Forecast reconciliation

2. Perspectives on forecast reconciliation

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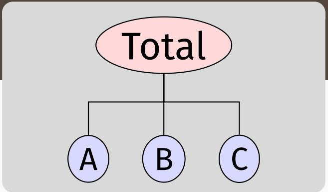
Outline

1. Reconciliation via constraints
2. The geometry of forecast reconciliation
3. Mean square error bounds
4. Other optimization approaches
5. Adding optimization constraints

Outline

Notation reminder



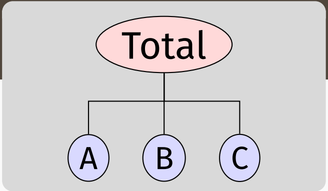
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = “summing matrix” containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Notation reminder



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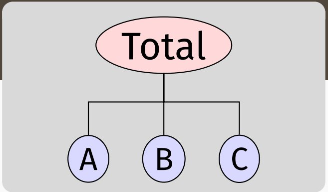
- Base forecasts: $\hat{\boldsymbol{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$
- MinT: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$ where $\boldsymbol{W}_h$ is covariance matrix of base forecast errors.

Notation



Aggregation matrix

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

$$\begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$\begin{pmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{pmatrix} \boldsymbol{b}_t$$

Constraint matrix

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

where

$$\begin{aligned} \boldsymbol{C} &= \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \boldsymbol{I}_{n_a} & -\boldsymbol{A} \end{bmatrix} \end{aligned}$$

Zero-constraint representation

Aggregation matrix A

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Zero-constraint representation

Aggregation matrix $\boldsymbol{A}$

$$\boldsymbol{y}_t = \begin{bmatrix} \boldsymbol{a}_t \\ \boldsymbol{b}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I}_{n_b} \end{bmatrix} \boldsymbol{b}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Constraint matrix $\boldsymbol{C}$

$$\boldsymbol{C}\boldsymbol{y}_t = \boldsymbol{0}$$

- Constraint matrix approach more general & more parsimonious.
- $\boldsymbol{C} = [\boldsymbol{I}_{n_a} \quad -\boldsymbol{A}]$.
- $\boldsymbol{S}$, $\boldsymbol{A}$ and $\boldsymbol{C}$ may contain any real values (not just 0s and 1s).

Zero-constraint representation

Assuming $\boldsymbol{C}$ is full rank

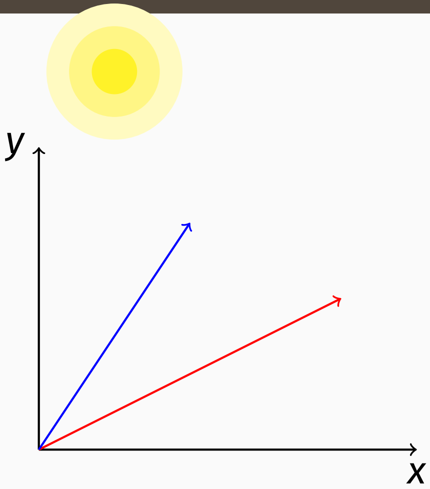
$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\hat{\boldsymbol{y}}_{T+h|T}$$

$$\text{where} \qquad \boldsymbol{M} = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$$

- Originally proved by Byron (1978, 1979) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- $\boldsymbol{M} = \boldsymbol{S}\boldsymbol{G}$ (the MinT solution)
- Leads to more efficient reconciliation than using $\boldsymbol{G}$.

Outline

Projections in linear algebra



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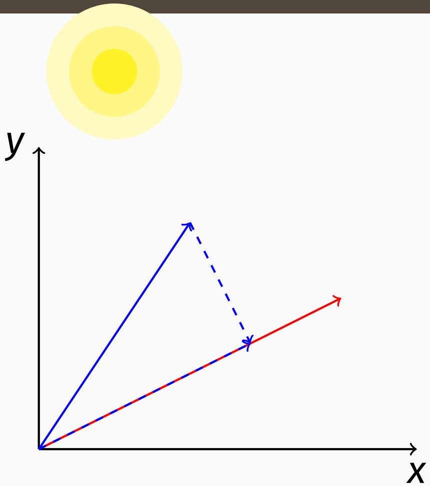

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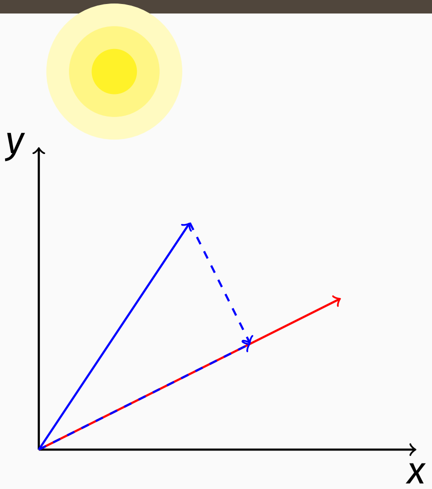

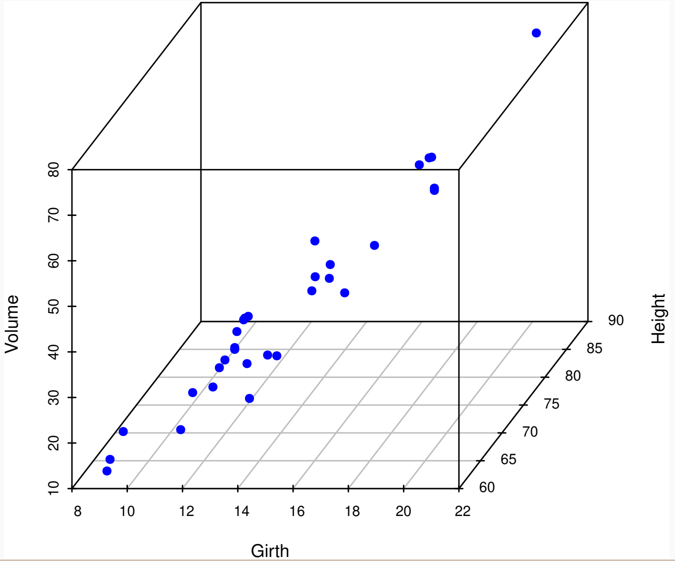

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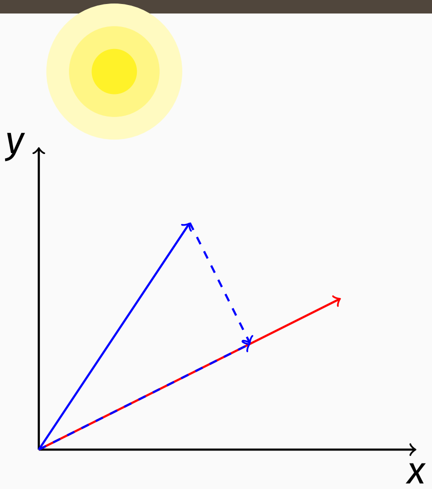

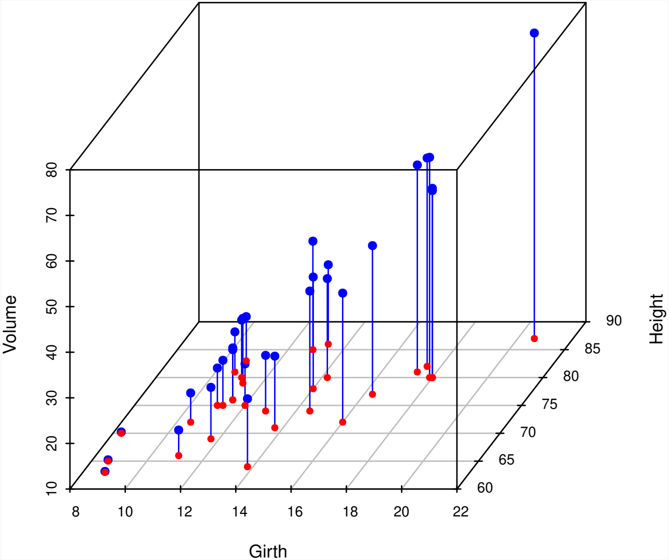

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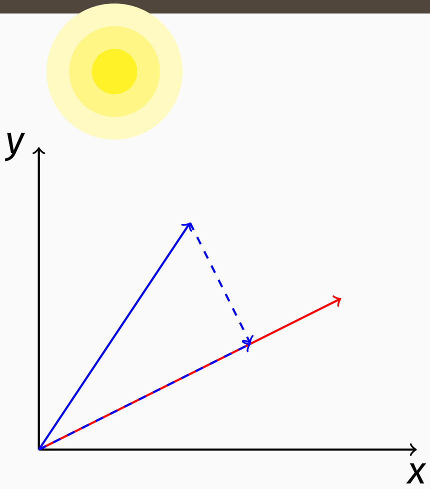
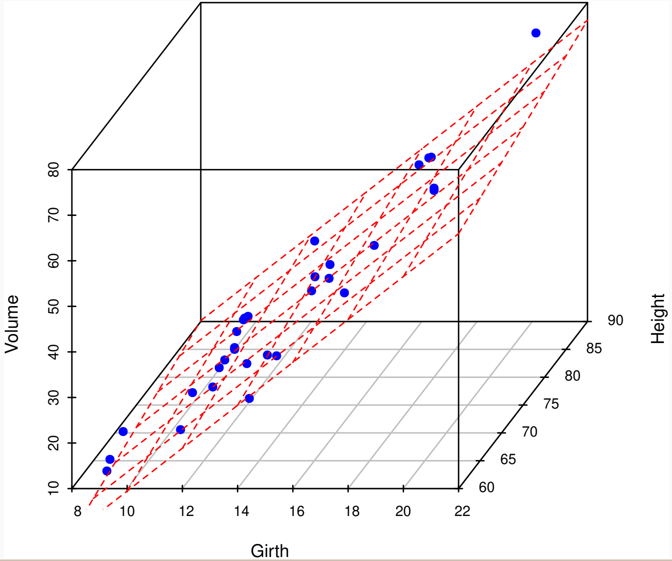
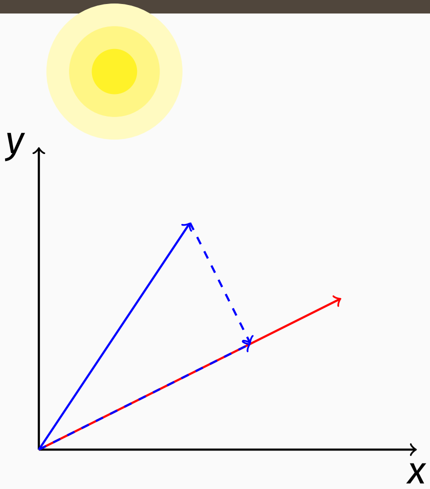
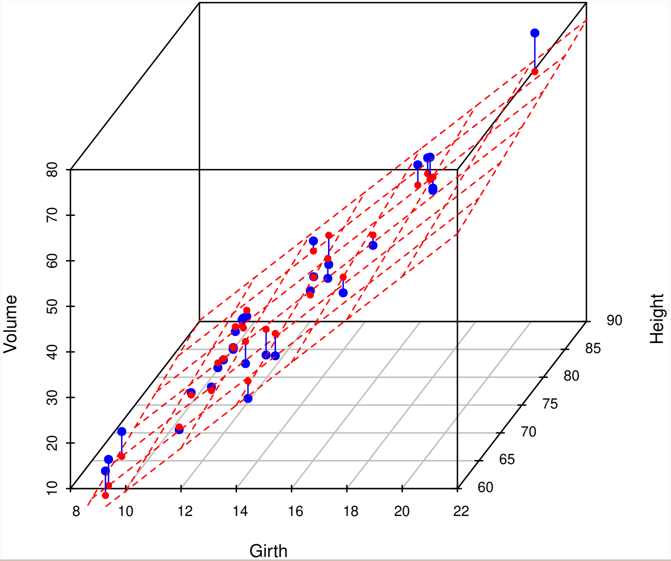




Projections in linear algebra





Projections in linear algebra

- A projection is a linear transformation $\boldsymbol{M}$ such that $\boldsymbol{M}^2 = \boldsymbol{M}$.
- i.e., M is idempotent: it leaves its image unchanged.
- $\boldsymbol{M}$ projects onto $\mathfrak{s}$ if $\boldsymbol{My} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- All eigenvalues of $\boldsymbol{M}$ are either 0 or 1.
- All singular values of $\boldsymbol{M}$ are greater than or equal to 1 (with equality iff $\boldsymbol{M}$ is orthogonal).
- A projection is *orthogonal* if $\boldsymbol{M}' = \boldsymbol{M}$.
- If a projection is not orthogonal, it is called *oblique*.
- In regression, OLS is an orthogonal projection onto space spanned by predictors.

The coherent subspace

Coherent subspace

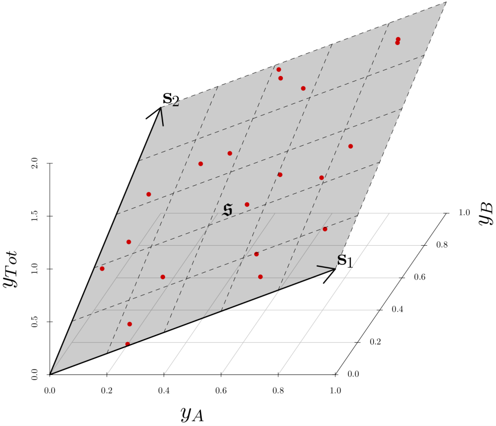
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

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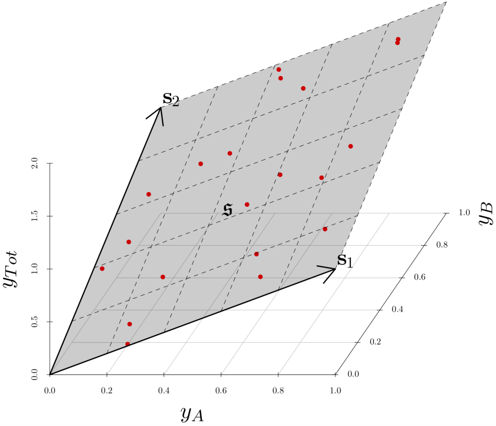
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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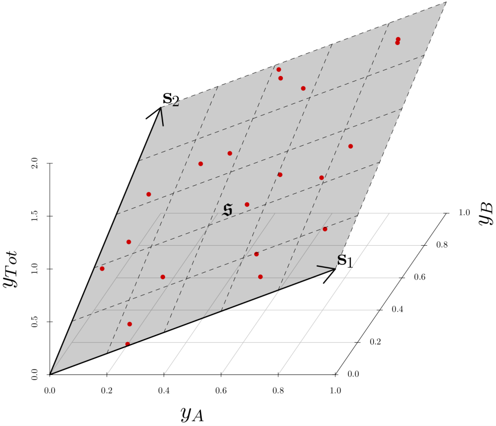


$y_{Tot} = y_A + y_B$

Base forecasts

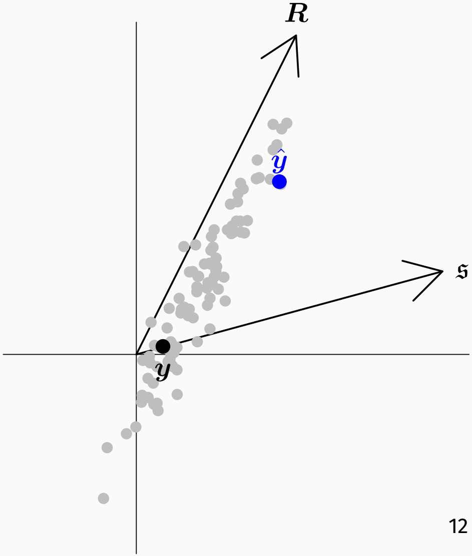
Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$.
$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

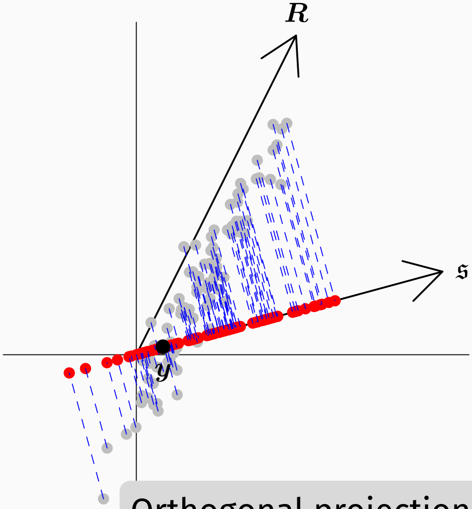
Linear projection reconciliation

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts



Linear projection reconciliation

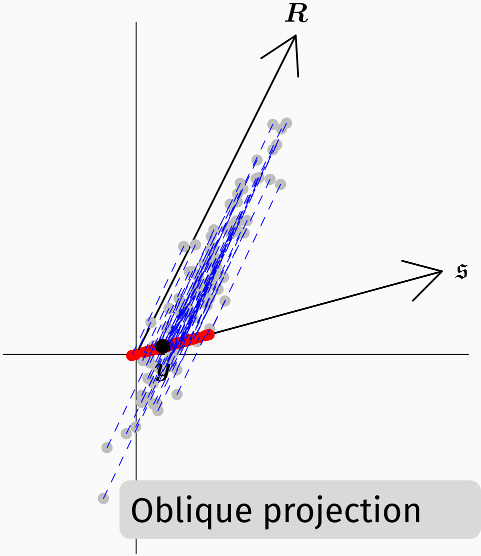
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.



Orthogonal projection

Linear projection reconciliation

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.



Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{M}$ is a projection onto $\mathfrak{s}$ if and only if $\boldsymbol{M}\boldsymbol{y} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- Coherent base forecasts are unchanged since $\boldsymbol{M}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}$
- If $\hat{\boldsymbol{y}}$ is unbiased, then $\tilde{\boldsymbol{y}}$ is also unbiased since

  $$\text{E}(\tilde{\boldsymbol{y}}_{t+h|t}) = \text{E}(\boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\text{E}(\hat{\boldsymbol{y}}_{t+h|t}) = \text{E}(\hat{\boldsymbol{y}}_{t+h|t}),$$

  and unbiased estimates must lie on $\mathfrak{s}$.
- The projection is orthogonal if and only if $\boldsymbol{M}' = \boldsymbol{M}$.
- $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$.
- Projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}, \qquad \text{where} \quad \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$$

OLS:	$\Psi = \boldsymbol{I}$	$\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$	$= \boldsymbol{I} - \boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}\boldsymbol{C}$
MinT:	$\Psi = \boldsymbol{W}_h$	$\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$	$= \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$

- $\boldsymbol{M}$ is orthogonal iff $\Psi = \boldsymbol{I}$.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$ is the covariance matrix of the base forecast errors.
- $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T] = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$ is minimized when $\Psi = \boldsymbol{W}_h$.

Outline

Mean square error bounds

Distance reducing property

Let $\|\boldsymbol{u}\|_\Psi = \boldsymbol{u}'\Psi\boldsymbol{u}$. Then

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\|_\Psi \leq \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|_\Psi$$

- Ψ-projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure*.
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.
- Other measures of forecast accuracy may be worse.

Mean square error bounds

Wickramasuriya (2021)

$$
\begin{aligned}
\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}\|_2^2 &= \|\boldsymbol{M}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h})\|_2^2 \\
&\leq \|\boldsymbol{M}\|_2^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2 \\
&= \sigma_{\text{max}}^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2
\end{aligned}
$$

- σ_{max} is the largest eigenvalue of $\boldsymbol{M}$
- $\sigma_{\text{max}} \geq 1$ as $\boldsymbol{M}$ is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

Mean square error bounds

Wickramasuriya (2021)

$$
\begin{aligned}
& \operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]\Big) \\
\le\ & \operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]\Big) \\
\le\ & \operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]\Big)
\end{aligned}
$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

Outline

A game theoretic perspective

van Erven and Cugliari (2015)

Find the solution to the minimax problem

$$V = \min_{\tilde{\boldsymbol{y}} \in \mathfrak{s}} \max_{\boldsymbol{y} \in \mathfrak{s}} \left\{ \ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) - \ell(\boldsymbol{y}, \hat{\boldsymbol{y}}) \right\},$$

where ℓ is a loss function, and $\mathfrak{s}$ is the coherent subspace.

- $V \leq 0$: reconciliation guaranteed to reduce loss.
- If $\ell(\boldsymbol{y}, \tilde{\boldsymbol{y}}) = \|\boldsymbol{y} - \tilde{\boldsymbol{y}}\|_\Psi = (\boldsymbol{y} - \tilde{\boldsymbol{y}})'\Psi(\boldsymbol{y} - \tilde{\boldsymbol{y}})$, where Ψ is any symmetric pd matrix, then:
  1. $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi\hat{\boldsymbol{y}}$ will always improve upon the base forecasts;
  2. The MinT solution $\tilde{\boldsymbol{y}} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}$ will optimise loss in expectation over any choice of Ψ.

Biased reconciliation

Ben Taieb and Koo (2019)

Regularized empirical risk minimization problem:

$$\min_{\boldsymbol{G}} \frac{1}{Nn}\|\boldsymbol{Y} - \hat{\boldsymbol{Y}}\boldsymbol{G}'\boldsymbol{S}'\|_F + \lambda\|\text{vec}\boldsymbol{G}\|_1,$$

- $N = T - T_1 - h + 1, \quad T_1$ is minimum training sample size
- $\|\cdot\|_F$ is the Frobenius norm
- $\boldsymbol{Y} = [\boldsymbol{y}_{T_1+h}, \dots, \boldsymbol{y}_T]'$
- $\hat{\boldsymbol{Y}} = [\hat{\boldsymbol{y}}_{T_1+h|T_1}, \dots, \hat{\boldsymbol{y}}_{T|T-h}]'$
- λ is a regularization parameter.

When $\lambda = 0$: $\quad \hat{\boldsymbol{G}} = \boldsymbol{B}'\hat{\boldsymbol{Y}}(\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}})^{-1}$ where $\boldsymbol{B} = [\boldsymbol{b}_{T_1+h}, \dots, \boldsymbol{b}_T]'$.

Unconstrained MinT

Wickramasuriya (2021)

Include?

Outline

Adding optimization constraints

Any approach to reconciliation based on optimisation uses a form of constrained optimisation since reconciled forecasts must lie on the coherent subspace. However, at times additional constraints may be implemented. The first is the case where reconciled forecasts must be non-negative. In general, even if base forecasts are constrained to be positive (which can be achieved by modelling on the log scale and back-transforming), there is no guarantee that the usual reconciliation approaches such as OLS and MinT will maintain the non-negativity of forecasts. To address this issue, the usual optimisation problem can be augmented with

ML and regularization

Bayesian versions

In-built coherence

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