

Forecast reconciliation

2. Perspectives on forecast reconciliation

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Outline

- 1 Time series reconciliation
- 2 Reconciliation via constraints
- 3 Example: reconciling GDP forecasts
- 4 The geometry of forecast reconciliation
- 5 Optimization and reconciliation
- 6 In-built coherence
- 7 Time series cross-validation

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Time series reconciliation

- Stone, Champernowne, and Meade (1942): reconciling national economic accounts (disaggregated into production, income, outlay, capital transactions, etc.)
- Byron (1978): extended Stone's work using more computationally efficient methods.
- 1984: Stone wins Nobel Prize in Economics.
- Same approach used for reconciling seasonally adjusted data.
- Chow and Lin (1971): Temporal reconciliation of monthly or quarterly estimates to sum to annual estimates.
- Di Fonzo (1990): Cross-temporal reconciliation of time series data.

Outline

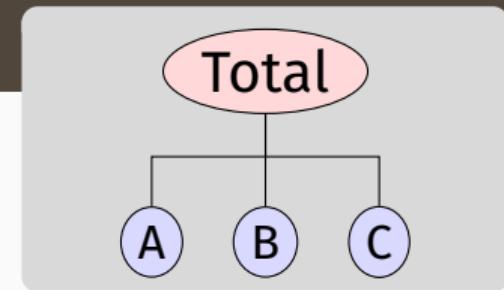
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Notation reminder

Every collection of time series with linear constraints can be written as

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

- \mathbf{y}_t = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t .
- $y_{X,t}$ = value of series X at time t .
- \mathbf{b}_t = vector of most disaggregated series at time t
- \mathbf{S} = “summing matrix” containing the linear constraints.



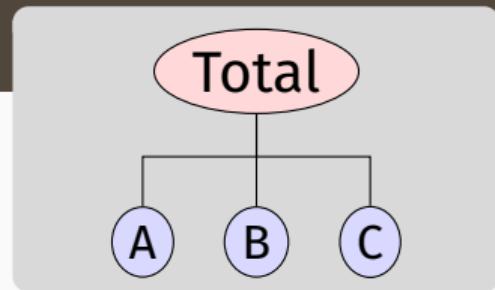
$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t} \end{aligned}$$

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- Base forecasts: $\hat{\mathbf{y}}_{T+h|T}$
- Reconciled forecasts:
$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$
- MinT:
$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}$$
 where \mathbf{W}_h is covariance matrix of base forecast errors.

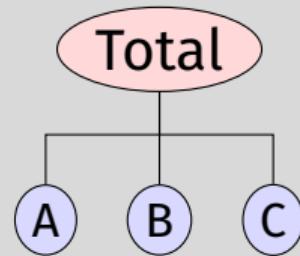
Notation

Aggregation matrix

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{pmatrix} \mathbf{b}_t$$



Constraint matrix

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

$$\begin{aligned} \text{where } \mathbf{C} &= [1 \ -1 \ -1 \ -1] \\ &= [\mathbf{I}_{n_a} \ -\mathbf{A}] \end{aligned}$$

Zero-constraint representation

Aggregation matrix A

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{bmatrix} \mathbf{b}_t = \mathbf{S}\mathbf{b}_t$$

Zero-constraint representation

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$$\mathbf{y}_t = \begin{bmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{bmatrix} \mathbf{b}_t = \mathbf{S}\mathbf{b}_t$$

Constraint matrix C

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

- Constraint matrix approach more general & more parsimonious.
- $\mathbf{C} = [\mathbf{I}_{n_a} \quad -\mathbf{A}]$.
- \mathbf{S}, \mathbf{A} and \mathbf{C} may contain any real values (not just 0s and 1s).

Zero-constraint representation

Assuming \mathbf{C} is full rank

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{M}\hat{\mathbf{y}}_{T+h|T}$$

where $\mathbf{M} = \mathbf{I} - \mathbf{W}_h\mathbf{C}'(\mathbf{C}\mathbf{W}_h\mathbf{C}')^{-1}\mathbf{C}$

- Originally proved by Byron (1978) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- $\mathbf{M} = \mathbf{S}\mathbf{G}$ (the MinT solution)
- Leads to more efficient reconciliation than using \mathbf{G} .

Zero-constraint representation

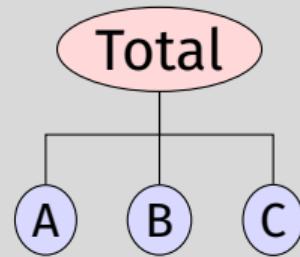
Suppose $\mathbf{W}_h = \mathbf{I}$. Then

$$\mathbf{M} = \mathbf{I} - \mathbf{W}_h \mathbf{C}' (\mathbf{C} \mathbf{W}_h \mathbf{C}')^{-1} \mathbf{C}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{4} (1 \quad -1 \quad -1 \quad -1)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$



$$\mathbf{A} = (1 \quad 1 \quad 1)$$

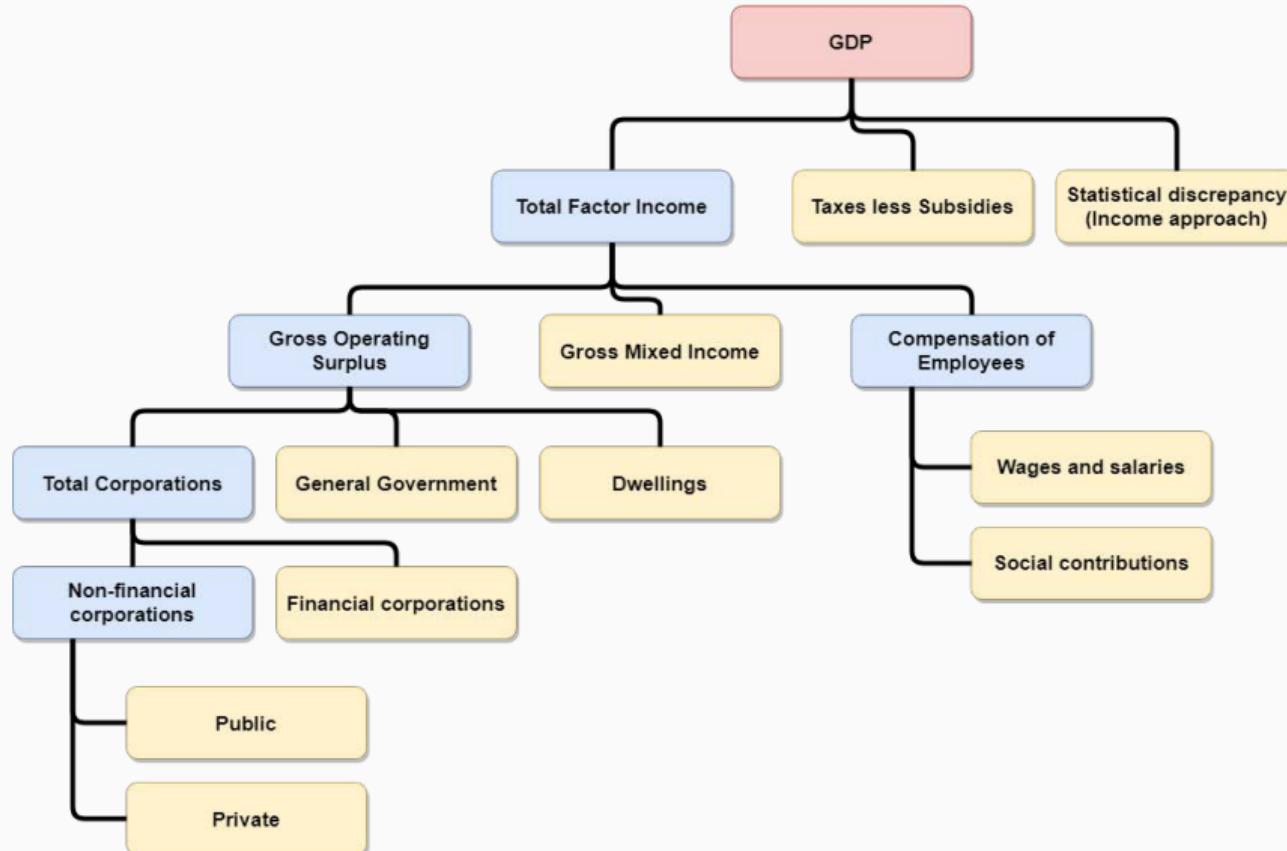
$$\mathbf{s} = \begin{pmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C} = (\mathbf{I}_{n_a} \quad -\mathbf{A}) = (1 \quad -1 \quad -1 \quad -1)$$

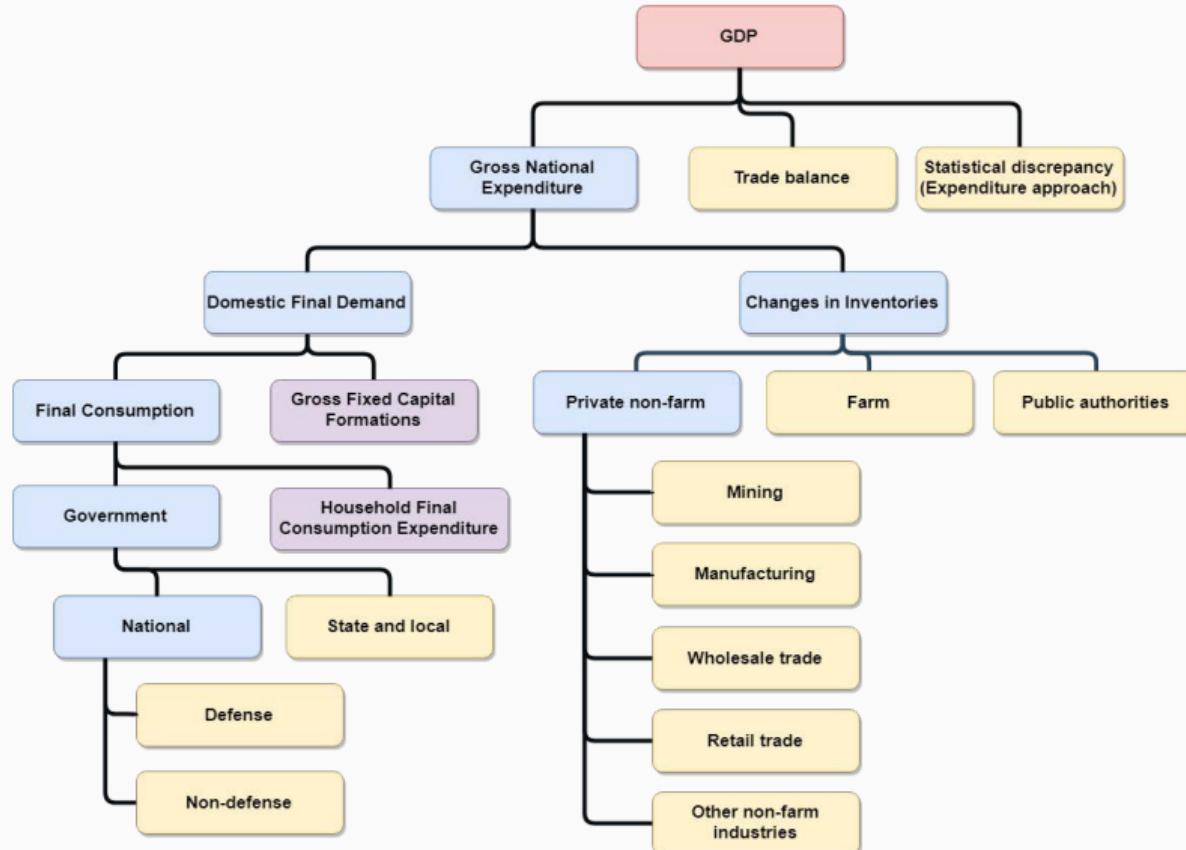
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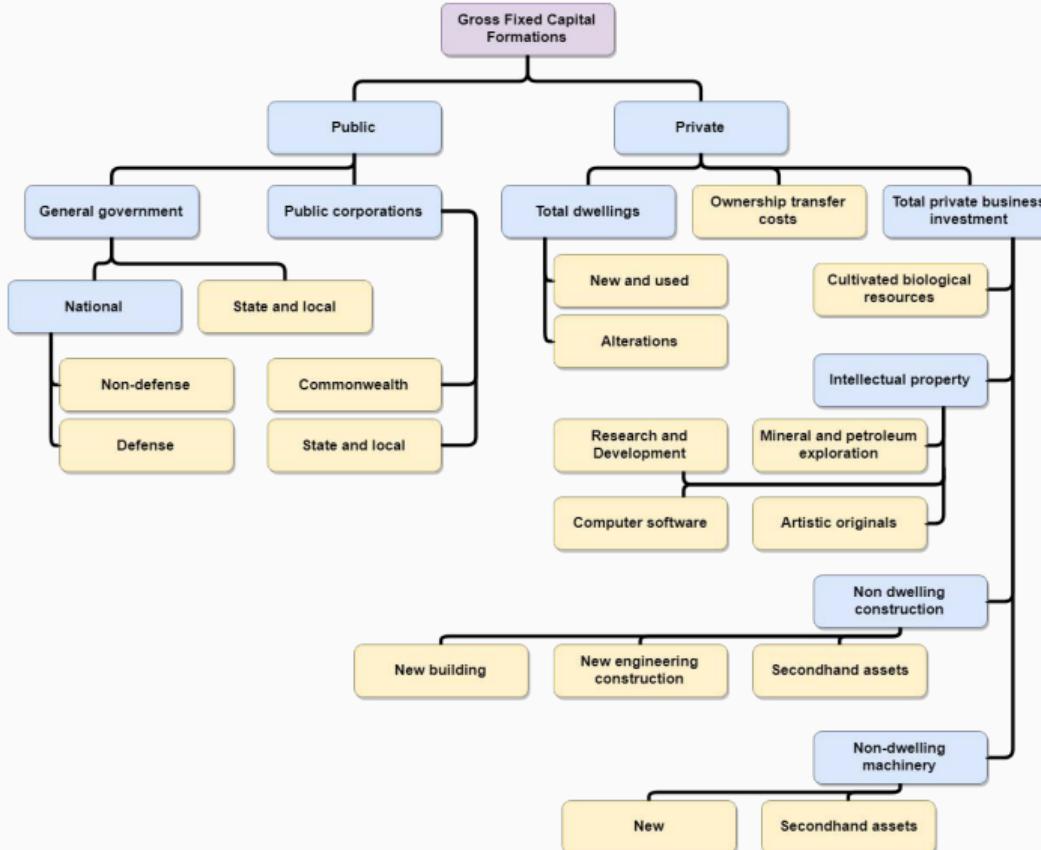
Example: reconciling GDP forecasts



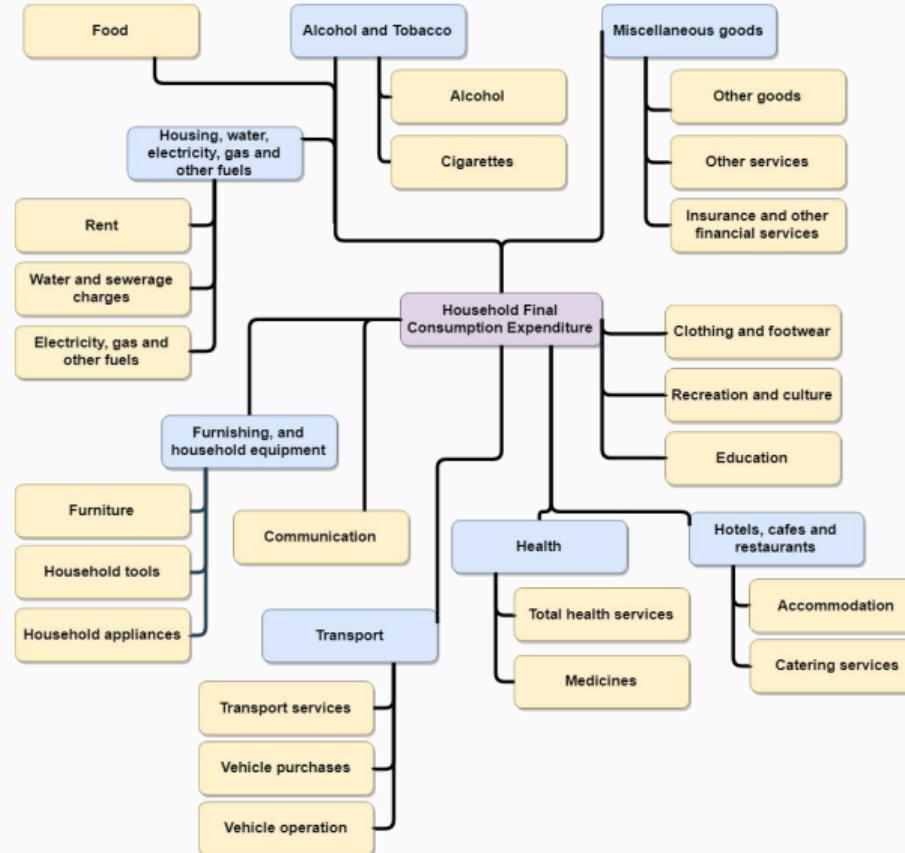
Example: reconciling GDP forecasts



Example: reconciling GDP forecasts



Example: reconciling GDP forecasts



Example: reconciling GDP forecasts

- No unique hierarchy.
- Several disaggregations with the same parent node
- Not possible to represent using structural **S** notation.
- Instead, we can use the constraint **C** notation.

Example: reconciling GDP forecasts

Using structural notation:

$$\mathbf{y}_t^I = \begin{bmatrix} X_t \\ \mathbf{a}_t^I \\ \mathbf{b}_t^I \end{bmatrix} = \mathbf{S}^I \mathbf{b}_t^I \quad \mathbf{y}_t^E = \begin{bmatrix} X_t \\ \mathbf{a}_t^E \\ \mathbf{b}_t^E \end{bmatrix} = \mathbf{S}^E \mathbf{b}_t^E$$

where

$$\mathbf{S}^I = \begin{bmatrix} \mathbf{1}'_{10} \\ \mathbf{A}' \\ \mathbf{I}_{10} \end{bmatrix} \quad \mathbf{S}^E = \begin{bmatrix} \mathbf{1}'_{53} \\ \mathbf{A}^E \\ \mathbf{I}_{53} \end{bmatrix}$$

- Can reconcile both trees, but the totals won't be equal.

Example: reconciling GDP forecasts

Using constraint notation:

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

where

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \mathbf{a}'_t \\ \mathbf{b}'_t \\ \mathbf{a}^E_t \\ \mathbf{b}^E_t \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & \mathbf{0}'_5 & -\mathbf{1}'_{10} & \mathbf{0}'_{26} & \mathbf{0}'_{53} \\ 1 & \mathbf{0}'_5 & \mathbf{0}'_{10} & \mathbf{0}'_{26} & -\mathbf{1}'_{53} \\ \mathbf{0}_5 & \mathbf{I}_5 & -\mathbf{A}' & \mathbf{0}_{5 \times 26} & \mathbf{0}_{5 \times 53} \\ \mathbf{0}_{26} & \mathbf{0}_{26 \times 5} & \mathbf{0}_{26 \times 10} & \mathbf{I}_{26} & -\mathbf{A}^E \end{bmatrix}$$

Ref: Bisaglia, Di Fonzo, and Girolimetto (2020)

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The coherent subspace

Coherent subspace

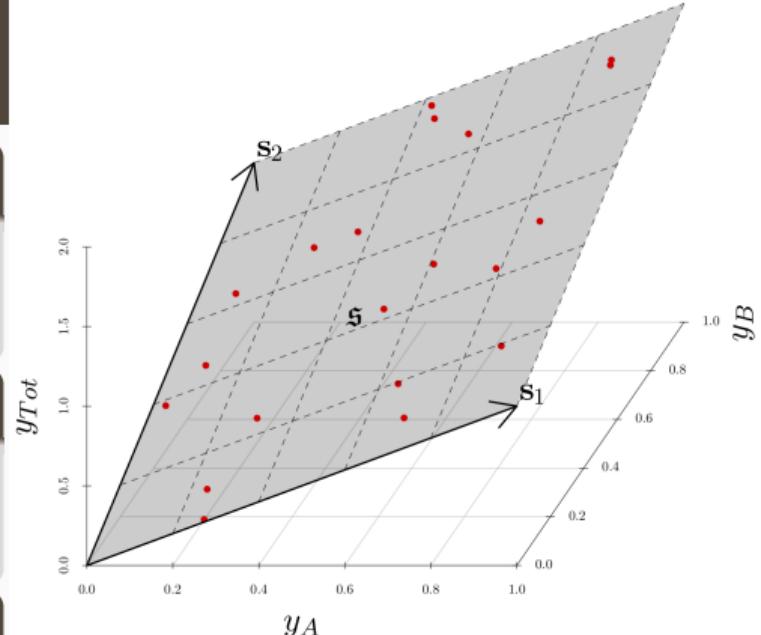
n_b -dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\mathbf{y} \in \mathfrak{s}$.

Hierarchical time series

An n -dimensional multivariate time series such that $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$ is coherent if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

The coherent subspace

Coherent subspace

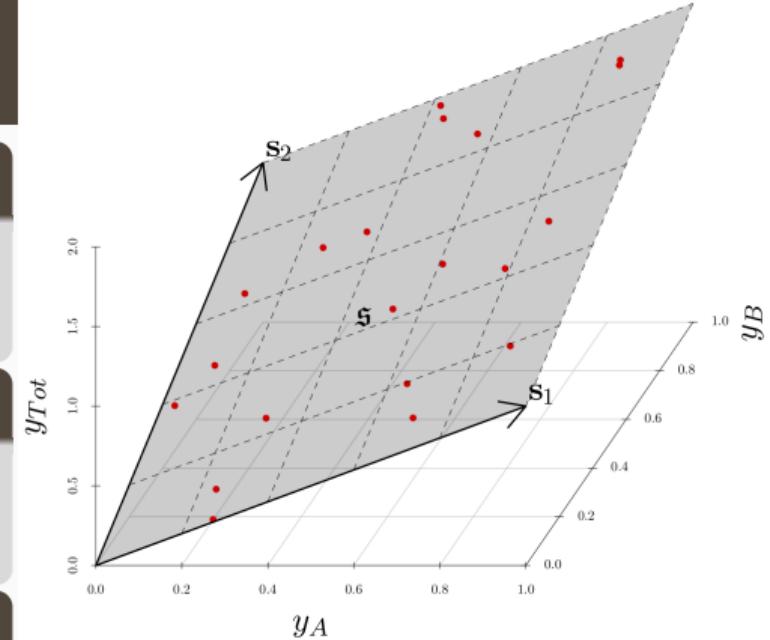
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$\tilde{\mathbf{y}}_{t+h|t}$ is *coherent* if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

Base forecasts

Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h -step forecasts.

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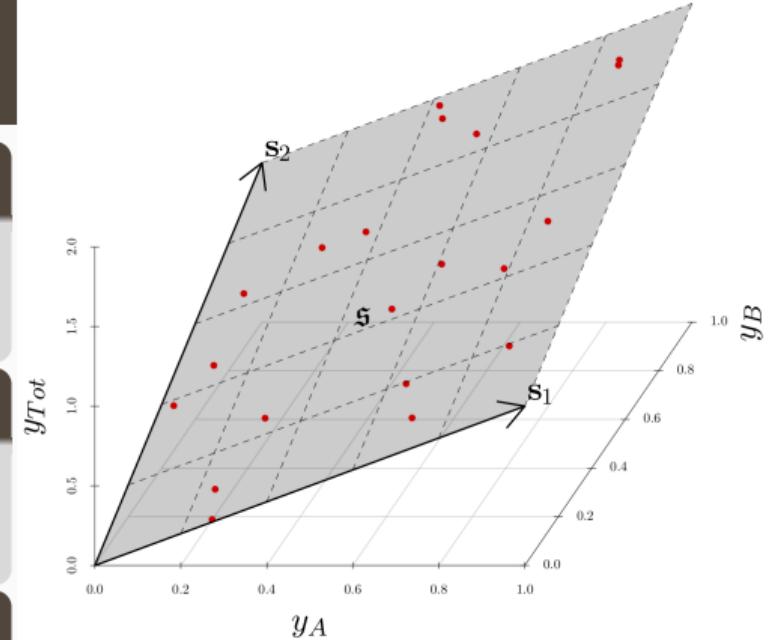
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Reconciled forecasts

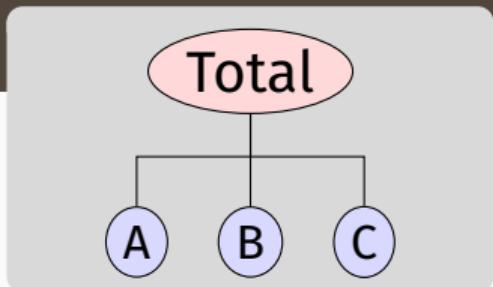
Let ψ be a mapping, $\psi : \chi^n \rightarrow \mathfrak{s}$.
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$ “reconciles” $\hat{\mathbf{y}}_{t+h|t}$.

The coherent subspace

The columns of \mathbf{S} form a basis set for \mathfrak{s} .

They are not unique.

Each corresponds to different vector of “bottom-level” series.

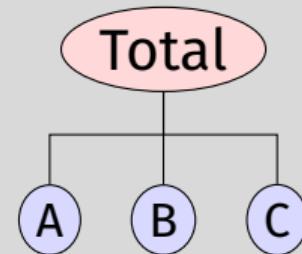


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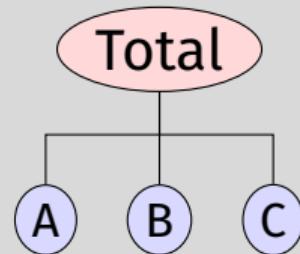
$$\mathbf{y} = \begin{pmatrix} \text{Total} \\ A \\ B \\ C \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

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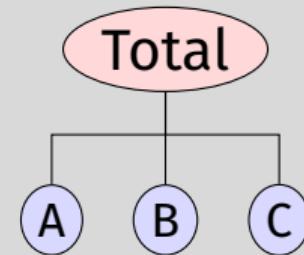
$$\mathbf{y} = \begin{pmatrix} \text{Total} \\ A \\ B \\ C \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & -1 & -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} \text{Total} \\ B \\ A \end{pmatrix}$$

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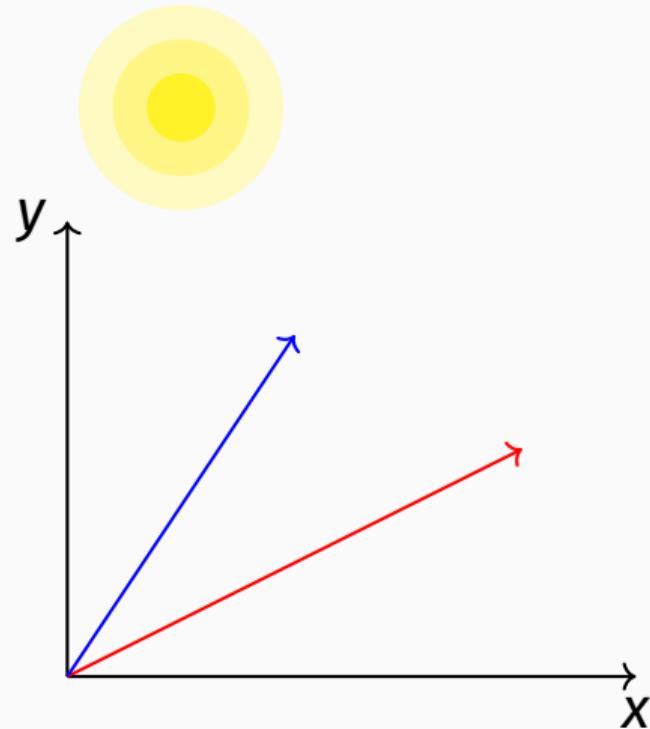
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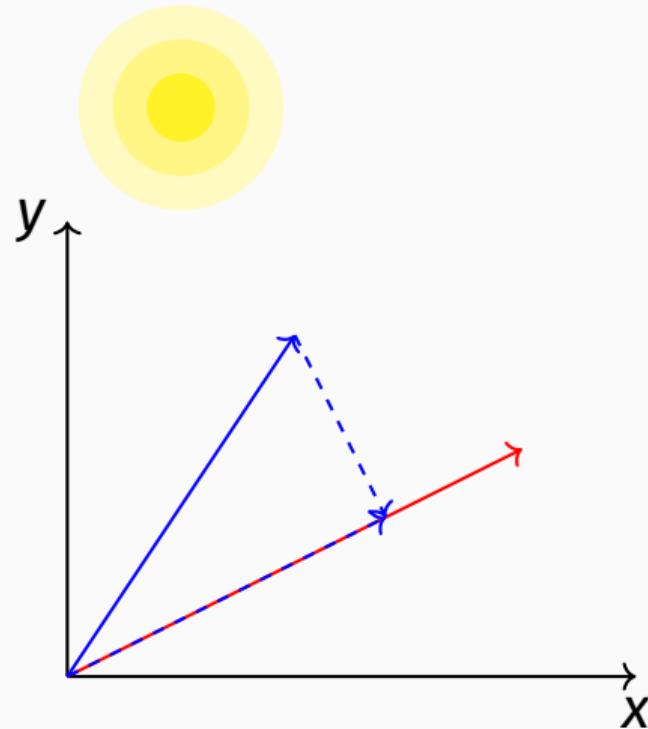


$$\mathbf{y} = \begin{pmatrix} \text{Total} \\ A \\ B \\ C \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} \text{Total} \\ B + A \\ C + B \end{pmatrix}$$

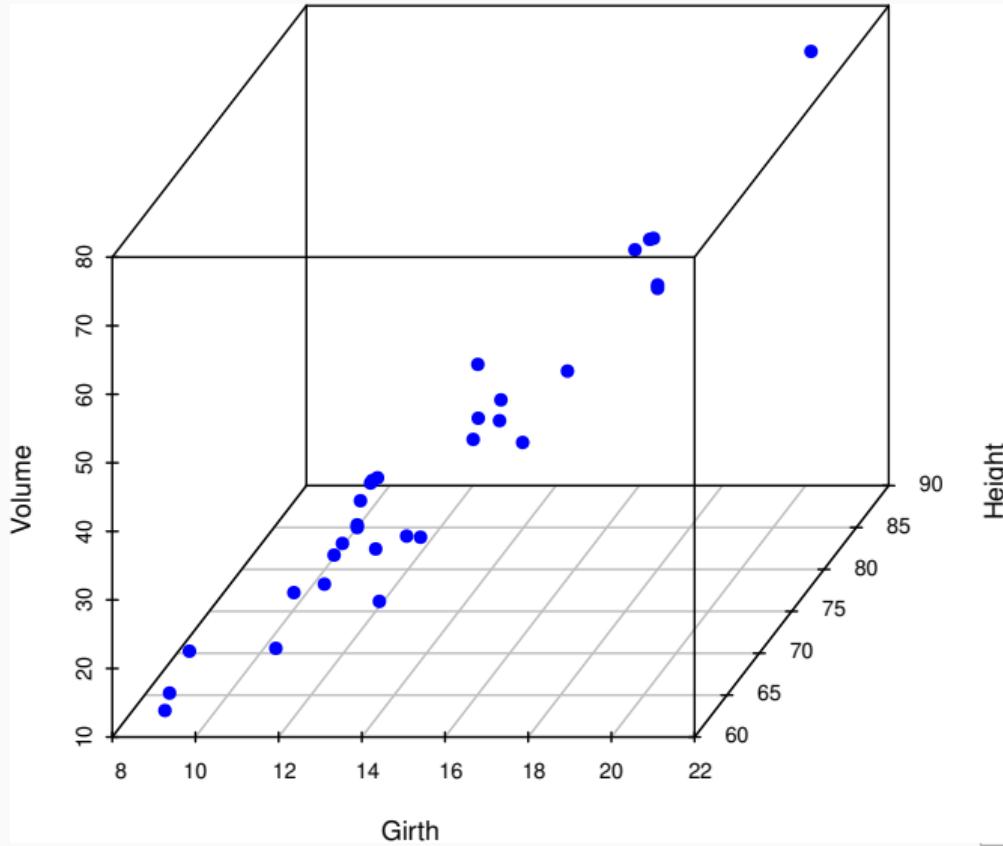
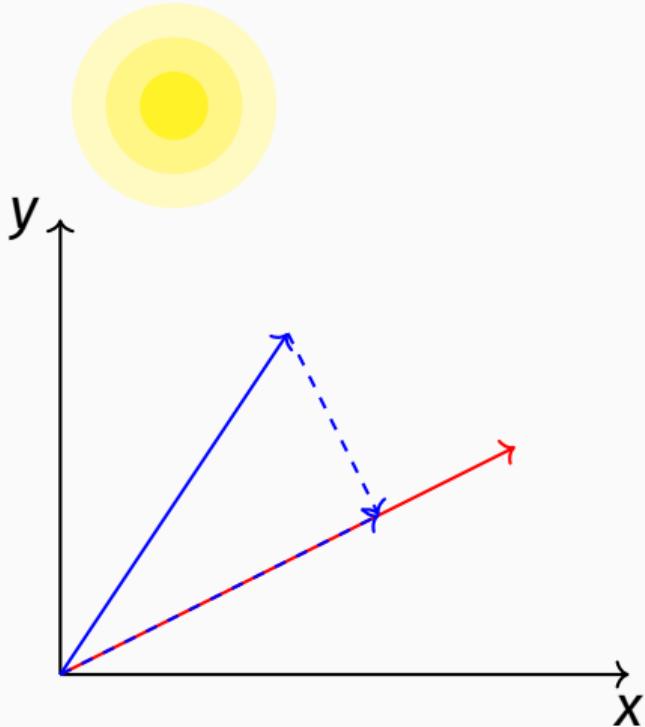
Projections in linear algebra



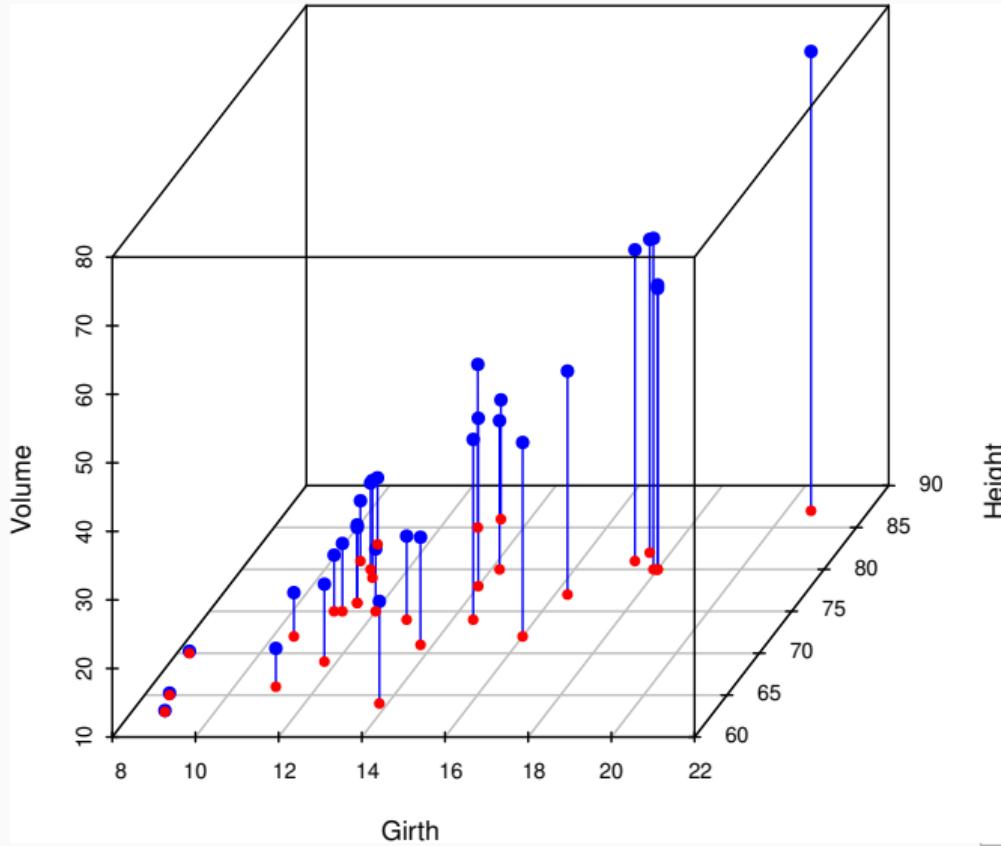
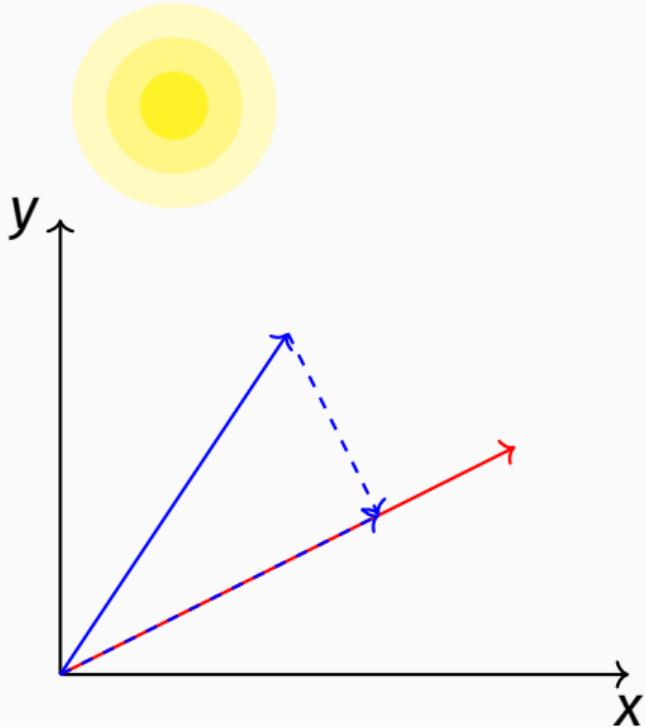
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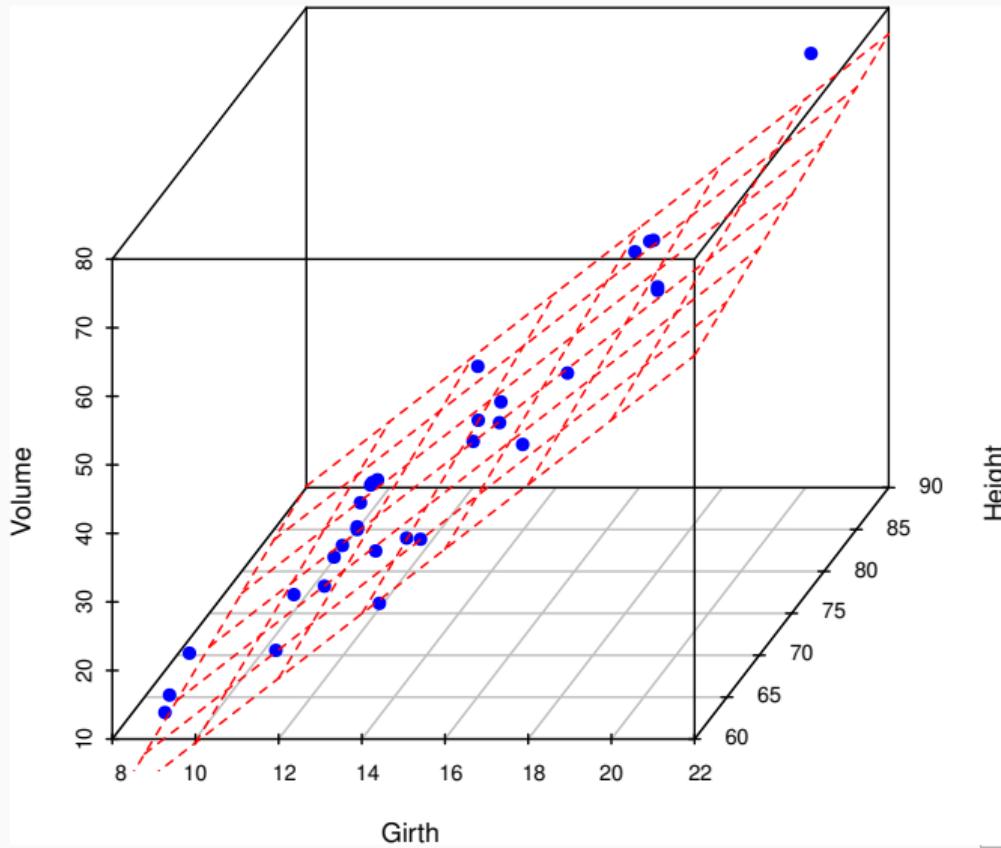
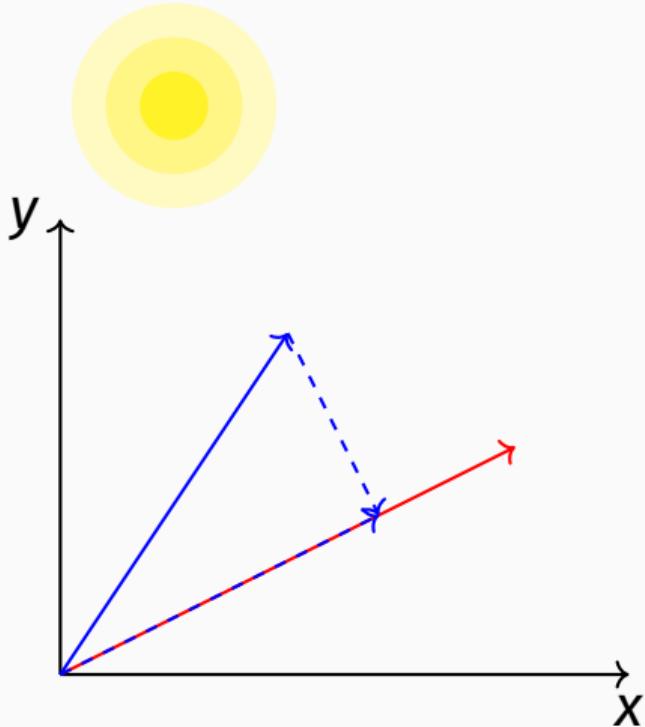
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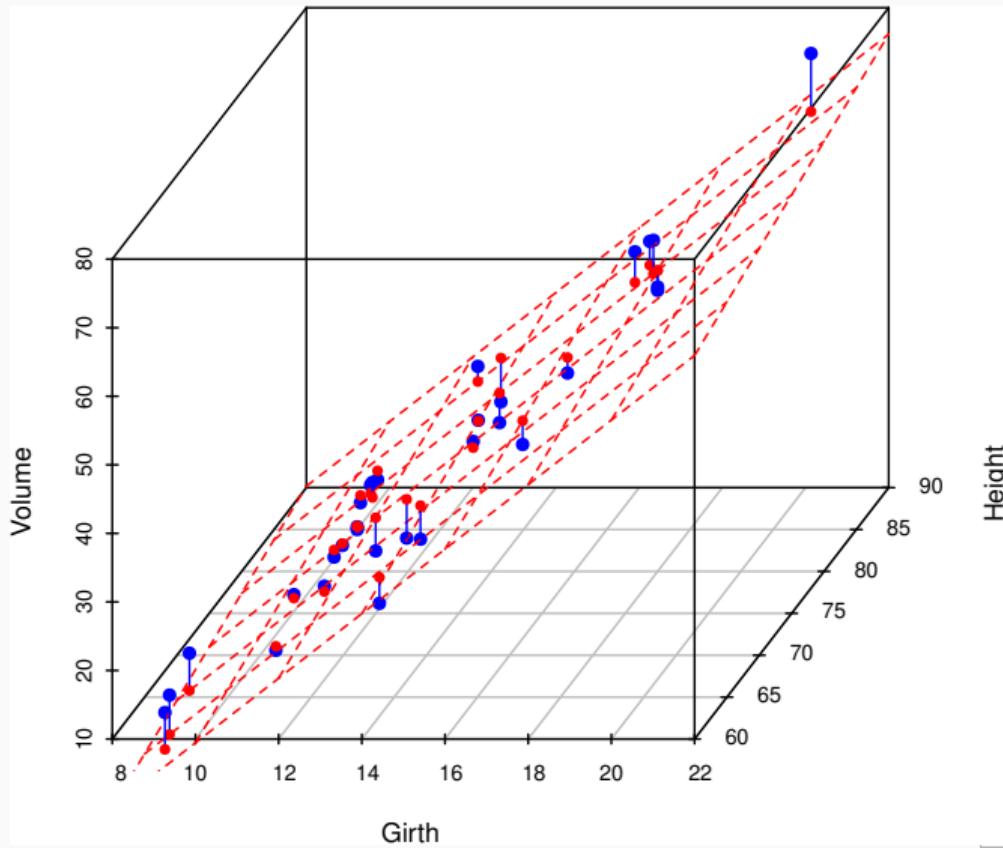
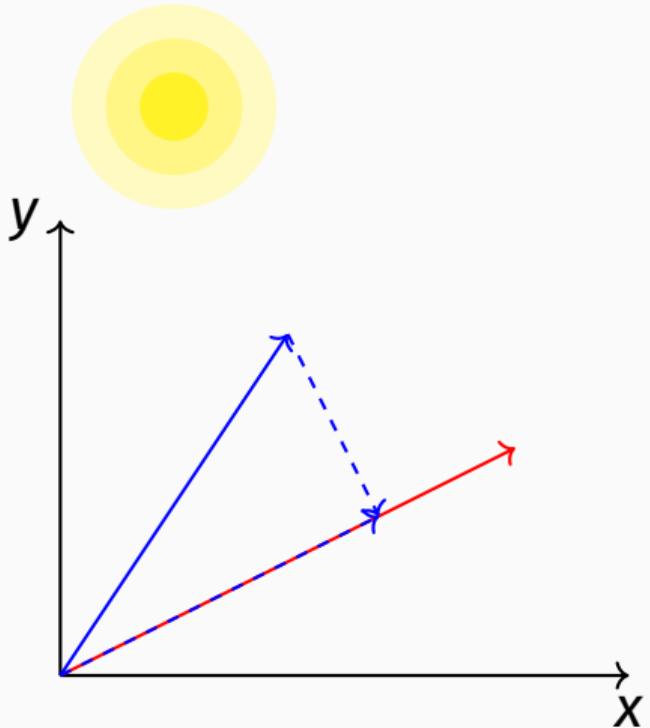
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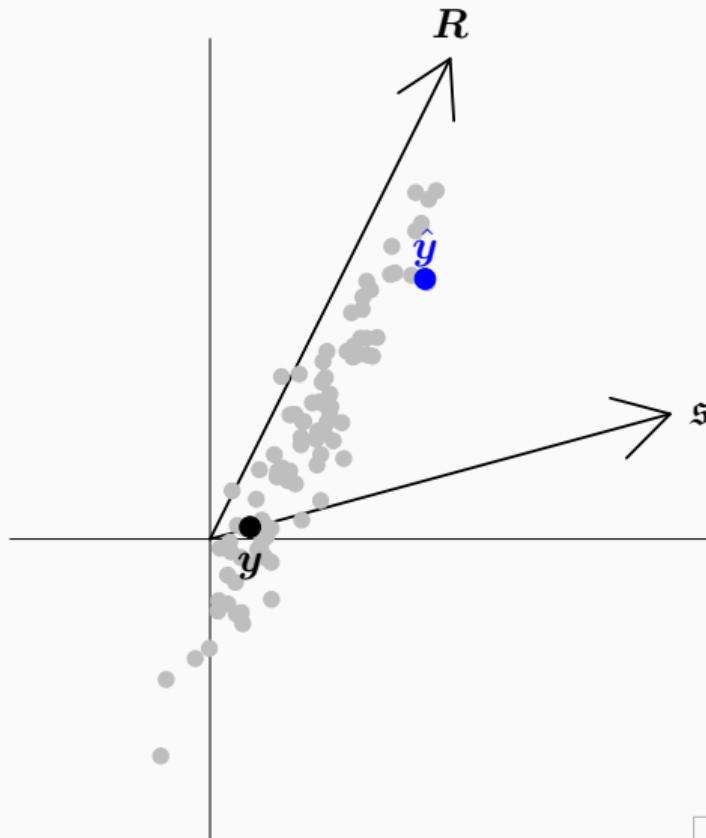


Projections in linear algebra

- A projection is a linear transformation \mathbf{M} such that $\mathbf{M}^2 = \mathbf{M}$.
- i.e., \mathbf{M} is idempotent: it leaves its image unchanged.
- \mathbf{M} projects onto \mathfrak{s} if $\mathbf{M}\mathbf{y} = \mathbf{y}$ for all $\mathbf{y} \in \mathfrak{s}$.
- All eigenvalues of \mathbf{M} are either 0 or 1.
- All singular values of \mathbf{M} are greater than or equal to 1 (with equality iff \mathbf{M} is orthogonal).
- A projection is *orthogonal* if $\mathbf{M}' = \mathbf{M}$.
- If a projection is not orthogonal, it is called *oblique*.
- In regression, OLS is an orthogonal projection onto space spanned by predictors.

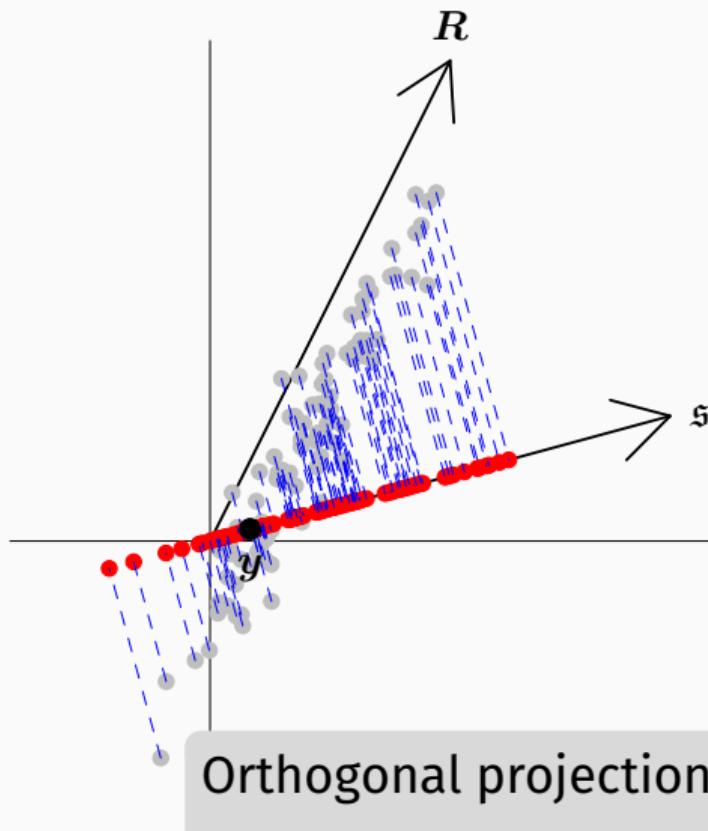
Linear projection reconciliation

- R is the most likely direction of deviations from \hat{s} .
- Grey: potential base forecasts



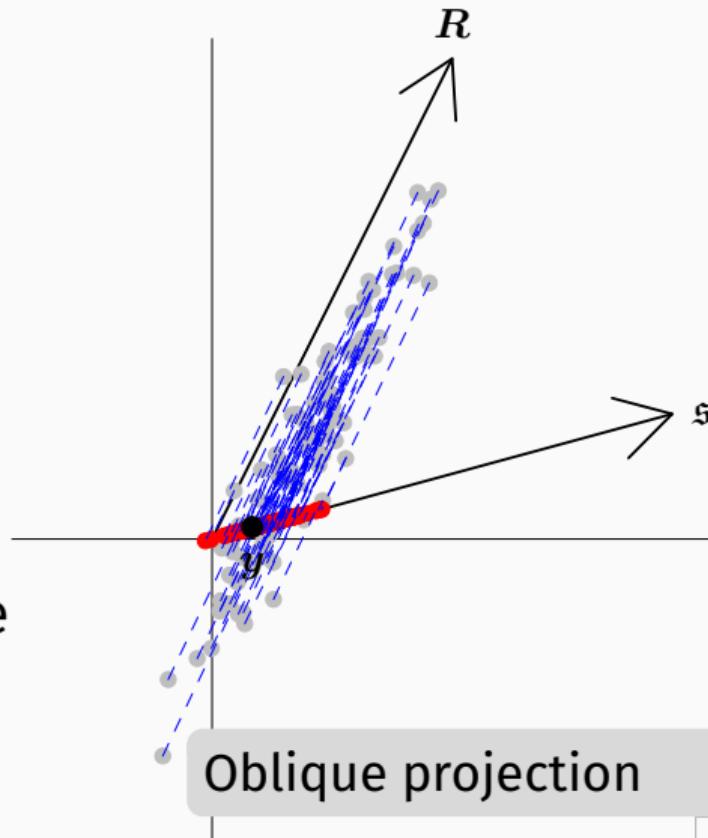
Linear projection reconciliation

- R is the most likely direction of deviations from \hat{s} .
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- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.



Linear projection reconciliation

- R is the most likely direction of deviations from \hat{s} .
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.



Linear projection reconciliation

$$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

- \mathbf{M} is a projection onto \mathfrak{s} if and only if $\mathbf{M}\mathbf{y} = \mathbf{y}$ for all $\mathbf{y} \in \mathfrak{s}$.
- Coherent base forecasts are unchanged since $\mathbf{M}\hat{\mathbf{y}} = \hat{\mathbf{y}}$
- If $\hat{\mathbf{y}}$ is unbiased, then $\tilde{\mathbf{y}}$ is also unbiased since

$$E(\tilde{\mathbf{y}}_{t+h|t}) = E(\mathbf{M}\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}E(\hat{\mathbf{y}}_{t+h|t}) = E(\hat{\mathbf{y}}_{t+h|t}),$$

and unbiased estimates must lie on \mathfrak{s} .

- The projection is orthogonal if and only if $\mathbf{M}' = \mathbf{M}$.
- If \mathbf{S} forms a basis set for \mathfrak{s} , then projections are of the form $\mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi$ where Ψ is a positive definite matrix.

Linear projection reconciliation

$$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}, \quad \text{where } \mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi$$

OLS: $\Psi = \mathbf{I}$ $\mathbf{M} = \mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}' = \mathbf{I} - \mathbf{C}'(\mathbf{C}\mathbf{C}')^{-1}\mathbf{C}$

MinT: $\Psi = \mathbf{W}_h$ $\mathbf{M} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1} = \mathbf{I} - \mathbf{W}_h\mathbf{C}'(\mathbf{C}\mathbf{W}_h\mathbf{C}')^{-1}\mathbf{C}$

- \mathbf{M} is orthogonal iff $\Psi = \mathbf{I}$.
- $\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T]$ is the covariance matrix of the base forecast errors.
- $\mathbf{V}_h = \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T] = \mathbf{M}\mathbf{W}_h\mathbf{M}'$ is minimized when $\Psi = \mathbf{W}_h$.

Distance reducing property

Let $\|\mathbf{u}\|_\Psi = \mathbf{u}'\Psi\mathbf{u}$. Then

$$\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}\|_\Psi \leq \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|_\Psi$$

- Ψ -projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure.*
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.
- Other measures of forecast accuracy may be worse.

$$\begin{aligned}\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h}\|_2^2 &= \|\mathbf{M}(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h})\|_2^2 \\ &\leq \|\mathbf{M}\|_2^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h}\|_2^2 \\ &= \sigma_{\max}^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h}\|_2^2\end{aligned}$$

- σ_{\max} is the largest eigenvalue of \mathbf{M}
- $\sigma_{\max} \geq 1$ as \mathbf{M} is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

$$\begin{aligned} & \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]'[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]\right) \end{aligned}$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

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- 7 Time series cross-validation

Minimum trace reconciliation

Minimum trace (MinT) reconciliation

If \mathbf{SG} is a projection, then the trace of $\mathbf{V}_h = \text{Var}(\tilde{\mathbf{y}}_{t+h|t} - \mathbf{y}_{t+h})$ is **minimized** when

$$\mathbf{G} = (\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1}$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S} (\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}$$

Reconciled forecasts

Base forecasts

- Trace of \mathbf{V}_h is sum of forecast variances.
- MinT solution is L_2 **optimal** amongst linear unbiased forecasts.

Find the solution to the minimax problem

$$V = \min_{\tilde{\mathbf{y}} \in \mathfrak{s}} \max_{\mathbf{y} \in \mathfrak{s}} \left\{ \ell(\mathbf{y}, \tilde{\mathbf{y}}) - \ell(\mathbf{y}, \hat{\mathbf{y}}) \right\},$$

where ℓ is a loss function, and \mathfrak{s} is the coherent subspace.

- $V \leq 0$: reconciliation guaranteed to reduce loss.
- If $\ell(\mathbf{y}, \tilde{\mathbf{y}}) = \|\mathbf{y} - \tilde{\mathbf{y}}\|_\Psi = (\mathbf{y} - \tilde{\mathbf{y}})' \Psi (\mathbf{y} - \tilde{\mathbf{y}})$, where Ψ is any symmetric pd matrix, then:
 - 1 $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}' \Psi \mathbf{S})^{-1} \mathbf{S}' \Psi \hat{\mathbf{y}}$ will always improve upon the base forecasts;
 - 2 The MinT solution $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}$ will optimise loss in expectation over any choice of Ψ .

Regularized empirical risk minimization problem:

$$\min_{\mathbf{G}} \frac{1}{Nn} \|\mathbf{Y} - \hat{\mathbf{Y}}\mathbf{G}'\mathbf{S}'\|_F + \lambda \|\text{vec}\mathbf{G}\|_1,$$

- $N = T - T_1 - h + 1$, T_1 is minimum training sample size
- $\|\cdot\|_F$ is the Frobenius norm
- $\mathbf{Y} = [\mathbf{y}_{T_1+h}, \dots, \mathbf{y}_T]'$
- $\hat{\mathbf{Y}} = [\hat{\mathbf{y}}_{T_1+h|T_1}, \dots, \hat{\mathbf{y}}_{T|T-h}]'$
- λ is a regularization parameter

When $\lambda = 0$: $\hat{\mathbf{G}} = \mathbf{B}'\hat{\mathbf{Y}}(\hat{\mathbf{Y}}'\hat{\mathbf{Y}})^{-1}$ where $\mathbf{B} = [\mathbf{b}_{T_1+h}, \dots, \mathbf{b}_T]'$.

Reference: Ben Taieb and Koo (2019)

MinT expressed as a regression

Since $\tilde{\mathbf{b}}_{t+h|t} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}$, we can write the MinT solution as a regression problem:

$$\begin{aligned}\tilde{\mathbf{b}}_{t+h|t} &= \arg \min_{\mathbf{b}} [\hat{\mathbf{y}}_{t+h|t} - \mathbf{S}\mathbf{b}]' \mathbf{W}_h^{-1} [\hat{\mathbf{y}}_{t+h|t} - \mathbf{S}\mathbf{b}] \\ &= \arg \min_{\mathbf{b}} [\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S}\mathbf{b} - 2\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t} + \hat{\mathbf{y}}_{t+h|t}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}] \\ &= \arg \min_{\mathbf{b}} [\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S}\mathbf{b} - 2\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}]\end{aligned}$$

- MinT solution is equivalent to a GLS regression of $\hat{\mathbf{y}}_{t+h|t}$ on \mathbf{S} with covariance weights \mathbf{W}_h^{-1} .
- The estimated coefficients are the forecasts of the bottom level series.

Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left(\mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Non-negative forecasts

$$\begin{aligned} & \min_{\mathbf{G}_h} \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]'[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]\right) \\ & \text{such that } \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Solve via quadratic programming:

$$\min_{\mathbf{b}} [\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - 2\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}] \quad \text{s.t. } \mathbf{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left(\mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Solve via quadratic programming:

$$\min_{\mathbf{b}} [\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - 2\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}] \quad \text{s.t. } \mathbf{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Set-negative-to-zero heuristic solution

- Negative reconciled forecasts at bottom level set to zero
- Remaining forecasts computed via aggregation
(Di Fonzo and Girolimetto, 2023)

$$\hat{\mathbf{y}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_{n_b-k} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix}$$

Suppose $\hat{\mathbf{u}}_{t+h|t}$ are fixed and let $\hat{\mathbf{w}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} - \mathbf{A}_2 \hat{\mathbf{u}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \end{bmatrix}$.

Optimization problem

$$\min_{\mathbf{v}} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}]' \mathbf{W}_{\mathbf{v}}^{-1} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}] \quad \text{where} \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{I}_{n_b-k} \end{bmatrix}$$

and $\mathbf{W}_{\mathbf{v}}$ contains elements of \mathbf{W}_h corresponding to $\hat{\mathbf{v}}_{t+h|t}$.

$$\hat{\mathbf{y}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_{n_b-k} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix}$$

Suppose $\hat{\mathbf{u}}_{t+h|t}$ are fixed and let $\hat{\mathbf{w}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} - \mathbf{A}_2 \hat{\mathbf{u}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \end{bmatrix}$.

Solve with non-negativity constraint

$$\min_{\mathbf{v}} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}]' \mathbf{W}_{\mathbf{v}}^{-1} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}] \quad \text{where} \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{I}_{n_b-k} \end{bmatrix}$$

such that $\mathbf{A}_3 \mathbf{v} \geq \begin{bmatrix} -\mathbf{A}_2 \hat{\mathbf{u}}_{t+h|t} \\ \mathbf{0} \end{bmatrix}$

Reconciliation and regularization

Mishchenko, Montgomery, and Vaggi (2019):
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{y}}_t} \sum_{t=1}^T \|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_2 + \lambda \sum_{t=1}^T \|\hat{\mathbf{y}}_t - \mathbf{s}_t \hat{\mathbf{b}}_t\|_2$$

Reconciliation and regularization

Mishchenko, Montgomery, and Vaggi (2019):
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{y}}_t} \sum_{t=1}^T \|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_2 + \lambda \sum_{t=1}^T \|\hat{\mathbf{y}}_t - \mathbf{s}_t \hat{\mathbf{b}}_t\|_2$$

Shiratori, Kobayashi, and Takano (2020):
Optimize bottom level forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{b}}_t} \sum_{t=1}^T \|\hat{\mathbf{b}}_t - \mathbf{b}_t\|_2 + \sum_{t=1}^T \Lambda \|\mathbf{a}_t - \mathbf{A}_t \hat{\mathbf{b}}_t\|_2$$

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- 2 Reconciliation via constraints
- 3 Example: reconciling GDP forecasts
- 4 The geometry of forecast reconciliation
- 5 Optimization and reconciliation
- 6 In-built coherence
- 7 Time series cross-validation

In-built coherence

Two-step approach: compute base forecasts \hat{y}_h , and then reconcile them to produce \tilde{y}_h .

One-step approaches: compute coherent \tilde{y}_h directly.

- Ashouri, Hyndman, and Shmueli (2022): linear regression models
- Pennings and Dalen (2017): state space models
- Villegas and Pedregal (2018): state space models

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}} \quad \mathbf{X}_{t+h}^* = \text{diag}(\mathbf{x}'_{t+h,1}, \dots, \mathbf{x}'_{t+h,n})$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}} \quad \mathbf{X}_{t+h}^* = \text{diag}(\mathbf{x}'_{t+h,1}, \dots, \mathbf{x}'_{t+h,n})$$

$$\tilde{\mathbf{y}}_{t+h} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h\hat{\mathbf{y}}_{t+h} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h\mathbf{X}_{t+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$\mathbf{V}_h = \sigma^2 \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h [1 + \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}_{T+h}^*)'] \mathbf{W}_h \mathbf{S}'(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'$$

Reference: Ashouri, Hyndman, and Shmueli (2022)

In-built coherence using state space models

Pennings and Dalen (2017) propose the state space model

$$\mathbf{y}_t = \mathbf{S}\mu_t + \mathbf{Z}_t\beta + \varepsilon_t, \quad \varepsilon_t \sim N(\mathbf{0}, \Sigma_\varepsilon),$$

$$\mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \sim N(\mathbf{0}, \Sigma_\eta).$$

- Coherent forecasts arise naturally using the Kalman filter
- Covariance matrices difficult to estimate except for small hierarchies.
- Requires the same model for all series
- A related approach proposed by Villegas and Pedregal (2018)

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Time series cross-validation

Traditional evaluation

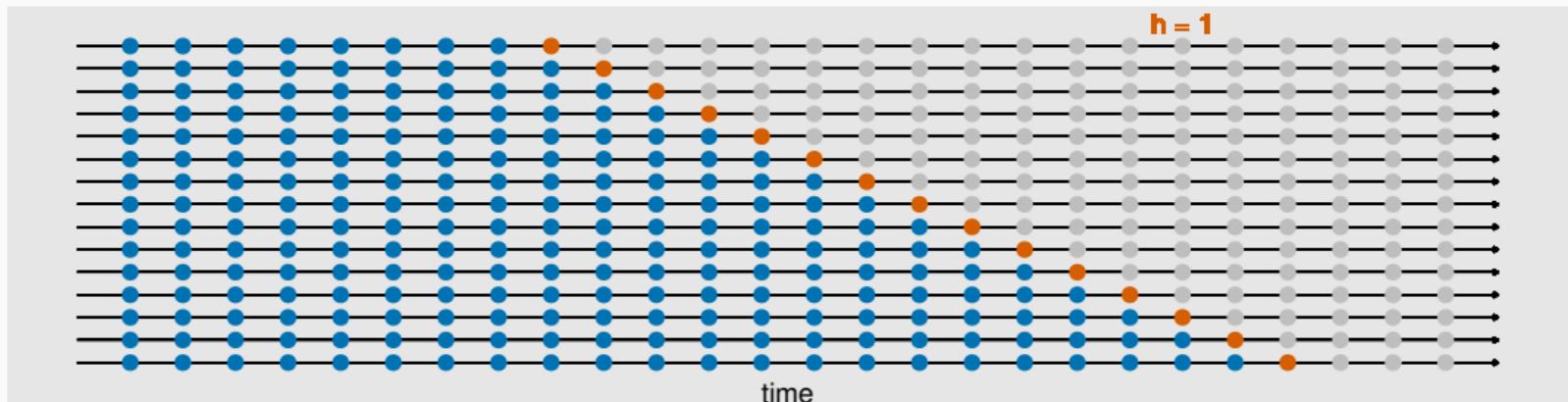


Time series cross-validation

Traditional evaluation



Time series cross-validation

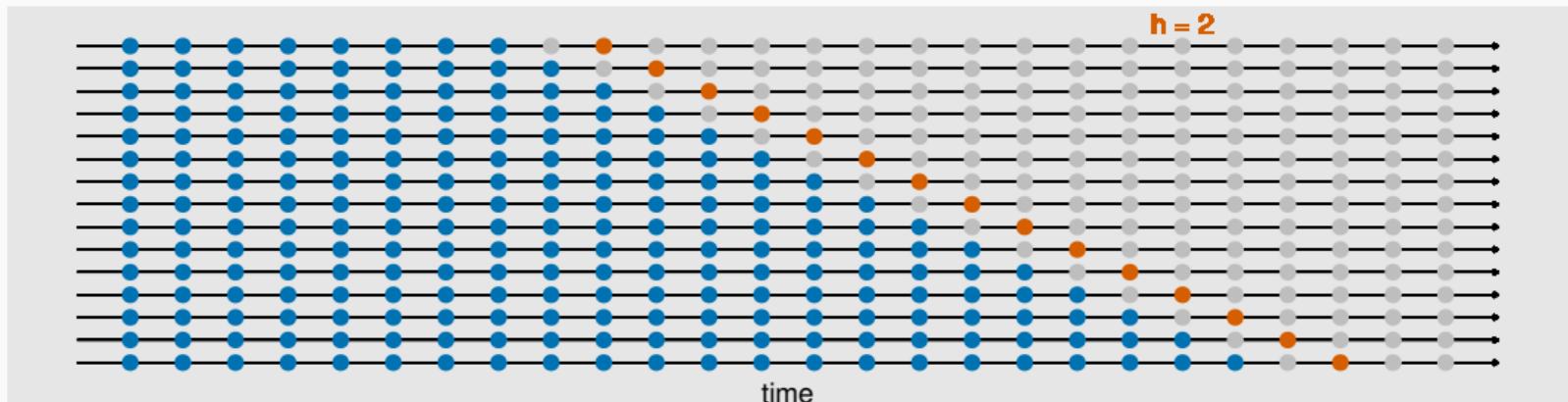


Time series cross-validation

Traditional evaluation



Time series cross-validation

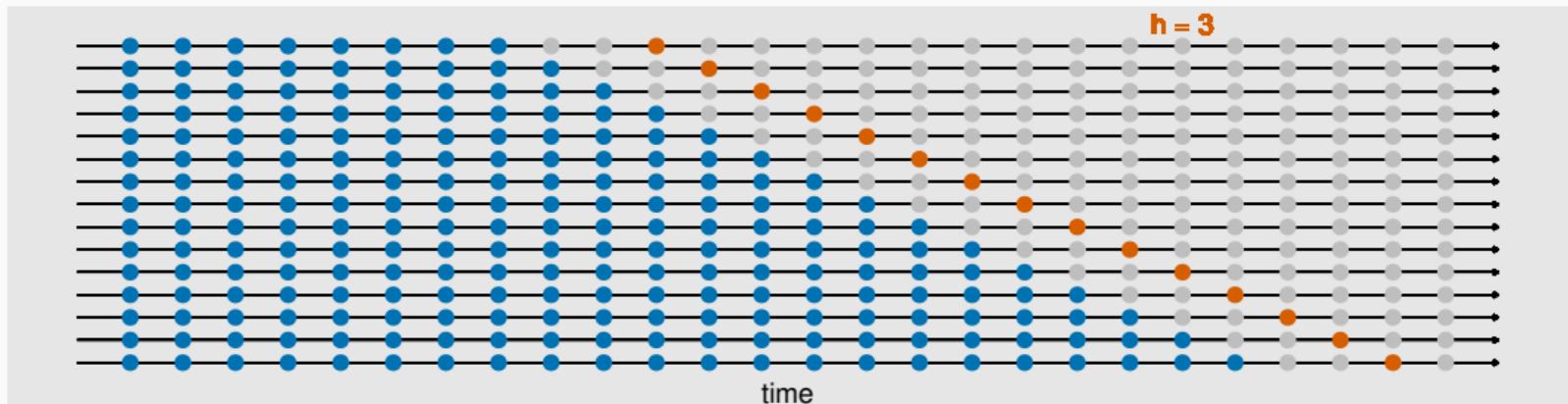


Time series cross-validation

Traditional evaluation



Time series cross-validation

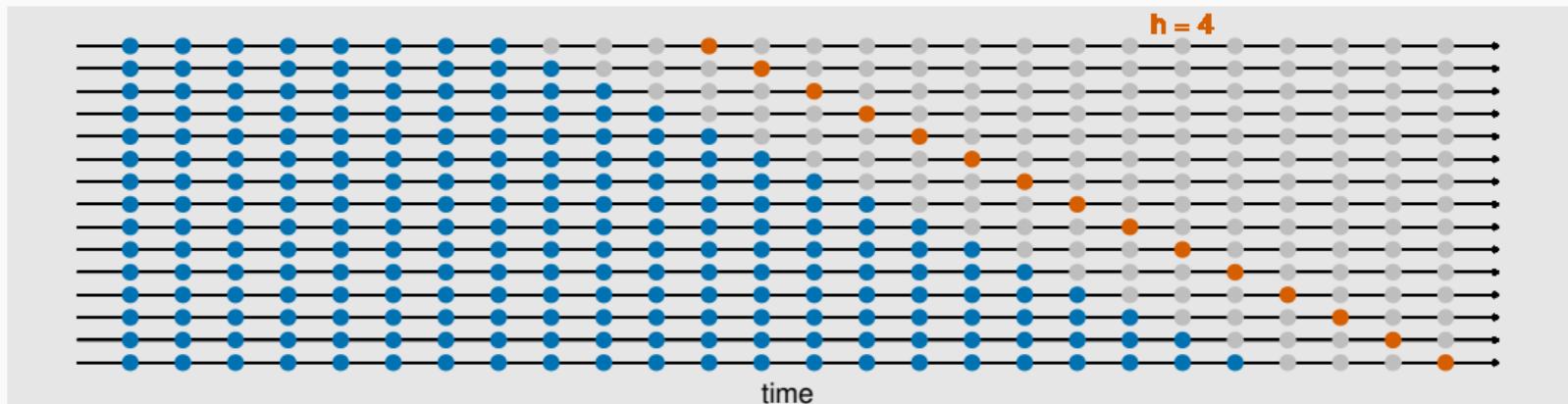


Time series cross-validation

Traditional evaluation

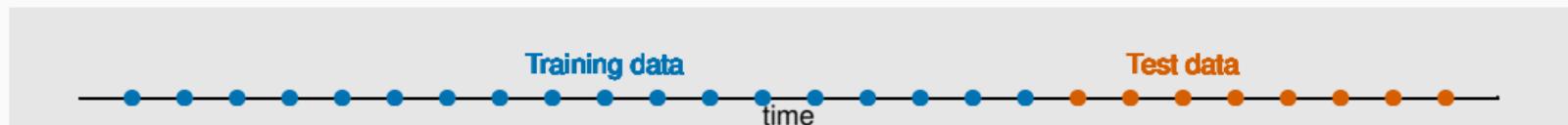


Time series cross-validation

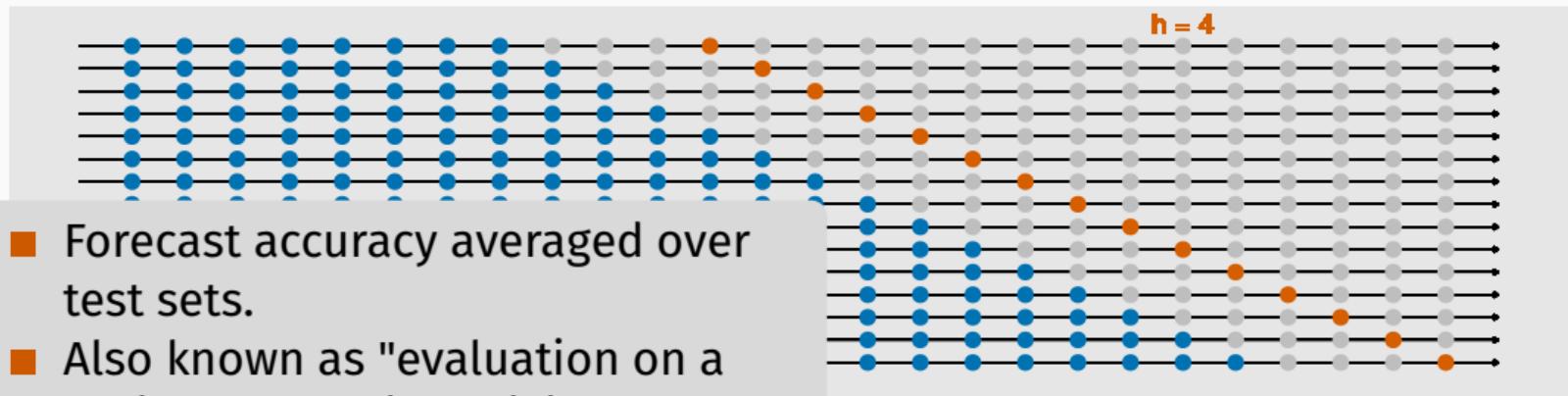


Time series cross-validation

Traditional evaluation



Time series cross-validation



Example: Australian tourism

```
# A tsibble: 18,000 x 5 [1M]
# Key:       state, zone, region [75]
    month state zone      region visitors
    <mth>  <chr> <chr>    <chr>     <dbl>
 1 1998   Jan NSW   Metro NSW Sydney     926.
 2 1998   Feb NSW   Metro NSW Sydney     647.
 3 1998   Mar NSW   Metro NSW Sydney     716.
 4 1998   Apr NSW   Metro NSW Sydney     621.
 5 1998   May NSW   Metro NSW Sydney     598.
 6 1998   Jun NSW   Metro NSW Sydney     601.
 7 1998   Jul NSW   Metro NSW Sydney     720.
 8 1998   Aug NSW   Metro NSW Sydney     645.
 9 1998   Sep NSW   Metro NSW Sydney     633.
10 1998   Oct NSW   Metro NSW Sydney     771.
# i 17,990 more rows
```

Example: Australian tourism

```
tourism_agg <- tourism |>
  aggregate_key(state/zone/region, visitors = sum(visitors))
tourism_stretch <- tourism_agg |>
  stretch_tsibble(.init = 48, .step=20*12)#.step = 1)
```

```
# A tsibble: 5,280 x 6 [1M]
# Key:      .id, state, zone, region [110]
#       month state  zone   region visitors   .id
#       <mth> <chr*> <chr*> <chr*>    <dbl> <int>
1 1998 Jan NSW   ACT    Canberra  210.     1
2 1998 Feb NSW   ACT    Canberra  156.     1
3 1998 Mar NSW   ACT    Canberra  185.     1
4 1998 Apr NSW   ACT    Canberra  178.     1
5 1998 May NSW   ACT    Canberra  134.     1
6 1998 Jun NSW   ACT    Canberra  105.     1
7 1998 Jul NSW   ACT    Canberra  142.     1
8 1998 Aug NSW   ACT    Canberra  137.     1
9 1998 Sep NSW   ACT    Canberra  166.     1
```

Example: Australian tourism

```
fit <- tourism_stretch |>  
model(  
  ets = ETS(visitors),  
  arima = ARIMA(visitors)  
) |>  
mutate(comb = (ets+arima)/2)
```

```
# A mable: 110 x 7  
# Key:   .id, state, zone, region [110]  
#       .id state  zone          region           ets  
#       <int> <chr*> <chr*>        <chr*>           <model>  
1     1 NSW    ACT      Canberra      <ETS(A,N,N)>  
2     1 NSW    ACT      <aggregated>  <ETS(A,N,N)>  
3     1 NSW  Metro NSW Central Coast <ETS(A,N,A)>  
4     1 NSW  Metro NSW   Sydney      <ETS(A,N,N)>  
5     1 NSW  Metro NSW      <aggregated> <ETS(M,N,M)>  
6     1 NSW North Coast NSW Hunter <ETS(A,N,N)>  
7     1 NSW North Coast NSW North Coast NSW <ETS(M,N,M)>
```

Example: Australian tourism

```
fc <- fit |>
  reconcile(
    ols = min_trace(comb, method = "ols"),
    wlsv = min_trace(comb, method = "wls_var"),
    wlss = min_trace(comb, method = "wls_struct"),
    mint_s = min_trace(comb, method = "mint_shrink"),
  ) |>
  forecast(h = "2 years")
```

```
# A fable: 18,480 x 8 [1M]
# Key:     .id, state, zone, region, .model [770]
  .id state  zone   region   .model      month     visitors .mean
  <int> <chr*> <chr*> <chr*>   <chr>      <mth>     <dist> <dbl>
1     1 NSW    ACT    Canberra  ets      2002 Jan N(169, 1553) 169.
2     1 NSW    ACT    Canberra  ets      2002 Feb N(169, 1553) 169.
3     1 NSW    ACT    Canberra  ets      2002 Mar N(169, 1553) 169.
4     1 NSW    ACT    Canberra  ets      2002 Apr N(169, 1553) 169.
5     1 NSW    ACT    Canberra  ets      2002 May N(169, 1553) 169.
```

Example: Australian tourism

```
fc |>  
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>  
  group_by(.model) |>  
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>  
  arrange(rmsse)
```

```
# A tibble: 7 x 3  
  .model    mase   rmsse  
  <chr>    <dbl>  <dbl>  
1 mint_s    0.798  0.823  
2 wlsv      0.818  0.840  
3 wlss      0.820  0.843  
4 comb      0.833  0.857  
5 ols       0.830  0.858  
6 ets       0.834  0.864  
7 arima     0.876  0.895
```

Example: Australian tourism

```
fc |>  
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>  
  group_by(.model) |>  
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>  
  arrange(rmsse)
```

```
# A tibble: 7 x 3  
  .model    mase   rmsse  
  <chr>    <dbl>  <dbl>  
1 mint_s  0.798  0.823  
2 wlsv    0.818  0.840  
3 wlss    0.820  0.843  
4 comb    0.833  0.857  
5 ols     0.830  0.858  
6 ets     0.834  0.864  
7 arima   0.876  0.895
```

- Combining improves forecast accuracy.
- Reconciling improves further.
- MinT-Shrink is almost always the best reconciliation method.

Example: Australian tourism

```
# A tibble: 28 x 4
# Groups:   .model [7]
  .model level      mase   rmsse
  <chr>  <fct>    <dbl> <dbl>
1 ets     National  0.835  0.823
2 mint_s  National  0.837  0.839
3 ols     National  0.846  0.848
4 comb    National  0.849  0.852
5 wlss   National  0.872  0.873
6 wlsv   National  0.873  0.876
7 arima   National  0.945  0.948
8 mint_s  State    0.801  0.827
9 wlsv   State    0.820  0.849
10 wlss  State    0.819  0.850
11 ols    State    0.818  0.860
12 ets    State    0.807  0.863
13 comb   State    0.827  0.871
14 arima   State   0.899  0.941
15 mint_s  Zone    0.801  0.828
16 wlss  Zone    0.820  0.836
17 wlsv  Zone    0.821  0.843
18 ols    Zone    0.828  0.850
19 ets    Zone    0.835  0.863
20 comb   Zone    0.841  0.865
21 arima   Zone   0.905  0.926
22 mint_s  Region  0.796  0.821
23 wlsv  Region  0.815  0.837
24 ets    Region  0.816  0.844
```

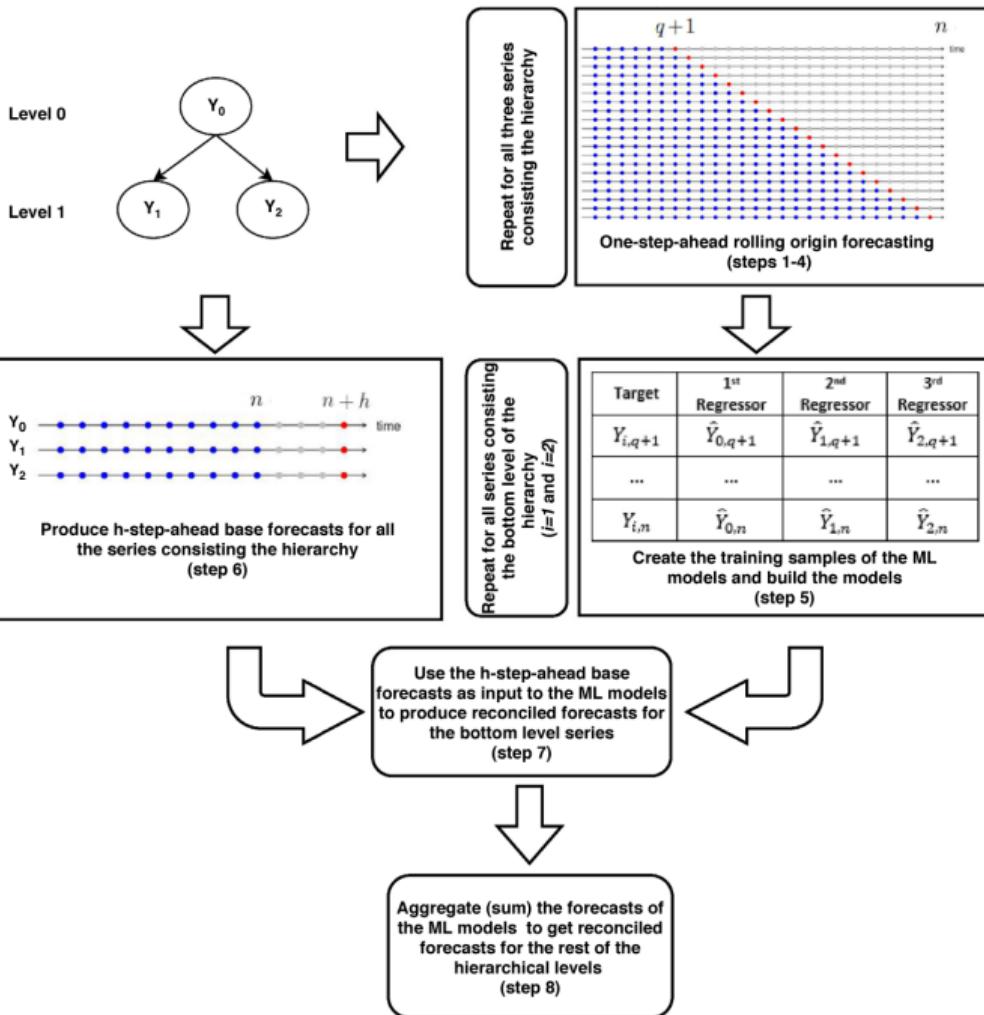
Example: Australian tourism

```
# A tibble: 28 x 4
# Groups:   .model [7]
  .model level      mase    rmsse
  <chr>  <fct>     <dbl>   <dbl>
1 ets    National  0.835  0.823
2 mint_s National  0.837  0.839
3 ols    National  0.846  0.848
4 comb   National  0.849  0.852
5 wlss   National  0.872  0.873
6 wlsv   National  0.873  0.876
7 arima  National  0.945  0.948
8 mint_s State     0.801  0.827
9 wlsv   State     0.820  0.849
10 wlss  State     0.819  0.850
11 ols   State     0.818  0.860
12 ets   State     0.807  0.863
13 comb  State     0.827  0.871
14 arima State     0.899  0.941
15 mint_s Zone     0.801  0.828
16 wlss  Zone     0.820  0.836
17 wlsv  Zone     0.821  0.843
18 ols   Zone     0.828  0.850
19 ets   Zone     0.835  0.863
20 comb  Zone     0.841  0.865
21 arima Zone     0.905  0.926
22 mint_s Region   0.796  0.821
23 wlsv  Region   0.815  0.837
24 ets   Region   0.816  0.841
```

■ MinT-Shrink is almost always the best reconciliation method.

ML reconciliation

- 1 Split all series using time series cross-validation
- 2 For each training set, compute one-step-ahead forecasts for all series
- 3 For each bottom-level series, use RF or XGB to predict values using forecasts of all series as inputs
- 4 Forecast all series
- 5 For each bottom-level series, apply ML model to improve forecasts
- 6 Aggregate bottom-level forecasts to obtain forecasts for other series.



ML reconciliation: tourism data

Method	Total	States	Zones	Regions	Average
MASE					
MinT-Struct	1.094	0.968	0.887	0.843	0.948
MinT-Shrink	1.047	0.956	0.872	0.824	0.925
ML-RF	1.045	0.964	0.859	0.812	0.920
ML-XGB	1.043	0.965	0.859	0.812	0.920
RMSSE					
MinT-Struct	1.308	1.225	1.137	1.109	1.195
MinT-Shrink	1.265	1.214	1.120	1.086	1.171
ML-RF	1.261	1.208	1.104	1.066	1.159
ML-XGB	1.255	1.208	1.101	1.064	1.157
AMSE					
MinT-Struct	0.988	0.611	0.426	0.349	0.593
MinT-Shrink	0.935	0.599	0.417	0.337	0.572
ML-RF	0.780	0.526	0.366	0.319	0.498
ML-XGB	0.779	0.526	0.365	0.317	0.497

- ML methods not significantly different.
- MinT methods significantly different from each other and from ML methods.

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