

# Forecast reconciliation

## 3. Probabilistic forecast reconciliation

Rob J Hyndman



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[robjhyndman.com/fr2023](http://robjhyndman.com/fr2023)

# Outline

- 1 Definition of probabilistic coherence
- 2 Evaluating probabilistic forecasts
- 3 Emergency Services Demand
- 4 Evaluating multivariate probabilistic forecasts
- 5 Example: Australian electricity generation

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# The coherent subspace

## Coherent subspace

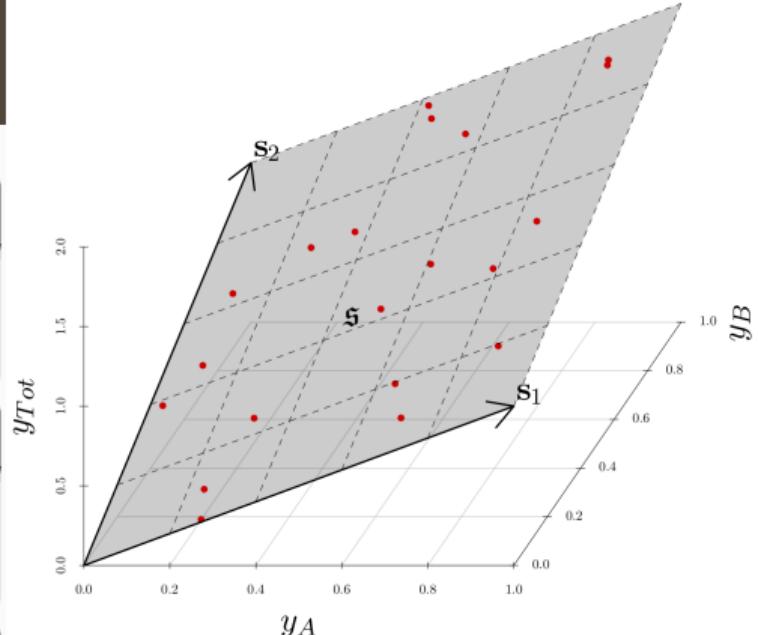
$m$ -dimensional linear subspace  $\mathfrak{s} \subset \mathbb{R}^n$  for which linear constraints hold for all  $\mathbf{y} \in \mathfrak{s}$ .

## Hierarchical time series

An  $n$ -dimensional multivariate time series such that  $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$ .

## Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$  is *coherent* if  $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$ .



$$y_{Tot} = y_A + y_B$$

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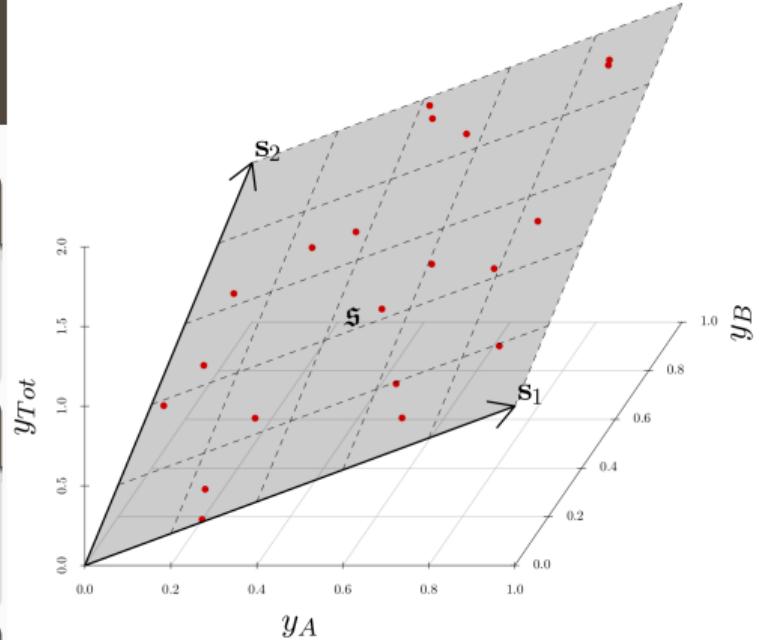
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## Base forecasts

Let  $\hat{\mathbf{y}}_{t+h|t}$  be vector of *incoherent* initial  $h$ -step forecasts.



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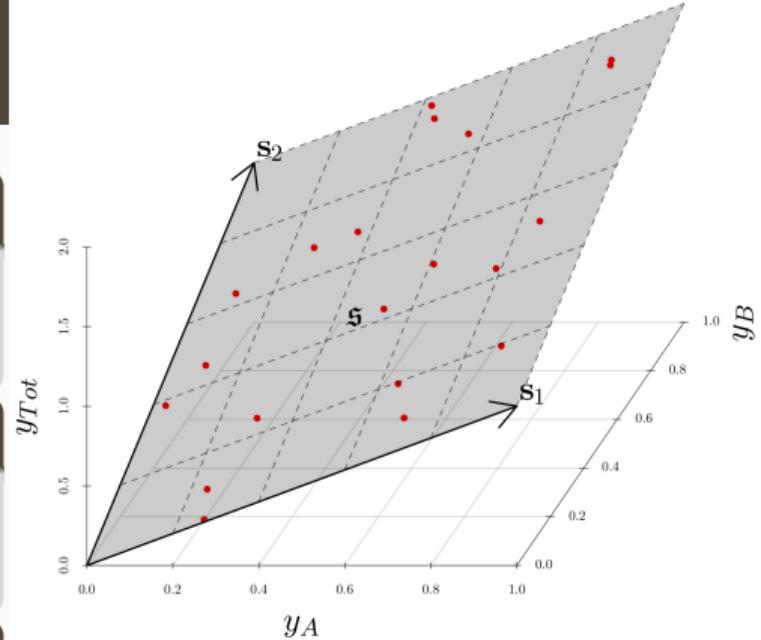
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$$y_{Tot} = y_A + y_B$$

## Reconciled forecasts

Let  $\psi$  be a mapping,  $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$ .  
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$  “reconciles”  $\hat{\mathbf{y}}_{t+h|t}$ .

# Coherent probabilistic forecasts

## Coherent probabilistic forecasts

Given the triple  $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$ , a coherent probability triple  $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \check{\nu})$  is such that

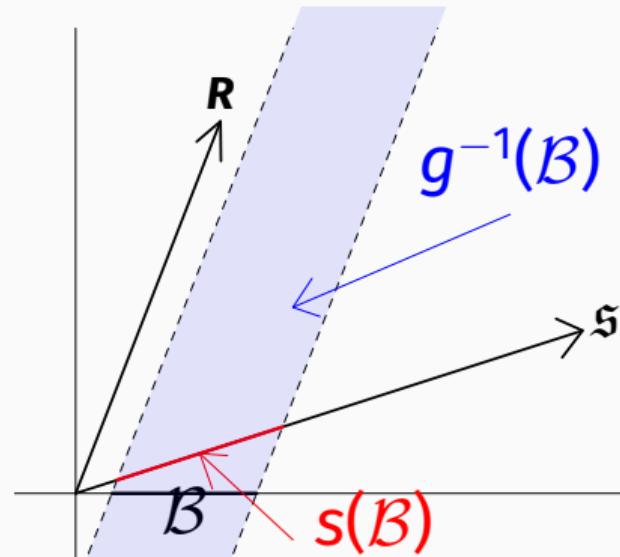
$$\check{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}^m}$$

## Probabilistic forecast reconciliation

The reconciled probability measure of  $\hat{\nu}$  wrt  $\psi(\cdot)$  is such that

$$\tilde{\nu}(\mathcal{B}) = \hat{\nu}(\psi^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathfrak{s}}$$

where  $\psi^{-1}(\mathcal{B}) := \{\mathbf{y} \in \mathbb{R}^n : \psi(\mathbf{y}) \in \mathcal{B}\}$  is the pre-image of  $\mathcal{B}$ , that is the set of all points in  $\mathbb{R}^n$  that  $\psi(\cdot)$  maps to a point in  $\mathcal{B}$ .



# Construction of reconciled distributions

## Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_b(\mathbf{b}) = |\mathbf{G}^*| \int \hat{f}(\mathbf{G}^- \mathbf{b} + \mathbf{G}_\perp \mathbf{a}) d\mathbf{a}$$

- $\hat{f}$  is density of incoherent base probabilistic forecasts
- $\mathbf{G}^-$  is  $n \times m$  generalised inverse of  $\mathbf{G}$  st  $\mathbf{G}\mathbf{G}^- = \mathbf{I}$
- $\mathbf{G}_\perp$  is  $n \times (n - m)$  orthogonal complement to  $\mathbf{G}$  st  $\mathbf{G}\mathbf{G}_\perp = \mathbf{0}$
- $\mathbf{G}^* = (\mathbf{G}^- : \mathbf{G}_\perp)$ , and  $\mathbf{b}$  and  $\mathbf{a}$  are obtained via

the change of variables  $\mathbf{y} = \mathbf{G}^* \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$

# Construction of reconciled distributions

## Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_y(\mathbf{y}) = |\mathbf{S}^*| \tilde{f}_b(\mathbf{S}^- \mathbf{y}) \mathbb{1}\{\mathbf{y} \in \mathfrak{s}\}$$

- $\mathbf{S}^* = (\mathbf{S}^{-'} \quad \mathbf{S}_{\perp})'$
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## Gaussian reconciliation

If the incoherent base forecasts are  $N(\hat{\mu}, \hat{\Sigma})$ , then the reconciled density is  $N(\mathbf{S}\mathbf{G}\hat{\mu}, \mathbf{S}\mathbf{G}\hat{\Sigma}\mathbf{G}'\mathbf{S}')$

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## Bootstrap reconciliation

Reconciling sample paths from incoherent distributions works.

# Simulation from a reconciled distribution

Suppose that  $(\hat{\mathbf{y}}^{[1]}, \dots, \hat{\mathbf{y}}^{[L]})$  is a sample drawn from an incoherent probability measure  $\hat{\nu}$ . Then  $(\tilde{\mathbf{y}}^{[1]}, \dots, \tilde{\mathbf{y}}^{[L]})$  where  $\tilde{\mathbf{y}}^{[\ell]} := \psi(\hat{\mathbf{y}}^{[\ell]})$  for  $\ell = 1, \dots, L$ , is a sample drawn from the reconciled probability measure  $\tilde{\nu}$ .

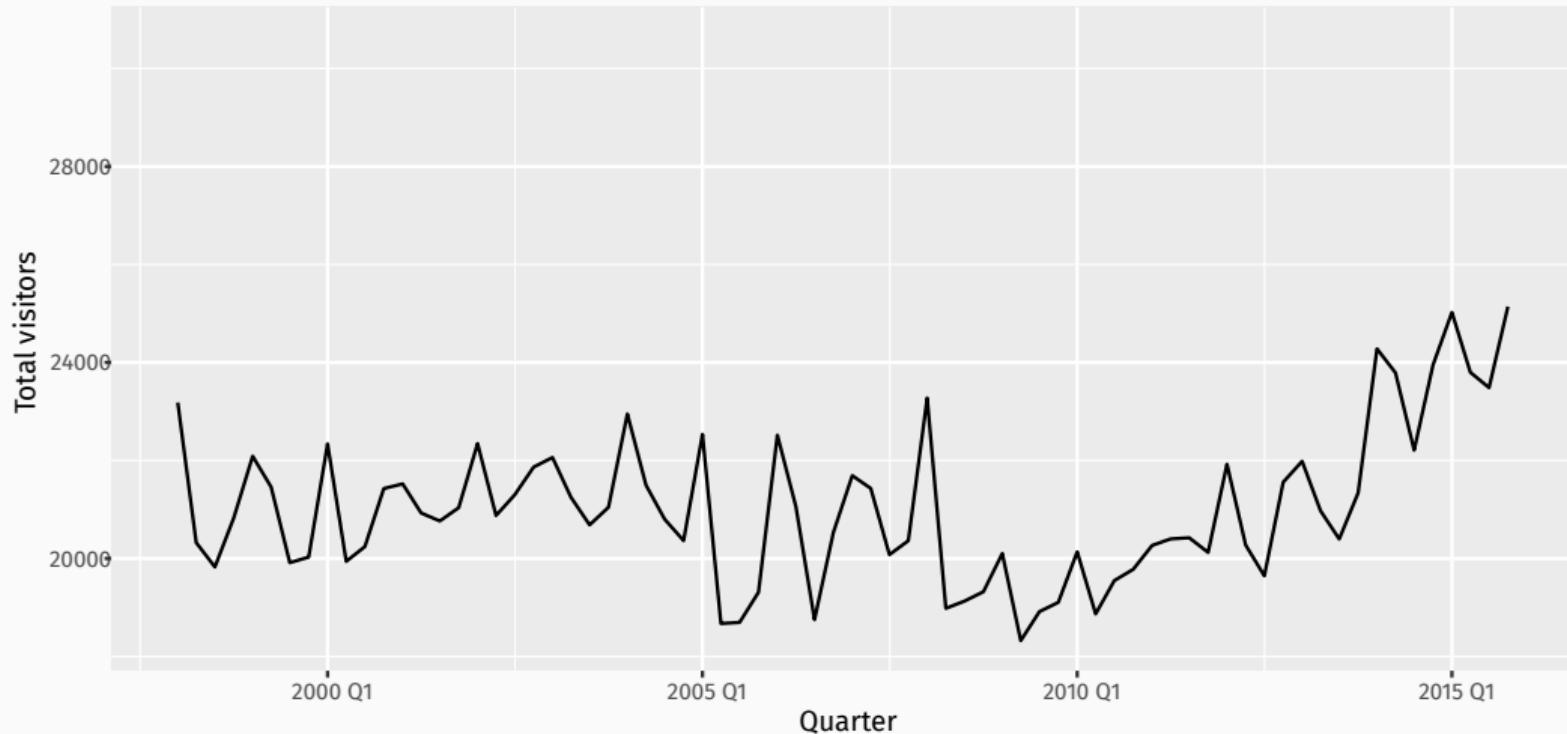
- Simulate future sample paths for each series using a multivariate bootstrap of the residuals from each model.
- Reconcile the sample paths.
- The reconciled sample paths are a sample from the reconciled distribution.

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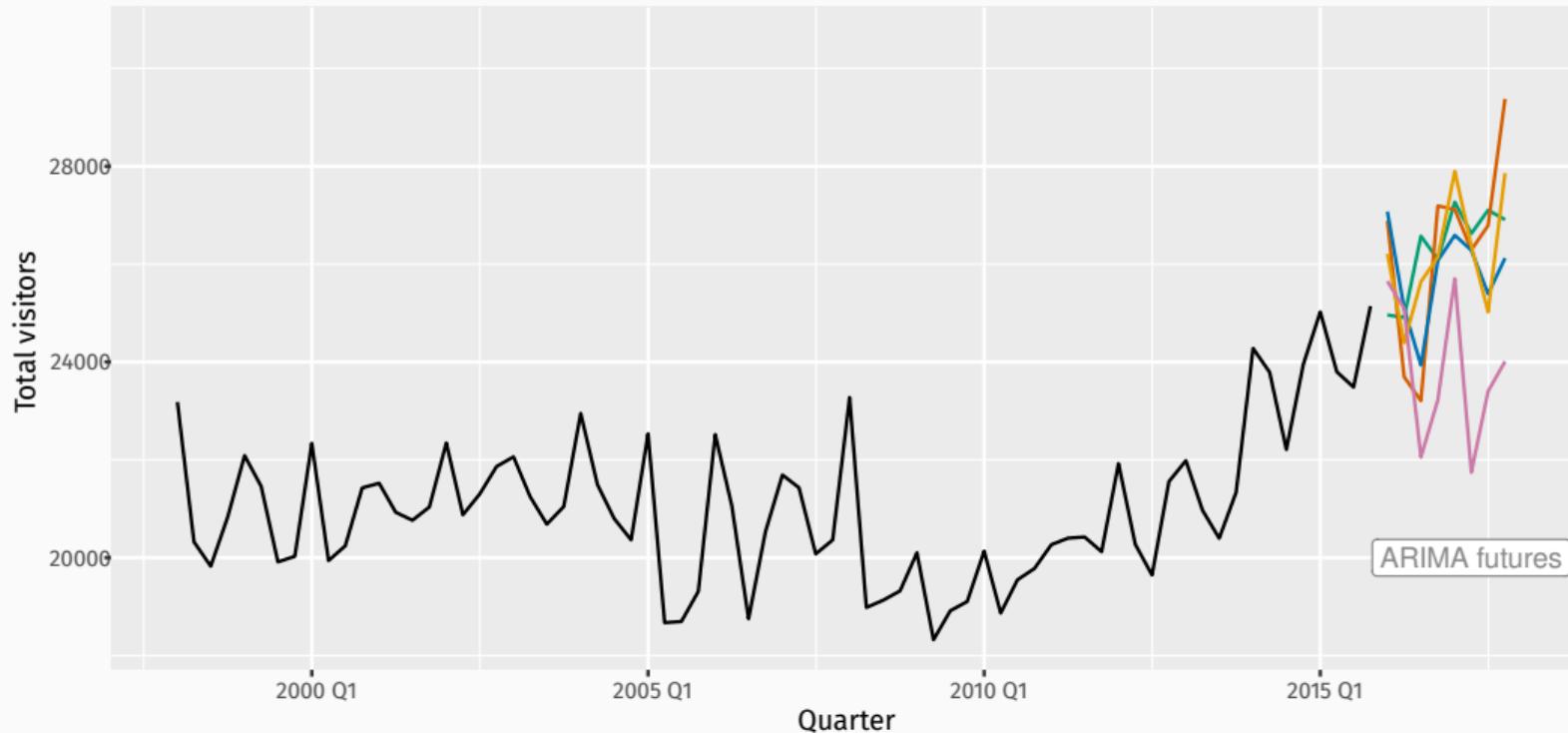
# Evaluating probabilistic forecasts

Australian domestic tourism



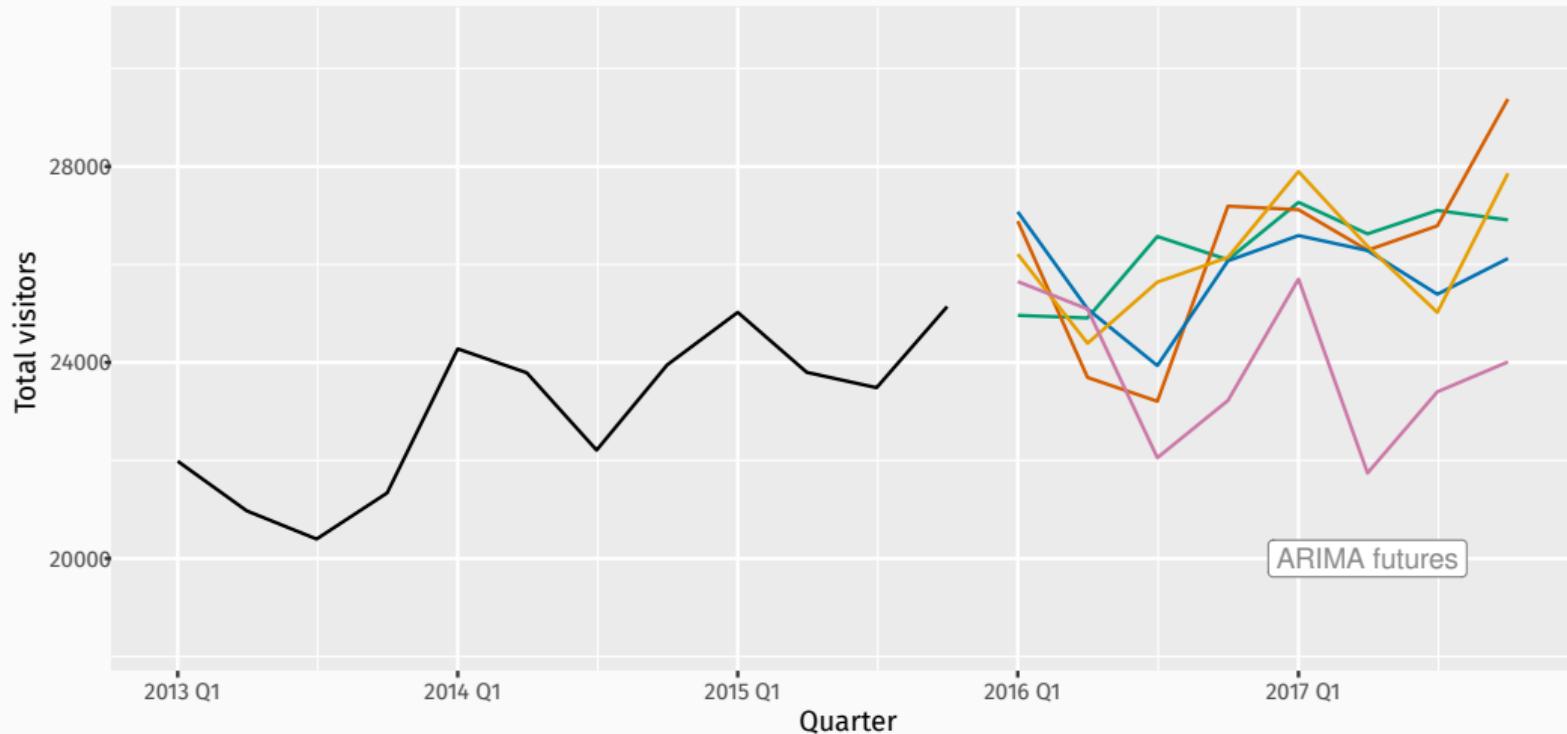
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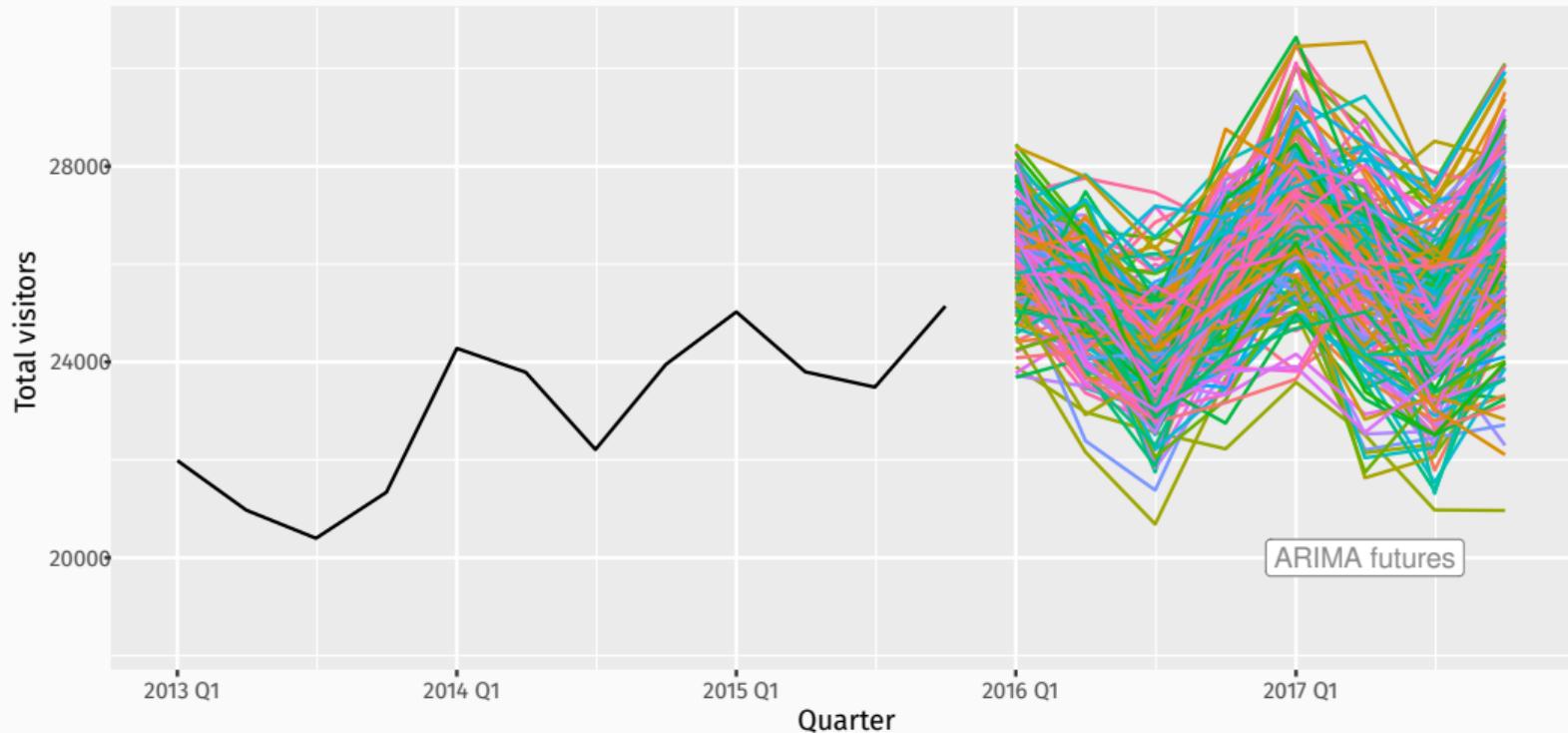
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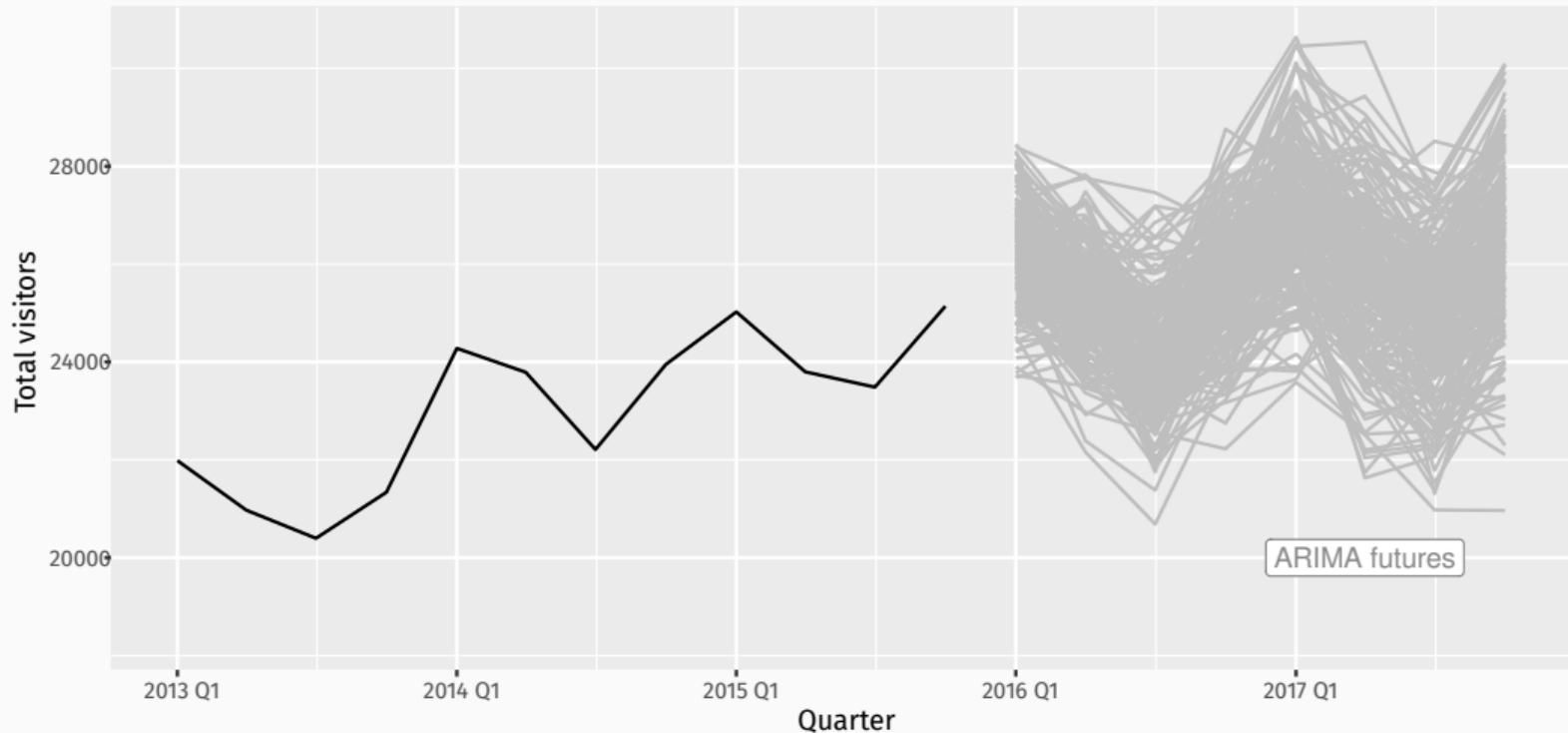
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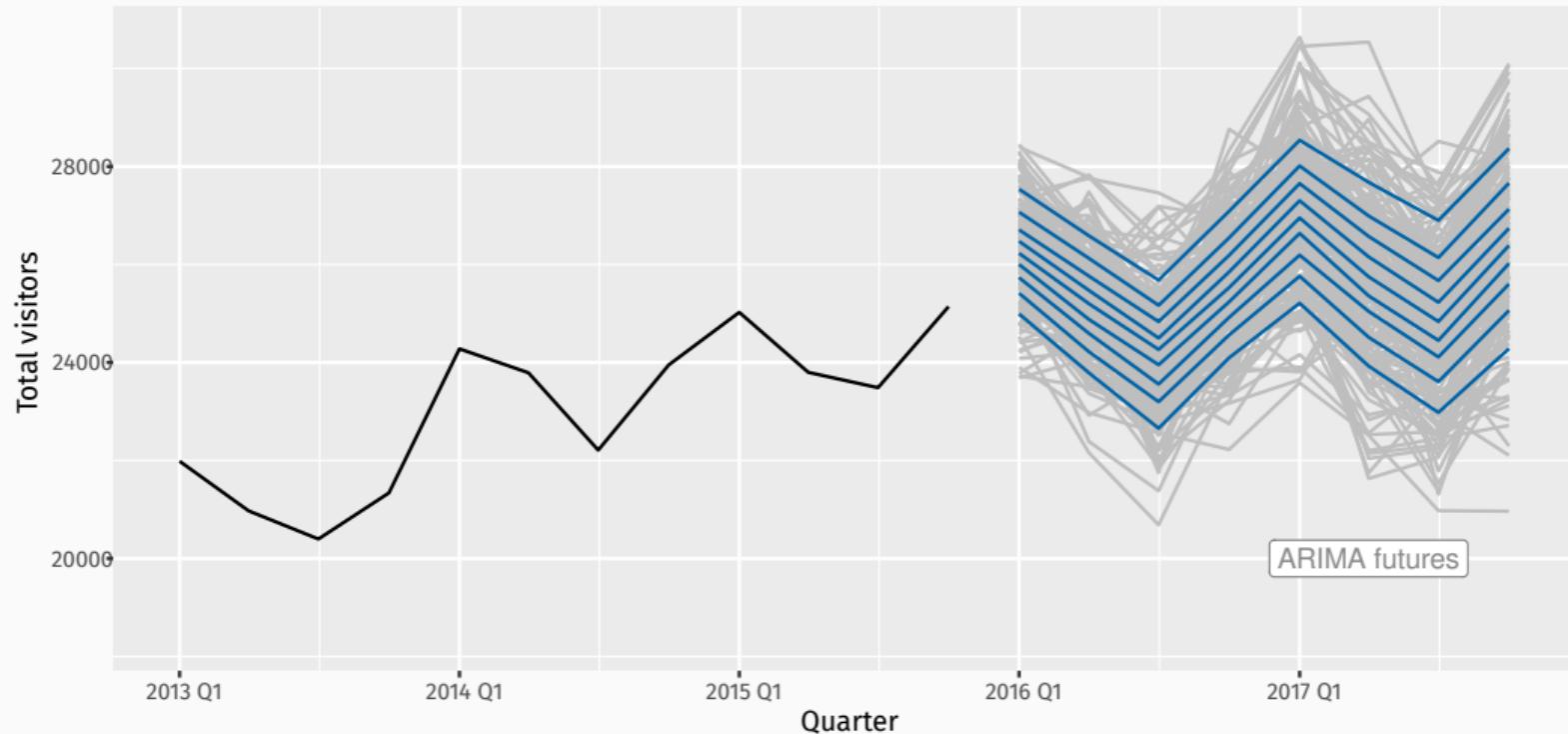
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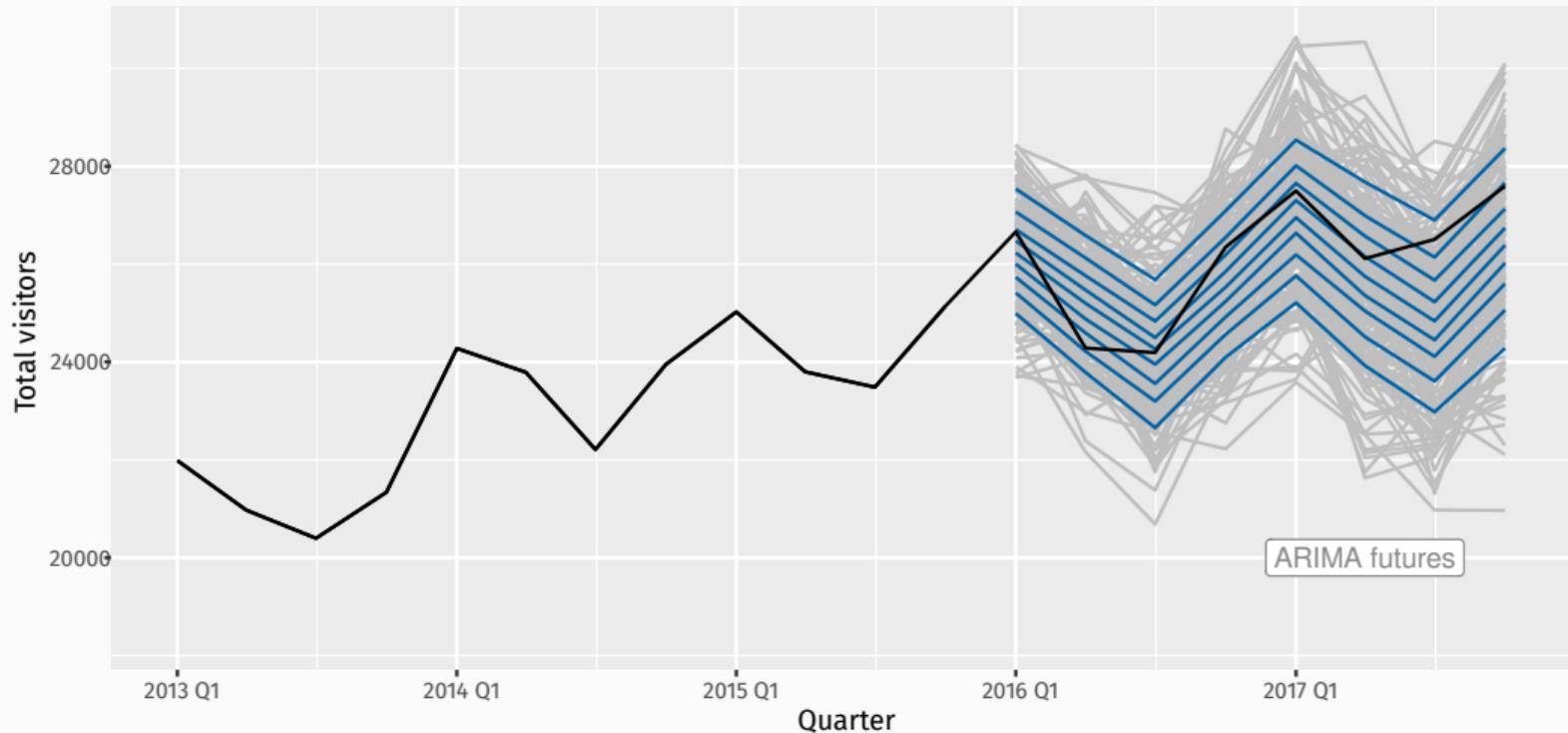
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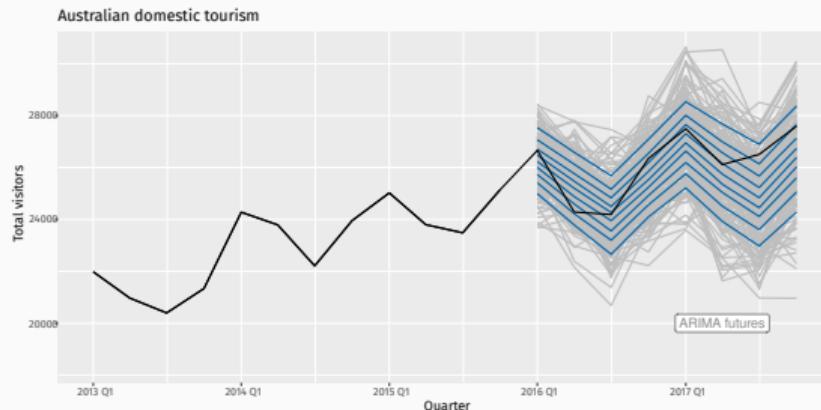
$y_t$  = observation at time  $t$

$q_{p,t}$  = quantile forecast: prob.  $p$ , time  $t$

## Quantile score

$$S_t(p, y) = \begin{cases} 2(1 - p)|y_t - q_{p,t}|, & \text{if } y_t < q_{p,t} \\ 2p|y_t - q_{p,t}|, & \text{if } y_t \geq q_{p,t} \end{cases}$$

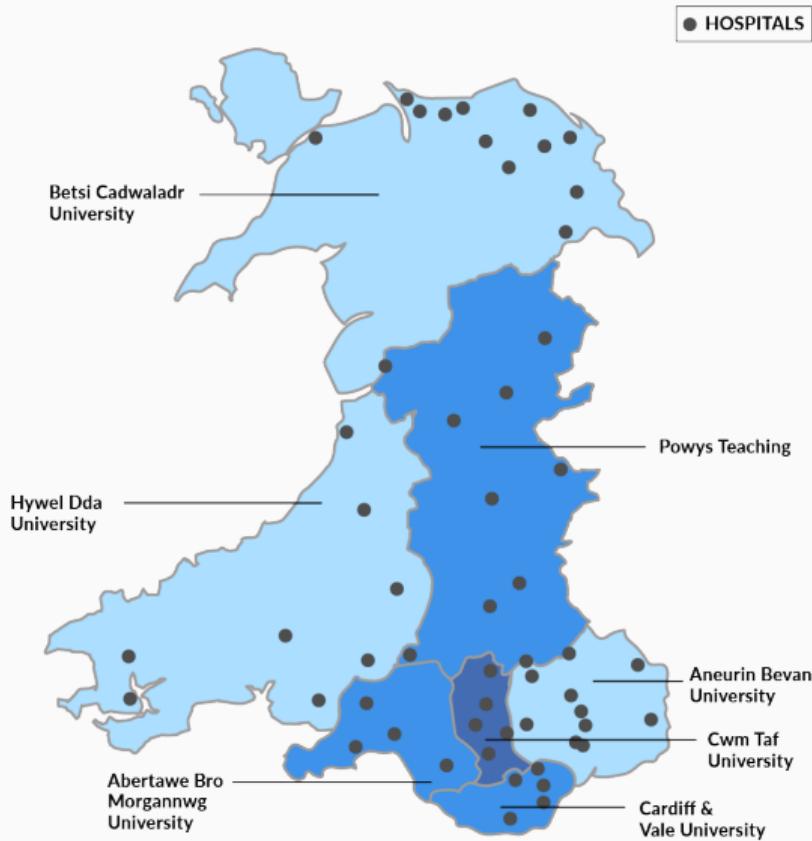
- Low  $S_t$  is good
- Multiplier of 2 often omitted, but useful for interpretation
- $S_t$  like absolute error, weighted to account for likely exceedance
- Average  $S_t(p, y)$  over  $p$  = CRPS (Continuous Rank Probability Score)



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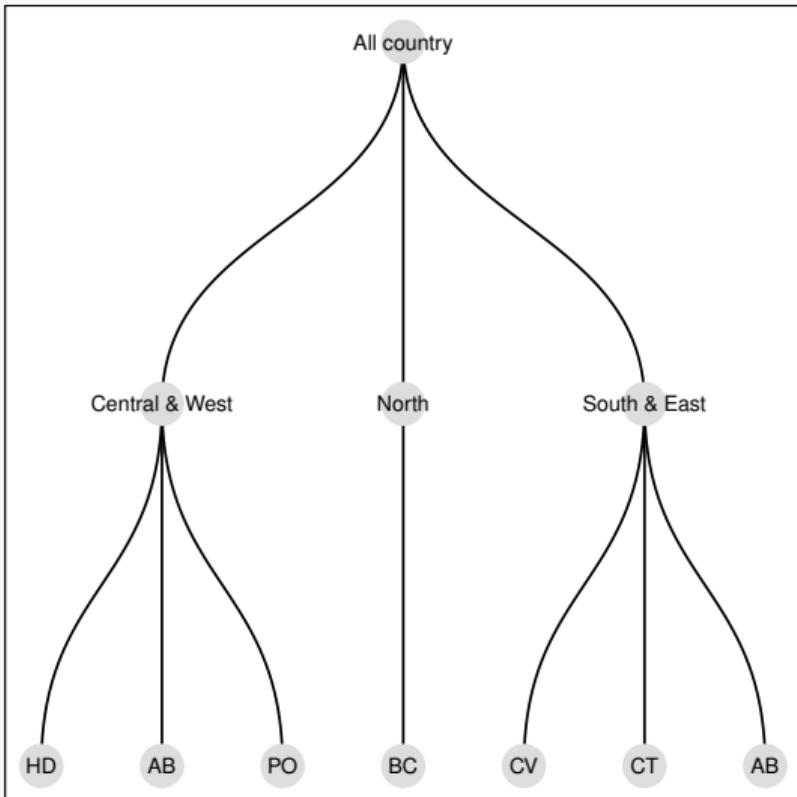
# Wales Health Board Areas



# Data

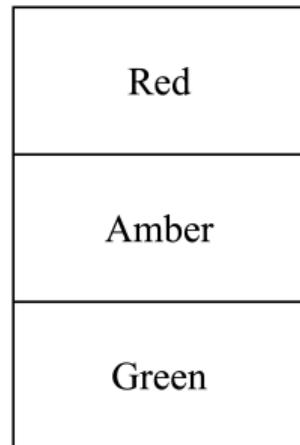
- Daily number of attended incidents:  
1 October 2015 – 31 July 2019
- Disaggregated by:
  - ▶ control area
  - ▶ health board
  - ▶ priority
  - ▶ nature of incidents
- 2,142,000 rows observations from 1,530 time series.

# Data structure



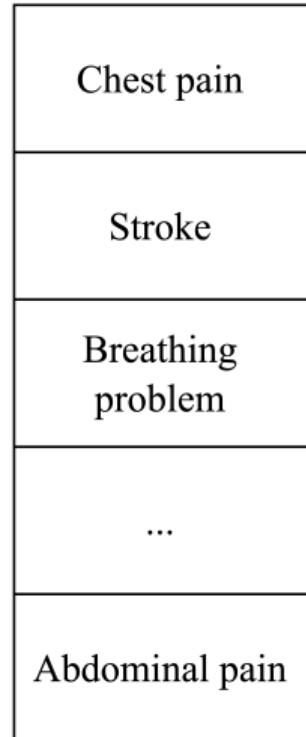
\*

## Priority



\*

## Nature of incident



# Data structure

Level	Number of series
All country	1
Control	3
Health board	7
Priority	3
Priority * Control	9
Priority * Health board	21
Nature of incident	35
Nature of incident * Control	105
Nature of incident * Health board	245
Priority * Nature of incident	104
Control * Priority * Nature of incident	306
Control * Health board * Priority * Nature of incident (Bottom level)	691
Total	1530

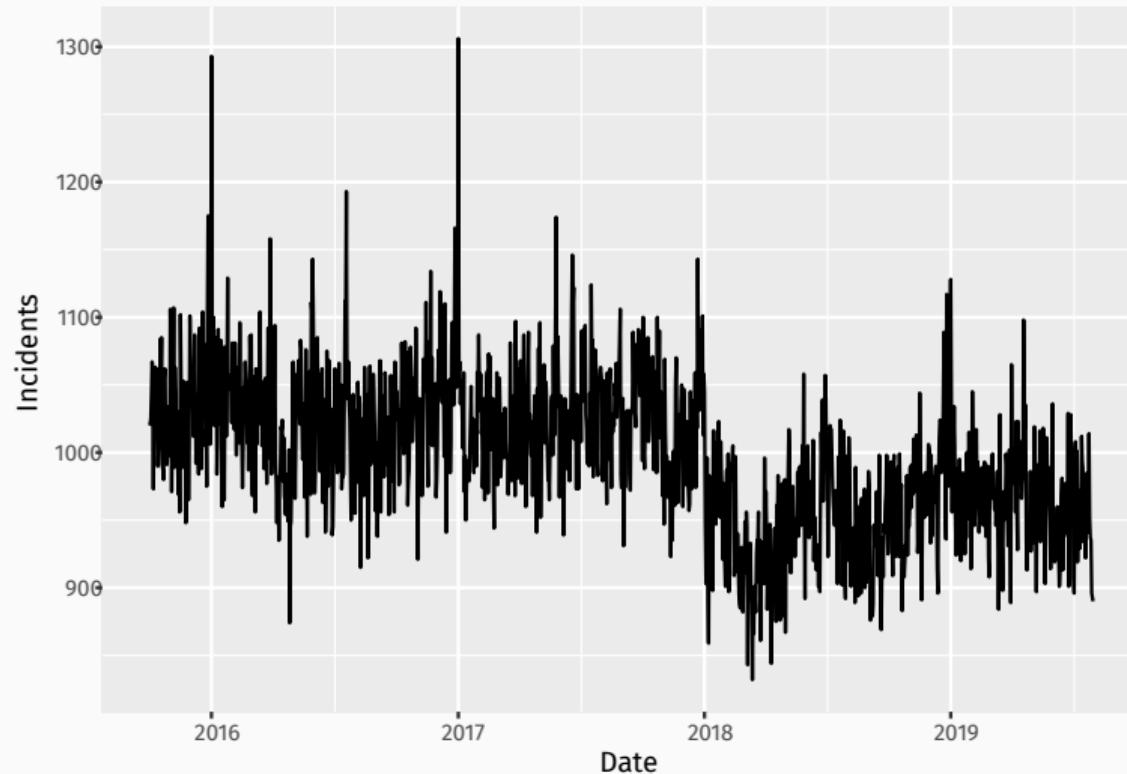
# Data

```
# A tsibble: 2,142,000 x 6 [1D]
# Key:      region, category, nature, lhb [1,530]
  date      region     category     nature      lhb      incident
  <date>    <chr*>     <chr*>     <chr*>     <chr*>     <dbl>
1 2015-10-01 <aggregated> <aggregated> <aggregated> <aggregated> 1020
2 2015-10-02 <aggregated> <aggregated> <aggregated> <aggregated> 1021
3 2015-10-03 <aggregated> <aggregated> <aggregated> <aggregated> 1025
4 2015-10-04 <aggregated> <aggregated> <aggregated> <aggregated> 1043
5 2015-10-05 <aggregated> <aggregated> <aggregated> <aggregated> 1067
6 2015-10-06 <aggregated> <aggregated> <aggregated> <aggregated> 1063
7 2015-10-07 <aggregated> <aggregated> <aggregated> <aggregated> 973
8 2015-10-08 <aggregated> <aggregated> <aggregated> <aggregated> 1057
9 2015-10-09 <aggregated> <aggregated> <aggregated> <aggregated> 1026
10 2015-10-10 <aggregated> <aggregated> <aggregated> <aggregated> 1063
# i 2,141,990 more rows
```

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# Key:      region, category, nature, lhb [1,530]
  date      region category nature    lhb      incident
  <date>    <chr*>  <chr*>   <chr*>  <chr*>    <dbl>
1 2015-10-01 C       Amber   ABDOMINAL HD        0
2 2015-10-01 C       Amber   ABDOMINAL PO        0
3 2015-10-01 C       Amber   ABDOMINAL SB        0
4 2015-10-01 C       Amber   ABDOMINAL <aggregated> 0
5 2015-10-01 C       Amber   ALLERGIES HD        0
6 2015-10-01 C       Amber   ALLERGIES PO        1
7 2015-10-01 C       Amber   ALLERGIES SB        0
8 2015-10-01 C       Amber   ALLERGIES <aggregated> 1
9 2015-10-01 C       Amber   ANIMALBIT HD        0
10 2015-10-01 C      Amber   ANIMALBIT PO       0
# i 2,141,990 more rows
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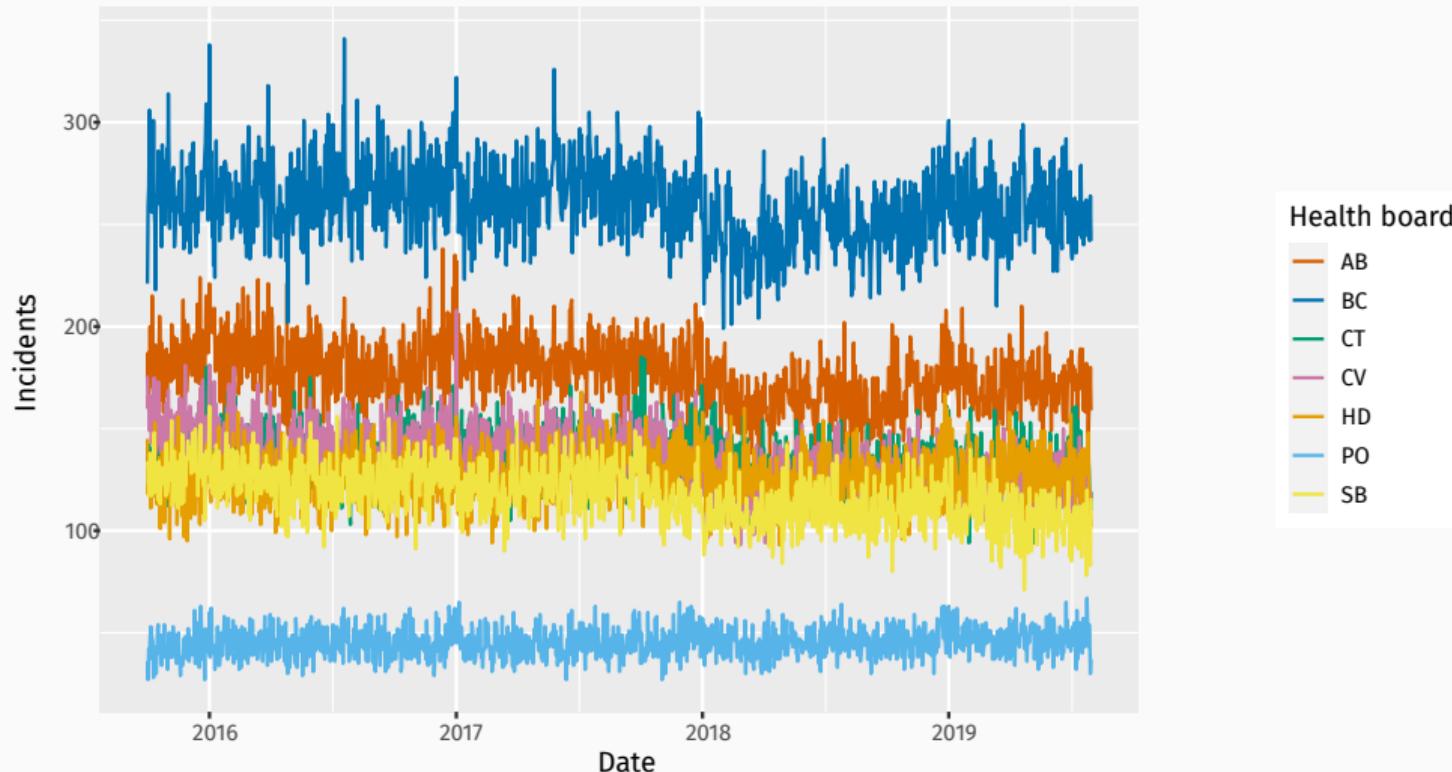
# Aggregated daily incidents



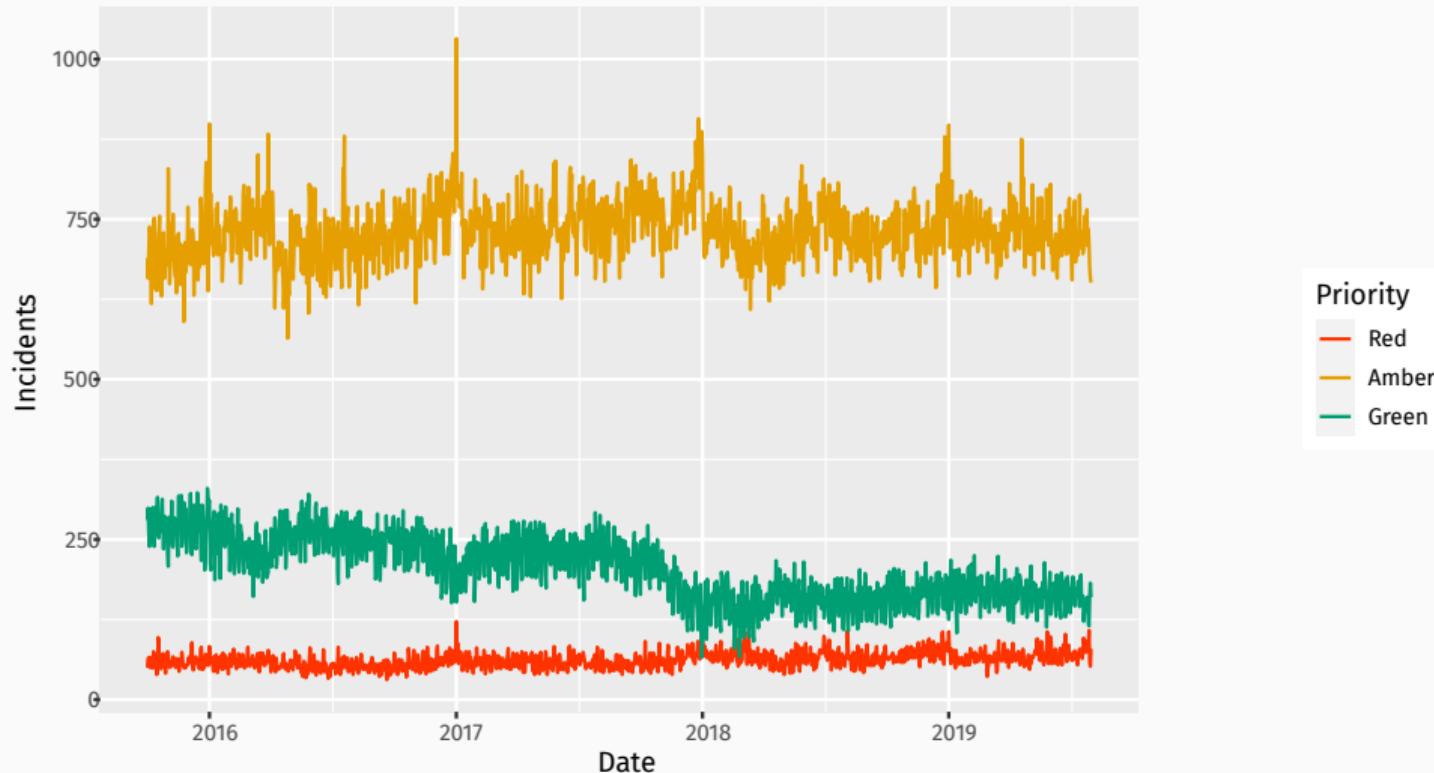
# Daily incidents by control area



# Data incidents by health board



# Data incidents by priority



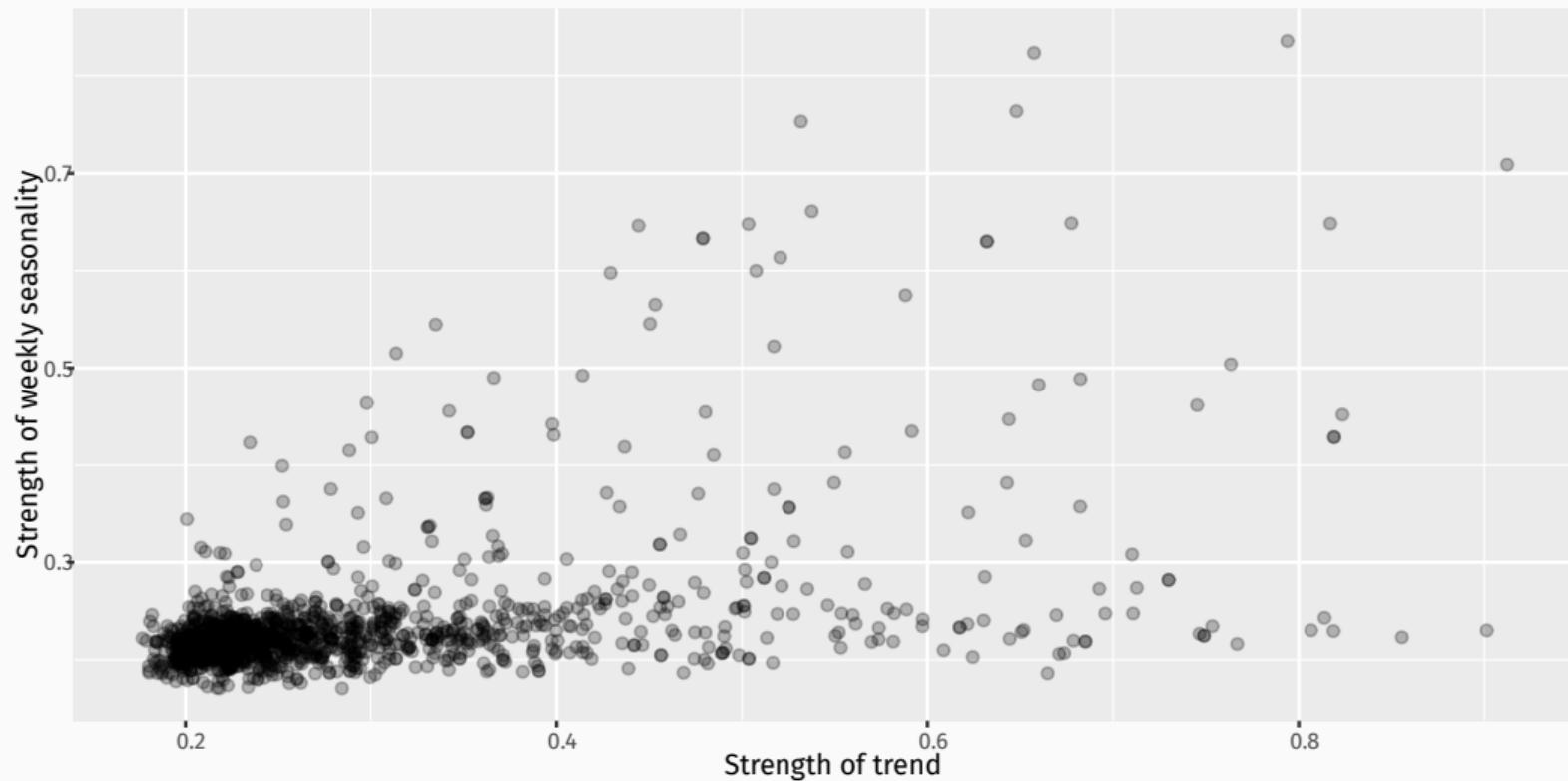
# Data incidents by nature of incident



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# Data features



# Forecasting methods

- 1 **Naïve:** Empirical distribution of past daily attended incidents.
- 2 **ETS:** Exponential Smoothing State Space models.
- 3 **GLM:** Poission Regression with spline trend, day of the week, annual Fourier seasonality, public holidays, school holidays, Christmas Day, New Year's Day.
- 4 **TSGLM:** Poisson Regression with same covariates plus three autoregressive terms.
- 5 **Ensemble:** Mixture distribution of 1–4.

# Forecasting methods

- 1 **Naïve:** Empirical distribution of past daily attended incidents.

$$y_{T+h|T} \sim \text{Empirical}(y_1, \dots, y_T)$$

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$$y_{T+h|T} \sim \text{Empirical}(y_1, \dots, y_T)$$

- 2 **ETS:** Exponential Smoothing State Space models.

$$y_{T+h|T} \sim \text{Normal}(\hat{y}_{T+h|T}, \hat{\sigma}_{T+h|T}^2)$$

# Forecasting methods

3

## GLM: Poisson Regression

$$y_{T+h|T} \sim \text{Poisson}(\hat{y}_{T+h|T}) \quad \text{where} \quad \hat{y}_{T+h|T} = \exp(\mathbf{x}'_{T+h}\boldsymbol{\beta})$$

and  $\mathbf{x}_{T+h}$  is a vector of covariates including

- spline trend
- day of the week
- annual Fourier seasonality
- public holidays
- school holidays
- Christmas Day
- New Year's Day

# Forecasting methods

	Estimate	Std. Error	z value	Pr(> z )	Significance
(Intercept)	6.998511	0.017412	401.93	< 2e-16	***
Spline_1	0.027859	0.004740	5.88	4.2e-09	***
Spline_2	-0.088244	0.006394	-13.80	< 2e-16	***
Spline_3	-0.075036	0.004784	-15.68	< 2e-16	***
Spline_4	-0.111854	0.010202	-10.96	< 2e-16	***
Spline_5	-0.043009	0.004462	-9.64	< 2e-16	***
Monday	0.019147	0.003174	6.03	1.6e-09	***
Tuesday	-0.016414	0.003180	-5.16	2.4e-07	***
Wednesday	-0.015479	0.003184	-4.86	1.2e-06	***
Thursday	-0.006804	0.003178	-2.14	0.03230	*
Friday	0.012235	0.003156	3.88	0.00011	***
Saturday	0.005293	0.003165	1.67	0.09438	.
Fourier_S1_365	0.005365	0.001294	4.15	3.4e-05	***
Fourier_C1_365	0.008263	0.001263	6.54	6.1e-11	***
Fourier_S2_365	0.004235	0.001271	3.33	0.00086	***
Fourier_C2_365	-0.010510	0.001216	-8.64	< 2e-16	***
Fourier_S3_365	-0.000556	0.001275	-0.44	0.66303	
Fourier_C3_365	0.002650	0.001243	2.13	0.03294	*
Public_holiday	0.033278	0.005697	5.84	5.2e-09	***
School_holiday	0.004857	0.002346	2.07	0.03843	*
Xmas	-0.051902	0.016772	-3.09	0.00197	**
New_years_day	0.120385	0.015573	7.73	1.1e-14	***
---					
Signif. codes:					
	---	•	••	•••	

# Forecasting methods

4

## TSGLM: Poisson Regression

$$y_{T+h|T} \sim \text{Poisson}(\hat{y}_{T+h|T})$$

where  $\hat{y}_{T+h|T} = \exp \left( \mathbf{x}'_{T+h} \boldsymbol{\beta} + \sum_{k=1}^3 \alpha_k \log(y_{T+h-k} + 1) \right)$

and  $\mathbf{x}_{T+h}$  is a vector of covariates including

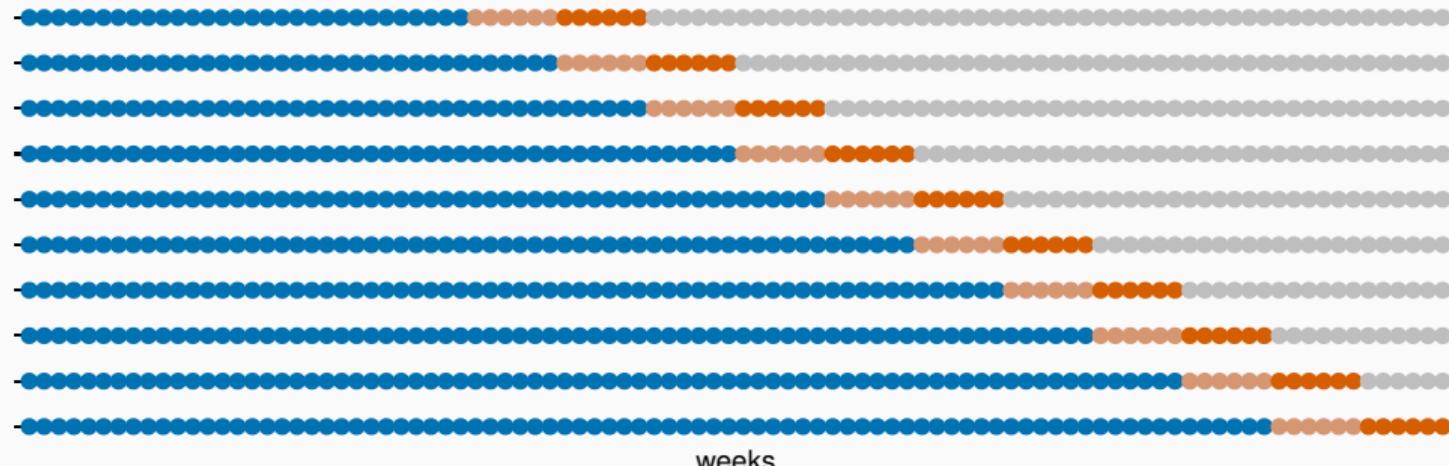
- spline trend
- day of the week
- annual Fourier seasonality
- public holidays
- school holidays
- Christmas Day
- New Year's Day

# Nonparametric bootstrap reconciliation

- Fit model to all series and store the residuals as  $\xi_t$ .
- These should be serially uncorrelated but cross-sectionally correlated.
- Draw iid samples from  $\xi_1, \dots, \xi_T$  with replacement.
- Simulate future sample paths for model using the bootstrapped residuals.
- Reconcile each sample path using MinT.
- Combine the reconciled sample paths to form a mixture distribution at each forecast horizon.

# Performance evaluation

- Ten-fold time series cross-validation
- Forecast horizon of 1–84 days
- Each training set contains an additional 42 days.
- Forecasts at 43–84 days correspond to planning horizon.



# Performance evaluation

$$\text{MASE} = \text{mean}(|q_j|)$$

$$q_j = \frac{e_j}{\frac{1}{T-m} \sum_{t=m+1}^T |y_t - y_{t-m}|}$$

- $y_t$  = observation for period  $t$
- $e_j$  = forecast error for forecast horizon  $j$
- $T$  = size of training set
- $m = 7$

# Performance evaluation

$$\text{MSSE} = \text{mean}(q_j^2)$$

$$q_j^2 = \frac{e_j^2}{\frac{1}{T-m} \sum_{t=m+1}^T (y_t - y_{t-m})^2}$$

- $y_t$  = observation for period  $t$
- $e_j$  = forecast error for forecast horizon  $j$
- $T$  = size of training set
- $m = 7$

# Performance evaluation

$$\text{CRPS} = \text{mean}(p_j)$$

$$p_j = \int_{-\infty}^{\infty} (G_j(x) - F_j(x))^2 dx,$$

- $G_j(x)$  = forecast distribution for forecast horizon  $j$
- $F_j(x)$  = true distribution for same period

# Forecast accuracy



# Forecast accuracy: 43–84 days ahead

		MSSE			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	1.169	1.056	1.062	1.031
Base	ETS	0.979	0.875	0.816	<b>0.975</b>
Base	GLM	0.813	0.897	0.875	1.009
Base	TSGLM	0.822	0.901	0.875	1.050
Base	Ensemble	0.599	0.729	0.774	0.993
MinT	Naïve	1.168	1.057	1.062	2.095
MinT	ETS	0.785	0.852	0.845	0.994
MinT	GLM	0.720	0.827	0.837	1.803
MinT	TSGLM	0.722	0.833	0.839	1.851
MinT	Ensemble	<b>0.560</b>	<b>0.706</b>	<b>0.765</b>	1.557

# Forecast accuracy: 43–84 days ahead

		MASE			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	1.139	1.059	1.047	1.019
Base	ETS	0.963	0.930	0.899	1.038
Base	GLM	0.910	0.940	0.923	<b>1.002</b>
Base	TSGLM	0.911	0.939	0.924	1.005
Base	Ensemble	0.782	0.856	0.876	1.008
MinT	Naïve	1.138	1.059	1.047	2.651
MinT	ETS	0.877	0.916	0.915	1.289
MinT	GLM	0.848	0.901	0.902	2.493
MinT	TSGLM	0.852	0.903	0.903	2.513
MinT	Ensemble	<b>0.753</b>	<b>0.844</b>	<b>0.872</b>	2.260

# Forecast accuracy: 43–84 days ahead

		CRPS			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	30.387	10.882	5.500	0.302
Base	ETS	14.309	6.074	3.476	0.244
Base	GLM	15.396	6.253	3.576	0.244
Base	TSGLM	15.316	6.227	3.575	0.245
Base	Ensemble	12.978	<b>5.727</b>	3.430	0.243
MinT	Naïve	30.368	10.902	5.498	0.313
MinT	ETS	13.515	5.967	3.547	<b>0.243</b>
MinT	GLM	13.839	5.917	3.453	0.246
MinT	TSGLM	14.000	5.947	3.455	0.248
MinT	Ensemble	<b>12.585</b>	5.728	<b>3.426</b>	0.247

# Conclusions

- Ensemble mixture distributions give better forecasts than any component methods.
- Forecast reconciliation improves forecast accuracy, even when some component methods are quite poor.
- The ensemble without the Naïve method was worse.
- Forecast reconciliation allows coordinated planning and resource allocation.

# Outline

- 1 Definition of probabilistic coherence
- 2 Evaluating probabilistic forecasts
- 3 Emergency Services Demand
- 4 Evaluating multivariate probabilistic forecasts
- 5 Example: Australian electricity generation

# Evaluating probabilistic forecasts

## Continuous Rank Probability Score (univariate forecasts)

Forecast distribution  $F_t$  and observation  $y_t$ .

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t) dp = E_F|Y - y_t| - \frac{1}{2}E_F|Y - Y^*|$$

- $Y$  and  $Y^*$  are iid draws from  $F_t$ .
- Optimal when  $F_t$  is truth (i.e., it is a proper score)

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## Energy score (multivariate forecasts)

$$\text{ES}(F_t, \mathbf{y}_t) = E_F||\mathbf{Y} - \mathbf{y}_t|| - \frac{1}{2}E_F||\mathbf{Y} - \mathbf{Y}^*||$$

# Evaluating probabilistic forecasts

## Continuous Rank Probability Score (univariate forecasts)

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## Energy score (multivariate forecasts)

$$\text{ES}(F_t, \mathbf{y}_t) = E_F\|\mathbf{Y} - \mathbf{y}_t\| - \frac{1}{2}E_F\|\mathbf{Y} - \mathbf{Y}^*\|$$

## Log score (multivariate forecasts)

$$\text{LS}(F_t, \mathbf{y}_t) = -\log f(\mathbf{y}_t)$$

# Evaluating probabilistic forecasts

## Proper scoring rule

optimized when true forecast distribution is used.

# Evaluating probabilistic forecasts

## Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule	Coherent v Incoherent	Coherent v Coherent
Log Score	Not proper	<ul style="list-style-type: none"><li>Ordering preserved if compared using bottom-level only</li></ul>
Energy Score	Proper	<ul style="list-style-type: none"><li>Full hierarchy should be used.</li><li>Rankings may change otherwise.</li></ul>

# Score optimal reconciliation

Algorithm proposed by Panagiotelis et al (2020) for optimizing  $\mathbf{G}$  using stochastic gradient descent to optimize Energy Score.

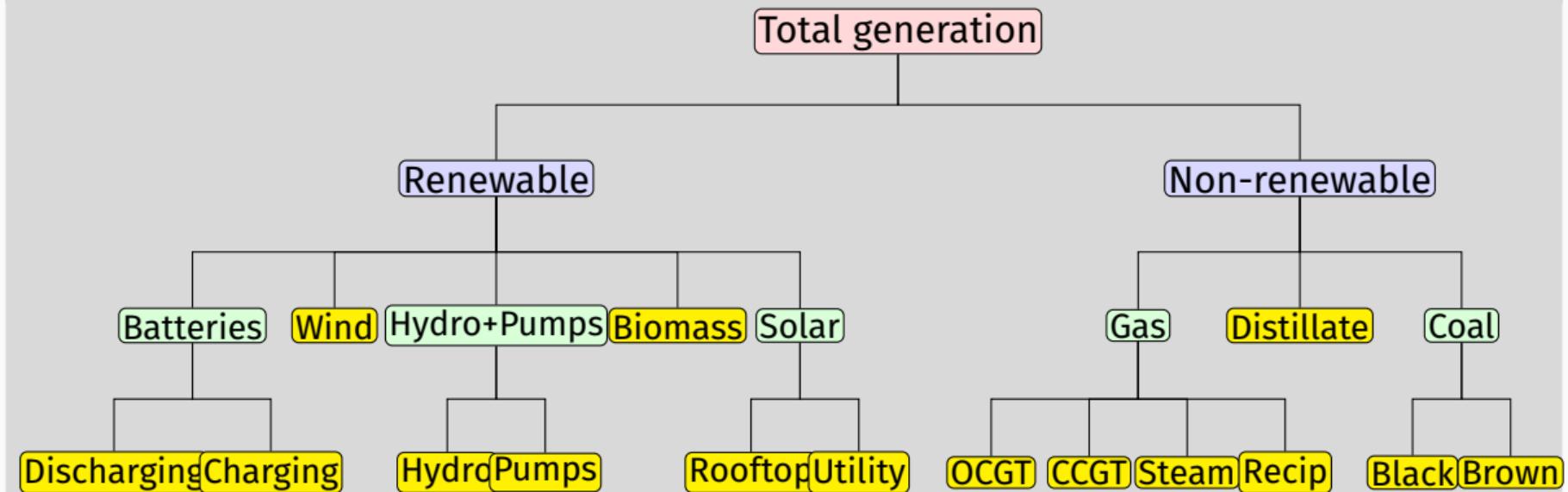
- 1 Compute base forecasts over a test set.
- 2 Compute OLS reconciliation:  $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
- 3 Iteratively update  $\mathbf{G}$  using SGD with Adam method and ES objective over a test set

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# Example: Australian electricity generation

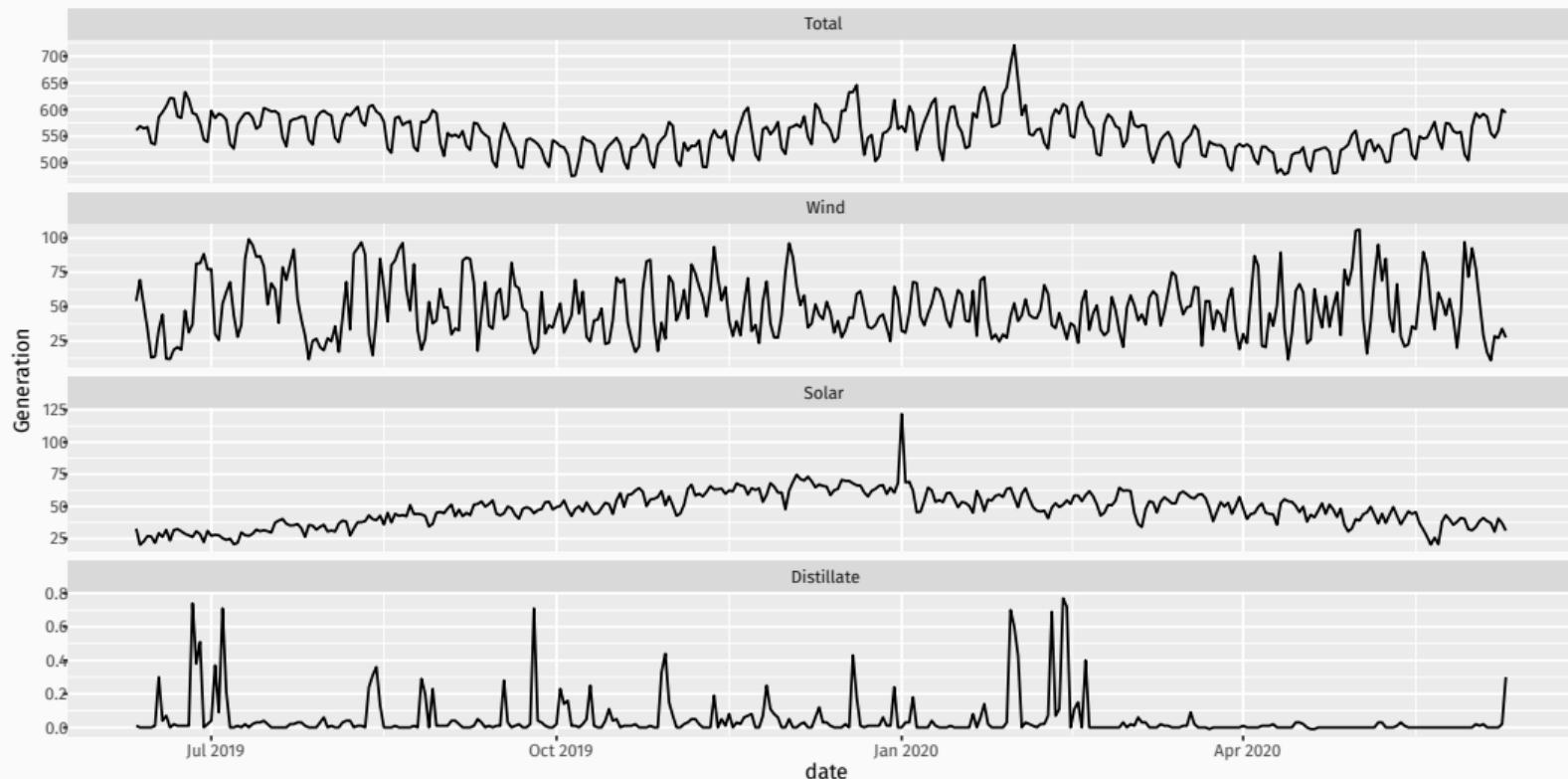
Daily time series from [opennem.org.au](https://opennem.org.au)



$n = 23$  series

$m = 15$  bottom-level series

# Example: Australian electricity generation

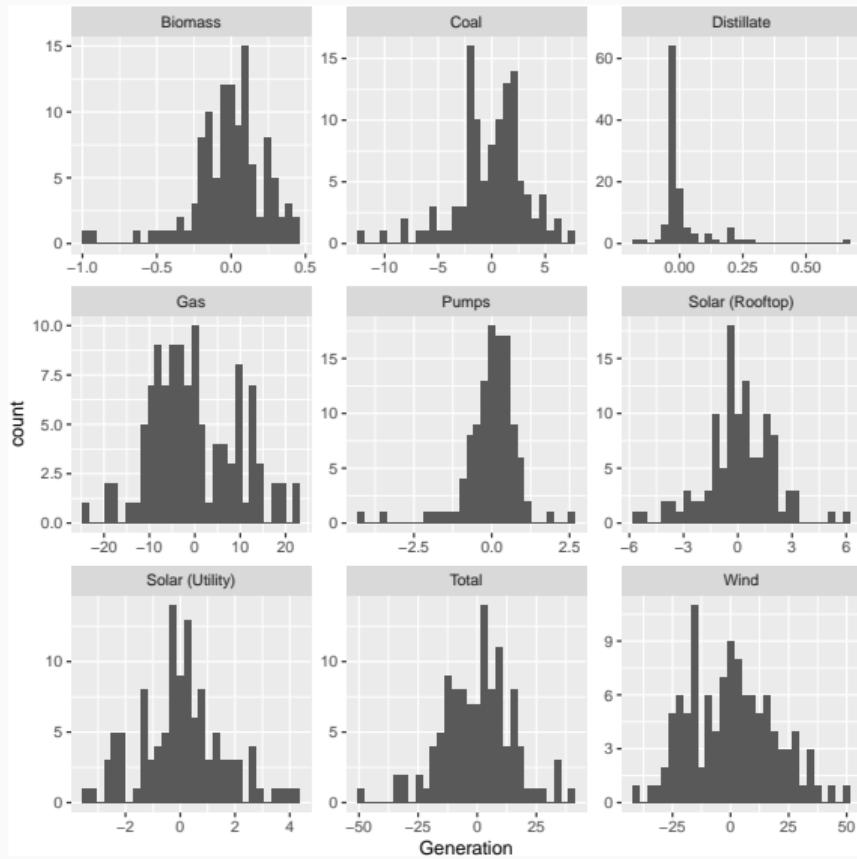


# Example: Australian electricity generation

## Forecast evaluation

- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using NNETAR() function in fable package.
- Model could be improved with temperature predictor.

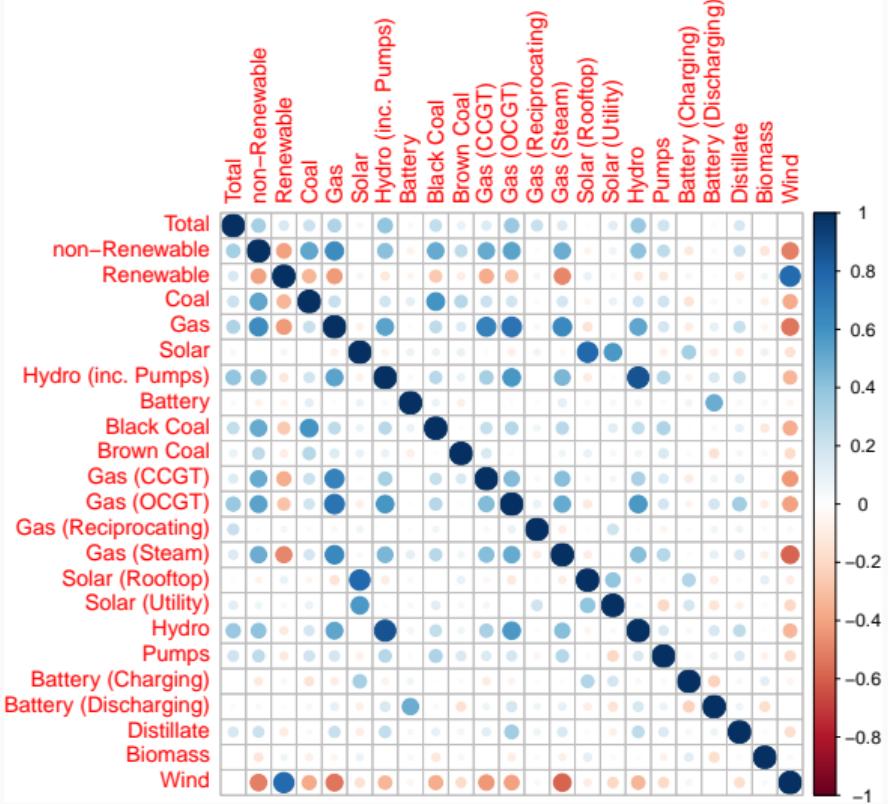
# Example: Australian electricity generation



Histogram of residuals:  
2 Oct 2019 – 21 Jan 2020

Clearly non-Gaussian

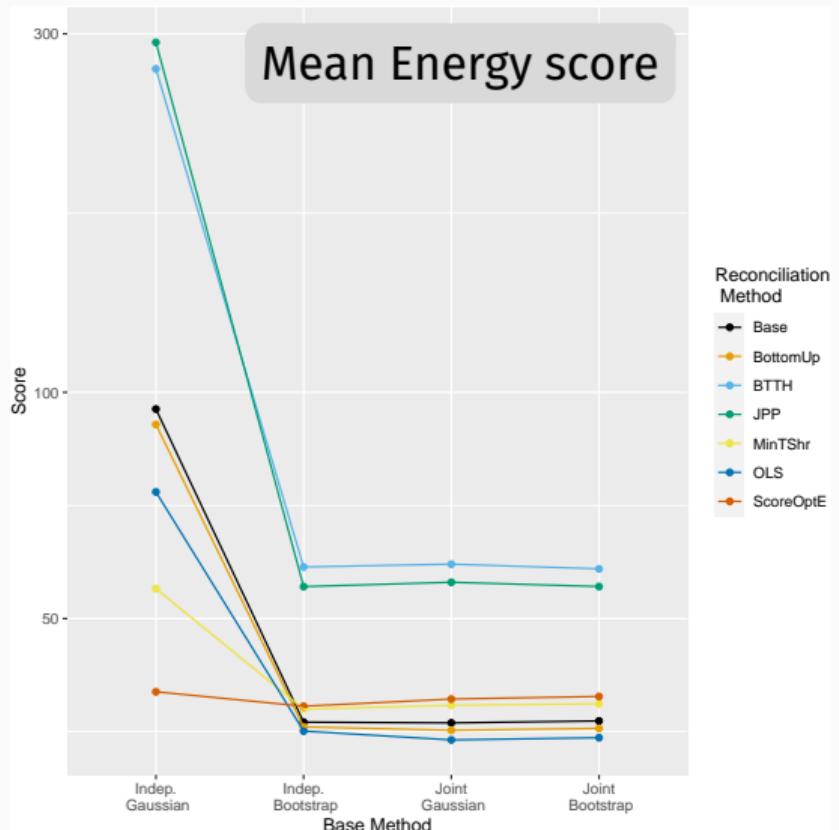
# Example: Australian electricity generation



**Correlations of residuals:  
2 Oct 2019 – 21 Jan 2020**

Blue = positive correlation.  
Red = negative correlation.  
Large = stronger correlations.

# Example: Australian electricity generation



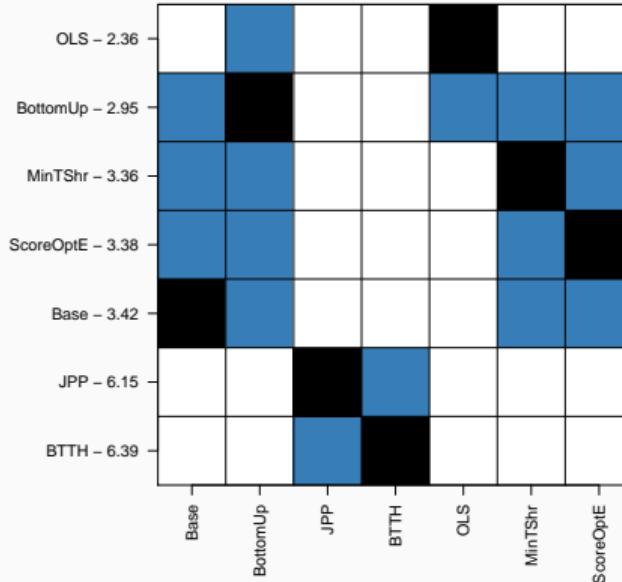
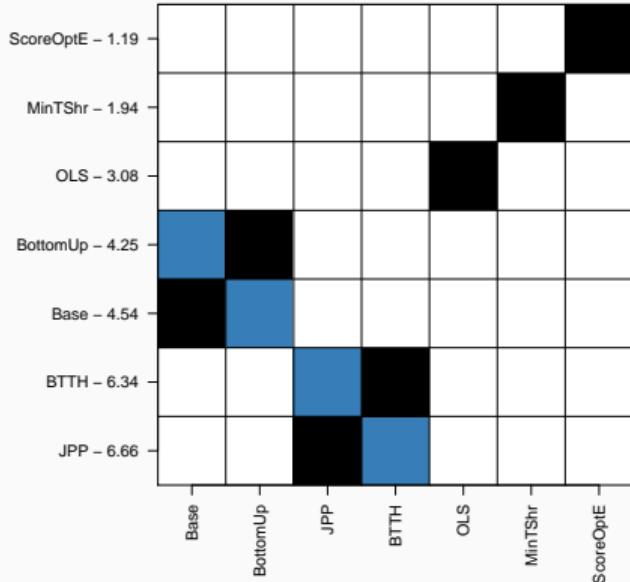
## Base residual assumptions

- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

## Reconciliation methods

- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS
- MinT(Shrink)
- Score Optimal Reconciliation

# Example: Australian electricity generation



Nemenyi test for different scores

Base forecasts are independent and Gaussian.

Nemenyi test for different scores

Base forecasts are obtained by jointly bootstrapping residuals.

# References

-  Ben Taieb, S, JW Taylor, and RJ Hyndman (2021). Hierarchical Probabilistic Forecasting of Electricity Demand with Smart Meter Data. *J American Statistical Association* **116**(533), 27–43.
-  Panagiotelis, A, P Gamakumara, G Athanasopoulos, and RJ Hyndman (2023). Probabilistic forecast reconciliation: properties, evaluation and score optimisation. *European J Operational Research* **306**(2), 693–706.
-  Rostami-Tabar, B and RJ Hyndman (2023). “Hierarchical Time Series Forecasting in Emergency Medical Services”. [robjhyndman.com/publications/fem](http://robjhyndman.com/publications/fem).