Forecast reconciliation

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1. Hierarchical time series and forecast reconciliation (Nov 6/7)
2. **Perspectives on forecast reconciliation** (Nov 9/10)
3. Probabilistic forecast reconciliation (Nov 13/14)
4. Temporal and cross-temporal forecast reconciliation (Nov 16/17)

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Outline

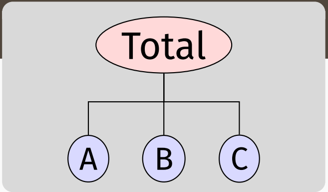
Outline

Time series reconciliation

- Stone, Champernowne, and Meade (1942): reconciling national economic accounts (disaggregated into production, income, outlay, capital transactions, etc.)
- Byron (1978): extended Stone's work using more computationally efficient methods.
- 1984: Stone wins Nobel Prize in Economics.
- Same approach used for reconciling seasonally adjusted data.
- Chow and Lin (1971): Temporal reconciliation of monthly or quarterly estimates to sum to annual estimates.
- Di Fonzo (1990): Cross-temporal reconciliation of time series data.

Outline

Notation reminder



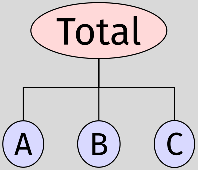
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Notation reminder



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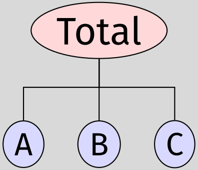
- Base forecasts: $\hat{\boldsymbol{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$
- MinT: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$ where $\boldsymbol{W}_h$ is covariance matrix of base forecast errors.

Notation



Aggregation matrix

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

$$\begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{pmatrix} \boldsymbol{b}_t$$

Constraint matrix

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

where

$$\begin{aligned} \boldsymbol{C} &= \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{I}_{n_a} & -\boldsymbol{A} \end{bmatrix} \end{aligned}$$

Zero-constraint representation

Aggregation matrix A

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Zero-constraint representation

Aggregation matrix A

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Constraint matrix C

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

- Constraint matrix approach more general & more parsimonious.
- $\boldsymbol{C} = [\boldsymbol{I}_{n_a} \quad -\boldsymbol{A}]$.
- $\boldsymbol{S}$, $\boldsymbol{A}$ and $\boldsymbol{C}$ may contain any real values (not just 0s and 1s).

Zero-constraint representation

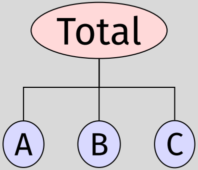
Assuming $\boldsymbol{C}$ is full rank

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\hat{\boldsymbol{y}}_{T+h|T}$$

$$\text{where} \qquad \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$$

- Originally proved by Byron (1978) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- $\boldsymbol{M} = \boldsymbol{S}\boldsymbol{G}$ (the MinT solution)
- Leads to more efficient reconciliation than using $\boldsymbol{G}$.

Zero-constraint representation



Suppose $\boldsymbol{W}_h = \boldsymbol{I}$. Then

$$
\begin{aligned}
\boldsymbol{M} &= \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}
\end{aligned}
$$

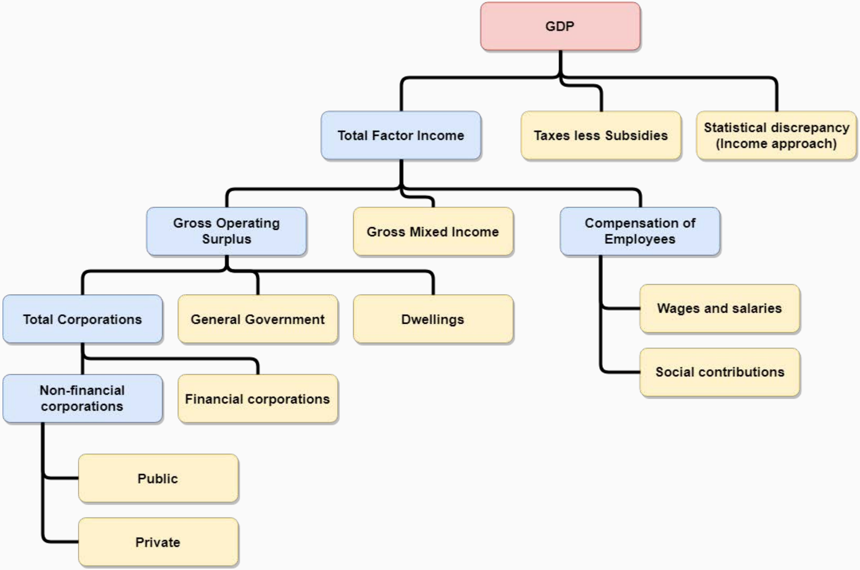
$$\boldsymbol{A} = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$

$$\boldsymbol{S} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

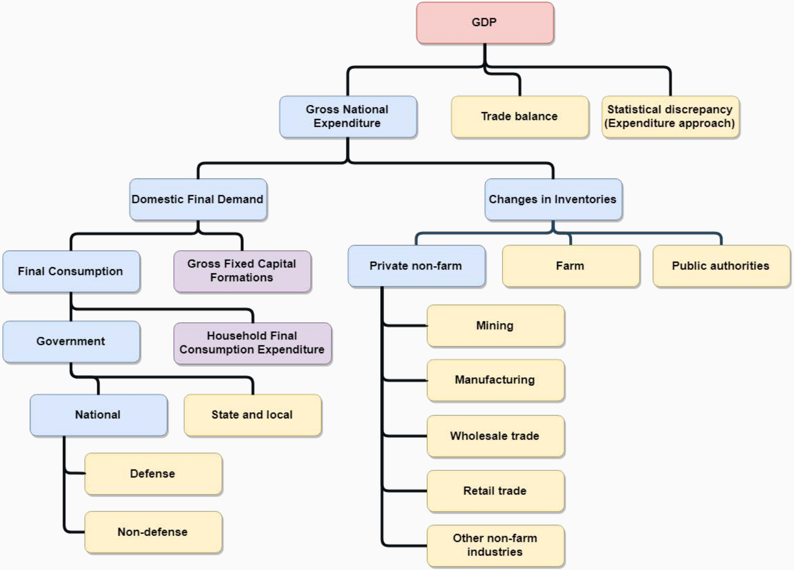
$$\boldsymbol{C} = \begin{pmatrix} \boldsymbol{I}_{n_a} & -\boldsymbol{A} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix}$$

Outline

Example: reconciling GDP forecasts



Example: reconciling GDP forecasts



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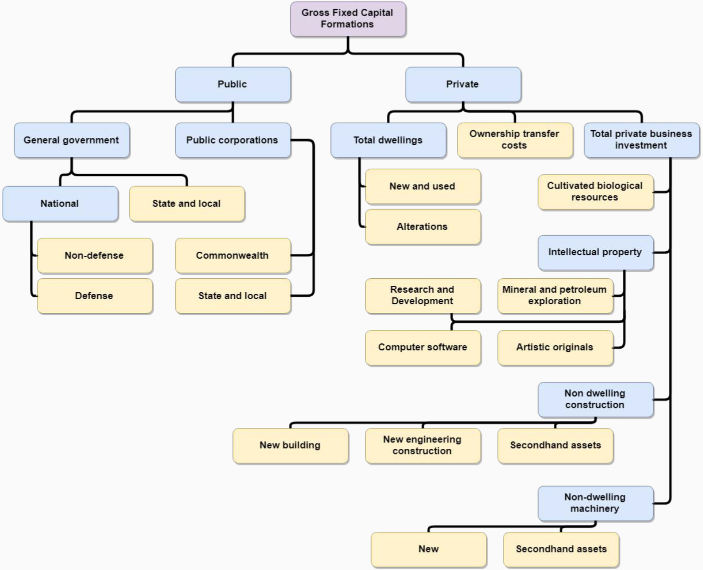
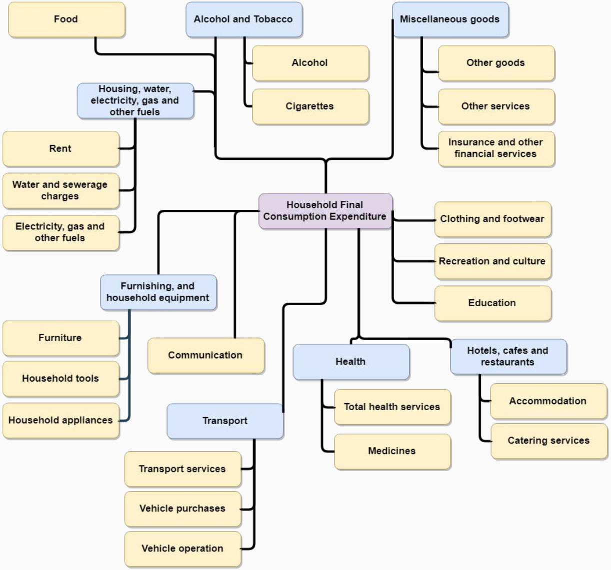


Example: reconciling GDP forecasts



Example: reconciling GDP forecasts

- No unique hierarchy.
- Several disaggregations with the same parent node
- Not possible to represent using structural $\boldsymbol{S}$ notation.
- Instead, we can use the constraint $\boldsymbol{C}$ notation.

Example: reconciling GDP forecasts

Using structural notation:

$$\boldsymbol{y}_t^I = \begin{bmatrix} x_t \\ \boldsymbol{a}_t^I \\ \boldsymbol{b}_t^I \end{bmatrix} = \boldsymbol{S}^I \boldsymbol{b}_t^I \qquad \boldsymbol{y}_t^E = \begin{bmatrix} x_t \\ \boldsymbol{a}_t^E \\ \boldsymbol{b}_t^E \end{bmatrix} = \boldsymbol{S}^E \boldsymbol{b}_t^E$$

where

$$\boldsymbol{S}^I = \begin{bmatrix} \mathbf{1}_{10}' \\ \boldsymbol{A}^I \\ \boldsymbol{I}_{10} \end{bmatrix} \qquad \boldsymbol{S}^E = \begin{bmatrix} \mathbf{1}_{53}' \\ \boldsymbol{A}^E \\ \boldsymbol{I}_{53} \end{bmatrix}$$

- Can reconcile both trees, but the totals won't be equal.

Example: reconciling GDP forecasts

Using constraint notation:

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

where

$$\boldsymbol{y}_t = \begin{bmatrix} x_t \\ \boldsymbol{a}_t^I \\ \boldsymbol{b}_t^I \\ \boldsymbol{a}_t^E \\ \boldsymbol{b}_t^E \end{bmatrix} \quad \text{and} \quad \boldsymbol{C} = \begin{bmatrix} 1 & \boldsymbol{0}_5' & -\boldsymbol{1}_{10}' & \boldsymbol{0}_{26}' & \boldsymbol{0}_{53}' \\ 1 & \boldsymbol{0}_5' & \boldsymbol{0}_{10}' & \boldsymbol{0}_{26}' & -\boldsymbol{1}_{53}' \\ \boldsymbol{0}_5 & \boldsymbol{I}_5 & -\boldsymbol{A}^I & \boldsymbol{0}_{5\times 26} & \boldsymbol{0}_{5\times 53} \\ \boldsymbol{0}_{26} & \boldsymbol{0}_{26\times 5} & \boldsymbol{0}_{26\times 10} & \boldsymbol{I}_{26} & -\boldsymbol{A}^E \end{bmatrix}$$

Ref: Bisaglia, Di Fonzo, and Girolimetto (2020)

Outline

The coherent subspace

Coherent subspace

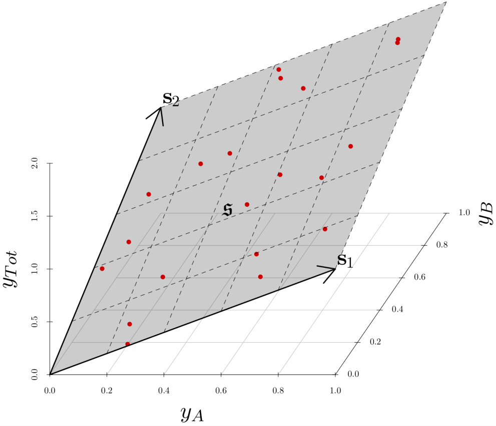
n_b-dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

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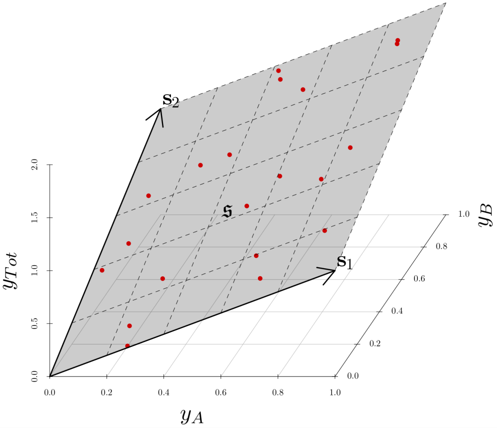
$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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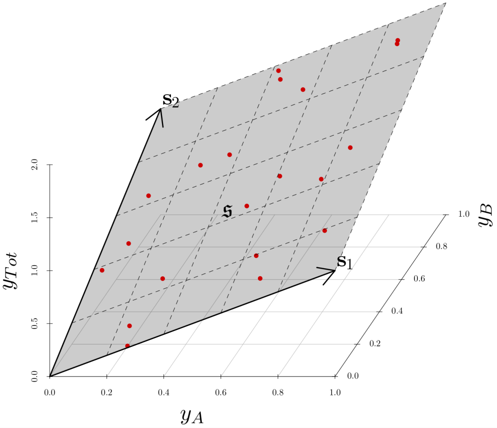


$y_{Tot} = y_A + y_B$

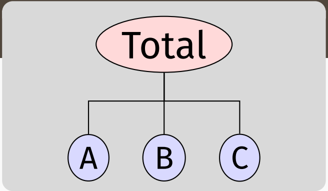
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \chi^n \rightarrow \mathfrak{s}$.
$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

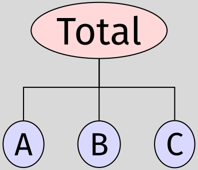
The coherent subspace



The columns of $\boldsymbol{S}$ form a basis set for $\mathfrak{s}$.

They are not unique.
Each corresponds to different vector of "bottom-level" series.

The coherent subspace



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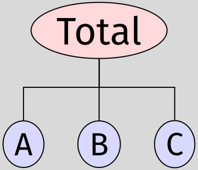

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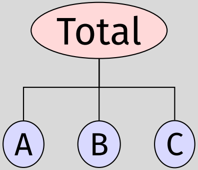
The coherent subspace



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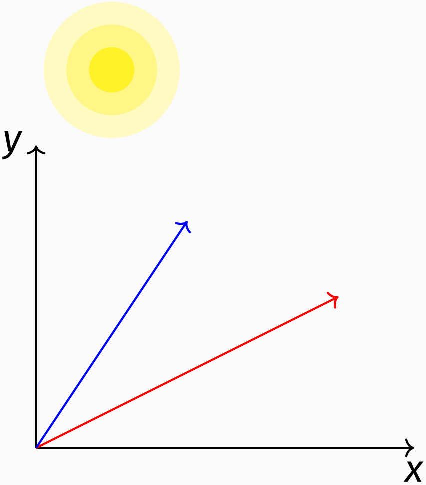
Projections in linear algebra



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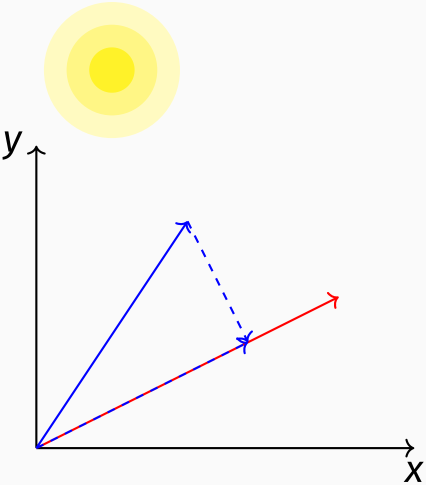

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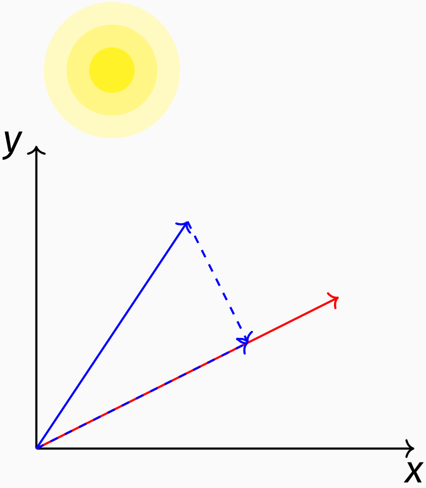

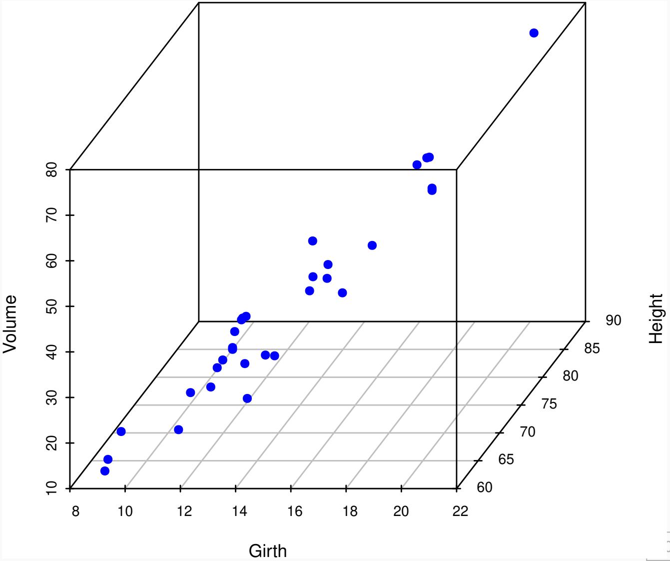

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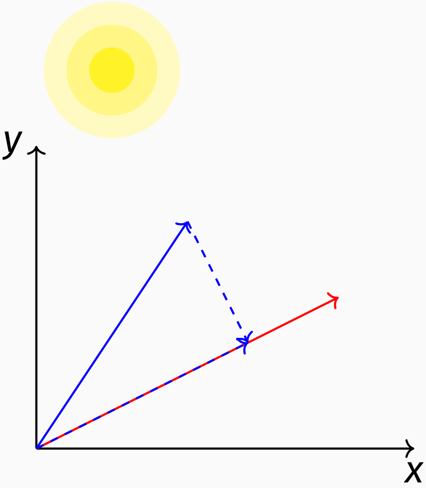

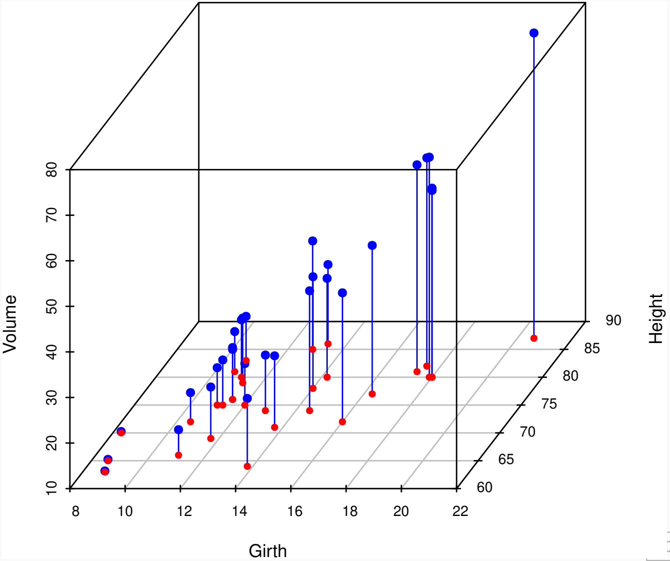

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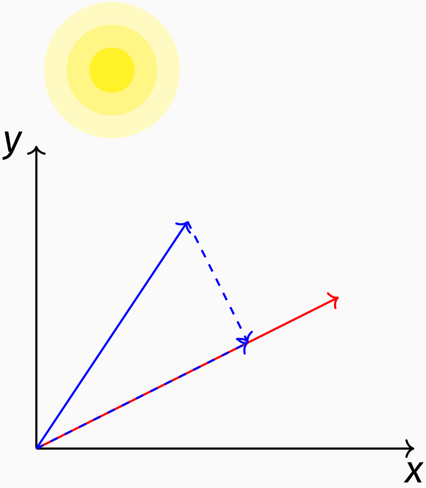

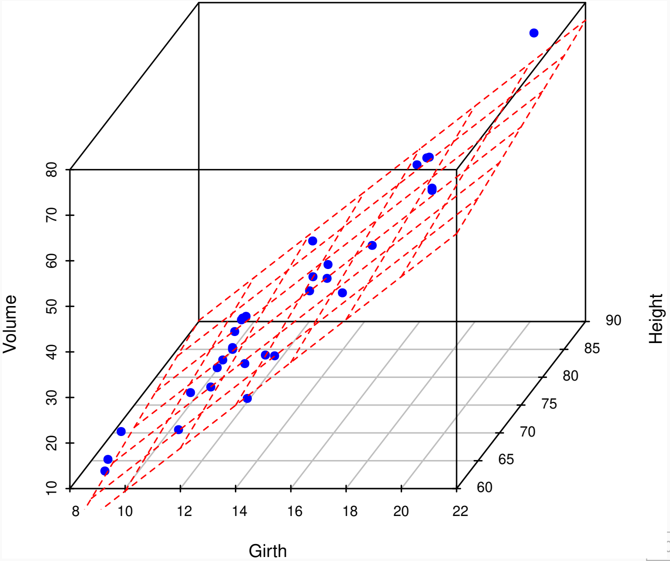

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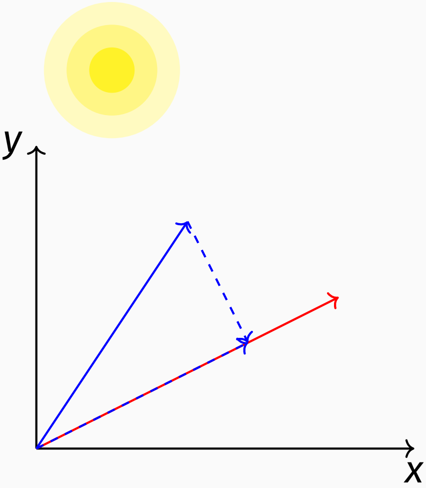
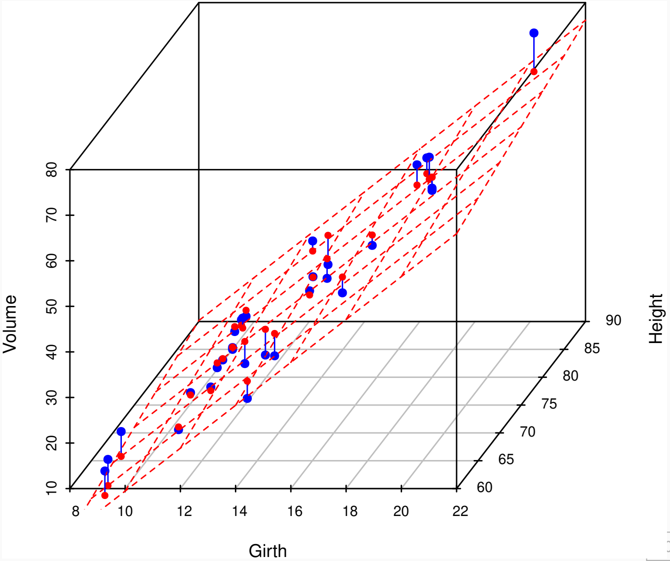


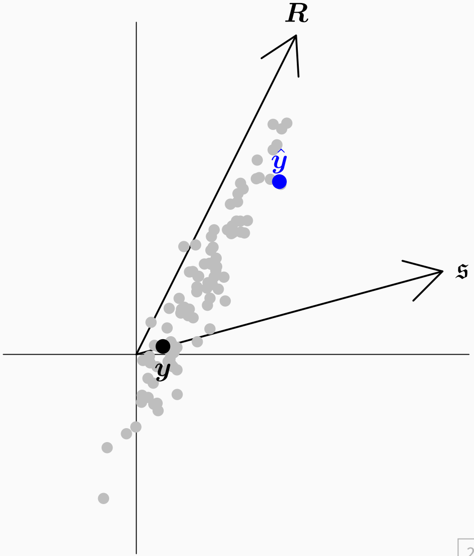


Projections in linear algebra

- A projection is a linear transformation $\boldsymbol{M}$ such that $\boldsymbol{M}^2 = \boldsymbol{M}$.
- i.e., M is idempotent: it leaves its image unchanged.
- $\boldsymbol{M}$ projects onto $\mathfrak{s}$ if $\boldsymbol{My} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- All eigenvalues of $\boldsymbol{M}$ are either 0 or 1.
- All singular values of $\boldsymbol{M}$ are greater than or equal to 1 (with equality iff $\boldsymbol{M}$ is orthogonal).
- A projection is *orthogonal* if $\boldsymbol{M}' = \boldsymbol{M}$.
- If a projection is not orthogonal, it is called *oblique*.
- In regression, OLS is an orthogonal projection onto space spanned by predictors.

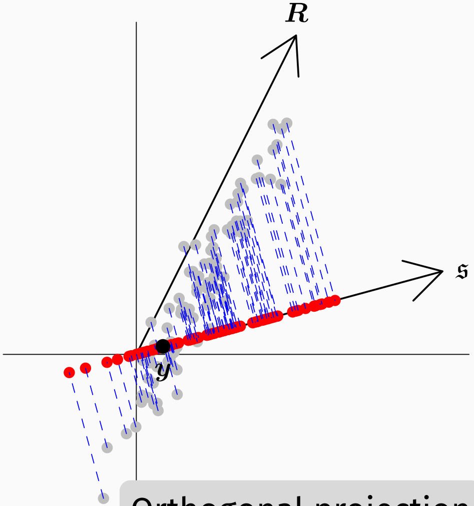
Linear projection reconciliation

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts



Linear projection reconciliation

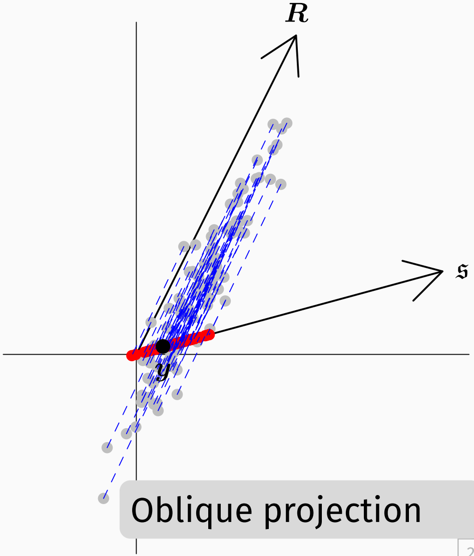
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.



Orthogonal projection

Linear projection reconciliation

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.



Oblique projection

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{M}$ is a projection onto $\mathfrak{s}$ if and only if $\boldsymbol{M}\boldsymbol{y} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- Coherent base forecasts are unchanged since $\boldsymbol{M}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}$
- If $\hat{\boldsymbol{y}}$ is unbiased, then $\tilde{\boldsymbol{y}}$ is also unbiased since

  $$\text{E}(\tilde{\boldsymbol{y}}_{t+h|t}) = \text{E}(\boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\text{E}(\hat{\boldsymbol{y}}_{t+h|t}) = \text{E}(\hat{\boldsymbol{y}}_{t+h|t}),$$

  and unbiased estimates must lie on $\mathfrak{s}$.
- The projection is orthogonal if and only if $\boldsymbol{M}' = \boldsymbol{M}$.
- If $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$, then projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}, \qquad \text{where} \quad \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$$

$$\begin{array}{llll} \text{OLS:} & \Psi = \boldsymbol{I} & \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}' & = \boldsymbol{I} - \boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}\boldsymbol{C} \\ \text{MinT:} & \Psi = \boldsymbol{W}_h & \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1} & = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C} \end{array}$$

- $\boldsymbol{M}$ is orthogonal iff $\Psi = \boldsymbol{I}$.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$ is the covariance matrix of the base forecast errors.
- $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T] = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$ is minimized when $\Psi = \boldsymbol{W}_h$.

Mean square error bounds

Panagiotelis et al. (2021)

Distance reducing property

Let $\|\boldsymbol{u}\|_\Psi = \boldsymbol{u}'\Psi\boldsymbol{u}$. Then

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\|_\Psi \le \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|_\Psi$$

- Ψ-projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure*.
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.
- Other measures of forecast accuracy may be worse.

Mean square error bounds

Wickramasuriya (2021)

$$
\begin{aligned}
\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}\|_2^2 &= \|\boldsymbol{M}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h})\|_2^2 \\
&\le \|\boldsymbol{M}\|_2^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2 \\
&= \sigma_{\text{max}}^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2
\end{aligned}
$$

- σ_{max} is the largest eigenvalue of $\boldsymbol{M}$
- $\sigma_{\text{max}} \geq 1$ as $\boldsymbol{M}$ is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

Mean square error bounds

Wickramasuriya (2021)

$$\begin{aligned}
&\text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]\Big) \\
\le\ &\text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]\Big) \\
\le\ &\text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]\Big)
\end{aligned}$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

Outline

Minimum trace reconciliation

Minimum trace (MinT) reconciliation

If $\boldsymbol{SG}$ is a projection, then the trace of $\boldsymbol{V}_h = \text{Var}(\tilde{\boldsymbol{y}}_{t+h|t} - \boldsymbol{y}_{t+h})$ is **minimized** when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$$

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciled forecasts Base forecasts

- Trace of $\boldsymbol{V}_h$ is sum of forecast variances.
- MinT solution is L_2 **optimal** amongst linear unbiased forecasts.

A game theoretic perspective

van Erven and Cugliari (2015)

Find the solution to the minimax problem

$$V = \min_{\tilde{\boldsymbol{y}} \in \mathfrak{s}} \max_{\boldsymbol{y} \in \mathfrak{s}} \left\{ \ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) - \ell(\boldsymbol{y}, \hat{\boldsymbol{y}}) \right\},$$

where ℓ is a loss function, and $\mathfrak{s}$ is the coherent subspace.

- $V \leq 0$: reconciliation guaranteed to reduce loss.
- If $\ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) = \|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_{\Psi} = (\boldsymbol{y} - \tilde{\boldsymbol{y}})'\Psi(\boldsymbol{y} - \tilde{\boldsymbol{y}})$, where Ψ is any symmetric pd matrix, then:
  1. $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi\hat{\boldsymbol{y}}$ will always improve upon the base forecasts;
  2. The MinT solution $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}$ will optimise loss in expectation over any choice of Ψ.

Biased reconciliation

Ben Taieb and Koo (2019)

Regularized empirical risk minimization problem:

$$\min_{\boldsymbol{G}} \frac{1}{Nn}\|\boldsymbol{Y} - \hat{\boldsymbol{Y}}\boldsymbol{G}'\boldsymbol{S}'\|_F + \lambda\|\text{vec}\boldsymbol{G}\|_1,$$

- $N = T - T_1 - h + 1, \quad T_1$ is minimum training sample size
- $\|\cdot\|_F$ is the Frobenius norm
- $\boldsymbol{Y} = [\boldsymbol{y}_{T_1+h}, \dots, \boldsymbol{y}_T]'$
- $\hat{\boldsymbol{Y}} = [\hat{\boldsymbol{y}}_{T_1+h|T_1}, \dots, \hat{\boldsymbol{y}}_{T|T-h}]'$
- λ is a regularization parameter

When $\lambda = 0$: $\quad \hat{\boldsymbol{G}} = \boldsymbol{B}'\hat{\boldsymbol{Y}}(\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}})^{-1}$ where $\boldsymbol{B} = [\boldsymbol{b}_{T_1+h}, \dots, \boldsymbol{b}_T]'$.

Reference: Ben Taieb and Koo (2019)

MinT expressed as a regression

Since $\tilde{\boldsymbol{b}}_{t+h|t} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{t+h|t}$, we can write the MinT solution as a regression problem:

$$\begin{aligned}
\tilde{\boldsymbol{b}}_{t+h|t} &= \arg\min_{\boldsymbol{b}} \left[\hat{\boldsymbol{y}}_{t+h|t} - \boldsymbol{S}\boldsymbol{b}\right]' \boldsymbol{W}_h^{-1} \left[\hat{\boldsymbol{y}}_{t+h|t} - \boldsymbol{S}\boldsymbol{b}\right] \\
&= \arg\min_{\boldsymbol{b}} \left[\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - 2\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{t+h|t} + \hat{\boldsymbol{y}}'_{t+h|t}\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{t+h|t}\right] \\
&= \arg\min_{\boldsymbol{b}} \left[\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - 2\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{t+h|t}\right]
\end{aligned}$$

- MinT solution is equivalent to a GLS regression of $\hat{\boldsymbol{y}}_{t+h|t}$ on $\boldsymbol{S}$ with covariance weights $\boldsymbol{W}_h^{-1}$.
- The estimated coefficients are the forecasts of the bottom level series.

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \operatorname{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$
$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$
$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Solve via quadratic programming:

$$\min_{\boldsymbol{b}}\Big[\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - 2\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T}\Big] \quad \text{s.t. } \boldsymbol{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$

$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Solve via quadratic programming:

$$\min_{\boldsymbol{b}}\left[\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - 2\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T}\right] \quad \text{s.t. } \boldsymbol{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Set-negative-to-zero heuristic solution

- Negative reconciled forecasts at bottom level set to zero
- Remaining forecasts computed via aggregation

(Di Fonzo and Girolimetto, 2023)

Immutable forecasts

Zhang et al. (2022)

$$\hat{\boldsymbol{y}}_{t+h|t} = \begin{bmatrix} \hat{\boldsymbol{a}}_{t+h|t} \\ \hat{\boldsymbol{v}}_{t+h|t} \\ \hat{\boldsymbol{u}}_{t+h|t} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_2 \\ \boldsymbol{I}_{n_b-k} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{I}_k \end{bmatrix} \begin{bmatrix} \hat{\boldsymbol{v}}_{t+h|t} \\ \hat{\boldsymbol{u}}_{t+h|t} \end{bmatrix}$$

Suppose $\hat{\boldsymbol{u}}_{t+h|t}$ are fixed and let $\hat{\boldsymbol{w}}_{t+h|t} = \begin{bmatrix} \hat{\boldsymbol{a}}_{t+h|t} - \boldsymbol{A}_2\hat{\boldsymbol{u}}_{t+h|t} \\ \hat{\boldsymbol{v}}_{t+h|t} \end{bmatrix}$.

Optimization problem

$$\min_{\boldsymbol{v}} \left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right]' \boldsymbol{W}_{\boldsymbol{v}}^{-1} \left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right] \qquad \text{where} \qquad \boldsymbol{A}_3 = \begin{bmatrix} \boldsymbol{A}_1 \\ \boldsymbol{I}_{n_b-k} \end{bmatrix}$$

and $\boldsymbol{W}_{\boldsymbol{v}}$ contains elements of $\boldsymbol{W}_h$ corresponding to $\hat{\boldsymbol{v}}_{t+h|t}$.

Immutable forecasts

Zhang et al. (2022)

$$\hat{\boldsymbol{y}}_{t+h|t} = \begin{bmatrix}\hat{\boldsymbol{a}}_{t+h|t}\\ \hat{\boldsymbol{v}}_{t+h|t}\\ \hat{\boldsymbol{u}}_{t+h|t}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{I}_{n_b-k} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{I}_k\end{bmatrix}\begin{bmatrix}\hat{\boldsymbol{v}}_{t+h|t}\\ \hat{\boldsymbol{u}}_{t+h|t}\end{bmatrix}$$

Suppose $\hat{\boldsymbol{u}}_{t+h|t}$ are fixed and let $\hat{\boldsymbol{w}}_{t+h|t} = \begin{bmatrix}\hat{\boldsymbol{a}}_{t+h|t} - \boldsymbol{A}_2\hat{\boldsymbol{u}}_{t+h|t}\\ \hat{\boldsymbol{v}}_{t+h|t}\end{bmatrix}$.

Solve with non-negativity constraint

$$\min_{\boldsymbol{v}}\left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right]'\boldsymbol{W}_{\boldsymbol{v}}^{-1}\left[\hat{\boldsymbol{w}}_{t+h|t} - \boldsymbol{A}_3\boldsymbol{v}\right] \qquad \text{where} \qquad \boldsymbol{A}_3 = \begin{bmatrix}\boldsymbol{A}_1\\ \boldsymbol{I}_{n_b-k}\end{bmatrix}$$

$$\text{such that} \qquad \boldsymbol{A}_3\boldsymbol{v} \geq \begin{bmatrix}-\boldsymbol{A}_2\hat{\boldsymbol{u}}_{t+h|t}\\ \boldsymbol{0}\end{bmatrix}$$

Reconciliation and regularization

Mishchenko, Montgomery, and Vaggi (2019):
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\boldsymbol{y}}_t} \sum_{t=1}^{T} \|\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t\|_2 + \lambda \sum_{t=1}^{T} \|\hat{\boldsymbol{y}}_t - \boldsymbol{S}_t\hat{\boldsymbol{b}}_t\|_2$$

Reconciliation and regularization

Mishchenko, Montgomery, and Vaggi (2019):
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\boldsymbol{y}}_t} \sum_{t=1}^{T} \|\boldsymbol{y}_t - \hat{\boldsymbol{y}}_t\|_2 + \lambda \sum_{t=1}^{T} \|\hat{\boldsymbol{y}}_t - \boldsymbol{S}_t \hat{\boldsymbol{b}}_t\|_2$$

Shiratori, Kobayashi, and Takano (2020):
Optimize bottom level forecasts with an incoherence penalty

$$\min_{\hat{\boldsymbol{b}}_t} \sum_{t=1}^{T} \|\hat{\boldsymbol{b}}_t - \boldsymbol{b}_t\|_2 + \sum_{t=1}^{T} \Lambda \|\boldsymbol{a}_t - \boldsymbol{A}_t \hat{\boldsymbol{b}}_t\|_2$$

Outline

In-built coherence

Two-step approach: compute base forecasts $\hat{\boldsymbol{y}}_h$, and then reconcile them to produce $\tilde{\boldsymbol{y}}_h$.

One-step approaches: compute coherent $\tilde{\boldsymbol{y}}_h$ directly.

- Ashouri, Hyndman, and Shmueli (2022): linear regression models
- Pennings and Dalen (2017): state space models
- Villegas and Pedregal (2018): state space models

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \qquad \hat{\boldsymbol{y}}_{t+h} = \boldsymbol{X}^*_{t+h}\hat{\boldsymbol{B}}$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \qquad \hat{\boldsymbol{y}}_{t+h} = \boldsymbol{X}^*_{t+h}\hat{\boldsymbol{B}} \qquad \boldsymbol{X}^*_{t+h} = \mathrm{diag}(\boldsymbol{x}'_{t+h,i}, \dots, \boldsymbol{x}'_{t+h,n})$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\boldsymbol{y}}_1 \\ \hat{\boldsymbol{y}}_2 \\ \vdots \\ \hat{\boldsymbol{y}}_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{X}_1 & 0 & \dots & 0 \\ 0 & \boldsymbol{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \boldsymbol{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \boldsymbol{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\boldsymbol{B}} = (\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y} \qquad \hat{\boldsymbol{y}}_{t+h} = \boldsymbol{X}^*_{t+h}\hat{\boldsymbol{B}} \qquad \boldsymbol{X}^*_{t+h} = \text{diag}(\boldsymbol{x}'_{t+h,i}, \dots, \boldsymbol{x}'_{t+h,n})$$

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h\hat{\boldsymbol{y}}_{t+h} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{X}^*_{t+h}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

$$\boldsymbol{V}_h = \sigma^2\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h\left[1 + \boldsymbol{X}^*_{T+h}(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}^*_{T+h})'\right]\boldsymbol{W}_h\boldsymbol{S}'(\boldsymbol{S}'\boldsymbol{W}_h\boldsymbol{S})^{-1}\boldsymbol{S}'$$

Reference: Ashouri, Hyndman, and Shmueli (2022)

In-built coherence using state space models

Pennings and Dalen (2017) propose the state space model

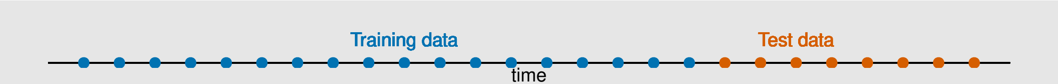
$$
\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{S}\boldsymbol{\mu}_t + \boldsymbol{Z}_t\beta + \varepsilon_t, & \varepsilon_t &\sim N(\mathbf{0}, \Sigma_\varepsilon),\\
\boldsymbol{\mu}_t &= \boldsymbol{\mu}_{t-1} + \boldsymbol{\eta}_t, & \eta_t &\sim N(\mathbf{0}, \Sigma_\eta).
\end{aligned}
$$

- Coherent forecasts arise naturally using the Kalman filter
- Covariance matrices difficult to estimate except for small hierarchies.
- Requires the same model for all series
- A related approach proposed by Villegas and Pedregal (2018)

Outline

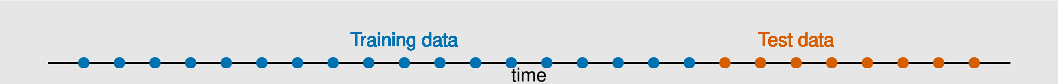
Time series cross-validation

Traditional evaluation

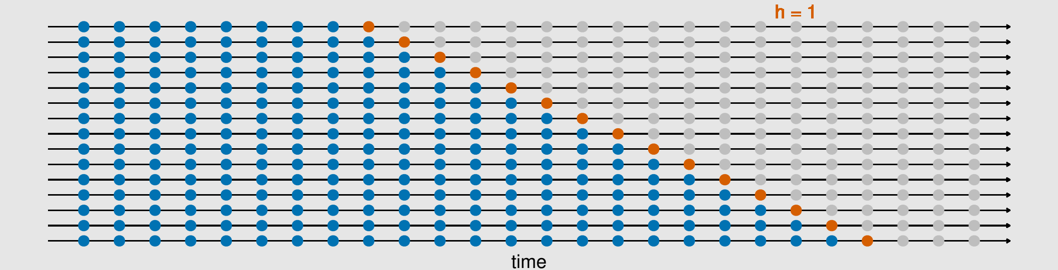


Time series cross-validation

Traditional evaluation

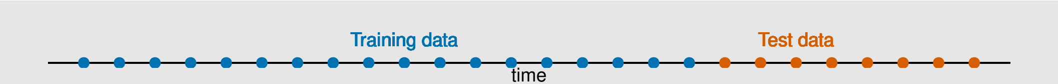


Time series cross-validation

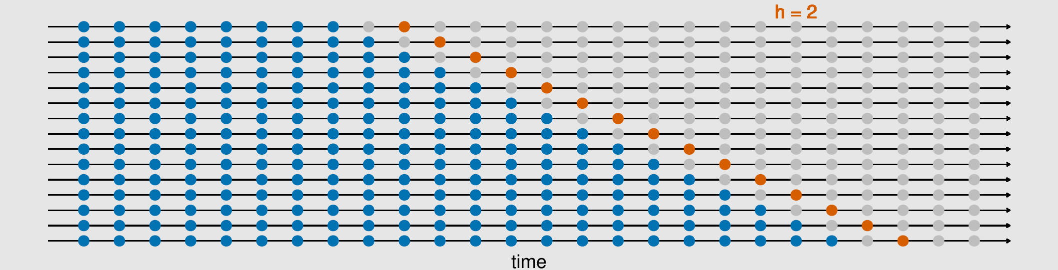


Time series cross-validation

Traditional evaluation

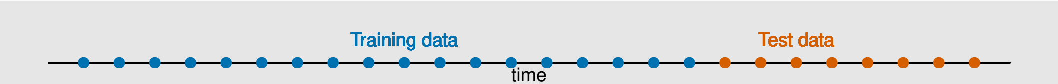


Time series cross-validation

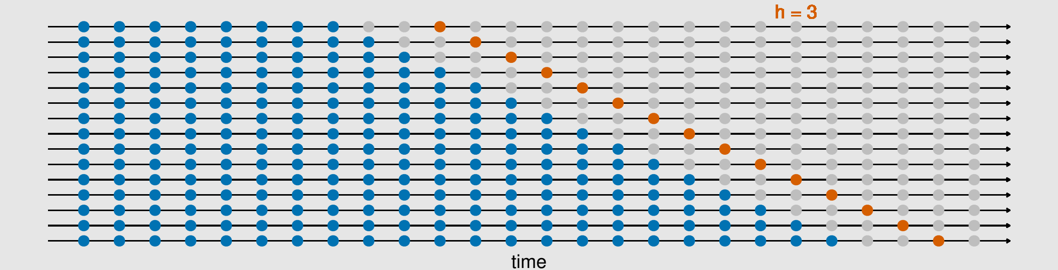


Time series cross-validation

Traditional evaluation

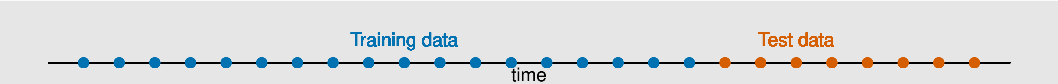


Time series cross-validation

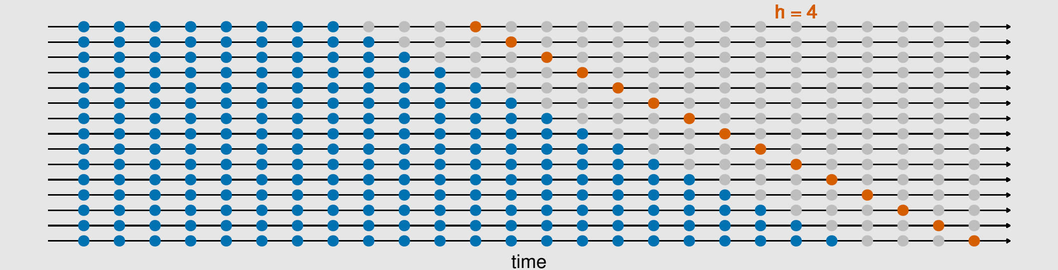


Time series cross-validation

Traditional evaluation

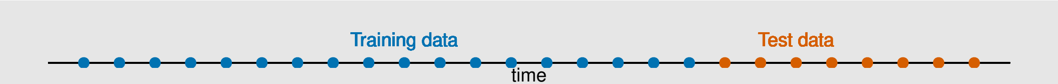


Time series cross-validation

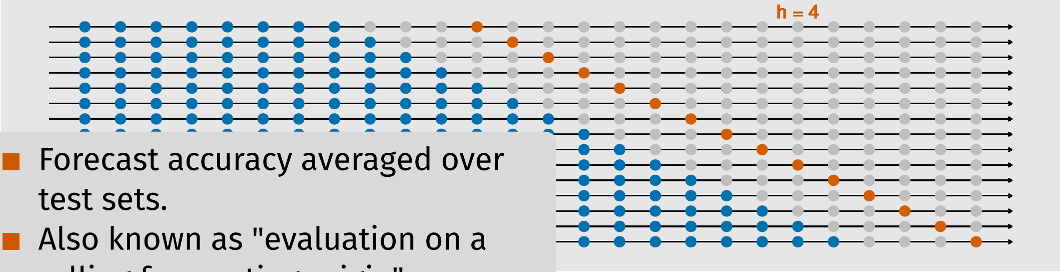


Time series cross-validation

Traditional evaluation



Time series cross-validation



- Forecast accuracy averaged over test sets.
- Also known as "evaluation on a rolling forecasting origin"

Example: Australian tourism

```
tourism
```

```
# A tsibble: 18,000 x 5 [1M]
# Key:       state, zone, region [75]
      month state zone      region visitors
      <mth> <chr> <chr>     <chr>     <dbl>
 1 1998 Jan NSW   Metro NSW Sydney     926.
 2 1998 Feb NSW   Metro NSW Sydney     647.
 3 1998 Mar NSW   Metro NSW Sydney     716.
 4 1998 Apr NSW   Metro NSW Sydney     621.
 5 1998 May NSW   Metro NSW Sydney     598.
 6 1998 Jun NSW   Metro NSW Sydney     601.
 7 1998 Jul NSW   Metro NSW Sydney     720.
 8 1998 Aug NSW   Metro NSW Sydney     645.
 9 1998 Sep NSW   Metro NSW Sydney     633.
10 1998 Oct NSW   Metro NSW Sydney     771.
# i 17,990 more rows
```

Example: Australian tourism

```r
tourism_agg <- tourism |>
  aggregate_key(state/zone/region, visitors = sum(visitors))
tourism_stretch <- tourism_agg |>
  stretch_tsibble(.init = 48, .step = 1)
tourism_stretch
```

```
# A tsibble: 3,057,120 x 6 [1M]
# Key:       .id, state, zone, region [21,230]
      month state  zone   region   visitors   .id
      <mth> <chr*> <chr*> <chr*>      <dbl> <int>
 1 1998 Jan NSW    ACT    Canberra     210.     1
 2 1998 Feb NSW    ACT    Canberra     156.     1
 3 1998 Mar NSW    ACT    Canberra     185.     1
 4 1998 Apr NSW    ACT    Canberra     178.     1
 5 1998 May NSW    ACT    Canberra     134.     1
 6 1998 Jun NSW    ACT    Canberra     105.     1
 7 1998 Jul NSW    ACT    Canberra     142.     1
 8 1998 Aug NSW    ACT    Canberra     137.     1
```

Example: Australian tourism

```r
fit <- tourism_stretch |>
  model(ets = ETS(visitors))
fit
```

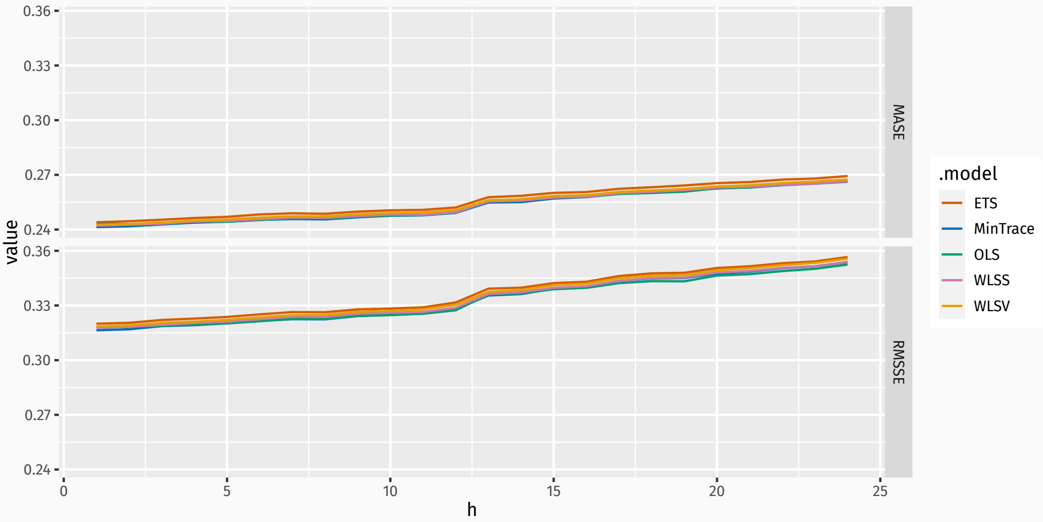
```
# A mable: 21,230 x 5
# Key:     .id, state, zone, region [21,230]
     .id state  zone            region                   ets
   <int> <chr*> <chr*>          <chr*>               <model>
 1     1 NSW    ACT             Canberra        <ETS(A,N,N)>
 2     1 NSW    ACT             <aggregated>    <ETS(A,N,N)>
 3     1 NSW    Metro NSW       Central Coast   <ETS(A,N,A)>
 4     1 NSW    Metro NSW       Sydney          <ETS(A,N,N)>
 5     1 NSW    Metro NSW       <aggregated>    <ETS(M,N,M)>
 6     1 NSW    North Coast NSW Hunter          <ETS(A,N,N)>
 7     1 NSW    North Coast NSW North Coast NSW <ETS(M,N,M)>
 8     1 NSW    North Coast NSW <aggregated>    <ETS(M,N,M)>
 9     1 NSW    North NSW       Blue Mountains  <ETS(M,N,N)>
10     1 NSW    North NSW       Central NSW     <ETS(A,N,N)>
# i 21,220 more rows
# i Use `print(n = ...)` to see more rows
```

Example: Australian tourism

```r
fc <- fit |>
  reconcile(
    ols = min_trace(ets, method = "ols"),
    wlsv = min_trace(ets, method = "wls_var"),
    wlss = min_trace(ets, method = "wls_struct"),
    mint_s = min_trace(ets, method = "mint_shrink"),
  ) |>
  forecast(h = "2 years")
fc
```

```
# A fable: 2,547,600 x 9 [1M]
# Key:     .id, state, zone, region, .model [106,150]
    .id state  zone   region   .model    month      visitors .mean     h
  <int> <chr*> <chr*> <chr*>   <chr>     <mth>        <dist> <dbl> <int>
1     1 NSW    ACT    Canberra ets    2002 Jan N(169, 1553)  169.     1
2     1 NSW    ACT    Canberra ets    2002 Feb N(169, 1553)  169.     2
3     1 NSW    ACT    Canberra ets    2002 Mar N(169, 1553)  169.     3
4     1 NSW    ACT    Canberra ets    2002 Apr N(169, 1553)  169.     4
```

Example: Australian tourism



Example: Australian tourism

Overall

Model	MASE	RMSSE
ETS	0.256	0.337
MinTrace	**0.253**	0.333
OLS	0.253	**0.333**
WLSS	0.253	0.334
WLSV	0.254	0.335

National

Model	MASE	RMSSE
ETS	**1.020**	**1.010**
MinTrace	1.072	1.061
OLS	1.026	1.016
WLSS	1.060	1.052
WLSV	1.078	1.069

State

Model	MASE	RMSSE
ETS	0.109	0.122
MinTrace	0.109	0.121
OLS	**0.107**	**0.120**
WLSS	0.108	0.121
WLSV	0.110	0.122

Zone

Model	MASE	RMSSE
ETS	0.147	0.191
MinTrace	0.145	0.189
OLS	**0.144**	**0.188**
WLSS	0.145	0.189
WLSV	0.146	0.190

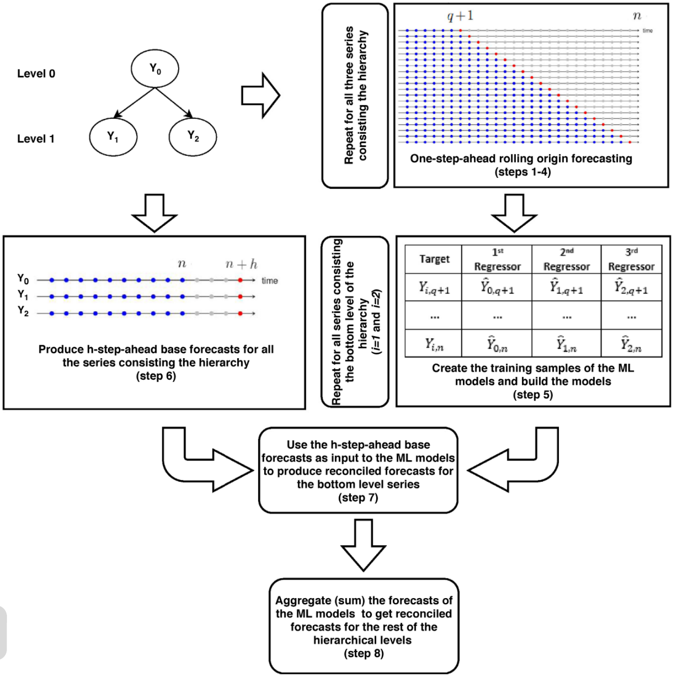
Region

Model	MASE	RMSSE
ETS	0.298	0.372
MinTrace	**0.294**	**0.366**
OLS	0.296	0.367
WLSS	0.295	0.367
WLSV	0.296	0.368

ML reconciliation

1. Split all series using time series cross-validation
2. For each training set, compute one-step-ahead forecasts for all series
3. For each bottom-level series, use RF or XGB to predict values using forecasts of all series as inputs
4. Forecast all series
5. For each bottom-level series, apply ML model to improve forecasts
6. Aggregate bottom-level forecasts to obtain forecasts for other series.

Spiliotis et al. (ASC, 2021)



ML reconciliation: tourism data

Method	Total	States	Zones	Regions	Average
			MASE		
MinT-Struct	1.094	0.968	0.887	0.843	0.948
MinT-Shrink	1.047	**0.956**	0.872	0.824	0.925
ML-RF	1.045	0.964	0.859	0.812	0.920
ML-XGB	**1.043**	0.965	**0.859**	**0.812**	**0.920**
			RMSSE		
MinT-Struct	1.308	1.225	1.137	1.109	1.195
MinT-Shrink	1.265	1.214	1.120	1.086	1.171
ML-RF	1.261	**1.208**	1.104	1.066	1.159
ML-XGB	1.255	1.208	**1.101**	**1.064**	**1.157**
			AMSE		
MinT-Struct	0.988	0.611	0.426	0.349	0.593
MinT-Shrink	0.935	0.599	0.417	0.337	0.572
ML-RF	0.780	**0.526**	0.366	0.319	0.498
ML-XGB	**0.779**	0.526	**0.365**	**0.317**	**0.497**

- ML methods not significantly different.
- MinT methods significantly different from each other and from ML methods.

References

- Ashouri, M, RJ Hyndman, and G Shmueli (2022). Fast forecast reconciliation using linear models. *J Computational & Graphical Statistics* **31**(1), 263–282.
- Ben Taieb, S and B Koo (2019). Regularized regression for hierarchical forecasting without unbiasedness conditions. In: *Proceedings of the 25th ACM SIGKDD International Conference on Knowledge Discovery & Data Mining*. KDD ’19. Anchorage, AK, USA: Association for Computing Machinery, pp.1337–1347.
- Bisaglia, L, T Di Fonzo, and D Girolimetto (2020). Fully reconciled GDP forecasts from income and expenditure sides. Ed. by A Pollice, N Salvati, and F Schirripa Spagnolo, 951–956.
- Byron, RP (1978). The estimation of large social account matrices. *Journal of the Royal Statistical Society, Series A* **141**(3), 359–367.
- Chow, GC and AI Lin (1971). Best linear unbiased interpolation, distribution, and extrapolation of time series by related series. *The Review of Economics and Statistics* **53**(4), 372–375.

References

- Di Fonzo, T (1990). The estimation of *M* disaggregate time series when contemporaneous and temporal aggregates are known. *The Review of Economics and Statistics* **72**(1), 178–182.
- Di Fonzo, T and D Girolimetto (2022). Forecast combination-based forecast reconciliation: Insights and extensions. *International Journal of Forecasting* **forthcoming**.
- Di Fonzo, T and D Girolimetto (2023). Spatio-temporal reconciliation of solar forecasts. *Solar Energy* **251**, 13–29.
- Mishchenko, K, M Montgomery, and F Vaggi (2019). A self-supervised approach to hierarchical forecasting with applications to groupwise synthetic controls. In: *Proceedings of the Time Series Workshop at the 36th International Conference on Machine Learning, Long Beach, California*. PMLR 97.
- Panagiotelis, A, P Gamakumara, G Athanasopoulos, and RJ Hyndman (2021). Forecast reconciliation: A geometric view with new insights on bias correction. *International J Forecasting* **37**(1), 343–359.

References

- Pennings, CL and J van Dalen (2017). Integrated hierarchical forecasting. *European Journal of Operational Research* **263**(2), 412–418.
- Shiratori, T, K Kobayashi, and Y Takano (2020). Prediction of hierarchical time series using structured regularization and its application to artificial neural networks. *PLOS ONE* **15**(11), e0242099.
- Spiliotis, E, M Abolghasemi, RJ Hyndman, F Petropoulos, and V Assimakopoulos (2021). Hierarchical forecast reconciliation with machine learning. *Applied Soft Computing* **112**, 107756.
- Stone, R, DG Champernowne, and JE Meade (1942). The precision of national income estimates. *The Review of Economic Studies* **9**(2), 111–125.
- van Erven, T and J Cugliari (2015). "Game-theoretically optimal reconciliation of contemporaneous hierarchical time series forecasts". In: *Modeling and Stochastic Learning for Forecasting in High Dimension*. Ed. by A Antoniadis, JM Poggi, and X Brossat. Cham: Springer International Publishing, pp.297–317.

References

- Villegas, MA and DJ Pedregal (2018). Supply chain decision support systems based on a novel hierarchical forecasting approach. *Decision Support Systems* **114**, 29–36.
- Wickramasuriya, SL (2021). Properties of point forecast reconciliation approaches. *arXiv preprint arXiv:2103.11129*.
- Wickramasuriya, SL, G Athanasopoulos, and RJ Hyndman (2019). Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization. *J American Statistical Association* **114**(526), 804–819.
- Wickramasuriya, SL, BA Turlach, and RJ Hyndman (2020). Optimal non-negative forecast reconciliation. *Statistics & Computing* **30**(5), 1167–1182.
- Zhang, B, Y Kang, A Panagiotelis, and F Li (2022). Optimal reconciliation with immutable forecasts. *European Journal of Operational Research* **forthcoming**.