Forecast reconciliation

2. Perspectives on forecast reconciliation

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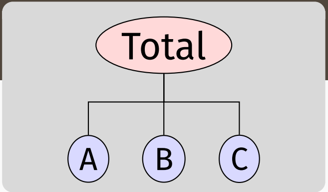
Outline

Outline

Notation reminder



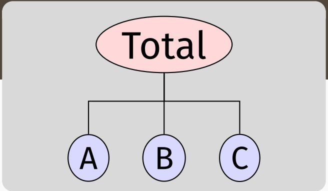
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = “summing matrix” containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Notation reminder



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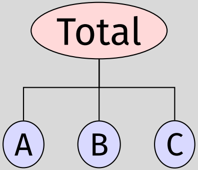
- Base forecasts: $\hat{\boldsymbol{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$
- MinT: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$ where $\boldsymbol{W}_h$ is covariance matrix of base forecast errors.

Notation



Aggregation matrix

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

$$\begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{pmatrix} \boldsymbol{b}_t$$

Constraint matrix

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

where
$$\boldsymbol{C} = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{n_a} & -\boldsymbol{A} \end{bmatrix}$$

Zero-constraint representation

Aggregation matrix A

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Zero-constraint representation

Aggregation matrix $\boldsymbol{A}$

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Constraint matrix $\boldsymbol{C}$

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

- Constraint matrix approach more general & more parsimonious.
- $\boldsymbol{C} = [\boldsymbol{I}_{n_a} \quad -\boldsymbol{A}]$.
- $\boldsymbol{S}$, $\boldsymbol{A}$ and $\boldsymbol{C}$ may contain any real values (not just 0s and 1s).

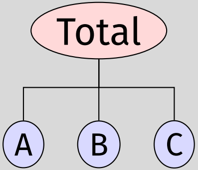
Zero-constraint representation

Assuming $\boldsymbol{C}$ is full rank

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\hat{\boldsymbol{y}}_{T+h|T}$$
$$\text{where} \qquad \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$$

- Originally proved by Byron (1978) & Byron (1979) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- $\boldsymbol{M} = \boldsymbol{S}\boldsymbol{G}$ (the MinT solution)
- Leads to more efficient reconciliation than using $\boldsymbol{G}$.

Zero-constraint representation



Suppose $\boldsymbol{W}_h = \boldsymbol{I}$. Then

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}
\end{aligned}
$$

$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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Outline

The coherent subspace

Coherent subspace

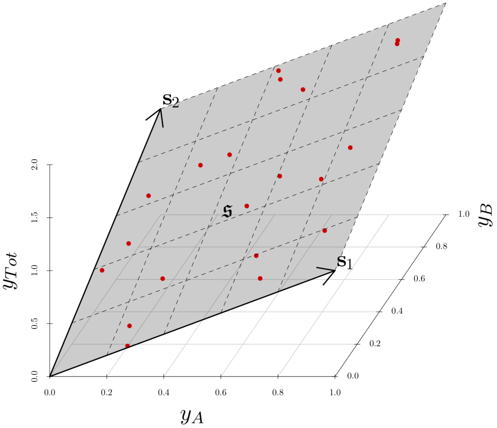
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An *n*-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

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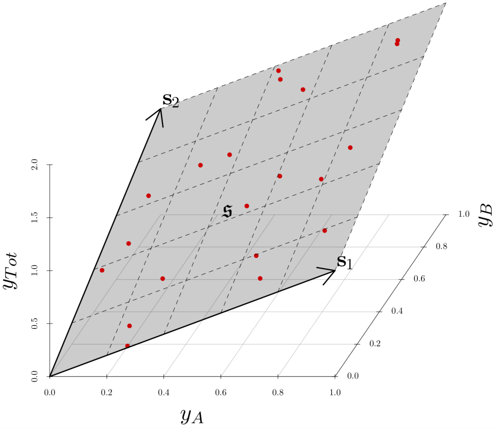
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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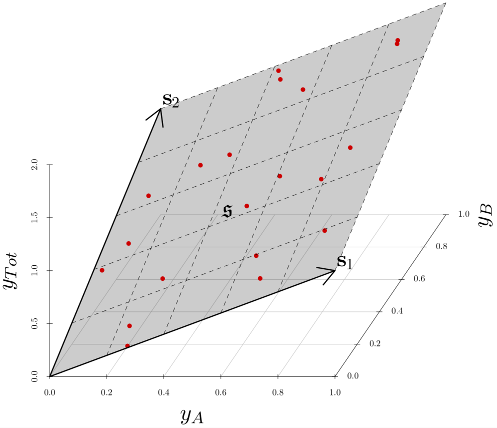


$y_{Tot} = y_A + y_B$

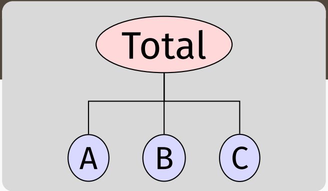
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \to \mathfrak{s}$. $\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

The coherent subspace

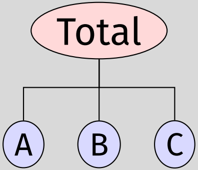


The columns of $\boldsymbol{S}$ form a basis set for $\mathfrak{s}$.

They are not unique.

Each corresponds to different vector of “bottom-level” series.

The coherent subspace



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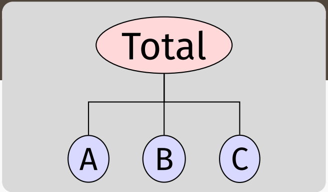

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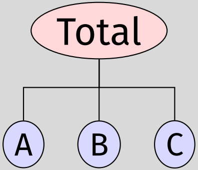

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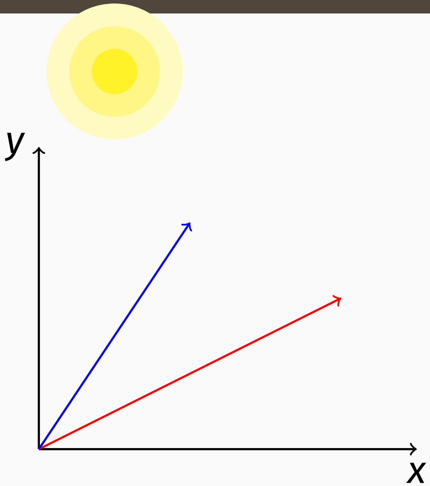
$$\boldsymbol{y} = \begin{pmatrix} \text{Total} \\ A \\ B \\ C \end{pmatrix} \qquad \boldsymbol{S} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \qquad \boldsymbol{b} = \begin{pmatrix} \text{Total} \\ B + A \\ C + B \end{pmatrix}$$

Projections in linear algebra



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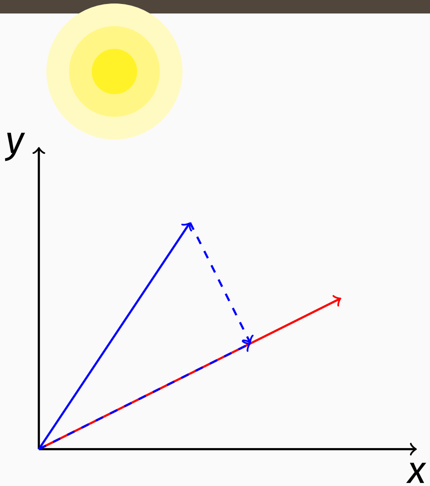

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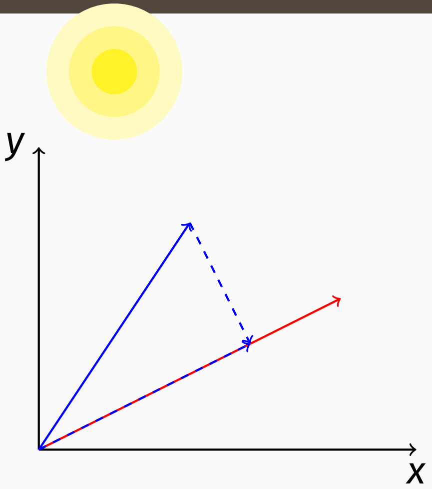

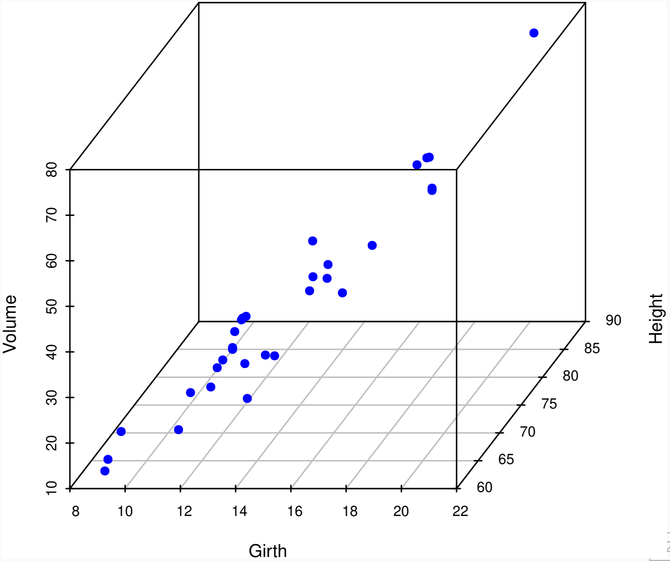

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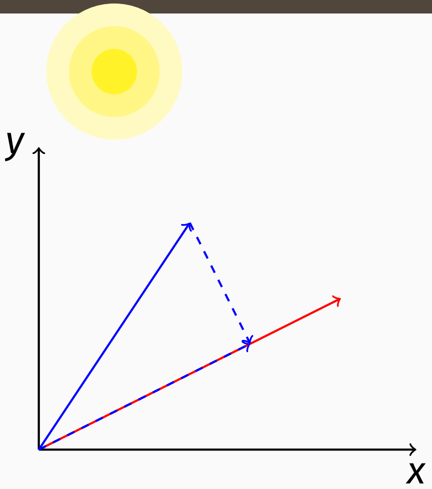

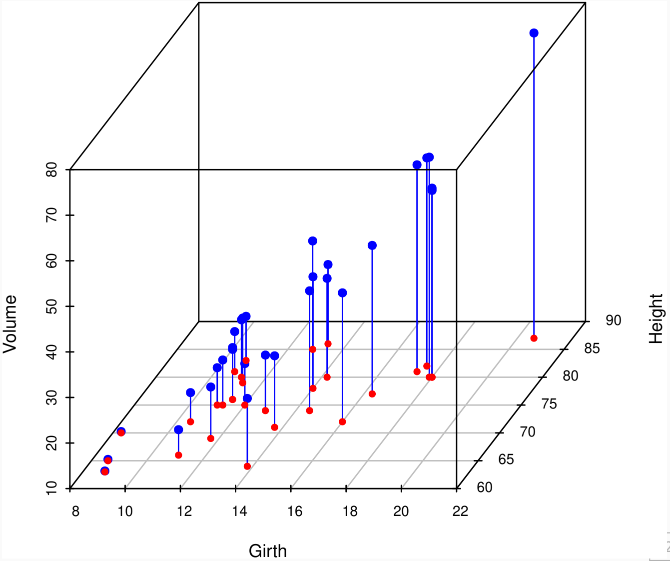

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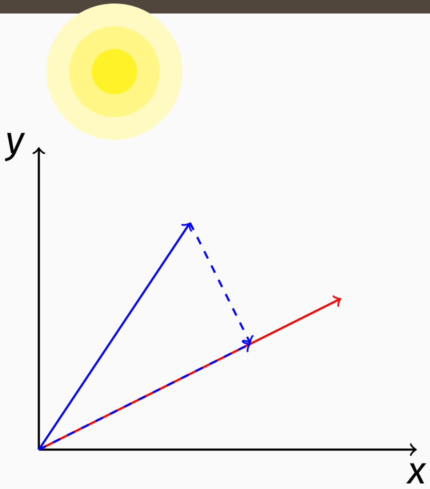

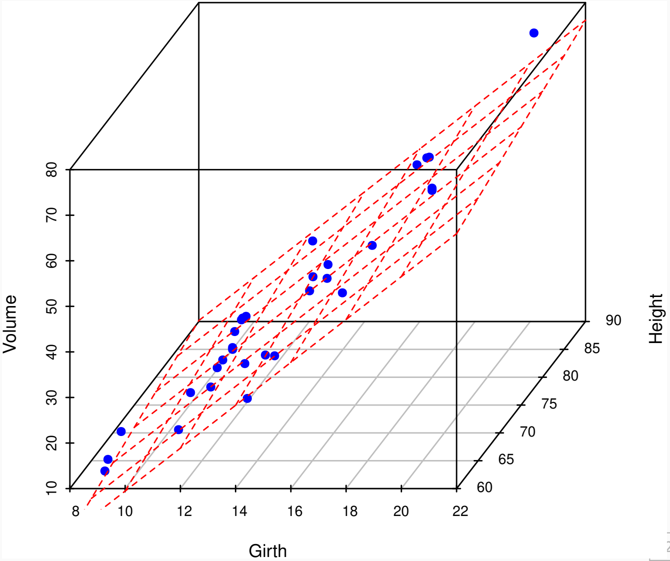

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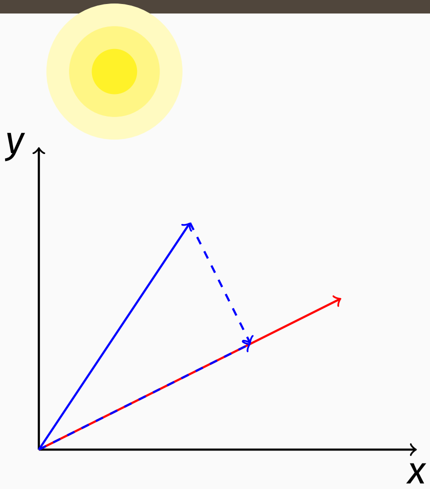
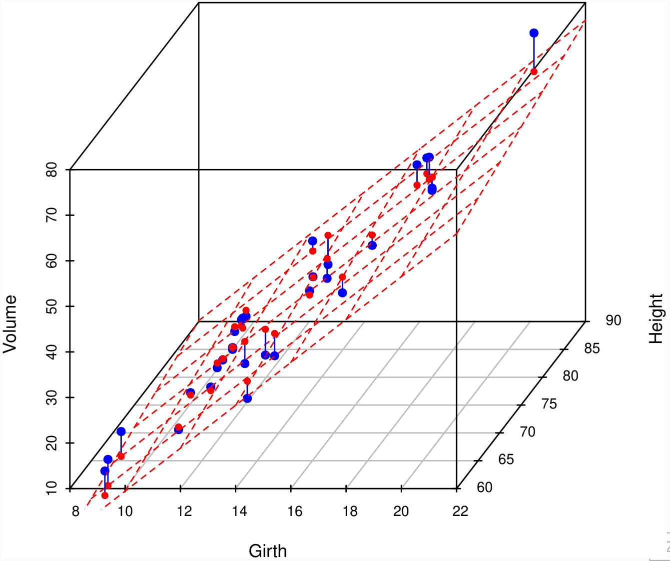


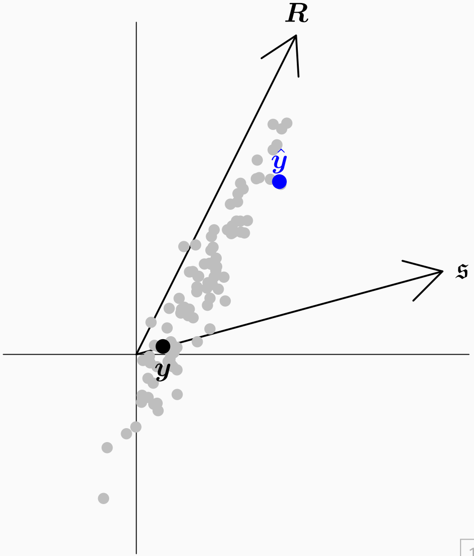


Projections in linear algebra

- A projection is a linear transformation $\boldsymbol{M}$ such that $\boldsymbol{M}^2 = \boldsymbol{M}$.
- i.e., M is idempotent: it leaves its image unchanged.
- $\boldsymbol{M}$ projects onto $\mathfrak{s}$ if $\boldsymbol{My} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- All eigenvalues of $\boldsymbol{M}$ are either 0 or 1.
- All singular values of $\boldsymbol{M}$ are greater than or equal to 1 (with equality iff $\boldsymbol{M}$ is orthogonal).
- A projection is *orthogonal* if $\boldsymbol{M}' = \boldsymbol{M}$.
- If a projection is not orthogonal, it is called *oblique*.
- In regression, OLS is an orthogonal projection onto space spanned by predictors.

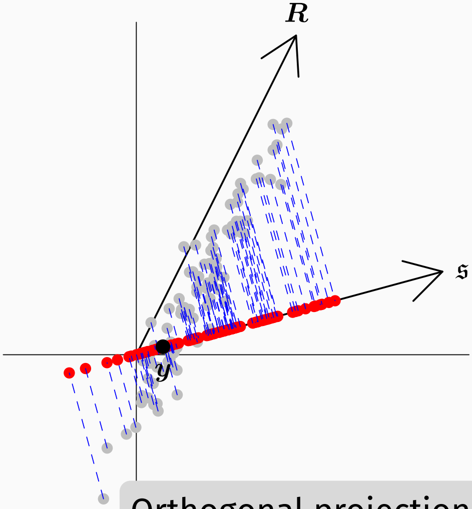
Linear projection reconciliation

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts



Linear projection reconciliation

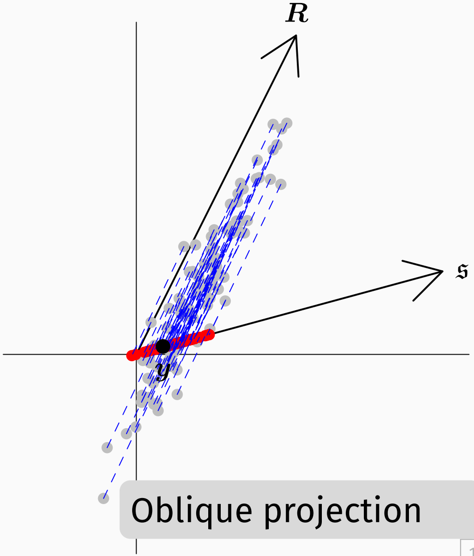
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.



Orthogonal projection

Linear projection reconciliation

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.



Oblique projection

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{M}$ is a projection onto $\mathfrak{s}$ if and only if $\boldsymbol{M}\boldsymbol{y} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- Coherent base forecasts are unchanged since $\boldsymbol{M}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}$
- If $\hat{\boldsymbol{y}}$ is unbiased, then $\tilde{\boldsymbol{y}}$ is also unbiased since

  $$\text{E}(\tilde{\boldsymbol{y}}_{t+h|t}) = \text{E}(\boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\text{E}(\hat{\boldsymbol{y}}_{t+h|t}) = \text{E}(\hat{\boldsymbol{y}}_{t+h|t}),$$

  and unbiased estimates must lie on $\mathfrak{s}$.
- The projection is orthogonal if and only if $\boldsymbol{M}' = \boldsymbol{M}$.
- If $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$, then projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}, \qquad \text{where} \quad \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$$

$$\begin{array}{llll}
\text{OLS:} & \Psi = \boldsymbol{I} & \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}' & = \boldsymbol{I} - \boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}\boldsymbol{C} \\
\text{MinT:} & \Psi = \boldsymbol{W}_h & \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1} & = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}
\end{array}$$

- $\boldsymbol{M}$ is orthogonal iff $\Psi = \boldsymbol{I}$.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$ is the covariance matrix of the base forecast errors.
- $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T] = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$ is minimized when $\Psi = \boldsymbol{W}_h$.

Mean square error bounds

Panagiotelis, Gamakumara, Athanasopoulos, and Hyndman (2021)

Distance reducing property

Let $\|\boldsymbol{u}\|_\Psi = \boldsymbol{u}'\Psi\boldsymbol{u}$. Then

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\|_\Psi \le \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|_\Psi$$

- Ψ-projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure*.
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.
- Other measures of forecast accuracy may be worse.

Mean square error bounds

Wickramasuriya (2021)

$$
\begin{aligned}
\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}\|_2^2 &= \|\boldsymbol{M}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h})\|_2^2 \\
&\leq \|\boldsymbol{M}\|_2^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2 \\
&= \sigma_{\text{max}}^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2
\end{aligned}
$$

- σ_{max} is the largest eigenvalue of $\boldsymbol{M}$
- $\sigma_{\text{max}} \geq 1$ as $\boldsymbol{M}$ is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

Mean square error bounds

Wickramasuriya (2021)

$$
\begin{aligned}
&\operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]\Big)\\
\le\ &\operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]\Big)\\
\le\ &\operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]\Big)
\end{aligned}
$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

Outline

Minimum trace reconciliation

Minimum trace (MinT) reconciliation

If $\boldsymbol{SG}$ is a projection, then the trace of $\boldsymbol{V}_h = \text{Var}(\tilde{\boldsymbol{y}}_{t+h|t} - \boldsymbol{y}_{t+h})$ is **minimized** when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$$

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciled forecasts

Base forecasts

- Trace of $\boldsymbol{V}_h$ is sum of forecast variances.
- MinT solution is L_2 **optimal** amongst linear unbiased forecasts.

A game theoretic perspective

van Erven and Cugliari (2015)

Find the solution to the minimax problem

$$V = \min_{\tilde{\boldsymbol{y}} \in \mathfrak{s}} \max_{\boldsymbol{y} \in \mathfrak{s}} \left\{ \ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) - \ell(\boldsymbol{y}, \hat{\boldsymbol{y}}) \right\},$$

where ℓ is a loss function, and $\mathfrak{s}$ is the coherent subspace.

- $V \leq 0$: reconciliation guaranteed to reduce loss.
- If $\ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) = \|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_{\Psi} = (\boldsymbol{y} - \tilde{\boldsymbol{y}})'\Psi(\boldsymbol{y} - \tilde{\boldsymbol{y}})$, where Ψ is any symmetric pd matrix, then:
  1. $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi\hat{\boldsymbol{y}}$ will always improve upon the base forecasts;
  2. The MinT solution $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}$ will optimise loss in expectation over any choice of Ψ.

Biased reconciliation

Ben Taieb and Koo (2019)

Regularized empirical risk minimization problem:

$$\min_{\boldsymbol{G}} \frac{1}{Nn}\|\boldsymbol{Y} - \hat{\boldsymbol{Y}}\boldsymbol{G}'\boldsymbol{S}'\|_F + \lambda\|\text{vec}\boldsymbol{G}\|_1,$$

- $N = T - T_1 - h + 1, \quad T_1$ is minimum training sample size
- $\|\cdot\|_F$ is the Frobenius norm
- $\boldsymbol{Y} = [\boldsymbol{y}_{T_1+h}, \dots, \boldsymbol{y}_T]'$
- $\hat{\boldsymbol{Y}} = [\hat{\boldsymbol{y}}_{T_1+h|T_1}, \dots, \hat{\boldsymbol{y}}_{T|T-h}]'$
- λ is a regularization parameter

When $\lambda = 0$: $\quad \hat{\boldsymbol{G}} = \boldsymbol{B}'\hat{\boldsymbol{Y}}(\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}})^{-1}$ where $\boldsymbol{B} = [\boldsymbol{b}_{T_1+h}, \dots, \boldsymbol{b}_T]'$.

MinT expressed as a regression

Since $\tilde{\bm{b}}_{t+h|t} = (\bm{S}'\bm{W}_h^{-1}\bm{S})^{-1}\bm{S}'\bm{W}_h^{-1}\hat{\bm{y}}_{t+h|t}$, we can write the MinT solution as a regression problem:

$$\begin{aligned}
\tilde{\bm{b}}_{t+h|t} &= \arg\min_{\bm{b}} \left[\hat{\bm{y}}_{t+h|t} - \bm{S}\bm{b}\right]' \bm{W}_h^{-1} \left[\hat{\bm{y}}_{t+h|t} - \bm{S}\bm{b}\right] \\
&= \arg\min_{\bm{b}} \left[\bm{b}'\bm{S}'\bm{W}_h^{-1}\bm{S}\bm{b} - 2\bm{b}'\bm{S}'\bm{W}_h^{-1}\hat{\bm{y}}_{t+h|t} + \hat{\bm{y}}'_{t+h|t}\bm{W}_h^{-1}\hat{\bm{y}}_{t+h|t}\right] \\
&= \arg\min_{\bm{b}} \left[\bm{b}'\bm{S}'\bm{W}_h^{-1}\bm{S}\bm{b} - 2\bm{b}'\bm{S}'\bm{W}_h^{-1}\hat{\bm{y}}_{t+h|t}\right]
\end{aligned}$$

- MinT solution is equivalent to a GLS regression of $\hat{\bm{y}}_{t+h|t}$ on $\bm{S}$ with covariance weights $\bm{W}_h^{-1}$.
- The estimated coefficients are the forecasts of the bottom level series.

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$
$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$

$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Solve via quadratic programming:

$$\min_{\boldsymbol{b}}\left[\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - 2\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T}\right] \quad \text{s.t. } \boldsymbol{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \operatorname{tr}\left(\mathrm{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\right)$$
$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Solve via quadratic programming:

$$\min_{\boldsymbol{b}}\left[\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - 2\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T}\right] \quad \text{s.t. } \boldsymbol{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Set-negative-to-zero heuristic solution

- Negative reconciled forecasts at bottom level set to zero
- Remaining forecasts computed via aggregation

(Di Fonzo and Girolimetto, 2023)

Immutable forecasts

Zhang, Kang, Panagiotelis, and Li (2022)

$$\hat{\boldsymbol{y}}_{t+h|t} = \begin{bmatrix}\hat{\boldsymbol{a}}_{t+h|t}\\ \hat{\boldsymbol{v}}_{t+h|t}\\ \hat{\boldsymbol{u}}_{t+h|t}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{I}_{n_b-k} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_k\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{v}}_{t+h|t}\\ \hat{\boldsymbol{u}}_{t+h|t}\end{bmatrix}$$

Suppose $\hat{\boldsymbol{u}}_{t+h|t}$ are fixed and let $\hat{\boldsymbol{w}}_{t+h|t} = \begin{bmatrix}\hat{\boldsymbol{a}}_{t+h|t} - \boldsymbol{A}_2\hat{\boldsymbol{u}}_{t+h|t}\\ \hat{\boldsymbol{v}}_{t+h|t}\end{bmatrix}$.

Optimization problem

$$\min_{\boldsymbol{v}}\left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right]'\boldsymbol{W}_{\boldsymbol{v}}^{-1}\left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right] \qquad \text{where} \qquad \boldsymbol{A}_3 = \begin{bmatrix}\boldsymbol{A}_1\\ \boldsymbol{I}_{n_b-k}\end{bmatrix}$$

and $\boldsymbol{W}_{\boldsymbol{v}}$ contains elements of $\boldsymbol{W}_h$ corresponding to $\hat{\boldsymbol{v}}_{t+h|t}$.

Immutable forecasts

Zhang, Kang, Panagiotelis, and Li (2022)

$$\hat{\boldsymbol{y}}_{t+h|t} = \begin{bmatrix} \hat{\boldsymbol{a}}_{t+h|t} \\ \hat{\boldsymbol{v}}_{t+h|t} \\ \hat{\boldsymbol{u}}_{t+h|t} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{I}_{n_b-k} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_k \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{v}}_{t+h|t} \\ \hat{\boldsymbol{u}}_{t+h|t} \end{bmatrix}$$

Suppose $\hat{\boldsymbol{u}}_{t+h|t}$ are fixed and let $\hat{\boldsymbol{w}}_{t+h|t} = \begin{bmatrix} \hat{\boldsymbol{a}}_{t+h|t} - \boldsymbol{A}_2\hat{\boldsymbol{u}}_{t+h|t} \\ \hat{\boldsymbol{v}}_{t+h|t} \end{bmatrix}$.

Solve with non-negativity constraint

$$\min_{\boldsymbol{v}} \left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right]' \boldsymbol{W}_{\boldsymbol{v}}^{-1} \left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right] \qquad \text{where} \qquad \boldsymbol{A}_3 = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{I}_{n_b-k} \end{bmatrix}$$

$$\text{such that} \qquad \boldsymbol{A}_3\boldsymbol{v} \geq \begin{bmatrix} -\boldsymbol{A}_2\hat{\boldsymbol{u}}_{t+h|t} \\ \boldsymbol{0} \end{bmatrix}$$

Outline

ML and regularization

- Replace the linear regression formulation with a less restrictive method to obtain combinations of forecasts from the various hierarchical levels.
- Coherence is achieved via a bottom-up approach, or by embedding coherence in the ML training.
-

Gleason (2020) attempts to overcome the lack of focus on coherence by adjusting the objective function. Using neural network forecasts, he includes a regularisation term that penalises incoherences in the generated forecasts. This

Outline

In-built coherence

Two-step approach: compute base forecasts $\hat{\boldsymbol{y}}_h$, and then reconcile them to produce $\tilde{\boldsymbol{y}}_h$.

One-step approaches: compute coherent $\tilde{\boldsymbol{y}}_h$ directly.

- Ashouri, Hyndman, and Shmueli (2022): linear regression models
- Pennings and Dalen (2017): state space models
- Villegas and Pedregal (2018): state space models

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \qquad \hat{\boldsymbol{y}}_{t+h} = \boldsymbol{X}^*_{t+h}\hat{\boldsymbol{B}}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$ $\qquad$ $\hat{\boldsymbol{y}}_{t+h} = \boldsymbol{X}^*_{t+h}\hat{\boldsymbol{B}}$ $\qquad$ $\boldsymbol{X}^*_{t+h} = \text{diag}(\boldsymbol{x}'_{t+h,i}, \dots, \boldsymbol{x}'_{t+h,n})$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \qquad \hat{\boldsymbol{y}}_{t+h} = \boldsymbol{X}^*_{t+h}\hat{\boldsymbol{B}} \qquad \boldsymbol{X}^*_{t+h} = \text{diag}(\boldsymbol{x}'_{t+h,i}, \dots, \boldsymbol{x}'_{t+h,n})$$

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h\hat{\boldsymbol{y}}_{t+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{X}^*_{t+h}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

$$\boldsymbol{V}_h = \sigma^2\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h\left[1 + \boldsymbol{X}^*_{T+h}(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}^*_{T+h})'\right]\boldsymbol{W}_h\boldsymbol{S}'(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'$$

Reference: Ashouri, Hyndman, and Shmueli (2022)

In-built coherence

Pennings and Dalen (2017) propose the state space model

$$
\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{S}\boldsymbol{\mu}_t + \boldsymbol{Z}_t\beta + \varepsilon_t, & \varepsilon_t &\sim N(\boldsymbol{0}, \Sigma_\varepsilon), & (1)\\
\boldsymbol{\mu}_t &= \boldsymbol{\mu}_{t-1} + \boldsymbol{\eta}_t, & \eta_t &\sim N(\boldsymbol{0}, \Sigma_\eta). & (2)
\end{aligned}
$$

- Coherent forecasts arise naturally using the Kalman filter
- Covariance matrices difficult to estimate except for small hierarchies.

In-build coherence

A related state space approach was proposed by Villegas and Pedregal (2018), who show that their formulation subsumes bottom-up, top-down, and some forms of forecast reconciliation and combination forecasting.

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