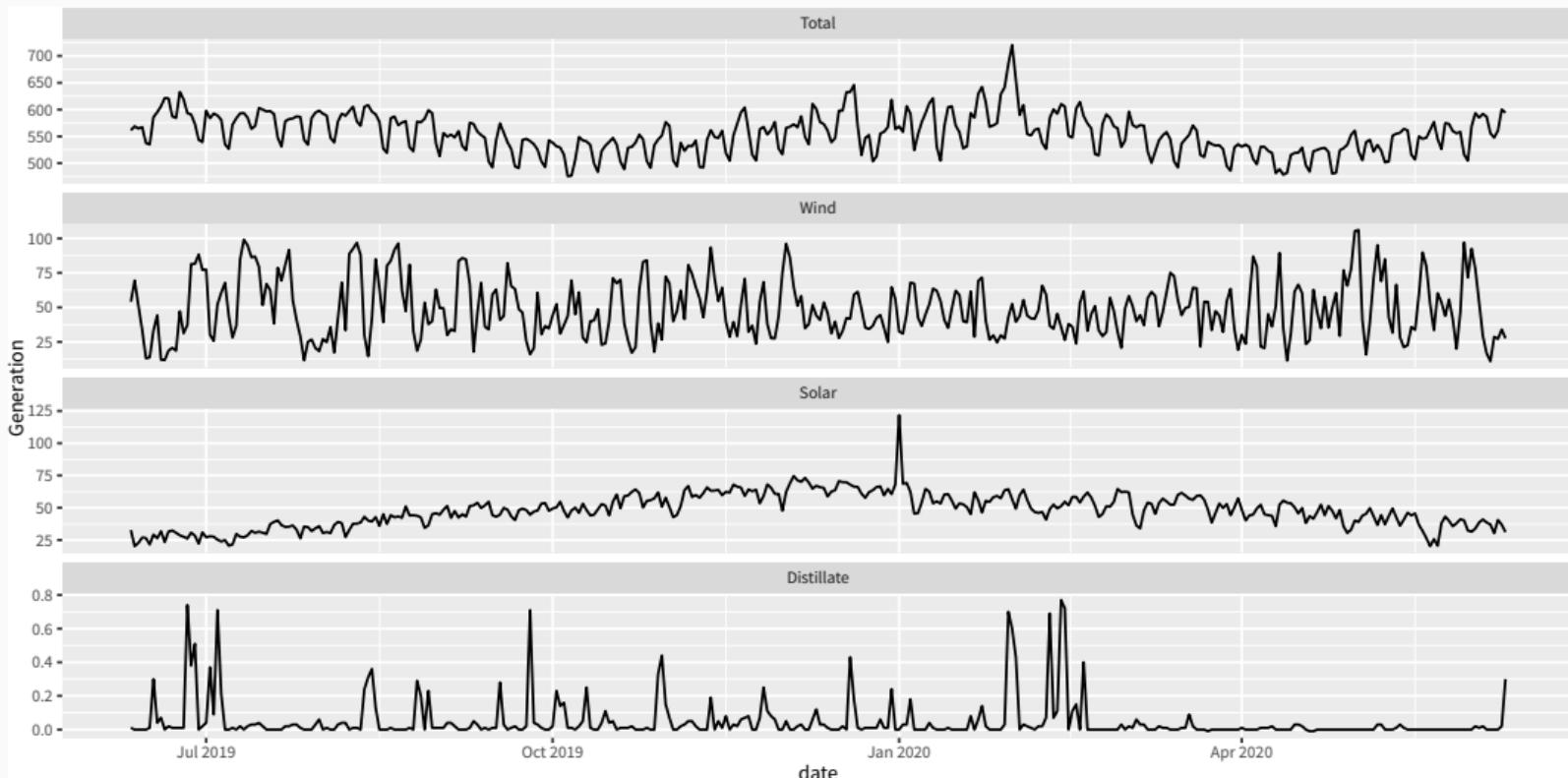
Daily time series from opennem.org.au



$n = 23$ series

$m = 15$ bottom-level series

Example: Australian electricity generation

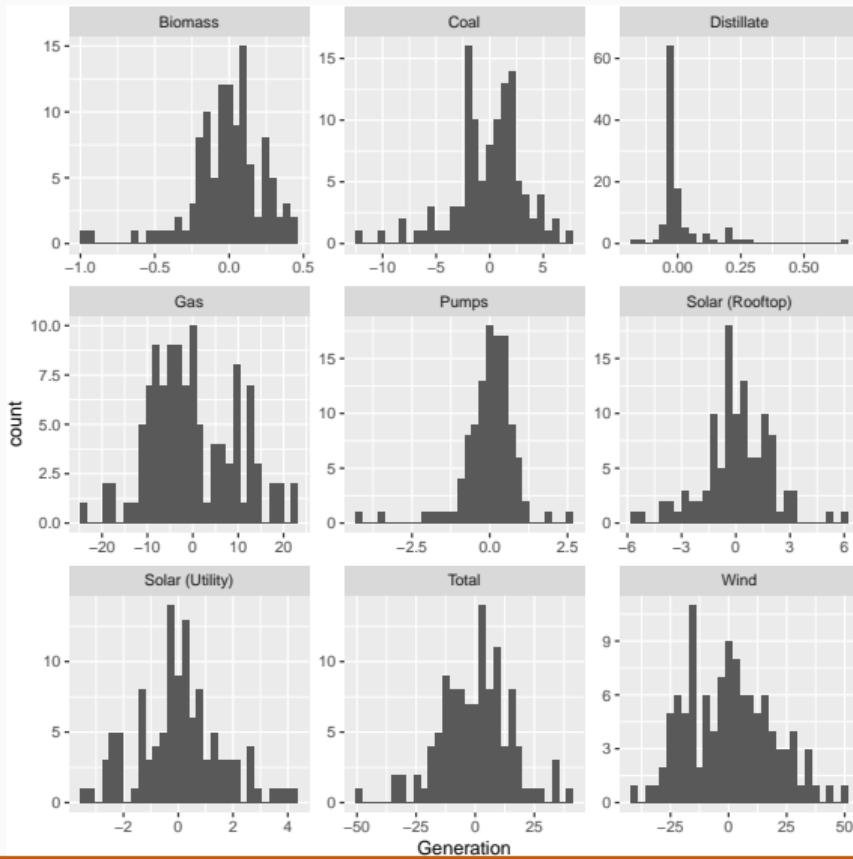


Example: Australian electricity generation

Forecast evaluation

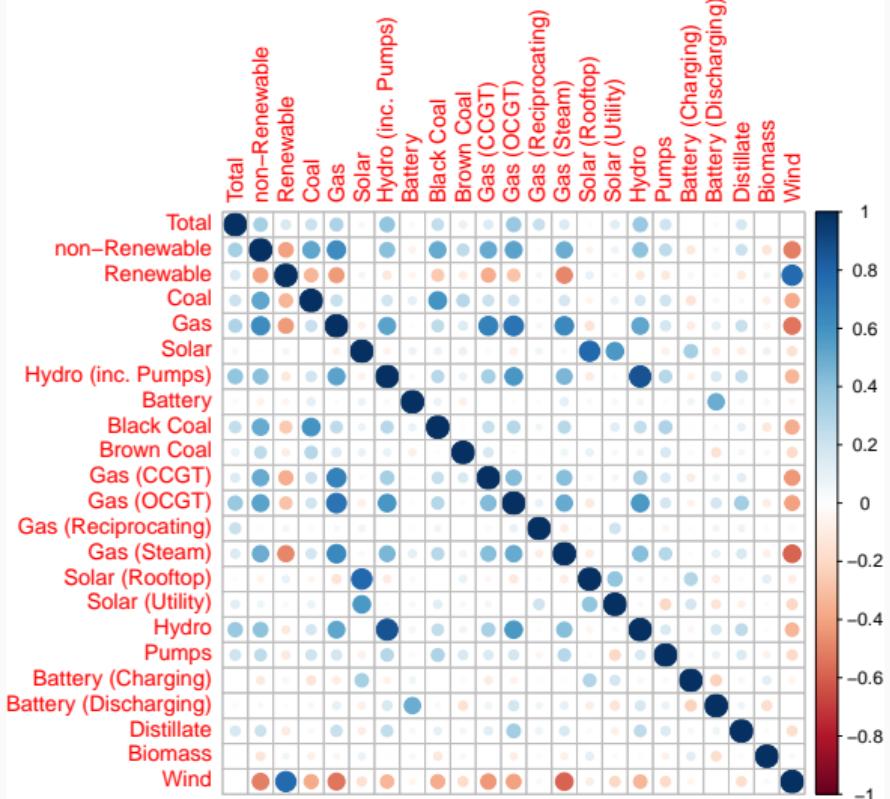
- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using `NNETAR()` function in `fable` package.
- Model could be improved with temperature predictor.

Example: Australian electricity generation



**Histogram of residuals:
2 Oct 2019 – 21 Jan 2020**
Clearly non-Gaussian

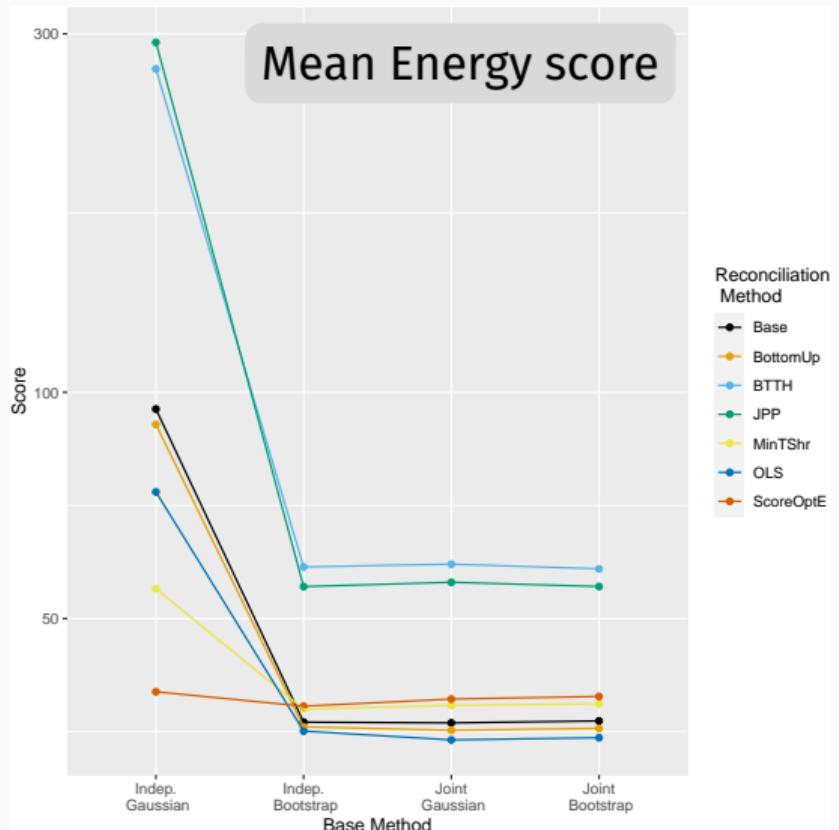
Example: Australian electricity generation



Correlations of residuals:
2 Oct 2019 – 21 Jan 2020

Blue = positive correlation.
Red = negative correlation.
Large = stronger correlations.

Example: Australian electricity generation



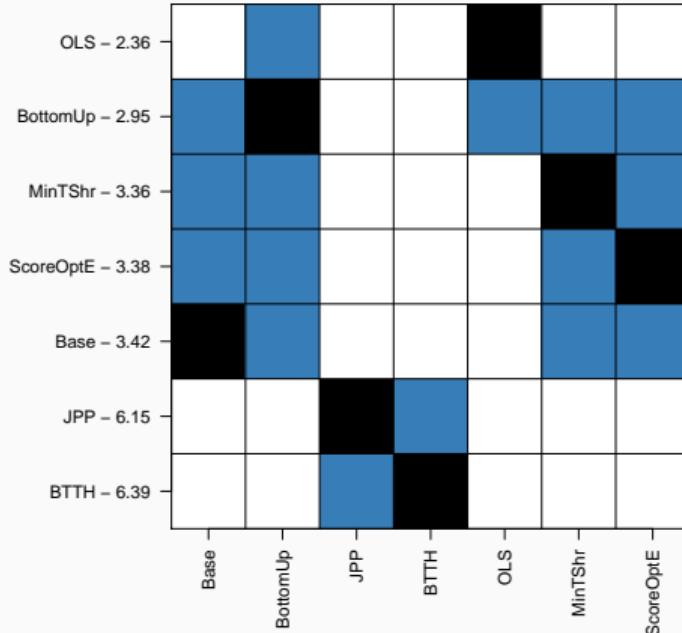
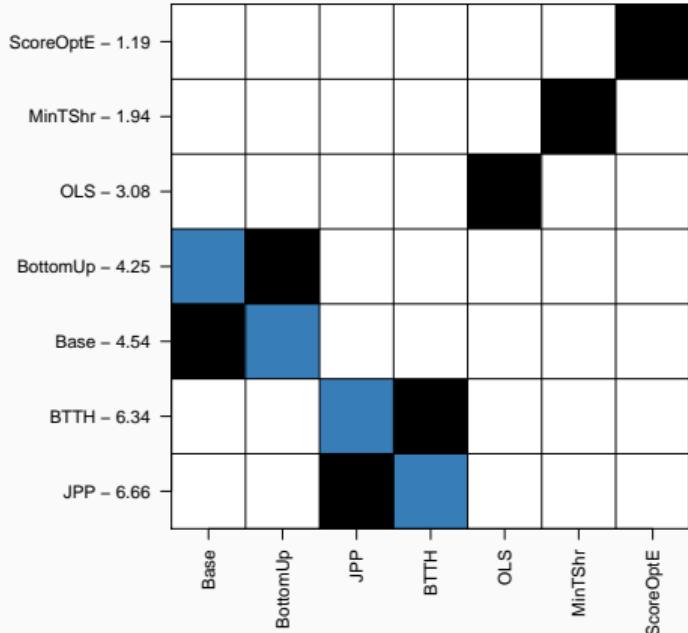
Base residual assumptions

- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

Reconciliation methods

- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS

Example: Australian electricity generation



Nemenyi test for different scores

Base forecasts are independent and

Nemenyi test for different scores

Base forecasts are obtained by jointly

Thanks!



More information

- Slides and papers: **robjhyndman.com**
- Packages: **tidyverts.org**
- Forecasting textbook using fable package:
OTexts.com/fpp3

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References

-  Ben Taieb, S, JW Taylor, and RJ Hyndman (2021). Hierarchical Probabilistic Forecasting of Electricity Demand with Smart Meter Data. *J American Statistical Association* **116**(533), 27–43.
-  Panagiotelis, A, P Gamakumara, G Athanasopoulos, and RJ Hyndman (2023). Probabilistic forecast reconciliation: properties, evaluation and score optimisation. *European J Operational Research* **306**(2), 693–706.