

# Forecast reconciliation

## 2. Perspectives on forecast reconciliation

Rob J Hyndman



MONASH University

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[robjhyndman.com/fr2023](http://robjhyndman.com/fr2023)

# Outline

- 1 Time series reconciliation
- 2 Reconciliation via constraints
- 3 Example: reconciling GDP forecasts
- 4 The geometry of forecast reconciliation
- 5 Optimization and reconciliation
- 6 Time series cross-validation
- 7 ML reconciliation
- 8 In-built coherence

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# Time series reconciliation

- Stone, Champernowne, and Meade (1942): reconciling national economic accounts (disaggregated into production, income, outlay, capital transactions, etc.)
- Byron (1978): extended Stone's work using more computationally efficient methods.
- 1984: Stone wins Nobel Prize in Economics.
- Same approach used for reconciling seasonally adjusted data.
- Chow and Lin (1971): Temporal reconciliation of monthly or quarterly estimates to sum to annual estimates.
- Di Fonzo (1990): Cross-temporal reconciliation of time series data.

# Outline

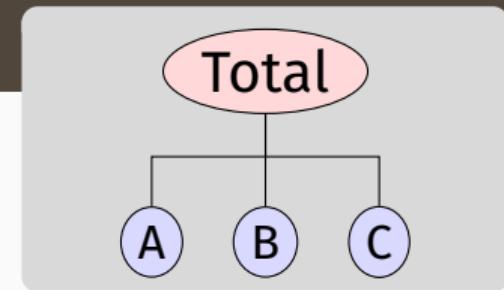
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# Notation reminder

Every collection of time series with linear constraints can be written as

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

- $\mathbf{y}_t$  = vector of all series at time  $t$
- $y_{\text{Total},t}$  = aggregate of all series at time  $t$ .
- $y_{X,t}$  = value of series  $X$  at time  $t$ .
- $\mathbf{b}_t$  = vector of most disaggregated series at time  $t$
- $\mathbf{S}$  = “summing matrix” containing the linear constraints.



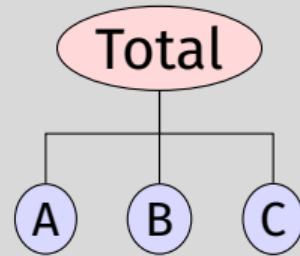
$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t} \end{aligned}$$

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- Base forecasts:  $\hat{\mathbf{y}}_{T+h|T}$
- Reconciled forecasts:  
$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$
- MinT:  
$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}$$

where  $\mathbf{W}_h$  is covariance matrix of base forecast errors.

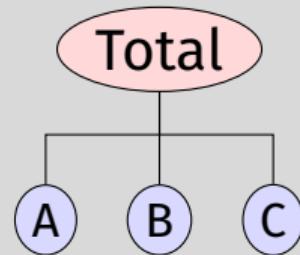
# Notation

## Aggregation matrix

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{pmatrix} \mathbf{b}_t$$



## Constraint matrix

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

$$\begin{aligned} \text{where } \mathbf{C} &= [1 \ -1 \ -1 \ -1] \\ &= [\mathbf{I}_{n_a} \ -\mathbf{A}] \end{aligned}$$

# Zero-constraint representation

## Aggregation matrix A

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{bmatrix} \mathbf{b}_t = \mathbf{S}\mathbf{b}_t$$

# Zero-constraint representation

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## Constraint matrix C

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

- Constraint matrix approach more general & more parsimonious.
- $\mathbf{C} = [\mathbf{I}_{n_a} \quad -\mathbf{A}]$ .
- $\mathbf{S}, \mathbf{A}$  and  $\mathbf{C}$  may contain any real values (not just 0s and 1s).

# Zero-constraint representation

Assuming  $\mathbf{C}$  is full rank

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{M}\hat{\mathbf{y}}_{T+h|T}$$

where  $\mathbf{M} = \mathbf{I} - \mathbf{W}_h\mathbf{C}'(\mathbf{C}\mathbf{W}_h\mathbf{C}')^{-1}\mathbf{C}$

- Originally proved by Byron (1978) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- $\mathbf{M} = \mathbf{S}\mathbf{G}$  (the MinT solution)
- Leads to more efficient reconciliation than using  $\mathbf{G}$ .

# Zero-constraint representation

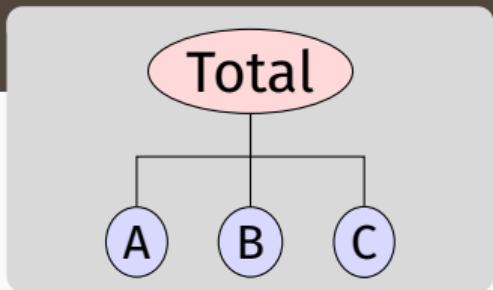
Suppose  $\mathbf{W}_h = \mathbf{I}$ . Then

$$\mathbf{M} = \mathbf{I} - \mathbf{W}_h \mathbf{C}' (\mathbf{C} \mathbf{W}_h \mathbf{C}')^{-1} \mathbf{C}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{4} (1 \quad -1 \quad -1 \quad -1)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$



$$\mathbf{A} = (1 \quad 1 \quad 1)$$

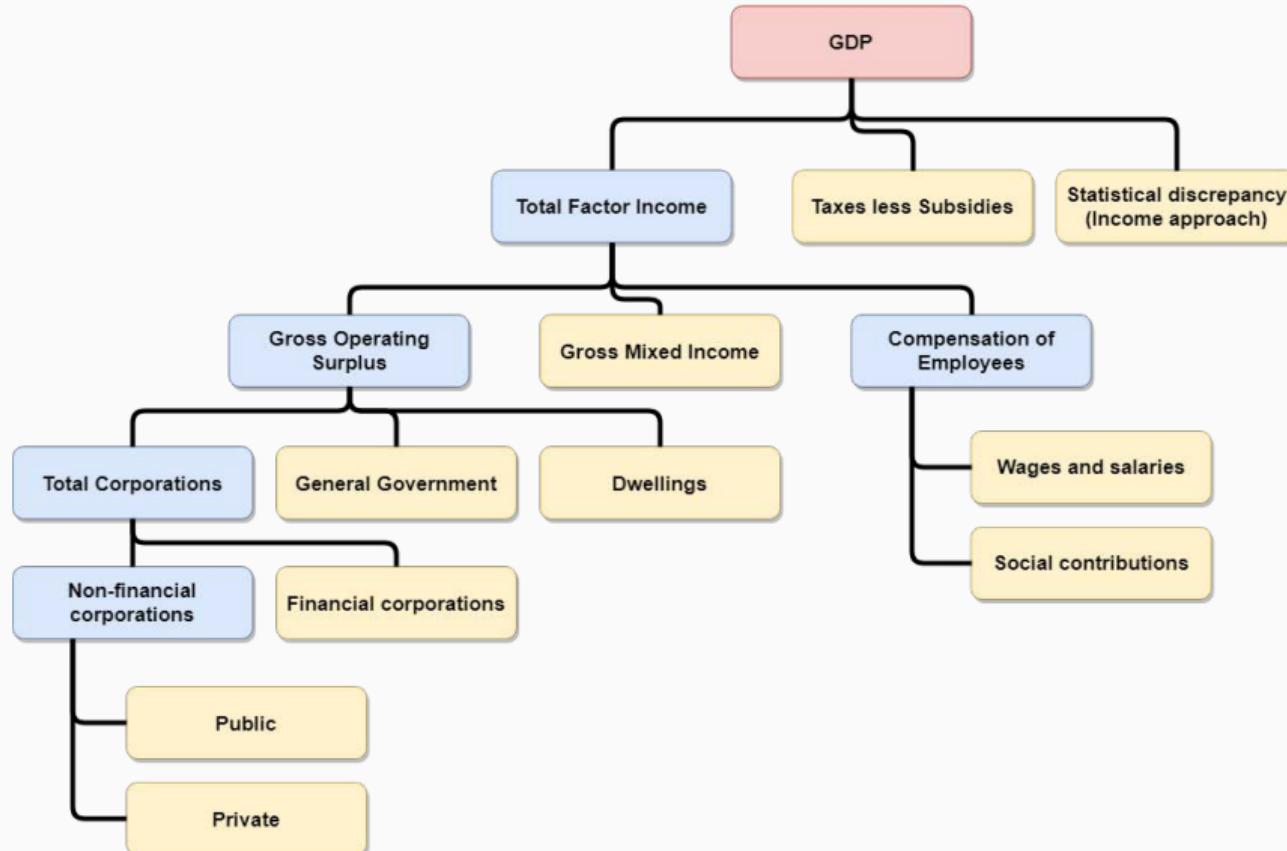
$$\mathbf{s} = \begin{pmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{C} = (\mathbf{I}_{n_a} \quad -\mathbf{A}) = (1 \quad -1 \quad -1 \quad -1)$$

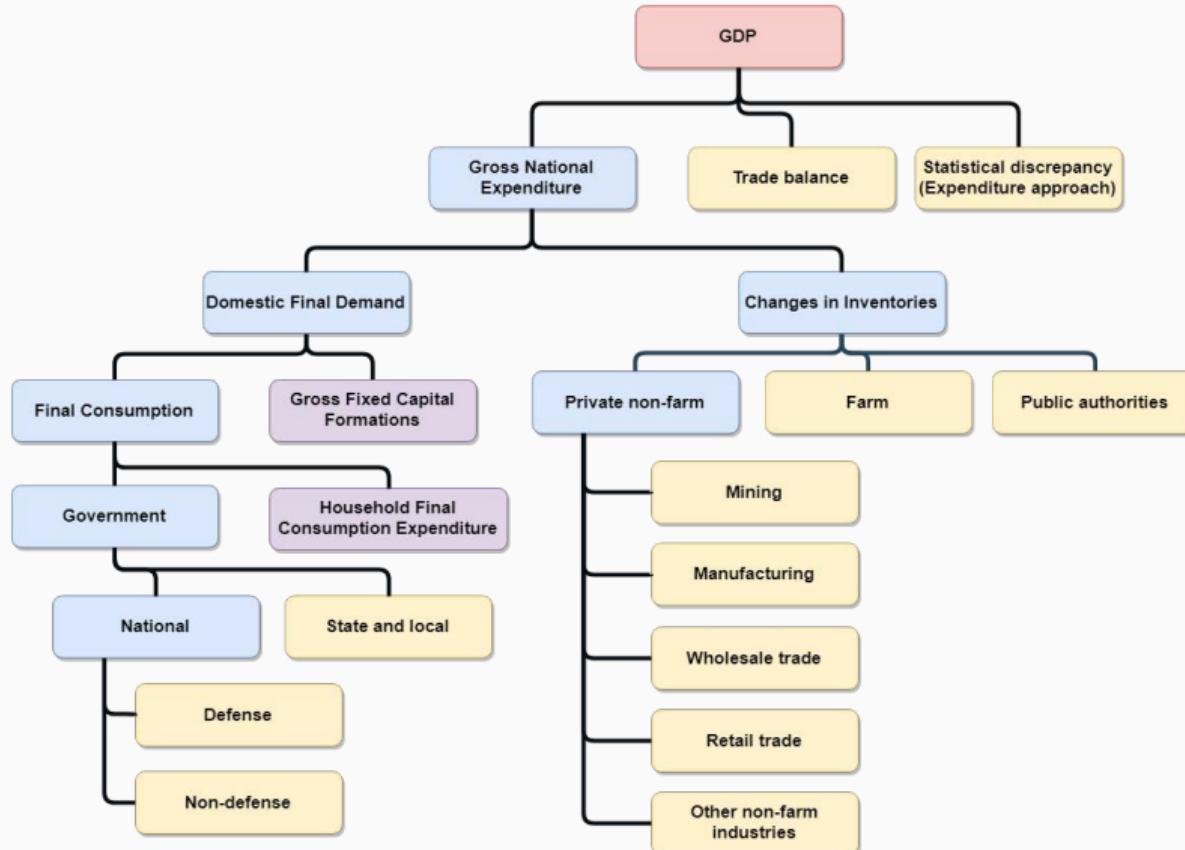
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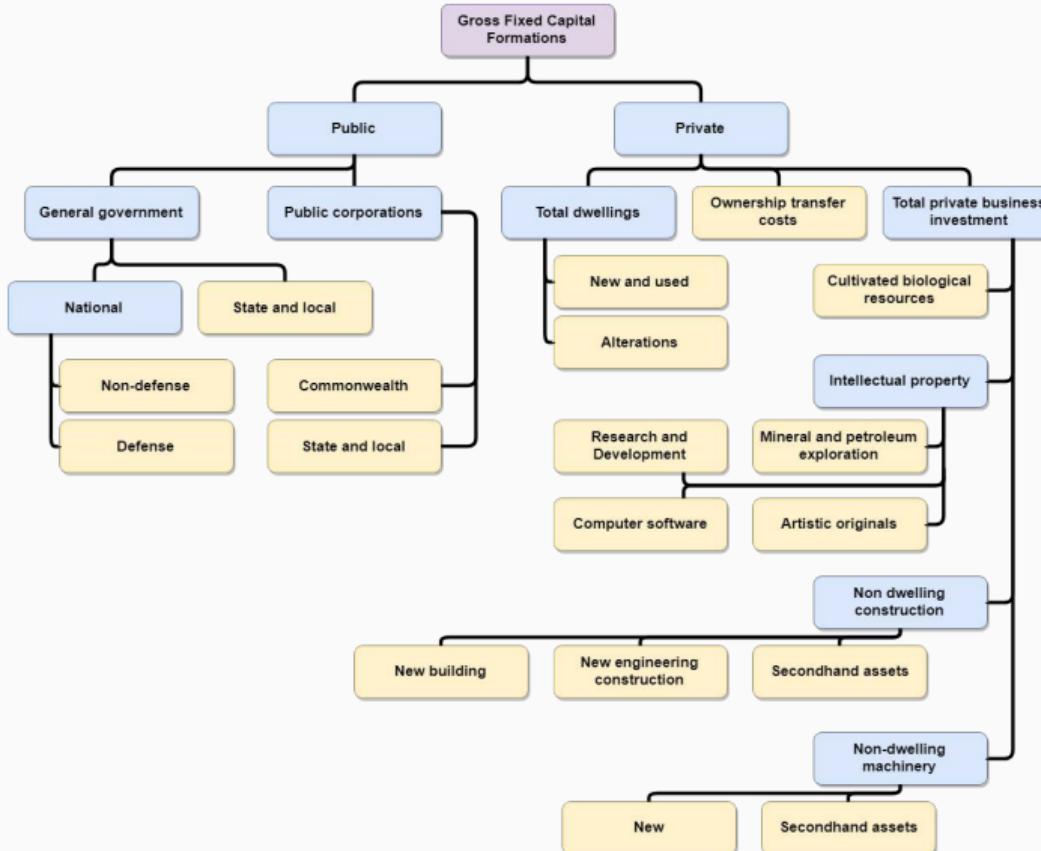
# Example: reconciling GDP forecasts



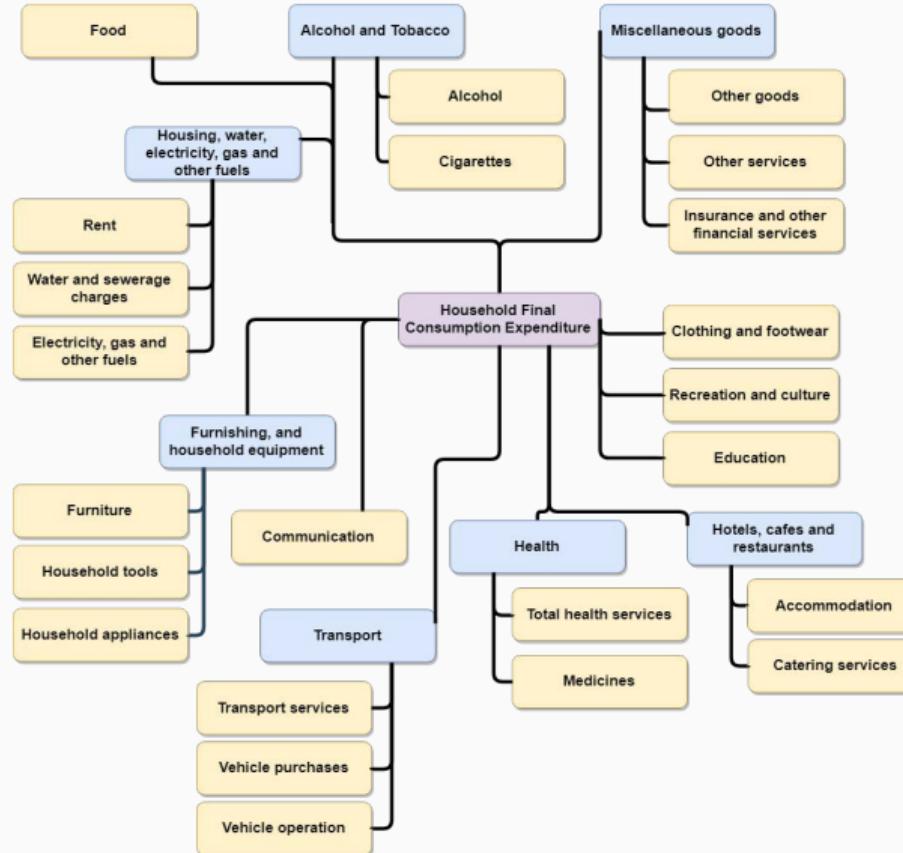
# Example: reconciling GDP forecasts



# Example: reconciling GDP forecasts



# Example: reconciling GDP forecasts



## Example: reconciling GDP forecasts

- No unique hierarchy.
- Several disaggregations with the same parent node
- Not possible to represent using structural **S** notation.
- Instead, we can use the constraint **C** notation.

# Example: reconciling GDP forecasts

Using structural notation:

$$\mathbf{y}_t^I = \begin{bmatrix} X_t \\ \mathbf{a}_t^I \\ \mathbf{b}_t^I \end{bmatrix} = \mathbf{S}^I \mathbf{b}_t^I \quad \mathbf{y}_t^E = \begin{bmatrix} X_t \\ \mathbf{a}_t^E \\ \mathbf{b}_t^E \end{bmatrix} = \mathbf{S}^E \mathbf{b}_t^E$$

where

$$\mathbf{S}^I = \begin{bmatrix} \mathbf{1}'_{10} \\ \mathbf{A}' \\ \mathbf{I}_{10} \end{bmatrix} \quad \mathbf{S}^E = \begin{bmatrix} \mathbf{1}'_{53} \\ \mathbf{A}^E \\ \mathbf{I}_{53} \end{bmatrix}$$

- Can reconcile both trees, but the totals won't be equal.

# Example: reconciling GDP forecasts

Using constraint notation:

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

where

$$\mathbf{y}_t = \begin{bmatrix} x_t \\ \mathbf{a}'_t \\ \mathbf{b}'_t \\ \mathbf{a}^E_t \\ \mathbf{b}^E_t \end{bmatrix} \quad \text{and} \quad \mathbf{C} = \begin{bmatrix} 1 & \mathbf{0}'_5 & -\mathbf{1}'_{10} & \mathbf{0}'_{26} & \mathbf{0}'_{53} \\ 1 & \mathbf{0}'_5 & \mathbf{0}'_{10} & \mathbf{0}'_{26} & -\mathbf{1}'_{53} \\ \mathbf{0}_5 & \mathbf{I}_5 & -\mathbf{A}' & \mathbf{0}_{5 \times 26} & \mathbf{0}_{5 \times 53} \\ \mathbf{0}_{26} & \mathbf{0}_{26 \times 5} & \mathbf{0}_{26 \times 10} & \mathbf{I}_{26} & -\mathbf{A}^E \end{bmatrix}$$

Ref: Bisaglia, Di Fonzo, and Girolimetto (2020)

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# The coherent subspace

## Coherent subspace

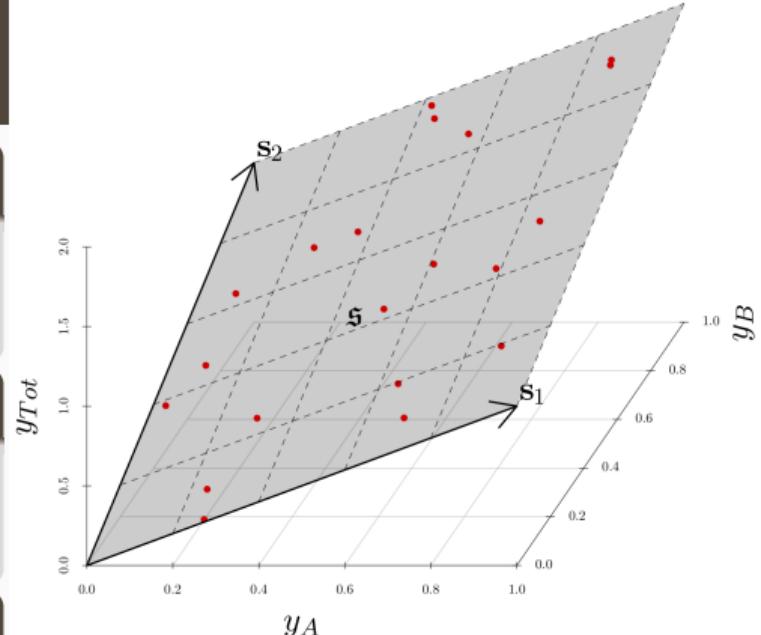
$n_b$ -dimensional linear subspace  $\mathfrak{s} \subset \chi^n$  for which linear constraints hold for all  $\mathbf{y} \in \mathfrak{s}$ .

## Hierarchical time series

An  $n$ -dimensional multivariate time series such that  $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$ .

## Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$  is coherent if  $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$ .



$$y_{Tot} = y_A + y_B$$

# The coherent subspace

## Coherent subspace

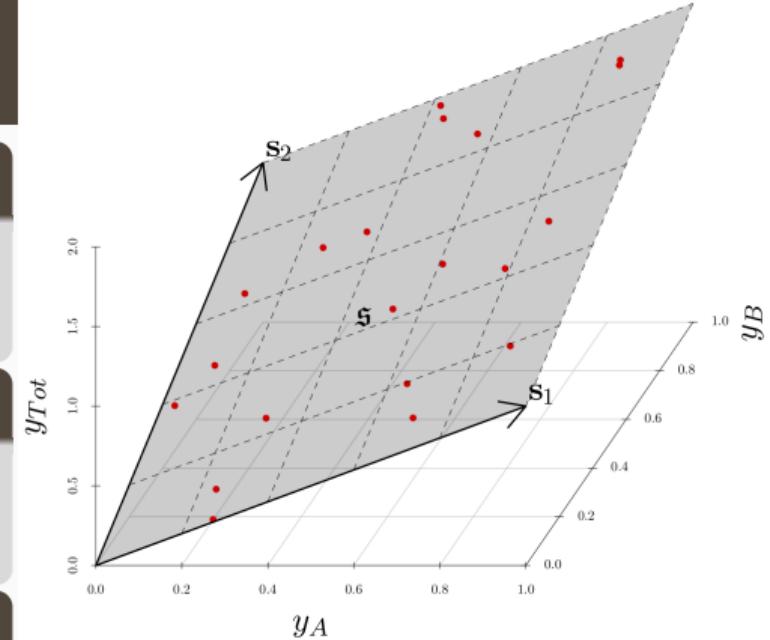
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$\tilde{\mathbf{y}}_{t+h|t}$  is *coherent* if  $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$ .



$$y_{Tot} = y_A + y_B$$

## Base forecasts

Let  $\hat{\mathbf{y}}_{t+h|t}$  be vector of *incoherent* initial  $h$ -step forecasts.

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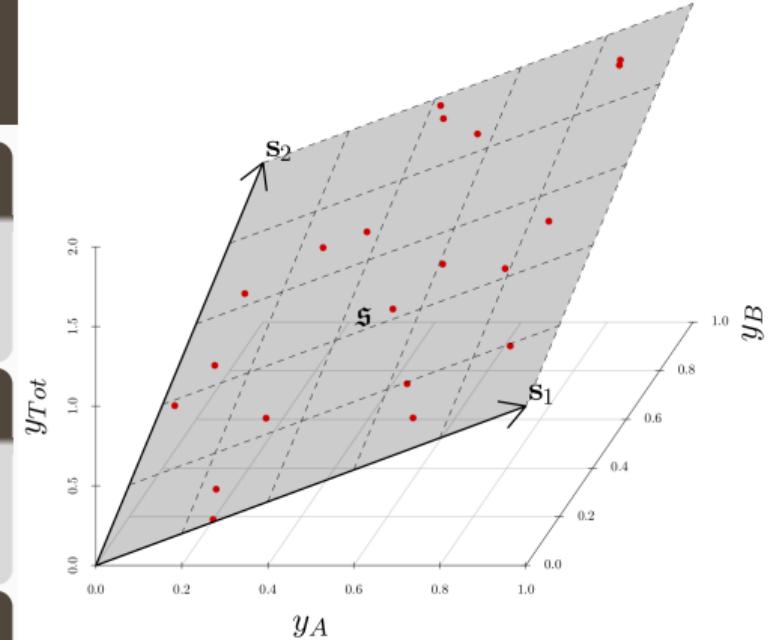
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## Base forecasts

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## Reconciled forecasts

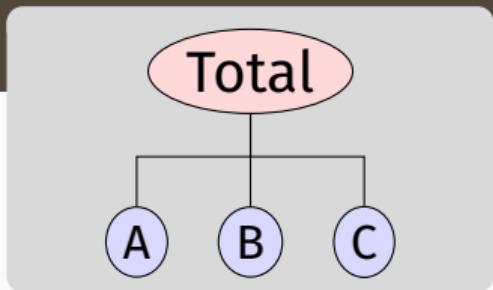
Let  $\psi$  be a mapping,  $\psi : \chi^n \rightarrow \mathfrak{s}$ .  
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$  “reconciles”  $\hat{\mathbf{y}}_{t+h|t}$ .

# The coherent subspace

The columns of  $\mathbf{S}$  form a basis set for  $\mathfrak{s}$ .

They are not unique.

Each corresponds to different vector of “bottom-level” series.

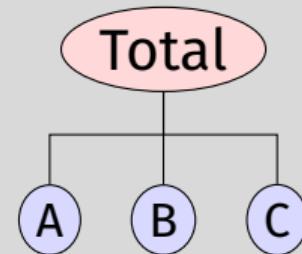


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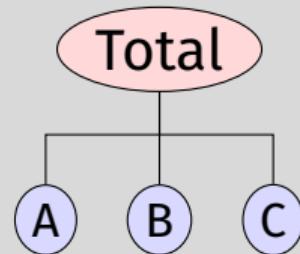
$$\mathbf{y} = \begin{pmatrix} \text{Total} \\ A \\ B \\ C \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

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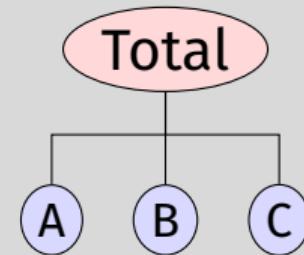
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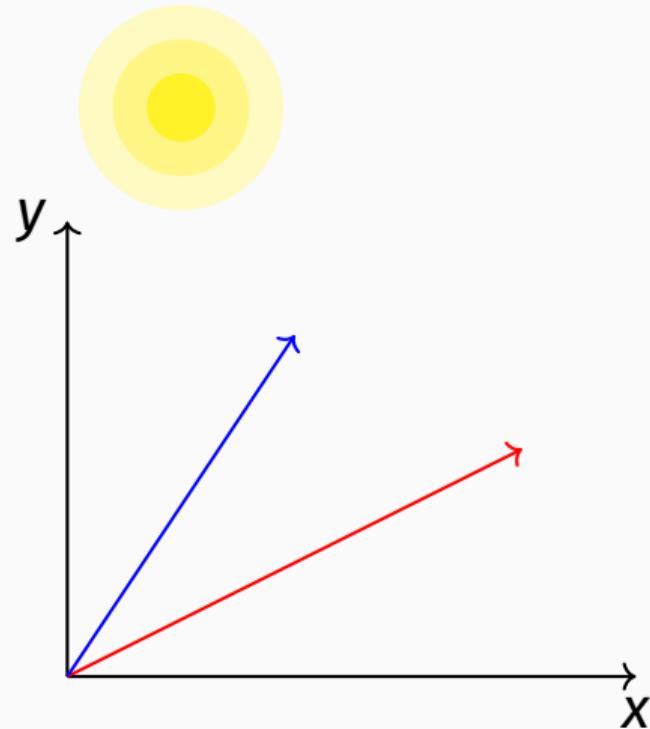
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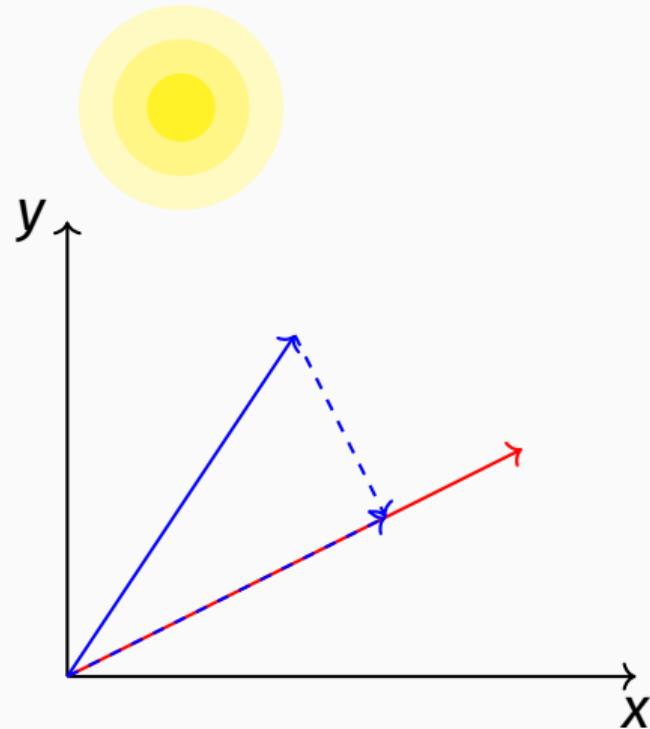


$$\mathbf{y} = \begin{pmatrix} \text{Total} \\ A \\ B \\ C \end{pmatrix} \quad \mathbf{S} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} \text{Total} \\ B + A \\ C + B \end{pmatrix}$$

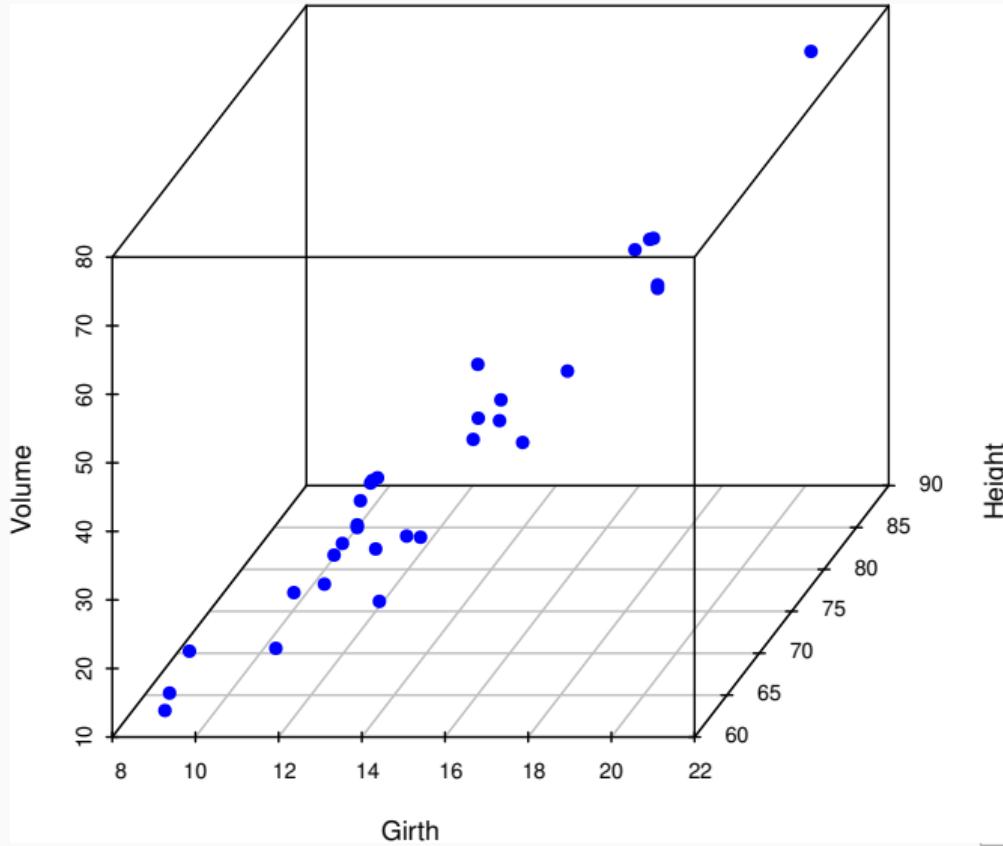
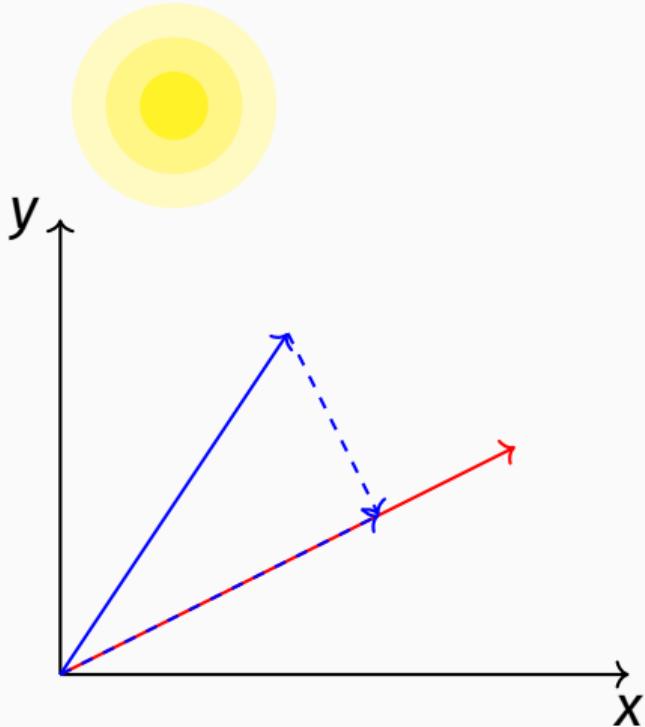
# Projections in linear algebra



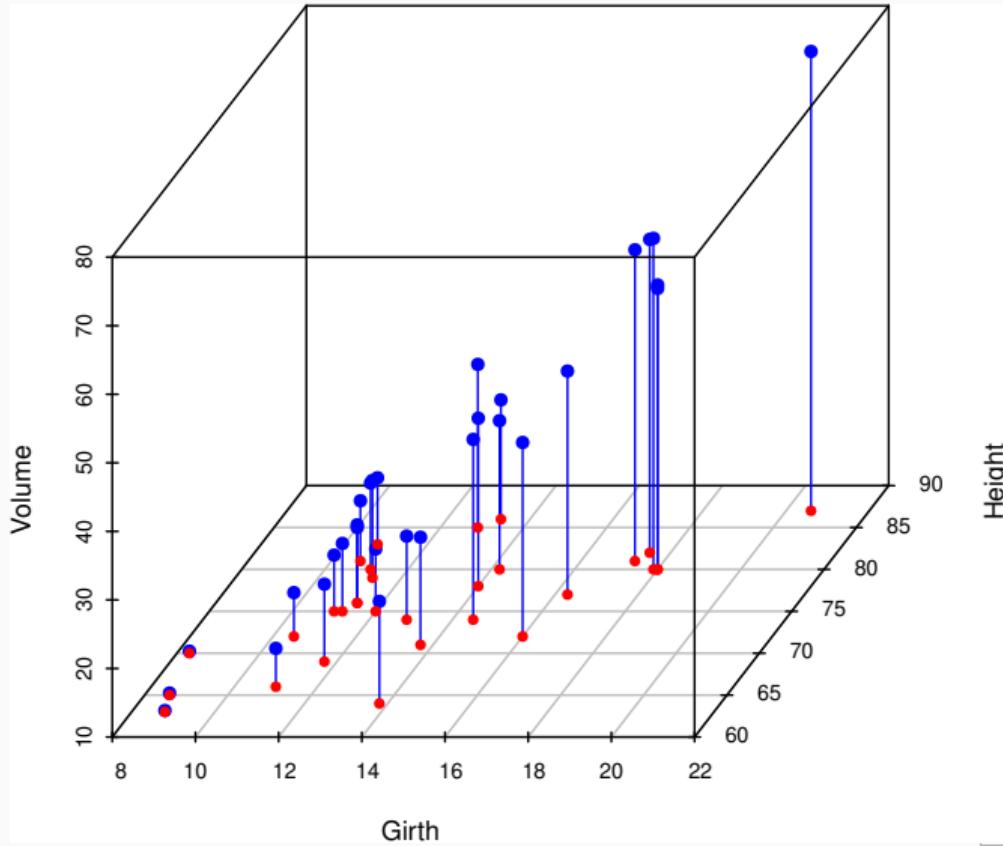
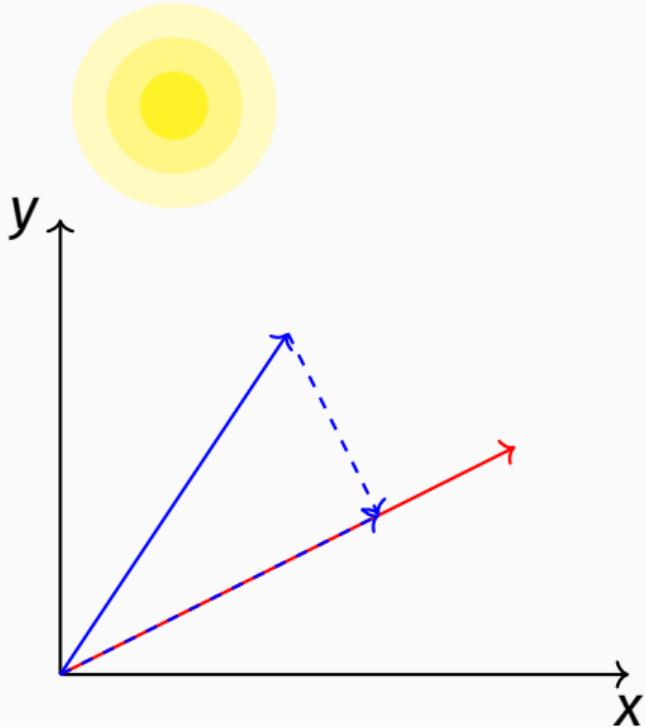
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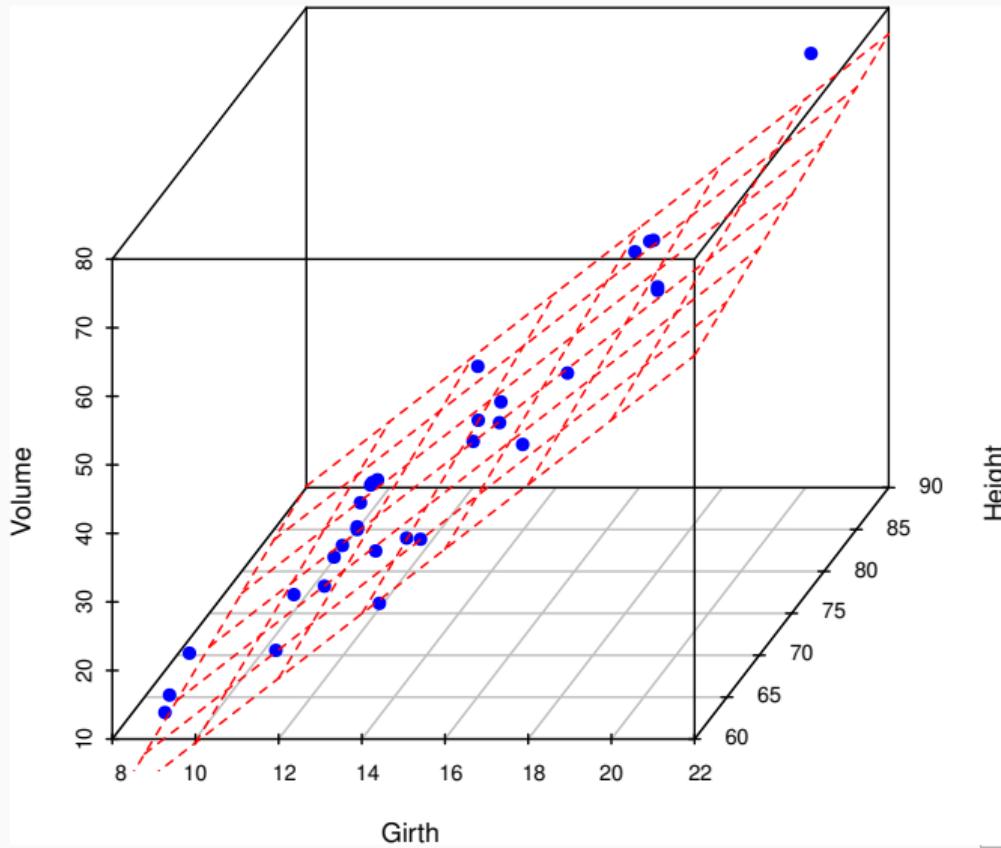
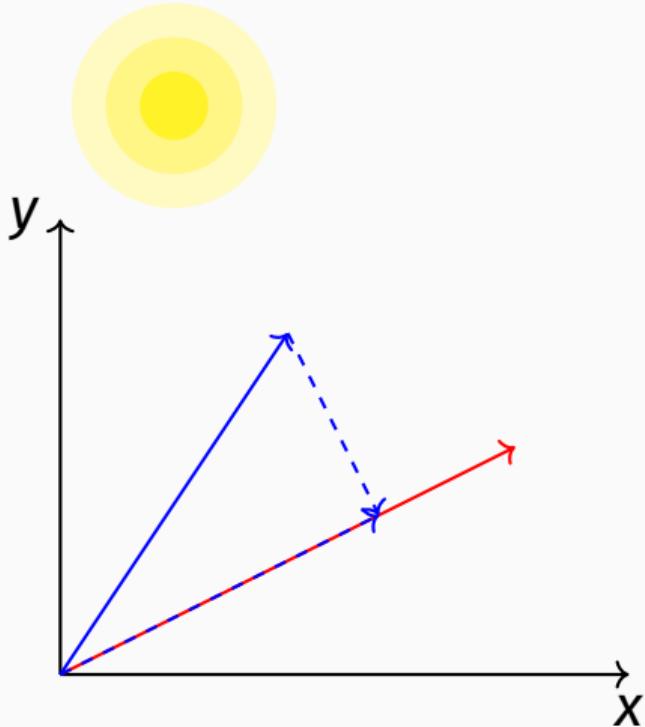
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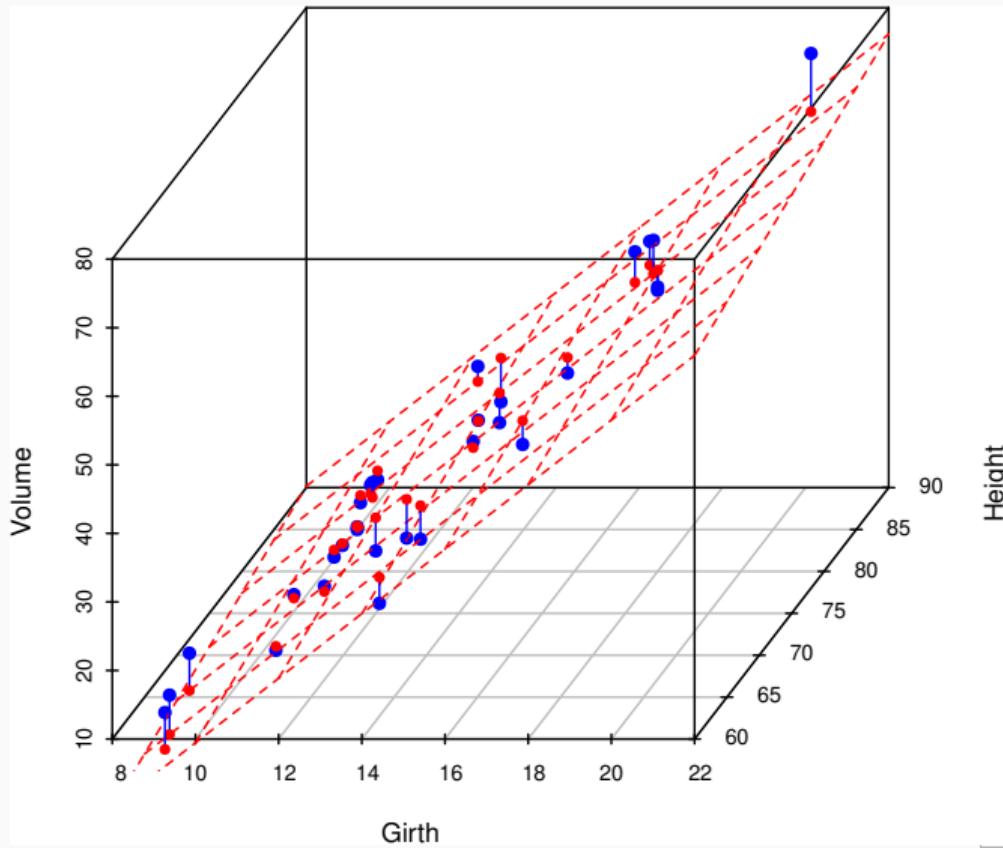
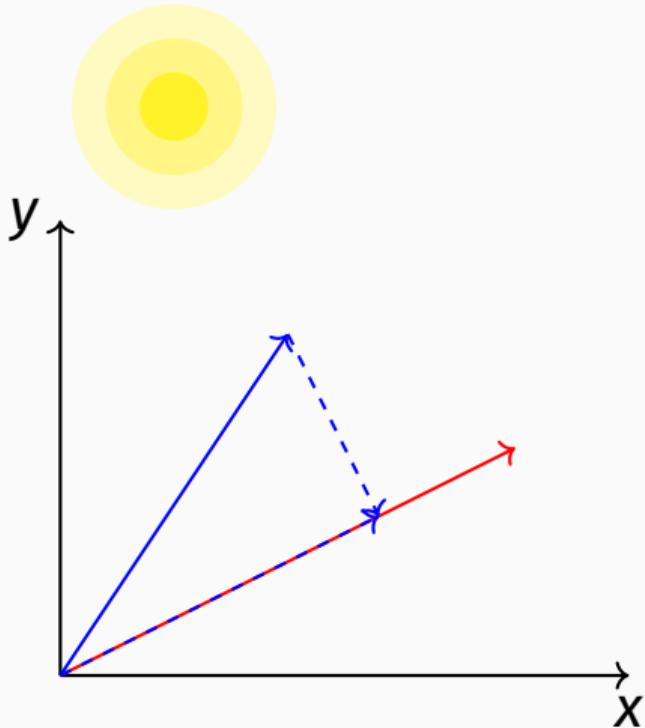
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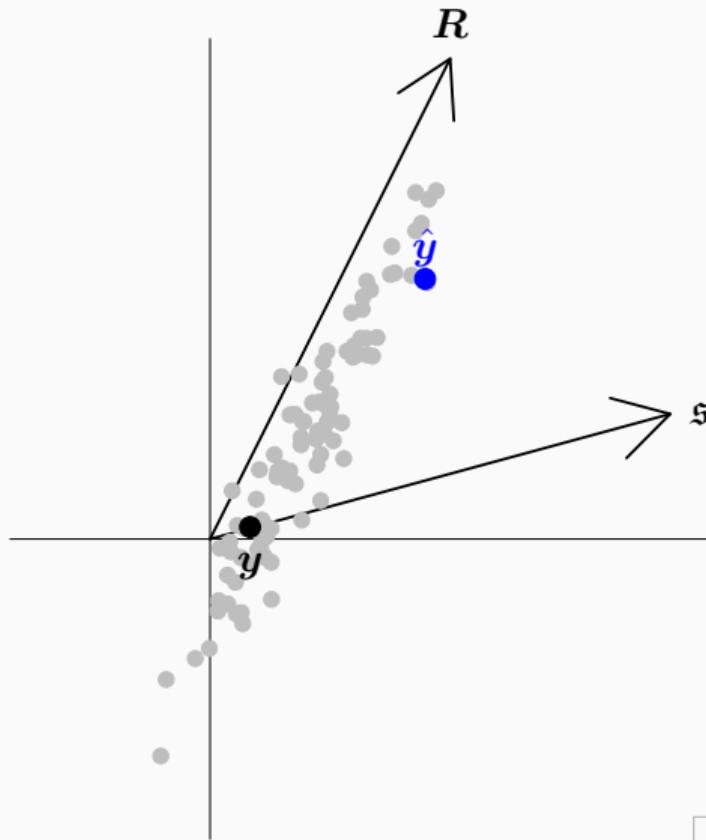


# Projections in linear algebra

- A projection is a linear transformation  $\mathbf{M}$  such that  $\mathbf{M}^2 = \mathbf{M}$ .
- i.e.,  $\mathbf{M}$  is idempotent: it leaves its image unchanged.
- $\mathbf{M}$  projects onto  $\mathfrak{s}$  if  $\mathbf{M}\mathbf{y} = \mathbf{y}$  for all  $\mathbf{y} \in \mathfrak{s}$ .
- All eigenvalues of  $\mathbf{M}$  are either 0 or 1.
- All singular values of  $\mathbf{M}$  are greater than or equal to 1 (with equality iff  $\mathbf{M}$  is orthogonal).
- A projection is *orthogonal* if  $\mathbf{M}' = \mathbf{M}$ .
- If a projection is not orthogonal, it is called *oblique*.
- In regression, OLS is an orthogonal projection onto space spanned by predictors.

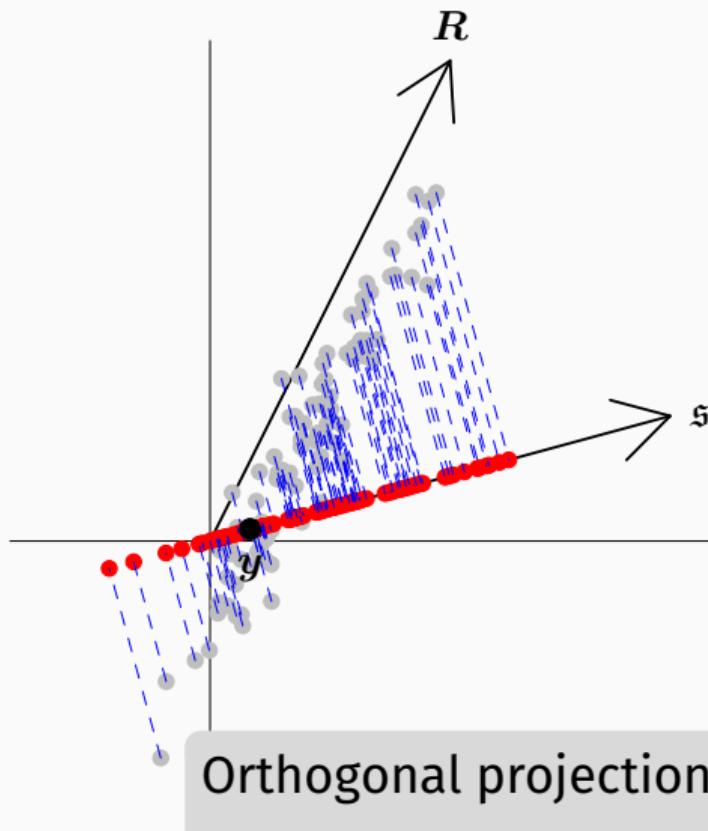
# Linear projection reconciliation

- $R$  is the most likely direction of deviations from  $\hat{s}$ .
- Grey: potential base forecasts



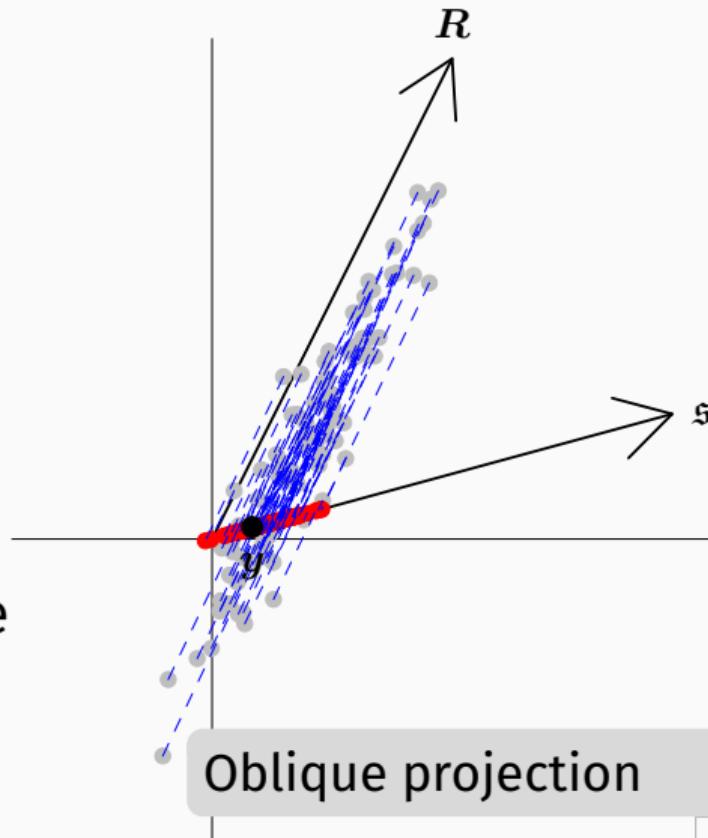
# Linear projection reconciliation

- $R$  is the most likely direction of deviations from  $\hat{s}$ .
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.



# Linear projection reconciliation

- $R$  is the most likely direction of deviations from  $\hat{s}$ .
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.



# Linear projection reconciliation

$$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

- $\mathbf{M}$  is a projection onto  $\mathfrak{s}$  if and only if  $\mathbf{M}\mathbf{y} = \mathbf{y}$  for all  $\mathbf{y} \in \mathfrak{s}$ .
- Coherent base forecasts are unchanged since  $\mathbf{M}\hat{\mathbf{y}} = \hat{\mathbf{y}}$
- If  $\hat{\mathbf{y}}$  is unbiased, then  $\tilde{\mathbf{y}}$  is also unbiased since

$$E(\tilde{\mathbf{y}}_{t+h|t}) = E(\mathbf{M}\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}E(\hat{\mathbf{y}}_{t+h|t}) = E(\hat{\mathbf{y}}_{t+h|t}),$$

and unbiased estimates must lie on  $\mathfrak{s}$ .

- The projection is orthogonal if and only if  $\mathbf{M}' = \mathbf{M}$ .
- If  $\mathbf{S}$  forms a basis set for  $\mathfrak{s}$ , then projections are of the form  $\mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi$  where  $\Psi$  is a positive definite matrix.

# Linear projection reconciliation

$$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}, \quad \text{where } \mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi$$

OLS:  $\Psi = \mathbf{I}$        $\mathbf{M} = \mathbf{S}(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}' = \mathbf{I} - \mathbf{C}'(\mathbf{C}\mathbf{C}')^{-1}\mathbf{C}$

MinT:  $\Psi = \mathbf{W}_h$        $\mathbf{M} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1} = \mathbf{I} - \mathbf{W}_h\mathbf{C}'(\mathbf{C}\mathbf{W}_h\mathbf{C}')^{-1}\mathbf{C}$

- $\mathbf{M}$  is orthogonal iff  $\Psi = \mathbf{I}$ .
- $\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T]$  is the covariance matrix of the base forecast errors.
- $\mathbf{V}_h = \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T] = \mathbf{M}\mathbf{W}_h\mathbf{M}'$  is minimized when  $\Psi = \mathbf{W}_h$ .

## Distance reducing property

Let  $\|\mathbf{u}\|_\Psi = \mathbf{u}'\Psi\mathbf{u}$ . Then

$$\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}\|_\Psi \leq \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|_\Psi$$

- $\Psi$ -projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure.*
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.
- Other measures of forecast accuracy may be worse.

$$\begin{aligned}\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h}\|_2^2 &= \|\mathbf{M}(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h})\|_2^2 \\ &\leq \|\mathbf{M}\|_2^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h}\|_2^2 \\ &= \sigma_{\max}^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h}\|_2^2\end{aligned}$$

- $\sigma_{\max}$  is the largest eigenvalue of  $\mathbf{M}$
- $\sigma_{\max} \geq 1$  as  $\mathbf{M}$  is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

$$\begin{aligned} & \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]'[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]\right) \end{aligned}$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

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# Minimum trace reconciliation

## Minimum trace (MinT) reconciliation

If  $\mathbf{SG}$  is a projection, then the trace of  $\mathbf{V}_h = \text{Var}(\tilde{\mathbf{y}}_{t+h|t} - \mathbf{y}_{t+h})$  is **minimized** when

$$\mathbf{G} = (\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1}$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}$$

Reconciled forecasts

Base forecasts

- Trace of  $\mathbf{V}_h$  is sum of forecast variances.
- MinT solution is  $L_2$  **optimal** amongst linear unbiased forecasts.

Find the solution to the minimax problem

$$V = \min_{\tilde{\mathbf{y}} \in \mathfrak{s}} \max_{\mathbf{y} \in \mathfrak{s}} \left\{ \ell(\mathbf{y}, \tilde{\mathbf{y}}) - \ell(\mathbf{y}, \hat{\mathbf{y}}) \right\},$$

where  $\ell$  is a loss function, and  $\mathfrak{s}$  is the coherent subspace.

- $V \leq 0$ : reconciliation guaranteed to reduce loss.
- If  $\ell(\mathbf{y}, \tilde{\mathbf{y}}) = \|\mathbf{y} - \tilde{\mathbf{y}}\|_\Psi = (\mathbf{y} - \tilde{\mathbf{y}})' \Psi (\mathbf{y} - \tilde{\mathbf{y}})$ , where  $\Psi$  is any symmetric pd matrix, then:
  - 1  $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}' \Psi \mathbf{S})^{-1} \mathbf{S}' \Psi \hat{\mathbf{y}}$  will always improve upon the base forecasts;
  - 2 The MinT solution  $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}$  will optimise loss in expectation over any choice of  $\Psi$ .

Regularized empirical risk minimization problem:

$$\min_{\mathbf{G}} \frac{1}{Nn} \|\mathbf{Y} - \hat{\mathbf{Y}}\mathbf{G}'\mathbf{S}'\|_F + \lambda \|\text{vec}\mathbf{G}\|_1,$$

- $N = T - T_1 - h + 1$ ,  $T_1$  is minimum training sample size
- $\|\cdot\|_F$  is the Frobenius norm
- $\mathbf{Y} = [\mathbf{y}_{T_1+h}, \dots, \mathbf{y}_T]'$
- $\hat{\mathbf{Y}} = [\hat{\mathbf{y}}_{T_1+h|T_1}, \dots, \hat{\mathbf{y}}_{T|T-h}]'$
- $\lambda$  is a regularization parameter

When  $\lambda = 0$ :  $\hat{\mathbf{G}} = \mathbf{B}'\hat{\mathbf{Y}}(\hat{\mathbf{Y}}'\hat{\mathbf{Y}})^{-1}$  where  $\mathbf{B} = [\mathbf{b}_{T_1+h}, \dots, \mathbf{b}_T]'$ .

Reference: Ben Taieb and Koo (2019)

# MinT expressed as a regression

Since  $\tilde{\mathbf{b}}_{t+h|t} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}$ , we can write the MinT solution as a regression problem:

$$\begin{aligned}\tilde{\mathbf{b}}_{t+h|t} &= \arg \min_{\mathbf{b}} [\hat{\mathbf{y}}_{t+h|t} - \mathbf{S}\mathbf{b}]' \mathbf{W}_h^{-1} [\hat{\mathbf{y}}_{t+h|t} - \mathbf{S}\mathbf{b}] \\ &= \arg \min_{\mathbf{b}} [\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S}\mathbf{b} - 2\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t} + \hat{\mathbf{y}}_{t+h|t}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}] \\ &= \arg \min_{\mathbf{b}} [\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S}\mathbf{b} - 2\mathbf{b}'\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}]\end{aligned}$$

- MinT solution is equivalent to a GLS regression of  $\hat{\mathbf{y}}_{t+h|t}$  on  $\mathbf{S}$  with covariance weights  $\mathbf{W}_h^{-1}$ .
- The estimated coefficients are the forecasts of the bottom level series.

# Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left( \mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

# Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left( \mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Solve via quadratic programming:

$$\min_{\mathbf{b}} [\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - 2\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}] \quad \text{s.t. } \mathbf{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

# Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left( \mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Solve via quadratic programming:

$$\min_{\mathbf{b}} [\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - 2\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}] \quad \text{s.t. } \mathbf{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Set-negative-to-zero heuristic solution

- Negative reconciled forecasts at bottom level set to zero
- Remaining forecasts computed via aggregation  
(Di Fonzo and Girolimetto, 2023)

$$\hat{\mathbf{y}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_{n_b-k} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix}$$

Suppose  $\hat{\mathbf{u}}_{t+h|t}$  are fixed and let  $\hat{\mathbf{w}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} - \mathbf{A}_2 \hat{\mathbf{u}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \end{bmatrix}$ .

## Optimization problem

$$\min_{\mathbf{v}} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}]' \mathbf{W}_{\mathbf{v}}^{-1} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}] \quad \text{where} \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{I}_{n_b-k} \end{bmatrix}$$

and  $\mathbf{W}_{\mathbf{v}}$  contains elements of  $\mathbf{W}_h$  corresponding to  $\hat{\mathbf{v}}_{t+h|t}$ .

$$\hat{\mathbf{y}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{I}_{n_b-k} & \mathbf{O} \\ \mathbf{O} & \mathbf{I}_k \end{bmatrix} \begin{bmatrix} \hat{\mathbf{v}}_{t+h|t} \\ \hat{\mathbf{u}}_{t+h|t} \end{bmatrix}$$

Suppose  $\hat{\mathbf{u}}_{t+h|t}$  are fixed and let  $\hat{\mathbf{w}}_{t+h|t} = \begin{bmatrix} \hat{\mathbf{a}}_{t+h|t} - \mathbf{A}_2 \hat{\mathbf{u}}_{t+h|t} \\ \hat{\mathbf{v}}_{t+h|t} \end{bmatrix}$ .

### Solve with non-negativity constraint

$$\min_{\mathbf{v}} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}]' \mathbf{W}_{\mathbf{v}}^{-1} [\hat{\mathbf{w}}_{t+h|t} - \mathbf{A}_3 \mathbf{v}] \quad \text{where} \quad \mathbf{A}_3 = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{I}_{n_b-k} \end{bmatrix}$$

such that  $\mathbf{A}_3 \mathbf{v} \geq \begin{bmatrix} -\mathbf{A}_2 \hat{\mathbf{u}}_{t+h|t} \\ \mathbf{0} \end{bmatrix}$

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# Time series cross-validation

## Traditional evaluation

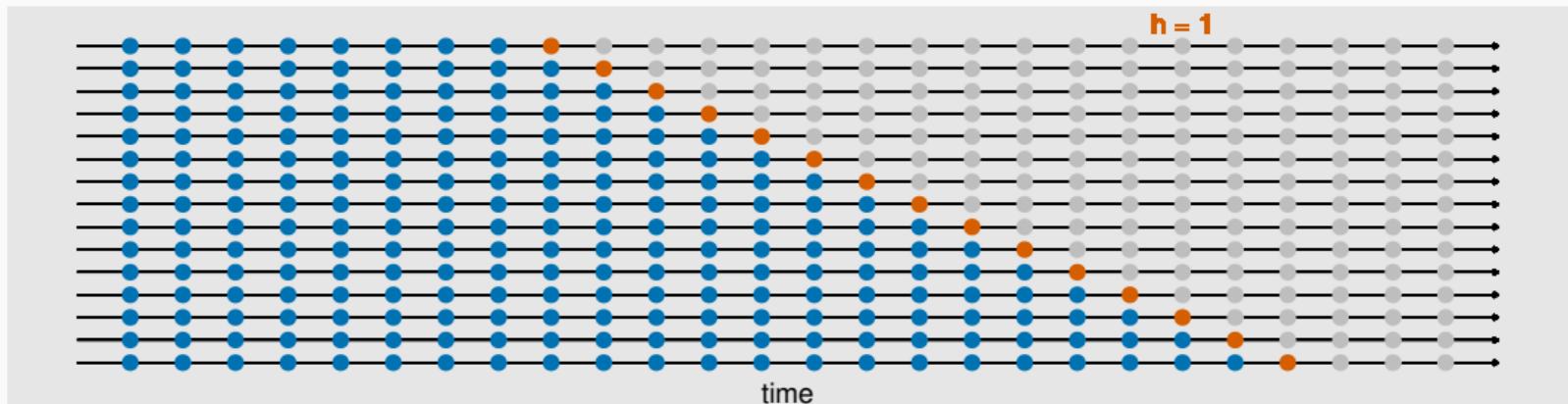


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

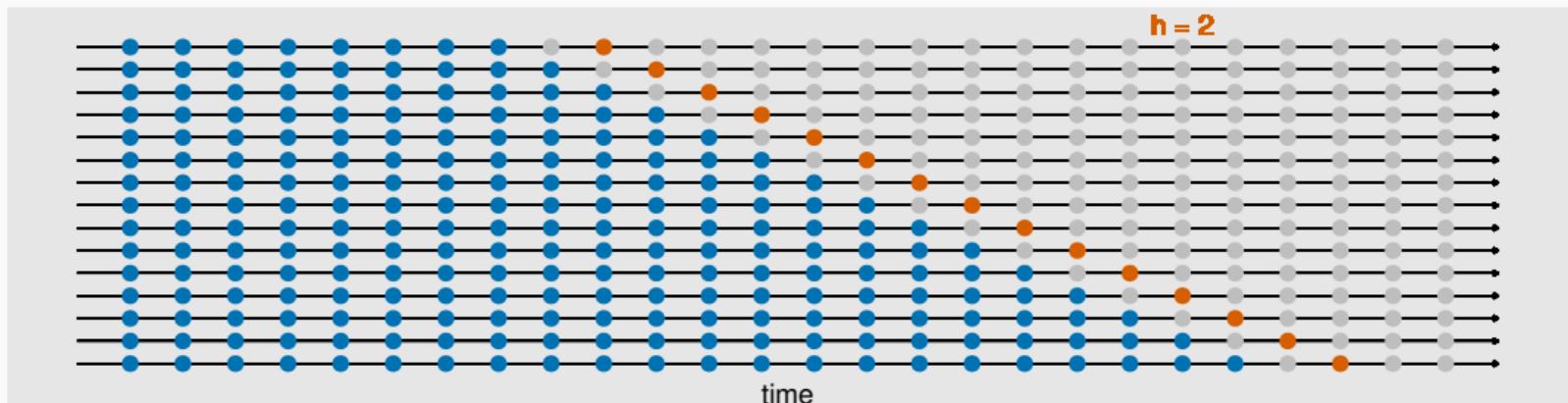


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

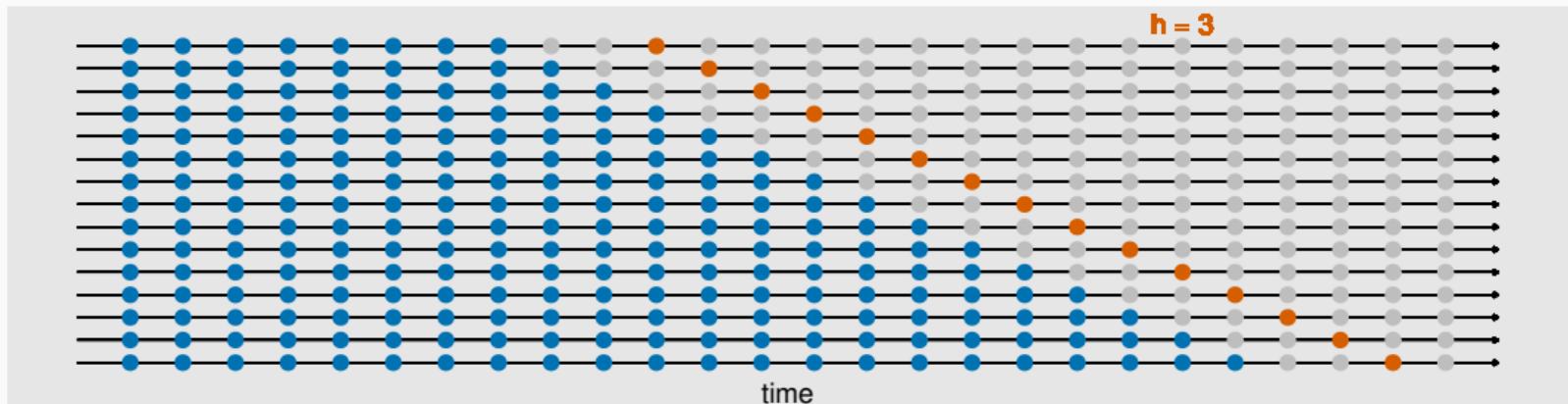


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

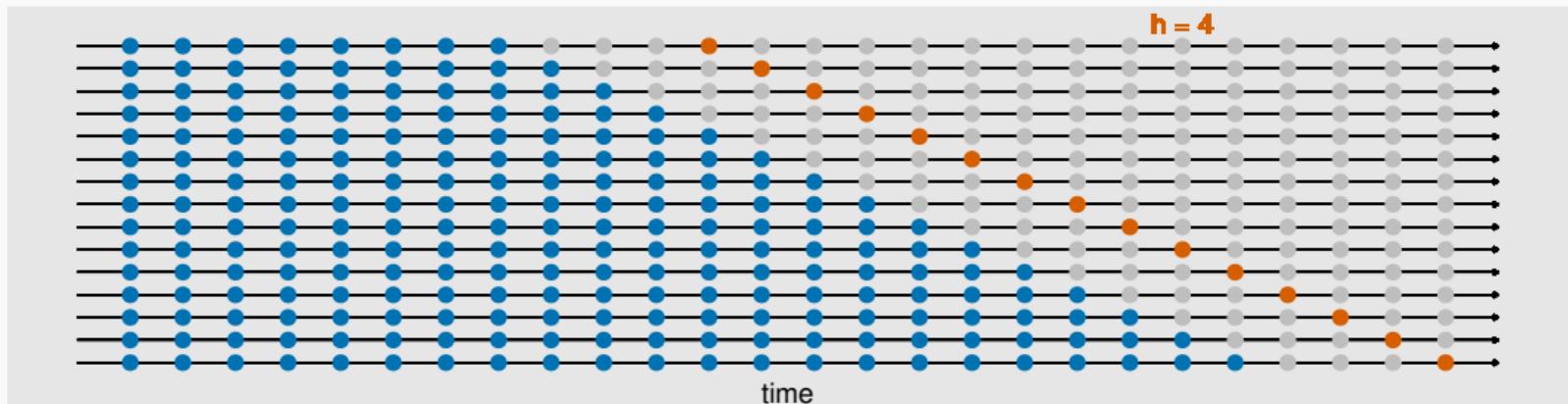


# Time series cross-validation

## Traditional evaluation

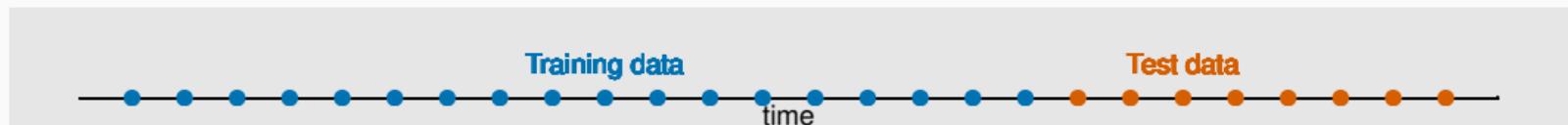


## Time series cross-validation

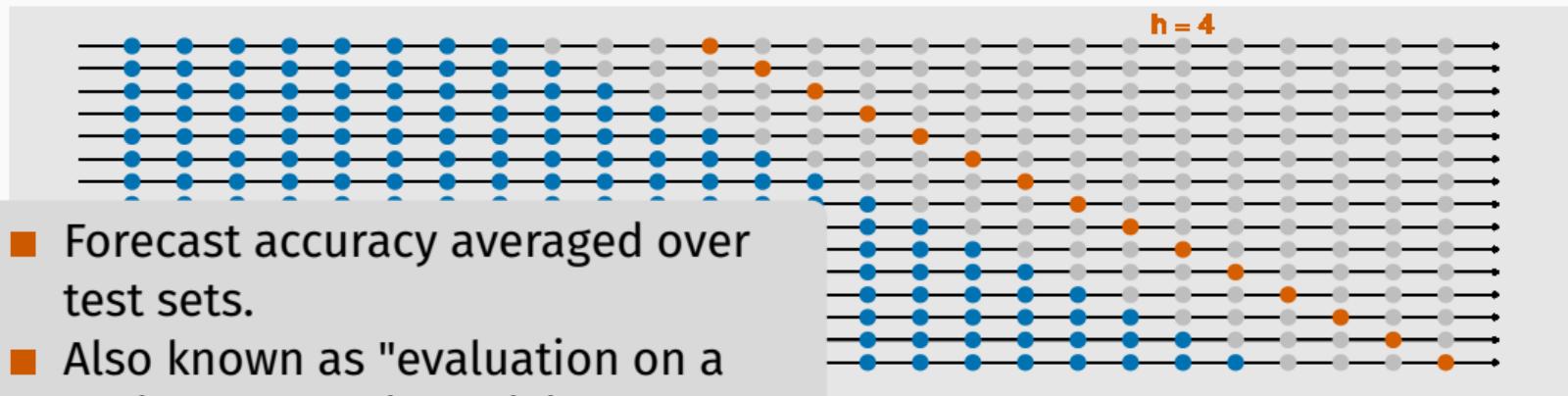


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

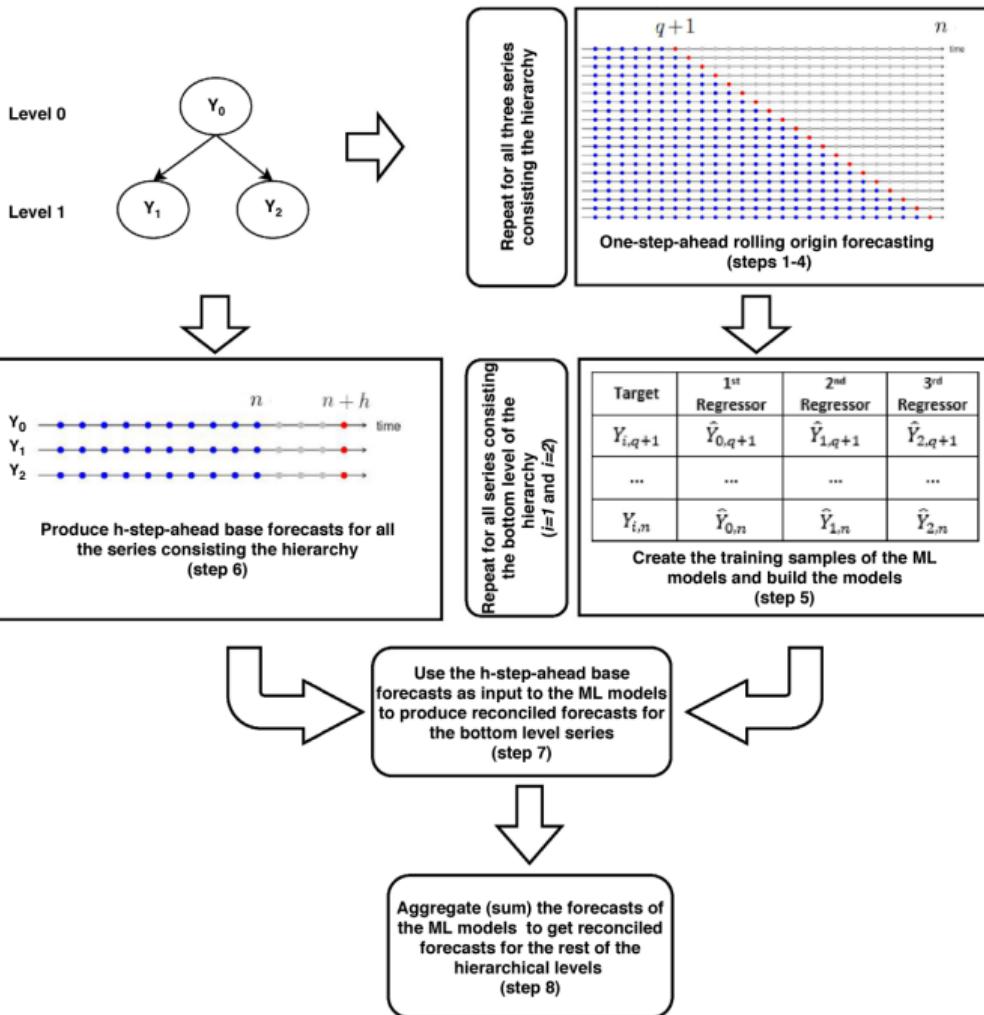


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# ML reconciliation

- 1 Split all series using time series cross-validation
- 2 For each training set, compute one-step-ahead forecasts for all series
- 3 For each bottom-level series, use RF or XGB to predict values using forecasts of all series as inputs
- 4 Forecast all series
- 5 For each bottom-level series, apply ML model to improve forecasts
- 6 Aggregate bottom-level forecasts to obtain forecasts for other series.



# ML reconciliation: tourism data

Method	Total	States	Zones	Regions	Average
MASE					
MinT-Struct	1.094	0.968	0.887	0.843	0.948
MinT-Shrink	1.047	<b>0.956</b>	0.872	0.824	0.925
ML-RF	1.045	0.964	0.859	0.812	0.920
ML-XGB	<b>1.043</b>	0.965	<b>0.859</b>	<b>0.812</b>	<b>0.920</b>
RMSSE					
MinT-Struct	1.308	1.225	1.137	1.109	1.195
MinT-Shrink	1.265	1.214	1.120	1.086	1.171
ML-RF	1.261	<b>1.208</b>	1.104	1.066	1.159
ML-XGB	1.255	1.208	<b>1.101</b>	<b>1.064</b>	<b>1.157</b>
AMSE					
MinT-Struct	0.988	0.611	0.426	0.349	0.593
MinT-Shrink	0.935	0.599	0.417	0.337	0.572
ML-RF	0.780	<b>0.526</b>	0.366	0.319	0.498
ML-XGB	<b>0.779</b>	0.526	<b>0.365</b>	<b>0.317</b>	<b>0.497</b>

- ML methods not significantly different.
- MinT methods significantly different from each other and from ML methods.

# ML and regularization

Mishchenko, Montgomery, and Vaggi (2019):  
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{y}}_t} \sum_{t=1}^T \|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_2 + \lambda \sum_{t=1}^T \|\hat{\mathbf{y}}_t - \mathbf{S}_t \hat{\mathbf{b}}_t\|_2$$

# ML and regularization

Mishchenko, Montgomery, and Vaggi (2019):  
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{y}}_t} \sum_{t=1}^T \|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_2 + \lambda \sum_{t=1}^T \|\hat{\mathbf{y}}_t - \mathbf{s}_t \hat{\mathbf{b}}_t\|_2$$

Shiratori, Kobayashi, and Takano (2020):  
Optimize bottom level forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{b}}_t} \sum_{t=1}^T \|\hat{\mathbf{b}}_t - \mathbf{b}_t\|_2 + \sum_{t=1}^T \Lambda \|\mathbf{a}_t - \mathbf{A}_t \hat{\mathbf{b}}_t\|_2$$

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# In-built coherence

**Two-step approach:** compute base forecasts  $\hat{y}_h$ , and then reconcile them to produce  $\tilde{y}_h$ .

**One-step approaches:** compute coherent  $\tilde{y}_h$  directly.

- Ashouri, Hyndman, and Shmueli (2022): linear regression models
- Pennings and Dalen (2017): state space models
- Villegas and Pedregal (2018): state space models

# In-built coherence using linear models

Suppose  $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$  with  $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$  &  $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$ .

# In-built coherence using linear models

Suppose  $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$  with  $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$  &  $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$ .

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

# In-built coherence using linear models

Suppose  $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$  with  $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$  &  $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$ .

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

# In-built coherence using linear models

Suppose  $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$  with  $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$  &  $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$ .

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}}$$

# In-built coherence using linear models

Suppose  $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$  with  $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$  &  $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$ .

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}} \quad \mathbf{X}_{t+h}^* = \text{diag}(\mathbf{x}'_{t+h,1}, \dots, \mathbf{x}'_{t+h,n})$$

# In-built coherence using linear models

Suppose  $\hat{y}_{t,i} = \hat{\beta}'_i \mathbf{x}_{t,i}$  with  $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \dots, x_{t,p,i})$  &  $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \dots, \hat{y}_{T,i})$ .

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \dots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \quad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \dots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \dots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \dots & x_{T,i,p} \end{pmatrix}$$

$$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \quad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}} \quad \mathbf{X}_{t+h}^* = \text{diag}(\mathbf{x}'_{t+h,1}, \dots, \mathbf{x}'_{t+h,n})$$

$$\tilde{\mathbf{y}}_{t+h} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h\hat{\mathbf{y}}_{t+h} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h\mathbf{X}_{t+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$\mathbf{V}_h = \sigma^2 \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h [1 + \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}_{T+h}^*)'] \mathbf{W}_h \mathbf{S}'(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'$$

Reference: Ashouri, Hyndman, and Shmueli (2022)

# In-built coherence using state space models

Pennings and Dalen (2017) propose the state space model

$$\mathbf{y}_t = \mathbf{S}\mu_t + \mathbf{Z}_t\beta + \varepsilon_t, \quad \varepsilon_t \sim N(\mathbf{0}, \Sigma_\varepsilon),$$

$$\mu_t = \mu_{t-1} + \eta_t, \quad \eta_t \sim N(\mathbf{0}, \Sigma_\eta).$$

- Coherent forecasts arise naturally using the Kalman filter
- Covariance matrices difficult to estimate except for small hierarchies.
- Requires the same model for all series
- A related approach proposed by Villegas and Pedregal (2018)

# References

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