Forecast reconciliation

2. Perspectives on forecast reconciliation

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Outline

1. Reconciliation via constraints
2. The geometry of forecast reconciliation
3. Game theory perspectives
4. Adding optimization constraints

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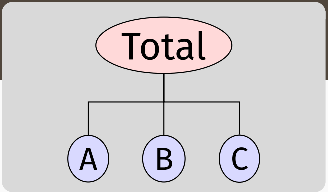
1 Reconciliation via constraints

2 The geometry of forecast reconciliation

3 Game theory perspectives

4 Adding optimization constraints

Notation reminder



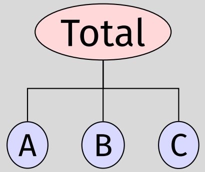
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- y_t = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Notation reminder



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- Base forecasts: $\hat{\boldsymbol{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$
- MinT: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$ where $\boldsymbol{W}_h$ is covariance matrix of base forecast errors.

Zero-constraint representation

Aggregation matrix $\boldsymbol{A}$

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Zero-constraint representation

Aggregation matrix A

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Constraint matrix C

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

- Constraint matrix approach more general & more parsimonious.
- $\boldsymbol{C} = [\boldsymbol{I}_{n_a} \quad -\boldsymbol{A}]$.
- $\boldsymbol{S}$, $\boldsymbol{A}$ and $\boldsymbol{C}$ may contain any real values (not just 0s and 1s).

Zero-constraint representation

Assuming $\boldsymbol{C}$ is full rank

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\hat{\boldsymbol{y}}_{T+h|T}$$

$$\text{where} \qquad \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$$

- Originally proved by Byron (1978, 1979) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- $\boldsymbol{M} = \boldsymbol{S}\boldsymbol{G}$.
- Leads to more efficient reconciliation than using $\boldsymbol{G}$.

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The coherent subspace

Coherent subspace

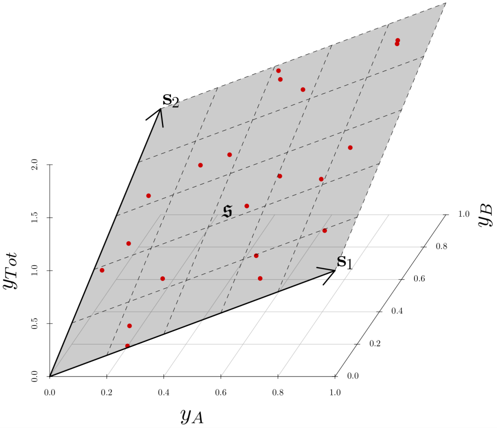
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

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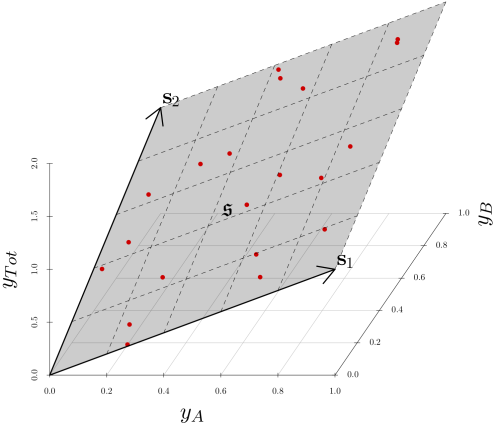
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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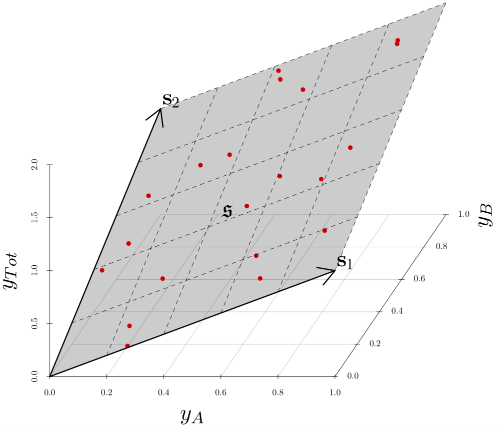

$y_{Tot} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \to \mathfrak{s}$.
$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear reconciliation

If $\psi(\boldsymbol{u}) = \boldsymbol{M}\boldsymbol{u}$ is a linear function, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

OLS:
$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}' \\ &= \boldsymbol{I} - \boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}\boldsymbol{C}\end{aligned}$$

MinT:
$$\begin{aligned}\boldsymbol{M} &= \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1} \\ &= \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}\end{aligned}$$

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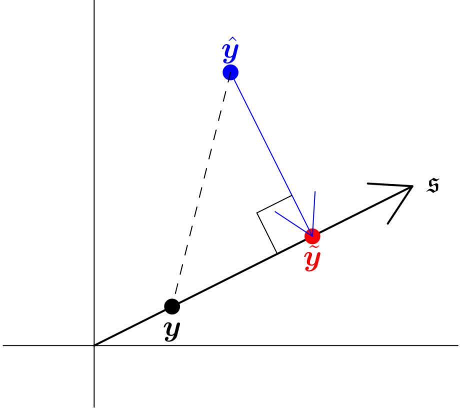

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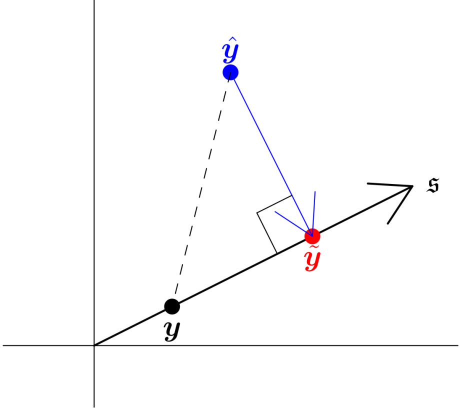

Projections

Suppose $\boldsymbol{M}$ is a projection onto $\mathfrak{s}$, then

- Coherent base forecasts are unchanged.
- Unbiased base forecasts remain unbiased.

- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.

Linear reconciliation

If $\psi(\boldsymbol{u}) = \boldsymbol{M}\boldsymbol{u}$ is a linear function, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

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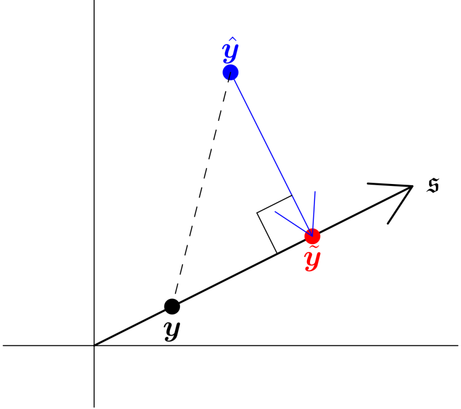
Distance reducing property

If $\boldsymbol{M}$ is an orthogonal projection onto $\mathfrak{s}$:

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\| \le \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|$$



- Distance reduction holds for any realisation and any forecast.
- Other measures of forecast accuracy may be worse.
- Not necessarily the optimal reconciliation.

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$$

where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{M}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1} = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$$

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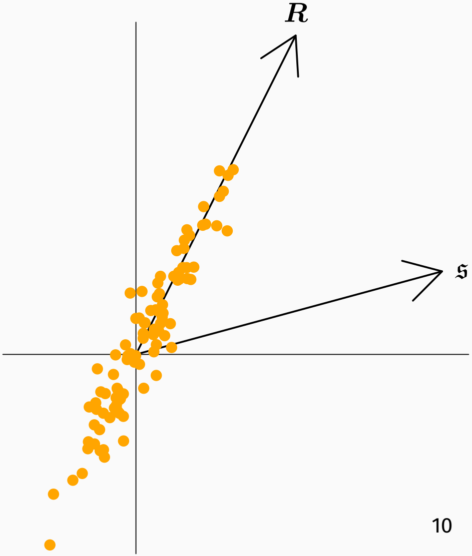
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Orange: in-sample errors



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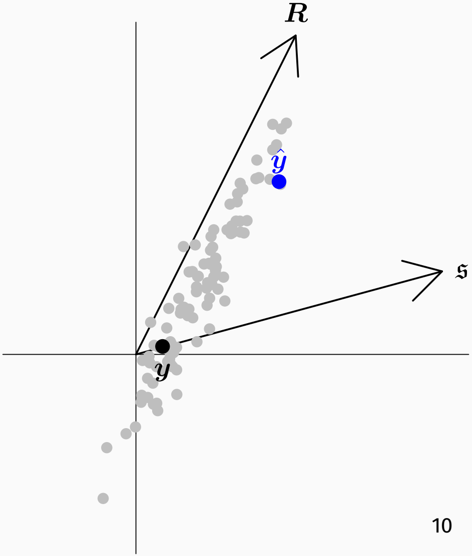

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts

Linear projections

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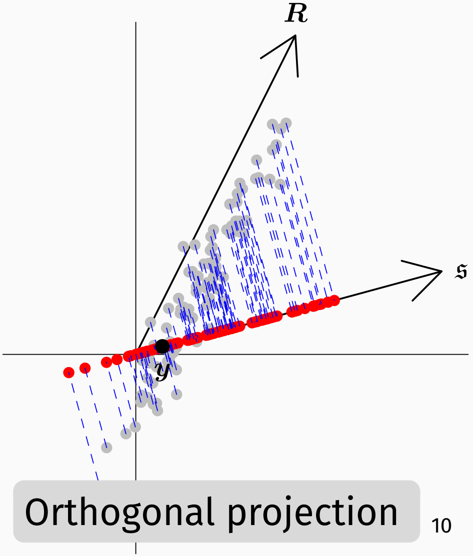

Orthogonal projection

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

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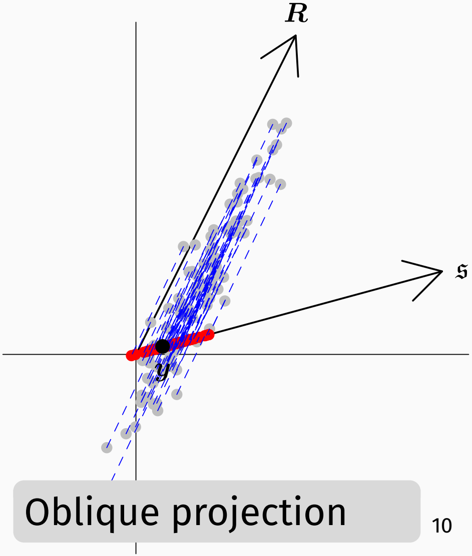

Oblique projection

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Game theory perspectives

Find the solution to the minimax problem

$$V = \min_{\tilde{\boldsymbol{y}} \in \mathfrak{s}} \max_{\boldsymbol{y} \in \mathfrak{s}} \left\{ \ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) - \ell(\boldsymbol{y}, \hat{\boldsymbol{y}}) \right\},$$

where ℓ is a loss function, and $\mathfrak{s}$ is the coherent subspace.

- $V \leq 0$: reconciliation guaranteed to reduce loss.
- If $\ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) = (\boldsymbol{y} - \tilde{\boldsymbol{y}})'\Psi(\boldsymbol{y} - \tilde{\boldsymbol{y}})$, where Ψ is any symmetric pd matrix, then:
  1. $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi\hat{\boldsymbol{y}}$ will always improve upon the base forecasts;
  2. The MinT solution $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}$ will optimise loss in expectation over any choice of Ψ.

Biased reconciliation

Regularized empirical risk minimization problem:

$$\min_{\boldsymbol{G}} \frac{1}{Nn} \|\boldsymbol{Y} - \hat{\boldsymbol{Y}}\boldsymbol{G}'\boldsymbol{S}'\|_F + \lambda \|\text{vec}\boldsymbol{G}\|_1,$$

- $N = T - T_1 - h + 1$, T_1 is minimum training sample size
- $\|\cdot\|_F$ is the Frobenius norm
- $\boldsymbol{Y} = [\boldsymbol{y}_{T_1+h}, \dots, \boldsymbol{y}_T]'$
- $\hat{\boldsymbol{Y}} = [\hat{\boldsymbol{y}}_{T_1+h|T_1}, \dots, \hat{\boldsymbol{y}}_{T|T-h}]'$
- λ is a regularization parameter.

When $\lambda = 0$, $\hat{\boldsymbol{G}} = \boldsymbol{B}'\hat{\boldsymbol{Y}}(\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}})^{-1}$ where $\boldsymbol{B} = [\boldsymbol{b}_{T_1+h}, \dots, \boldsymbol{b}_T]'$.

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Adding optimization constraints

Any approach to reconciliation based on optimisation uses a form of constrained optimisation since reconciled forecasts must lie on the coherent subspace. However, at times additional constraints may be implemented. The first is the case where reconciled forecasts must be non-negative. In general, even if base forecasts are constrained to be positive (which can be achieved by modelling on the log scale and back-transforming), there is no guarantee that the usual reconciliation approaches such as OLS and MinT will maintain the non-negativity of forecasts. To address this issue, the usual optimisation problem can be augmented with

ML and regularization

Bayesian versions

In-built coherence

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