Forecast reconciliation

2. Temporal & cross-temporal forecast reconciliation

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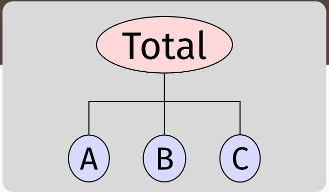
Notation reminder

Every collection of time series with linear constraints can be written as

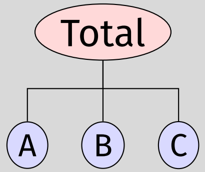
$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.



$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Notation reminder



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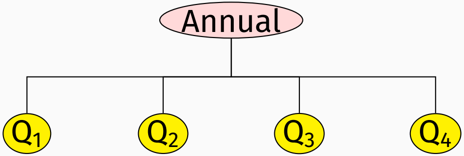
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

- Base forecasts: $\hat{\boldsymbol{y}}_{T+h|T}$
- Reconciled forecasts: $\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$
- MinT: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$ where $\boldsymbol{W}_h$ is covariance matrix of base forecast errors.

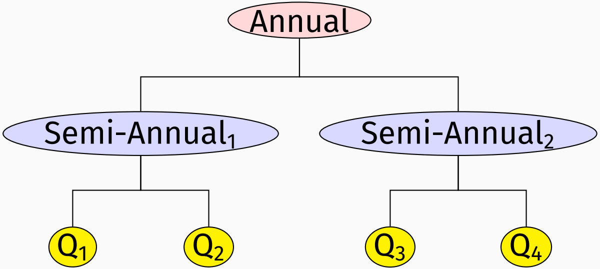
Outline

Temporal reconciliation



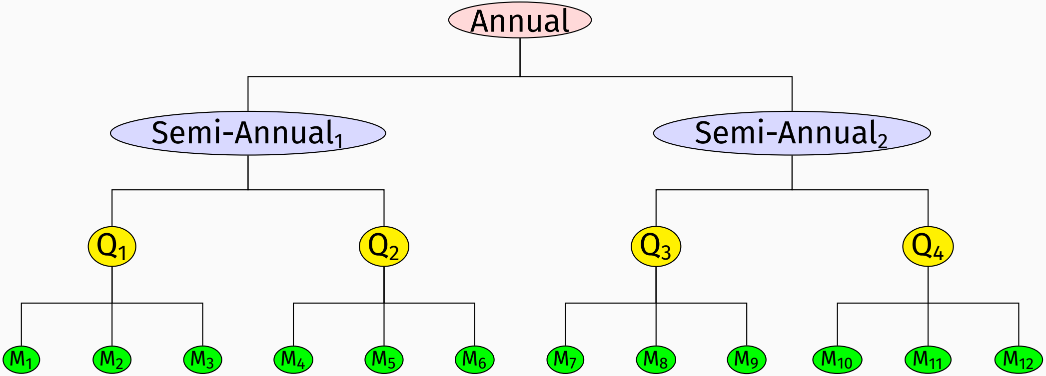
$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[2]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,2}^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Temporal reconciliation

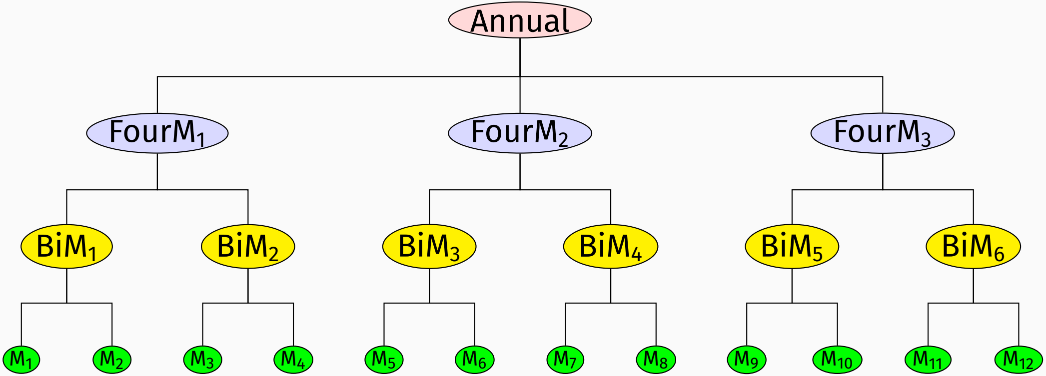


$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[2]} \\ x_{\tau,1}^{[2]} \\ x_{\tau,2}^{[2]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,2}^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

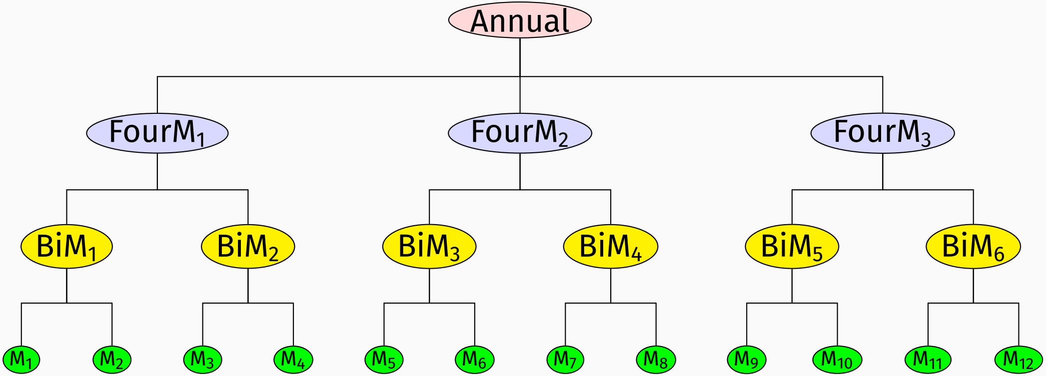
Temporal reconciliation



Temporal reconciliation



Temporal reconciliation



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation

$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[12]} \\ \boldsymbol{x}_\tau^{[6]} \\ \boldsymbol{x}_\tau^{[4]} \\ \boldsymbol{x}_\tau^{[3]} \\ \boldsymbol{x}_\tau^{[2]} \\ \boldsymbol{x}_\tau^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & \boldsymbol{I}_{12} & & & & & \end{bmatrix}$$

Temporal reconciliation

For a time series $y_1, \dots, y_T$, observed at frequency m:

$$x_j^{[k]} = \sum_{t=(j-1)k+1}^{jk} y_t \qquad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in \mathcal{K} = \{k_1, \dots, k_p\}$ denote the p factors of m in ascending order, where $k_1 = 1$ and $k_p = m$
- $x_j^{[1]} = y_t$
- A single unique hierarchy is only possible when there are no coprime pairs in $\mathcal{K}$.
- $M_k = m/k$ is seasonal period of aggregated series.

Temporal reconciliation

$$\boldsymbol{x}_\tau = \boldsymbol{S}\boldsymbol{x}_\tau^{[1]}, \qquad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I} \end{bmatrix}$$

where

$$\boldsymbol{x}_\tau = \begin{bmatrix} \boldsymbol{x}_\tau^{[k_p]} \\ \boldsymbol{x}_\tau^{[k_{p-1}]} \\ \vdots \\ \boldsymbol{x}_\tau^{[k_1]} \end{bmatrix} \qquad \boldsymbol{x}_\tau^{[k]} = \begin{bmatrix} x_{M_k(\tau-1)+1}^{[k]} \\ x_{M_k(\tau-1)+2}^{[k]} \\ \vdots \\ x_{M_k\tau}^{[k]} \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{1}'_m \\ \boldsymbol{I}_{m/k_{p-1}} \otimes \boldsymbol{1}'_{k_{p-1}} \\ \vdots \\ \boldsymbol{I}_{m/k_2} \otimes \boldsymbol{1}'_{k_2} \end{bmatrix}$$

τ is time index for most aggregated series,

$k \in \mathcal{K} = \{k_1, \dots, k_p\}$, $\quad k_1 = 1$, $\quad k_p = m$, $\quad \tau = 1, \dots, T/m$.

Outline

Temporal reconciliation: thief package

```r
library(thief)
USAccDeaths
```

```
       Jan   Feb   Mar   Apr   May   Jun   Jul   Aug   Sep   Oct   Nov   Dec
1973  9007  8106  8928  9137 10017 10826 11317 10744  9713  9938  9161  8927
1974  7750  6981  8038  8422  8714  9512 10120  9823  8743  9129  8710  8680
1975  8162  7306  8124  7870  9387  9556 10093  9620  8285  8466  8160  8034
1976  7717  7461  7767  7925  8623  8945 10078  9179  8037  8488  7874  8647
1977  7792  6957  7726  8106  8890  9299 10625  9302  8314  8850  8265  8796
1978  7836  6892  7791  8192  9115  9434 10484  9827  9110  9070  8633  9240
```

```r
thief(USAccDeaths, usemodel = "arima", comb = "struc")
```

```
       Jan   Feb   Mar   Apr   May   Jun   Jul   Aug   Sep   Oct   Nov   Dec
1979  8185  7381  8132  8516  9393  9764 10881 10060  9182  9405  8869  9361
1980  8406  7602  8352  8737  9614  9985 11101 10280  9403  9626  9090  9581
```

Temporal reconciliation: thief package

```r
# Construct aggregates
aggts <- tsaggregates(USAccDeaths)
aggts[[1]]
```

```
      Jan   Feb   Mar   Apr   May   Jun   Jul   Aug   Sep   Oct   Nov   Dec
1973  9007  8106  8928  9137 10017 10826 11317 10744  9713  9938  9161  8927
1974  7750  6981  8038  8422  8714  9512 10120  9823  8743  9129  8710  8680
1975  8162  7306  8124  7870  9387  9556 10093  9620  8285  8466  8160  8034
1976  7717  7461  7767  7925  8623  8945 10078  9179  8037  8488  7874  8647
1977  7792  6957  7726  8106  8890  9299 10625  9302  8314  8850  8265  8796
1978  7836  6892  7791  8192  9115  9434 10484  9827  9110  9070  8633  9240
```

Temporal reconciliation: thief package

```
aggts[[2]]
```

```
Time Series:
Start = c(1973, 1)
End = c(1978, 6)
Frequency = 6
 [1] 17113 18065 20843 22061 19651 18088 14731 16460 18226 19943 17872 17390 15468 1
[15] 18943 19713 16751 16194 15178 15692 17568 19257 16525 16521 14749 15832 18189 1
[29] 17164 17061 14728 15983 18549 20311 18180 17873
```

Temporal reconciliation: thief package

```
aggts[[3]]
```

```
      Qtr1  Qtr2  Qtr3  Qtr4
1973 26041 29980 31774 28026
1974 22769 26648 28686 26519
1975 23592 26813 27998 24660
1976 22945 25493 27294 25009
1977 22475 26295 28241 25911
1978 22519 26741 29421 26943
```

Temporal reconciliation: thief package

```
aggts[[4]]
```

```
Time Series:
Start = c(1973, 1)
End = c(1978, 3)
Frequency = 3
 [1] 35178 42904 37739 31191 38169 35262 31462 38656 32945 30870 36825 33046 30581
[15] 34225 30711 38860 36053
```

Temporal reconciliation: thief package

```r
aggts[[5]]
```

```
Time Series:
Start = c(1973, 1)
End = c(1978, 2)
Frequency = 2
 [1] 56021 59800 49417 55205 50405 52658 48438 52303 48770 54152 49260 56364
```

Temporal reconciliation: thief package

```r
aggts[[6]]
```

```
Time Series:
Start = 1973
End = 1978
Frequency = 1
[1] 115821 104622 103063 100741 102922 105624
```

Temporal reconciliation: thief package

```r
# Compute forecasts
fc <- list()
for(i in seq_along(aggts)) {
  fc[[i]] <- forecast(auto.arima(aggts[[i]]), h=2*frequency(aggts[[i]]))
}
reconciled <- reconcilethief(fc)
reconciled[[1]]
```

```
         Point Forecast Lo 80 Hi 80 Lo 95 Hi 95
Jan 1979           8185  7774  8597  7556  8814
Feb 1979           7381  6908  7854  6657  8105
Mar 1979           8132  7604  8660  7324  8939
Apr 1979           8516  7938  9094  7632  9400
May 1979           9393  8770 10017  8440 10347
Jun 1979           9764  9098 10430  8746 10783
Jul 1979          10881 10175 11587  9801 11961
Aug 1979          10060  9316 10804  8922 11197
```

Temporal reconciliation: thief package

```r
reconciled[[2]]
```

```
        Point Forecast Lo 80 Hi 80 Lo 95 Hi 95
1979.00          15566 14789 16344 14377 16756
1979.17          16648 15634 17661 15098 18197
1979.33          19157 17954 20361 17317 20998
1979.50          20941 19573 22308 18849 23032
1979.67          18587 17073 20101 16272 20903
1979.83          18230 16583 19877 15710 20749
1980.00          16007 14159 17856 13181 18834
1980.17          17089 15077 19101 14013 20165
1980.33          19599 17436 21761 16291 22906
1980.50          21382 19078 23686 17858 24906
1980.67          19029 16592 21466 15302 22756
1980.83          18671 16108 21234 14751 22591
```

Temporal reconciliation: thief package

```
reconciled[[3]]
```

```
        Point Forecast Lo 80 Hi 80 Lo 95 Hi 95
1979 Q1          23698 22461 24935 21806 25590
1979 Q2          27673 25924 29423 24997 30349
1979 Q3          30123 27980 32266 26846 33401
1979 Q4          27635 25161 30110 23851 31419
1980 Q1          24360 20860 27859 19008 29712
1980 Q2          28335 24049 32621 21780 34890
1980 Q3          30785 25836 35734 23216 38354
1980 Q4          28297 22764 33830 19835 36759
```

Temporal reconciliation: thief package

```
reconciled[[4]]
```

```
        Point Forecast Lo 80 Hi 80 Lo 95 Hi 95
1979.00          32214 30197 34231 29129 35299
1979.33          40098 37245 42951 35735 44461
1979.67          36817 33324 40311 31474 42161
1980.00          33096 27760 38433 24935 41258
1980.33          40980 34290 47671 30749 51212
1980.67          37700 29888 45512 25752 49648
```

Temporal reconciliation: thief package

```
reconciled[[5]]
```

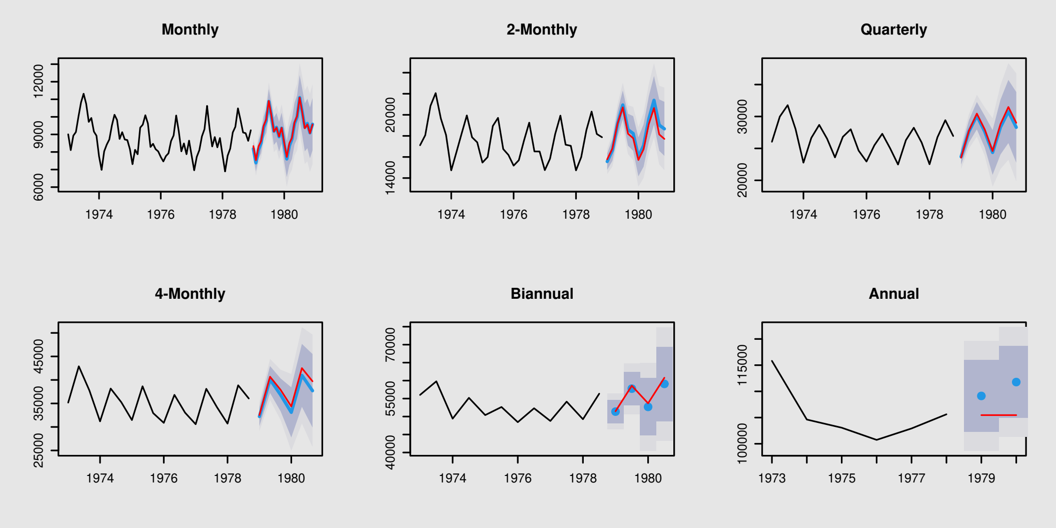
```
        Point Forecast Lo 80 Hi 80 Lo 95 Hi 95
1979.00          51371 48111 54631 46386 56357
1979.50          57758 53148 62368 50707 64809
1980.00          52695 44710 60680 40483 64907
1980.50          59082 48773 69391 43316 74848
```

Temporal reconciliation: thief package

```
reconciled[[6]]
```

```
     Point Forecast  Lo 80  Hi 80  Lo 95  Hi 95
1979         109129 102287 115972  98665 119594
1980         111777 104934 118619 101312 122241
```

Temporal reconciliation: thief package



Outline

Temporal reconciliation: M3 monthly series

- Apply temporal reconciliation to all 1428 monthly series from M3 competition
- Forecast horizon $h = 18$ months
- ETS and ARIMA models
- Measure percentage difference to base forecasts
- Reconciliation methods:
  - WLS_H (diagonal)
  - WLS_V (diagonal with common variances for same frequency)
  - WLS_S (diagonal/structural)

Temporal reconciliation: M3 monthly series

Aggregation level	h	ETS Base	ETS BU	ETS WLS_H	ETS WLS_V	ETS WLS_S	ARIMA Base	ARIMA BU	ARIMA WLS_H	ARIMA WLS_V	ARIMA WLS_S
					RMAE						
Annual	1	1.0	-19.6	-22.0	-22.0	**-25.1**	1.0	-28.6	-33.1	-32.8	**-33.4**
Semi-annual	3	1.0	0.6	-4.0	-3.6	**-5.4**	1.0	-3.4	-8.2	-8.3	**-9.9**
Four-monthly	4	1.0	2.0	-2.4	-2.2	**-3.0**	1.0	-1.7	-5.5	-5.9	**-6.7**
Quarterly	6	1.0	2.4	-1.6	-1.7	**-2.8**	1.0	-3.6	-7.2	-8.1	**-9.1**
Bi-monthly	9	1.0	0.7	-2.9	-3.3	**-4.3**	1.0	-1.5	-4.4	-5.3	**-6.3**
Monthly	18	1.0	0.0	-2.2	-3.2	**-3.9**	1.0	0.0	-0.9	-2.9	**-3.4**
Average			*-2.3*	*-5.9*	*-6.0*	***-7.4***		*-6.5*	*-9.9*	*-10.5*	***-11.5***
					MASE						
Annual	1	1.1	-12.1	-17.9	-17.8	**-18.5**	1.3	-25.4	-29.9	-29.9	**-30.2**
Semi-annual	3	1.0	0.0	-6.3	-6.0	**-6.9**	1.1	-2.9	-8.1	-8.2	**-9.4**
Four-monthly	4	0.9	3.1	-3.2	-3.0	**-3.4**	0.9	-1.8	-6.2	-6.5	**-7.1**
Quarterly	6	0.9	3.2	-2.8	-2.7	**-3.4**	1.0	-2.6	-6.9	-7.4	**-8.1**
Bi-monthly	9	0.9	2.7	-2.9	-3.0	**-3.7**	0.9	-1.3	-5.0	-5.5	**-6.3**
Monthly	18	0.9	0.0	-3.7	-4.6	**-5.0**	0.9	0.0	-1.9	-3.2	**-3.7**
Average			*-0.5*	*-6.1*	*-6.2*	***-6.8***		*-5.7*	*-9.7*	*-10.1*	***-10.8***

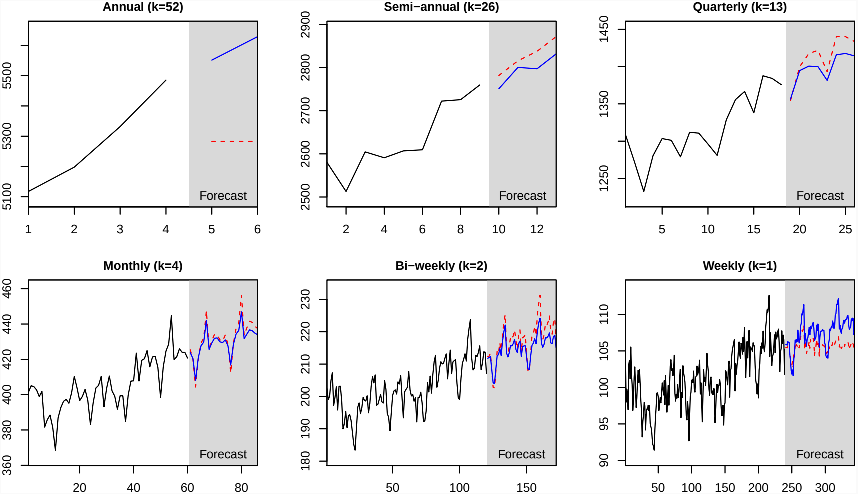
Example: Accident & emergency services demand

Weekly A&E demand data: 7 November 2010 to 7 June 2015.

Type 1 Departments — Major A&E
Type 2 Departments — Single Specialty
Type 3 Departments — Other A&E/Minor Injury Unit
Total Attendances
Type 1 Departments — Major A&E $>$ 2 hours
Type 2 Departments — Single Specialty $>$ 2 hours
Type 3 Departments — Other A&E/Minor Injury Unit $>$ 2 hours
Total Attendances $>$ 2 hours
Emergency Admissions via Type 1 A&E
Total Emergency Admissions via A&E
Other Emergency Admissions (i.e not via A&E)
Total Emergency Admissions
Number of patients spending $>$ 2 hours from decision to admit to admission

Example: Accident & emergency services demand

Total emergency admissions via A&E



Example: Accident & emergency services demand

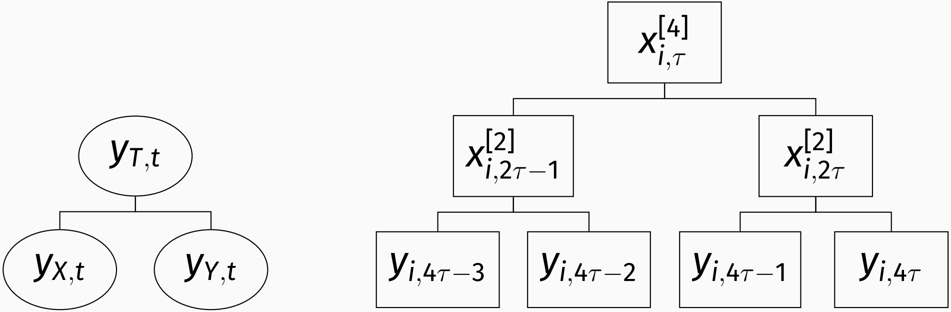
Test set: last 52 weeks

MASE comparison (ARIMA models)

Aggr. Level	h	Base	Reconciled	Change
Annual	1	3.4	1.9	−42.9%
Weekly	1–52	2.0	1.9	−5.0%
Weekly	13	2.3	1.9	−16.2%
Weekly	4	1.9	1.5	−18.6%
Weekly	1	1.6	1.3	−17.2%

Outline

Cross-temporal forecast reconciliation



- $n = 3$, $n_a = 1$, $n_b = 2$
- Quarterly series: $m = 2$, $\mathcal{K} = \{1, 2, 4\}$

Cross-temporal reconciliation

- $\boldsymbol{y}_t$ = series at most temporally disaggregated level, including all cross-sectionally disaggregated and aggregated series.
- $y_{i,t}$ = ith element of $\boldsymbol{y}_t$, $i = 1, \dots, n$.
- For each i, we can expand $y_{i,t}$ to include all temporally aggregated variants:

$$\boldsymbol{x}_{i,\tau} = \begin{bmatrix} x_{i,\tau}^{[k_p]} \\ \vdots \\ \boldsymbol{x}_{i,\tau}^{[k_1]} \end{bmatrix} \qquad \boldsymbol{x}_{\tau} = \begin{bmatrix} \boldsymbol{x}_{1,\tau} \\ \vdots \\ \boldsymbol{x}_{n,\tau} \end{bmatrix}.$$

Cross-temporal reconciliation

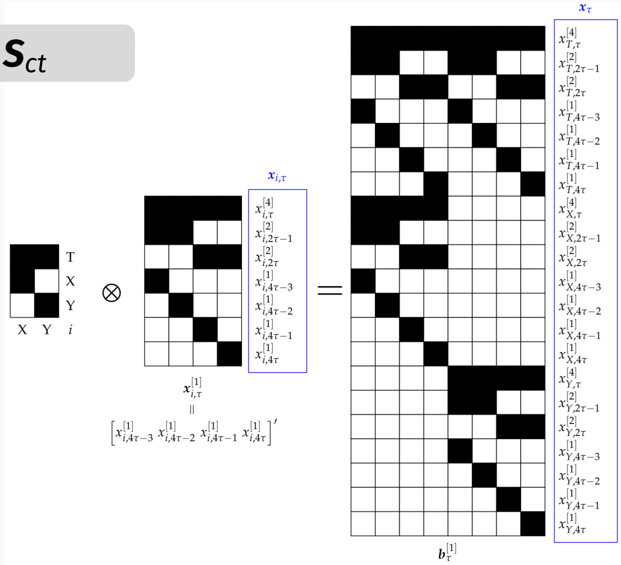
Structural matrix approach

- $\boldsymbol{S}_{cs}$ = structural cross-sectional matrix
- $\boldsymbol{S}_{te}$ = structural temporal matrix
- $\boldsymbol{S}_{ct} = \boldsymbol{S}_{cs} \otimes \boldsymbol{S}_{te}$

$$\boldsymbol{x}_{\tau} = \boldsymbol{S}_{ct}\boldsymbol{b}_{\tau}, \qquad \text{where} \qquad \boldsymbol{b}_{\tau} = \begin{bmatrix} \boldsymbol{x}_{1,\tau}^{[1]} \\ \vdots \\ \boldsymbol{x}_{n,\tau}^{[1]} \end{bmatrix}.$$

Cross-temporal forecast reconciliation

$$\boldsymbol{S}_{cs} \otimes \boldsymbol{S}_{te} = \boldsymbol{S}_{ct}$$



Cross-temporal reconciliation

Constraint matrix approach

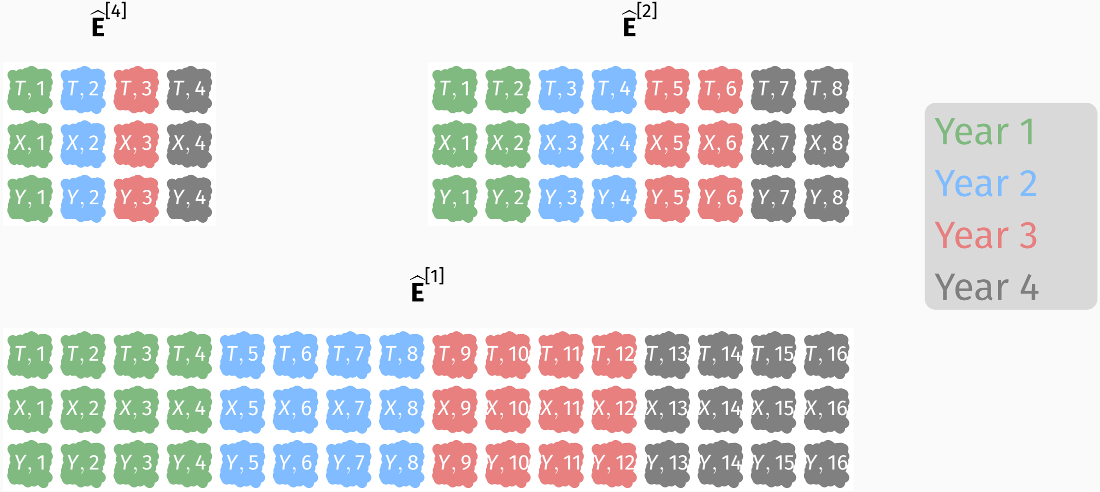
- $\boldsymbol{C}_{cs}$ = cross-sectional constraint matrix
- $\boldsymbol{C}_{te}$ = temporal constraint matrix

$$\boldsymbol{C}_{ct}\boldsymbol{x}_{\tau} = \boldsymbol{0} \qquad \text{where} \qquad \boldsymbol{C}_{ct} = \begin{bmatrix} (\boldsymbol{0}_{(n_a m \times nk^*)} \quad \boldsymbol{I}_m \otimes \boldsymbol{C}_{cs})\boldsymbol{P}' \\ \boldsymbol{I}_n \otimes \boldsymbol{C}_{te} \end{bmatrix}$$

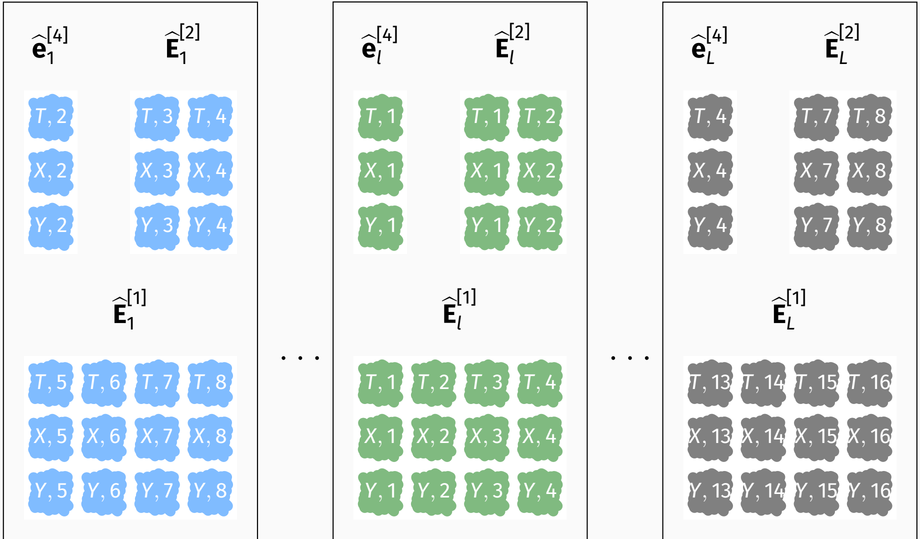
- $k^* = \displaystyle\sum_{k \in \mathcal{K} \setminus \{k_1\}} \frac{m}{k}$
- $\boldsymbol{P}$ = the commutation matrix such that $\boldsymbol{P}\text{vec}(\boldsymbol{Y}_{\tau}) = \text{vec}(\boldsymbol{Y}_{\tau}')$.

Outline

Cross-temporal probabilistic forecast reconciliation



Cross-temporal probabilistic forecast reconciliation



Thanks!

More information

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References

- Athanasopoulos, G, RJ Hyndman, N Kourentzes, and F Petropoulos (2017). Forecasting with temporal hierarchies. *European J Operational Research* **262**(1), 60–74.
- Di Fonzo, T and D Girolimetto (2023). Cross-temporal forecast reconciliation: Optimal combination method and heuristic alternatives. *International Journal of Forecasting* **39**(1), 39–57.