Forecast reconciliation

2. Perspectives on forecast reconciliation

Rob J Hyndman

Outline

Outline

Notation reminder

Every collection of time series with linear constraints can be written as

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

- $\mathbf{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\mathbf{b}_t$ = vector of most disaggregated series at time t
- $\mathbf{S}$ = "summing matrix" containing the linear constraints.

$$\mathbf{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t}$$

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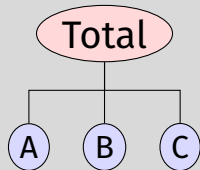
- Base forecasts: $\hat{\boldsymbol{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$
- MinT: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$ where $\boldsymbol{W}_h$ is covariance matrix of base forecast errors.
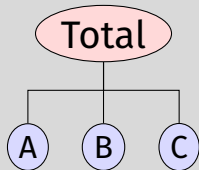
Notation

Aggregation matrix

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{pmatrix} \mathbf{b}_t$$

Constraint matrix

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

where $\quad \mathbf{C} = \begin{bmatrix} 1 & -1 & -1 & -1 \end{bmatrix}$

$$= \begin{bmatrix} \mathbf{I}_{n_a} & -\mathbf{A} \end{bmatrix}$$

Zero-constraint representation

Aggregation matrix A

$$y_t = \begin{bmatrix} a_t \\ b_t \end{bmatrix} = \begin{bmatrix} A \\ I_{n_b} \end{bmatrix} b_t = Sb_t$$

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$$y_t = \begin{bmatrix} a_t \\ b_t \end{bmatrix} = \begin{bmatrix} A \\ I_{n_b} \end{bmatrix} b_t = Sb_t$$

Constraint matrix C

$$Cy_t = 0$$

- Constraint matrix approach more general & more parsimonious.
- $C = [I_{n_a} \quad -A]$.
- S, A and C may contain any real values (not just 0s and 1s).

Zero-constraint representation

Assuming **C** is full rank

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{M}\hat{\mathbf{y}}_{T+h|T}$$

where $\quad \mathbf{M} = \mathbf{I} - \mathbf{W}_h\mathbf{C}'(\mathbf{C}\mathbf{W}_h\mathbf{C}')^{-1}\mathbf{C}$

- Originally proved by Byron (1978) & Byron (1979) for reconciling data.
- Re-discovered by Wickramasuriya, Athanasopoulos, and Hyndman (2019) for reconciling forecasts.
- **M** = **SG** (the MinT solution)
- Leads to more efficient reconciliation than using **G**.

Zero-constraint representation

Suppose $W_h = I$. Then

$$M = I - W_h C'(C W_h C')^{-1} C$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} \frac{1}{4} \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & \frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} & \frac{3}{4} \end{pmatrix}$$

Total — A, B, C
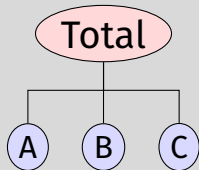
$$A = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$$
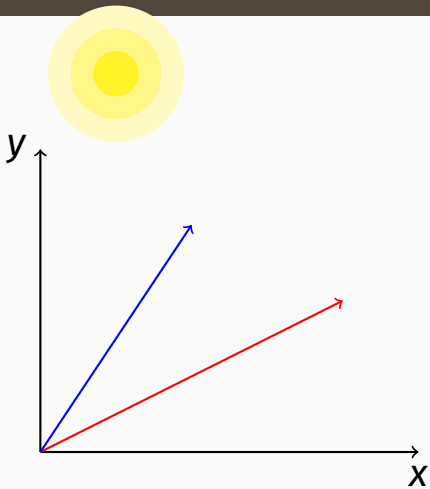
$$S = \begin{pmatrix} A \\ I_{n_b} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} I_{n_a} & -A \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 & -1 \end{pmatrix}$$

Outline

Projections in linear algebra

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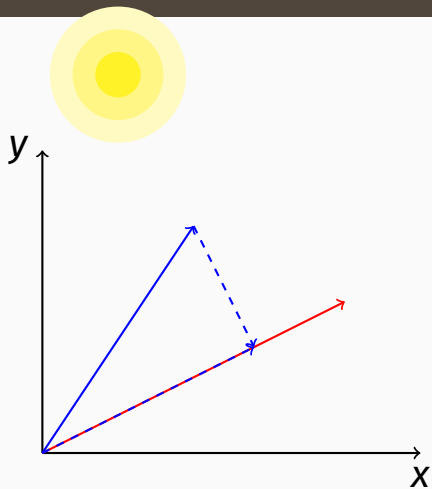
Projections in linear algebra
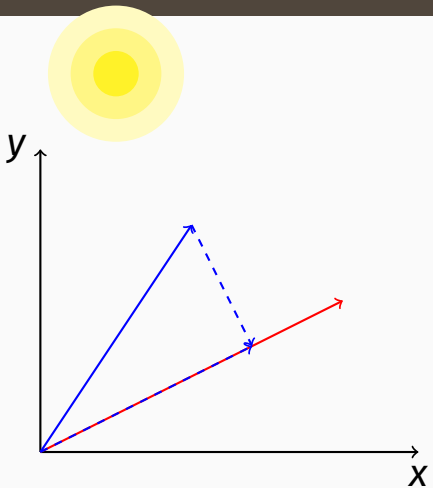
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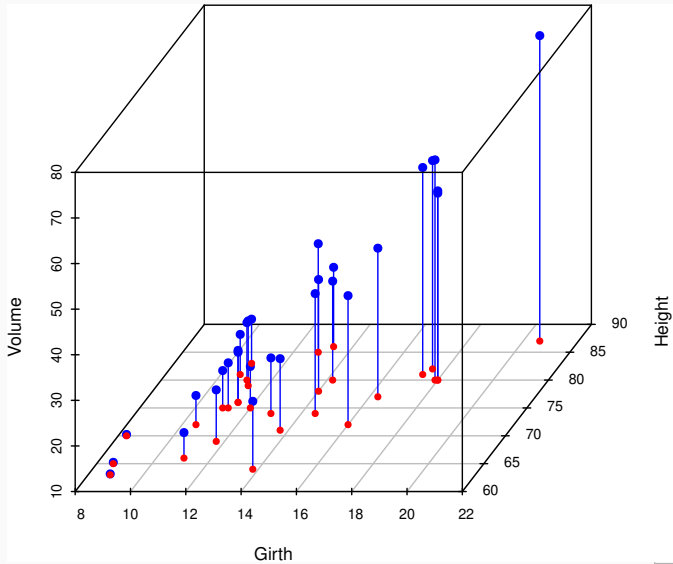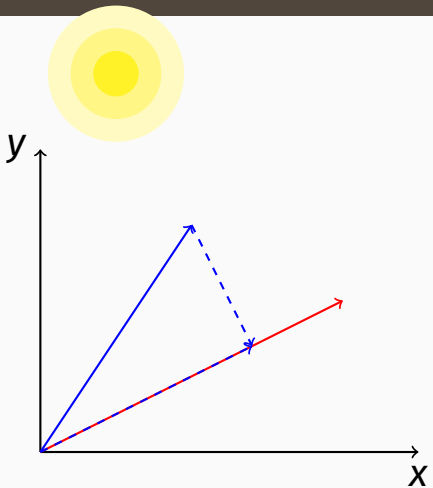
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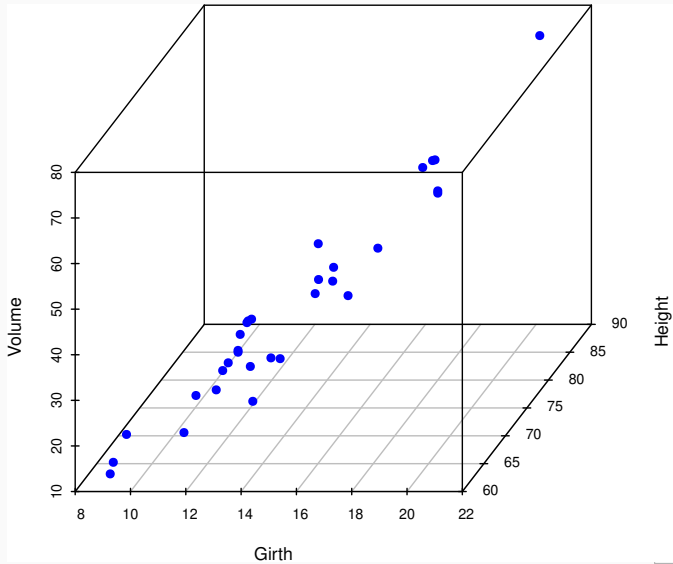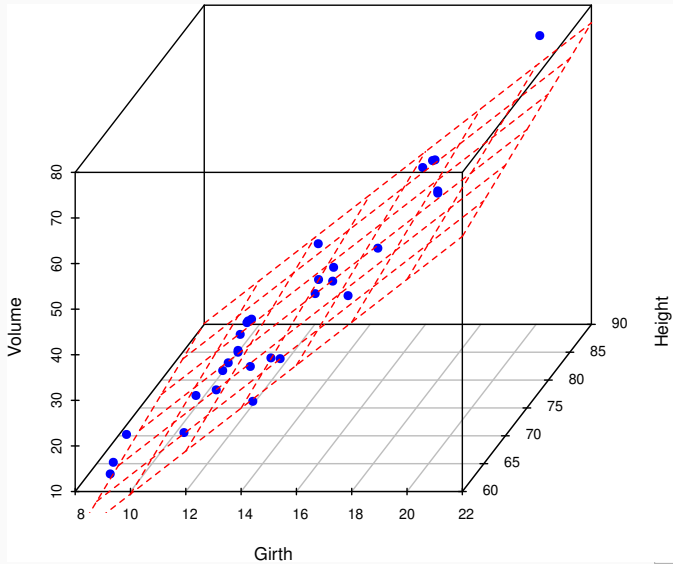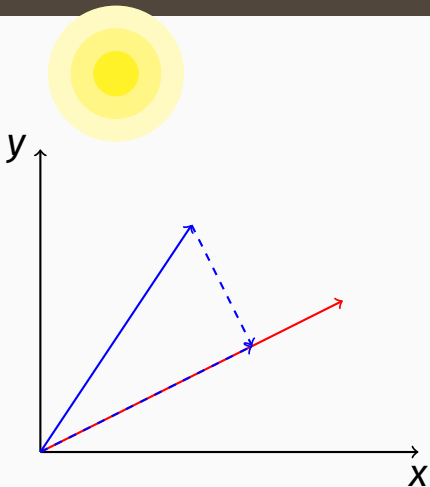
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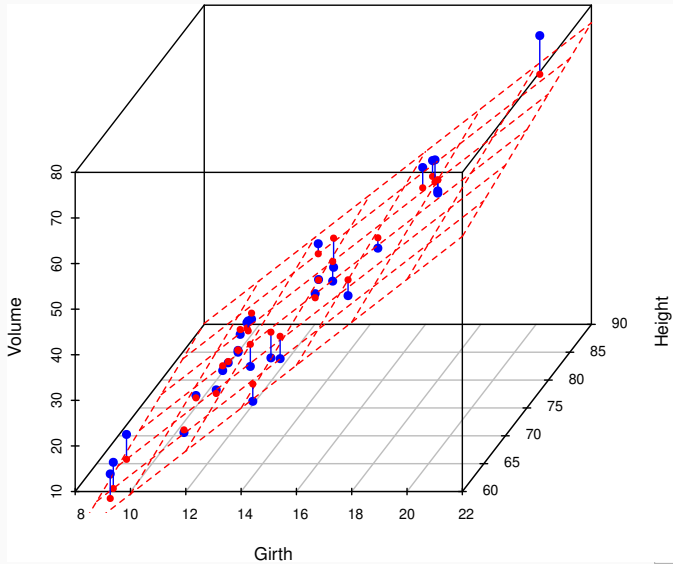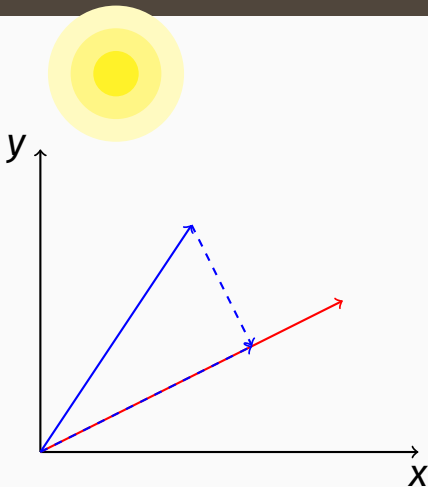
Projections in linear algebra

- A projection is a linear transformation M such that $M^2 = M$.
- i.e., M is idempotent: it leaves its image unchanged.
- M projects onto $\mathfrak{s}$ if $My = y$ for all $y \in \mathfrak{s}$.
- All eigenvalues of M are either 0 or 1.
- All singular values of M are greater than or equal to 1 (with equality iff M is orthogonal).
- A projection is *orthogonal* if $M' = M$.
- If a projection is not orthogonal, it is called *oblique*.
- In regression, OLS is an orthogonal projection onto space spanned by predictors.

The coherent subspace

Coherent subspace

m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\mathbf{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$ is *coherent* if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.
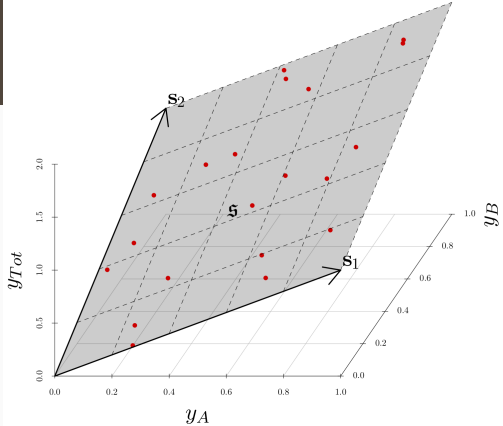
$$y_{Tot} = y_A + y_B$$

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Base forecasts

Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

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Base forecasts

Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \to \mathfrak{s}$.
$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$ "reconciles" $\hat{\mathbf{y}}_{t+h|t}$.
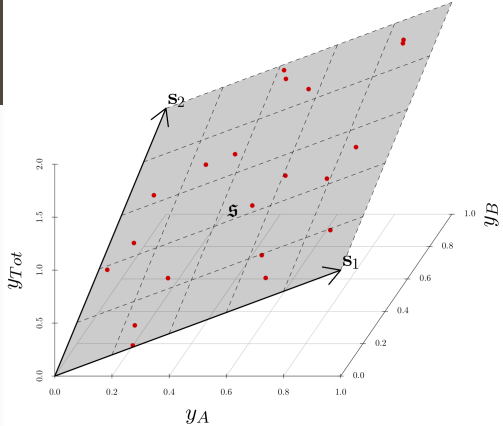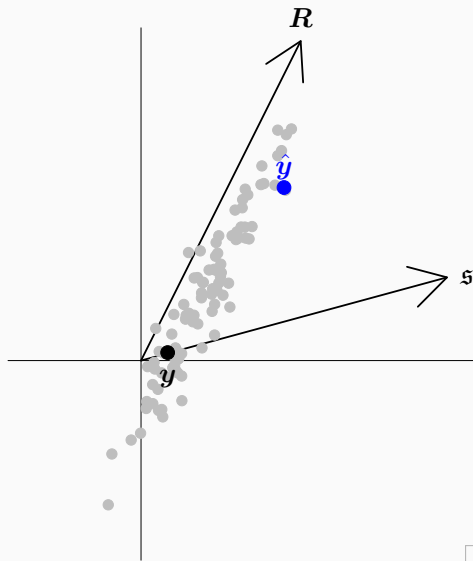
Linear projection reconciliation

- *R* is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts

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- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.

Orthogonal projection
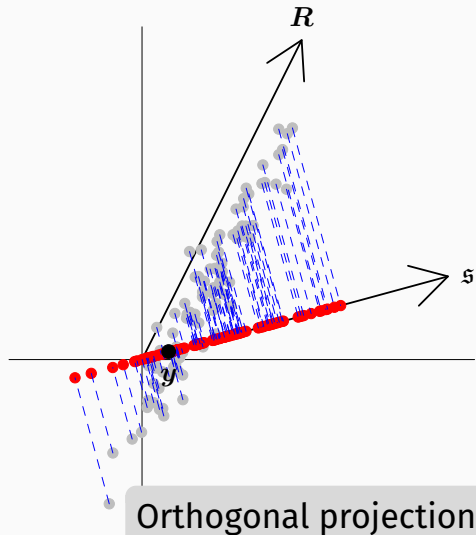
Linear projection reconciliation

- *R* is the most likely direction of deviations from 𝔰.
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.

Oblique projection
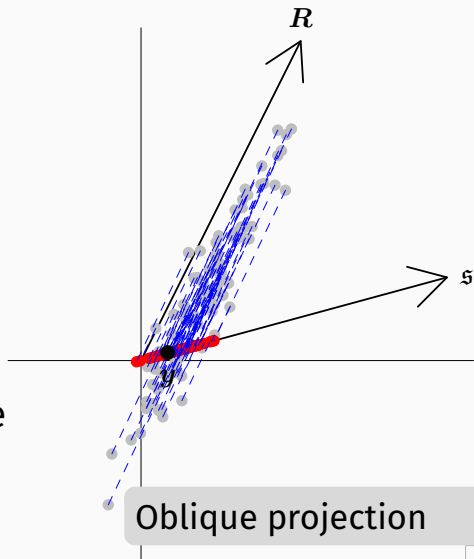
Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{M}$ is a projection onto $\mathfrak{s}$ if and only if $\boldsymbol{M}\boldsymbol{y} = \boldsymbol{y}$ for all $\boldsymbol{y} \in \mathfrak{s}$.
- Coherent base forecasts are unchanged since $\boldsymbol{M}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}$
- If $\hat{\boldsymbol{y}}$ is unbiased, then $\tilde{\boldsymbol{y}}$ is also unbiased since

$$\text{E}(\tilde{\boldsymbol{y}}_{t+h|t}) = \text{E}(\boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\text{E}(\hat{\boldsymbol{y}}_{t+h|t}) = \text{E}(\hat{\boldsymbol{y}}_{t+h|t}),$$

  and unbiased estimates must lie on $\mathfrak{s}$.
- The projection is orthogonal if and only if $\boldsymbol{M}' = \boldsymbol{M}$.
- If $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$, then projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}, \qquad \text{where} \qquad \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$$

OLS: $\quad \Psi = \boldsymbol{I} \qquad \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}' \qquad\qquad = \boldsymbol{I} - \boldsymbol{C}'(\boldsymbol{C}\boldsymbol{C}')^{-1}\boldsymbol{C}$

MinT: $\quad \Psi = \boldsymbol{W}_h \qquad \boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1} \quad = \boldsymbol{I} - \boldsymbol{W}_h\boldsymbol{C}'(\boldsymbol{C}\boldsymbol{W}_h\boldsymbol{C}')^{-1}\boldsymbol{C}$

- $\boldsymbol{M}$ is orthogonal iff $\Psi = \boldsymbol{I}$.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \ldots, \boldsymbol{y}_T]$ is the covariance matrix of the base forecast errors.
- $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \ldots, \boldsymbol{y}_T] = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$ is minimized when $\Psi = \boldsymbol{W}_h$.

Mean square error bounds

Distance reducing property

Let $\|\boldsymbol{u}\|_\Psi = \boldsymbol{u}'\Psi\boldsymbol{u}$. Then
$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\|_\Psi \leq \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|_\Psi$$

- Ψ-projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure.*
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.
- Other measures of forecast accuracy may be worse.

Mean square error bounds

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}\|_2^2 = \|\boldsymbol{M}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h})\|_2^2$$
$$\leq \|\boldsymbol{M}\|_2^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2$$
$$= \sigma_{\max}^2 \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2$$

- $\sigma_{\max}$ is the largest eigenvalue of $\boldsymbol{M}$
- $\sigma_{\max} \geq 1$ as $\boldsymbol{M}$ is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

$$\text{tr}\Big(\mathsf{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}^{\mathsf{MinT}}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}^{\mathsf{MinT}}]\Big)$$

$$\leq \text{tr}\Big(\mathsf{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}^{\mathsf{OLS}}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}^{\mathsf{OLS}}]\Big)$$

$$\leq \text{tr}\Big(\mathsf{E}[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]\Big)$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

Outline

Minimum trace reconciliation

Minimum trace (MinT) reconciliation

If SG is a projection, then the trace of $V_h = \text{Var}(\tilde{y}_{t+h|t} - y_{t+h})$ is **minimized** when

$$G = (S'W_h^{-1}S)^{-1}S'W_h^{-1}$$

$$\textcolor{red}{\tilde{y}_{T+h|T}} = S(S'W_h^{-1}S)^{-1}S'W_h^{-1}\textcolor{blue}{\hat{y}_{T+h|T}}$$

Reconciled forecasts   Base forecasts

- Trace of V_h is sum of forecast variances.
- MinT solution is L_2 **optimal** amongst linear unbiased forecasts.

Find the solution to the minimax problem

$$V = \min_{\tilde{\mathbf{y}} \in \mathfrak{s}} \max_{\mathbf{y} \in \mathfrak{s}} \left\{ \ell(\mathbf{y}, \tilde{\mathbf{y}}) - \ell(\mathbf{y}, \hat{\mathbf{y}}) \right\},$$

where ℓ is a loss function, and $\mathfrak{s}$ is the coherent subspace.

- $V \leq 0$: reconciliation guaranteed to reduce loss.

- If $\ell(\mathbf{y}, \tilde{\mathbf{y}}) = \|\mathbf{y} - \tilde{\mathbf{y}}\|_{\Psi} = (\mathbf{y} - \tilde{\mathbf{y}})' \Psi (\mathbf{y} - \tilde{\mathbf{y}})$, where Ψ is any symmetric pd matrix, then:

  1. $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi\hat{\mathbf{y}}$ will always improve upon the base forecasts;
  2. The MinT solution $\tilde{\mathbf{y}} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}$ will optimise loss in expectation over any choice of Ψ.

Regularized empirical risk minimization problem:

$$\min_{\boldsymbol{G}} \frac{1}{Nn} \|\boldsymbol{Y} - \hat{\boldsymbol{Y}}\boldsymbol{G}'\boldsymbol{S}'\|_F + \lambda\|\text{vec}\boldsymbol{G}\|_1,$$

- $N = T - T_1 - h + 1$, T_1 is minimum training sample size
- $\|\cdot\|_F$ is the Frobenius norm
- $\boldsymbol{Y} = [\boldsymbol{y}_{T_1+h}, \ldots, \boldsymbol{y}_T]'$
- $\hat{\boldsymbol{Y}} = [\hat{\boldsymbol{y}}_{T_1+h|T_1}, \ldots, \hat{\boldsymbol{y}}_{T|T-h}]'$
- λ is a regularization parameter.

When $\lambda = 0$: $\hat{\boldsymbol{G}} = \boldsymbol{B}'\hat{\boldsymbol{Y}}(\hat{\boldsymbol{Y}}'\hat{\boldsymbol{Y}})^{-1}$ where $\boldsymbol{B} = [\boldsymbol{b}_{T_1+h}, \ldots, \boldsymbol{b}_T]'$.

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$
$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \operatorname{tr}\Big(\mathsf{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$

such that $\boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$

Solve via quadratic programming:

$$\min_{\boldsymbol{b}} \frac{1}{2}\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - \boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T} \quad \text{s.t.} \ \ \boldsymbol{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Non-negative forecasts

$$\min_{\boldsymbol{G}_h} \operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{S}\boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t}]\Big)$$
$$\text{such that } \boldsymbol{b}_{t+h|t} = \boldsymbol{G}_h\hat{\boldsymbol{y}}_{t+h|t} \geq 0$$

Solve via quadratic programming:

$$\min_{\boldsymbol{b}} \frac{1}{2}\boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S}\boldsymbol{b} - \boldsymbol{b}'\boldsymbol{S}'\boldsymbol{W}_h^{-1}\hat{\boldsymbol{y}}_{T+h|T} \quad \text{s.t. } \boldsymbol{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Set-negative-to-zero heuristic solution

- Negative reconciled forecasts at bottom level set to zero
- Remaining forecasts computed via aggregation

(Di Fonzo and Girolimetto, 2023)

Immutable forecasts

$$\min_{\boldsymbol{G}_h} \text{tr}\Big(\text{E}[\boldsymbol{y}_{t+h} - \boldsymbol{SG}_h \hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \boldsymbol{SG}_h \hat{\boldsymbol{y}}_{t+h|t}] \Big)$$
$$\text{such that } \boldsymbol{CSG}_h \hat{\boldsymbol{y}}_{t+h|t} = \boldsymbol{d}$$

- Differs from top-down approaches in that it can be done while also preserving the unbiasedness of base forecasts.

See also Di Fonzo and Girolimetto (2022). Zhang, Kang, Panagiotelis, and Li (2022).

Outline

ML and regularization

- Replace the linear regression formulation with a less restrictive method to obtain combinations of forecasts from the various hierarchical levels.
- Coherence is achieved via a bottom-up approach, or by embedding coherence in the ML training.
-

Gleason (2020) attempts to overcome the lack of focus on coherence by adjusting the objective function. Using neural network forecasts, he includes a regularisation term that penalises incoherences in the generated forecasts. This

Outline

In-built coherence

Two-step approach: compute base forecasts $\hat{\boldsymbol{y}}_h$, and then reconcile them to produce $\tilde{\boldsymbol{y}}_h$.

One-step approaches: compute coherent $\tilde{\boldsymbol{y}}_h$ directly.

- Ashouri, Hyndman, and Shmueli (2022): linear regression models
- Pennings and Dalen (2017): state space models
- Villegas and Pedregal (2018): state space models

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}'_i \boldsymbol{x}_{t,i}$ with $\boldsymbol{x}_{t,i} = (1, x_{t,1,i}, \ldots, x_{t,p,i})$ & $\hat{\boldsymbol{y}}_i = (\hat{y}_{1,i}, \ldots, \hat{y}_{T,i})$.

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$$
\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \ldots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \ldots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \ldots & x_{T,i,p} \end{pmatrix}
$$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \ldots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \ldots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \ldots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \ldots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \ldots & x_{T,i,p} \end{pmatrix}$$

$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \ldots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \ldots, \hat{y}_{T,i})$.

$$
\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \ldots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \ldots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \ldots & x_{T,i,p} \end{pmatrix}
$$

$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ $\qquad$ $\hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}}$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \ldots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \ldots, \hat{y}_{T,i})$.

$$
\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \ldots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \ldots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \ldots & x_{T,i,p} \end{pmatrix}
$$

$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ $\qquad \hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}}$ $\qquad \mathbf{X}_{t+h}^* = \mathrm{diag}(\mathbf{x}_{t+h,i}', \ldots, \mathbf{x}_{t+h,n}')$

In-built coherence using linear models

Suppose $\hat{y}_{t,i} = \hat{\beta}_i' \mathbf{x}_{t,i}$ with $\mathbf{x}_{t,i} = (1, x_{t,1,i}, \ldots, x_{t,p,i})$ & $\hat{\mathbf{y}}_i = (\hat{y}_{1,i}, \ldots, \hat{y}_{T,i})$.

$$\begin{pmatrix} \hat{\mathbf{y}}_1 \\ \hat{\mathbf{y}}_2 \\ \vdots \\ \hat{\mathbf{y}}_n \end{pmatrix} = \begin{pmatrix} \mathbf{X}_1 & 0 & \ldots & 0 \\ 0 & \mathbf{X}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \mathbf{X}_n \end{pmatrix} \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_2 \\ \vdots \\ \hat{\beta}_n \end{pmatrix}, \qquad \mathbf{X}_i = \begin{pmatrix} 1 & x_{1,i,1} & x_{1,i,2} & \ldots & x_{1,i,p} \\ 1 & x_{2,i,1} & x_{2,i,2} & \ldots & x_{2,i,p} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{T,i,1} & x_{T,i,2} & \ldots & x_{T,i,p} \end{pmatrix}$$

$\hat{\mathbf{B}} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$ $\hat{\mathbf{y}}_{t+h} = \mathbf{X}_{t+h}^* \hat{\mathbf{B}}$ $\mathbf{X}_{t+h}^* = \text{diag}(\mathbf{x}_{t+h,i}', \ldots, \mathbf{x}_{t+h,n}')$

$$\tilde{\mathbf{y}}_{t+h} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h\hat{\mathbf{y}}_{t+h} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h\mathbf{X}_{t+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

$$\mathbf{V}_h = \sigma^2 \mathbf{S}(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h \left[1 + \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}_{T+h}^*)' \right] \mathbf{W}_h\mathbf{S}'(\mathbf{S}'\mathbf{W}_h\mathbf{S})^{-1}\mathbf{S}'$$

Reference: Ashouri, Hyndman, and Shmueli (2022)

In-built coherence

Pennings and Dalen (2017) propose the state space model

$$\boldsymbol{y}_t = \boldsymbol{S}\mu_t + \boldsymbol{Z}_t\beta + \varepsilon_t, \qquad \varepsilon_t \sim N(\boldsymbol{0}, \Sigma_\varepsilon), \qquad (1)$$

$$\boldsymbol{\mu}_t = \boldsymbol{\mu}_{t-1} + \boldsymbol{\eta}_t, \qquad \boldsymbol{\eta}_t \sim N(\boldsymbol{0}, \Sigma_\eta). \qquad (2)$$

- Coherent forecasts arise naturally using the Kalman filter
- Covariance matrices difficult to estimate except for small hierarchies.

In-build coherence

A related state space approach was proposed by Villegas and Pedregal (2018), who show that their formulation subsumes bottom-up, top-down, and some forms of forecast reconciliation and combination forecasting.

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