Forecast reconciliation

A brief overview

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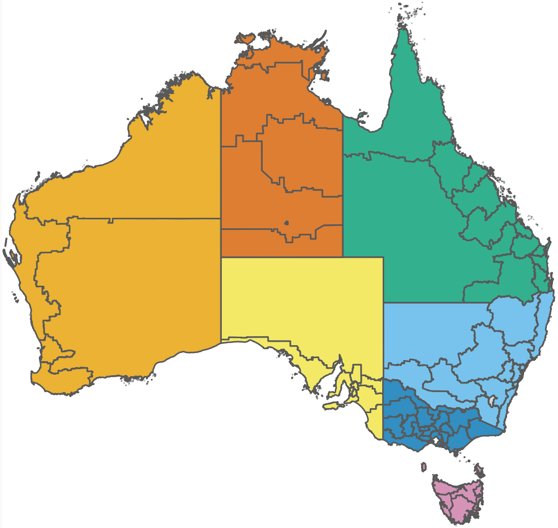
Outline

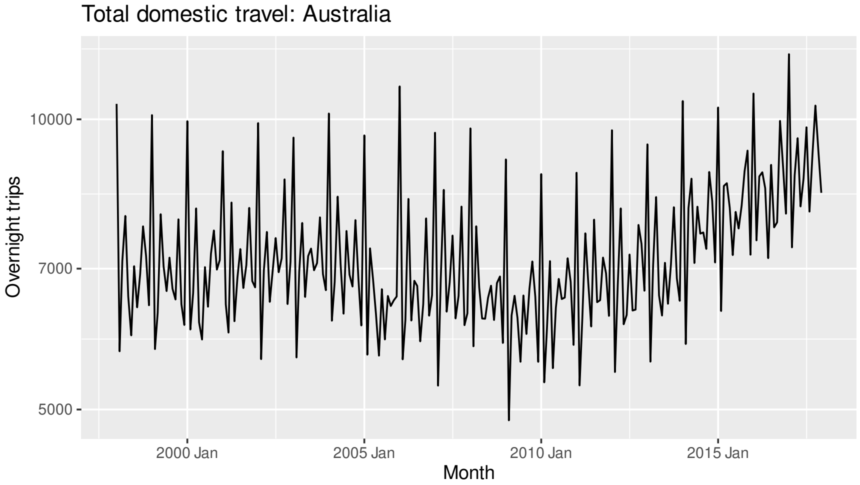
Outline

Australian tourism regions

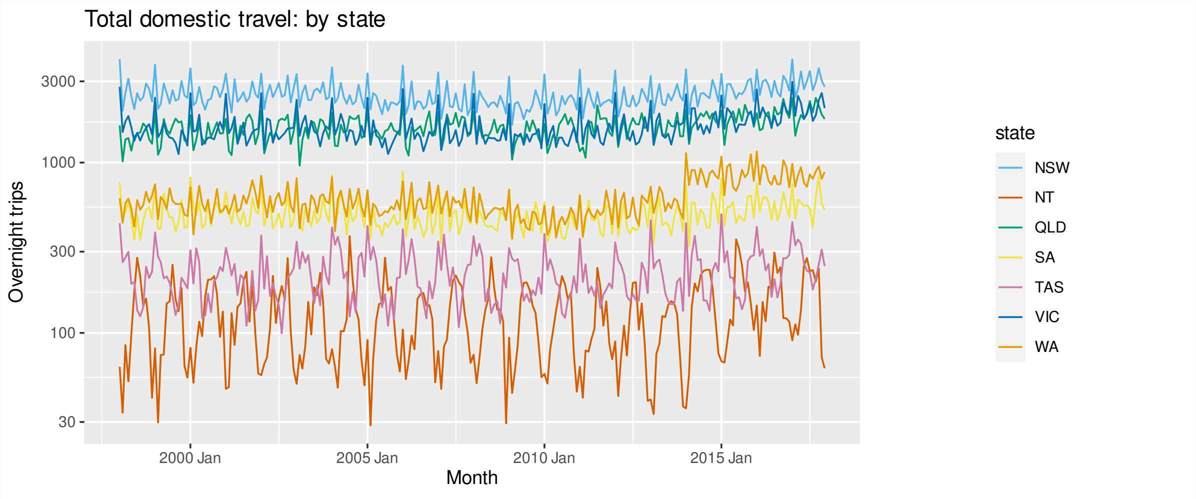


- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - 7 states
  - 27 zones
  - 75 regions

Australian tourism data



Australian tourism data



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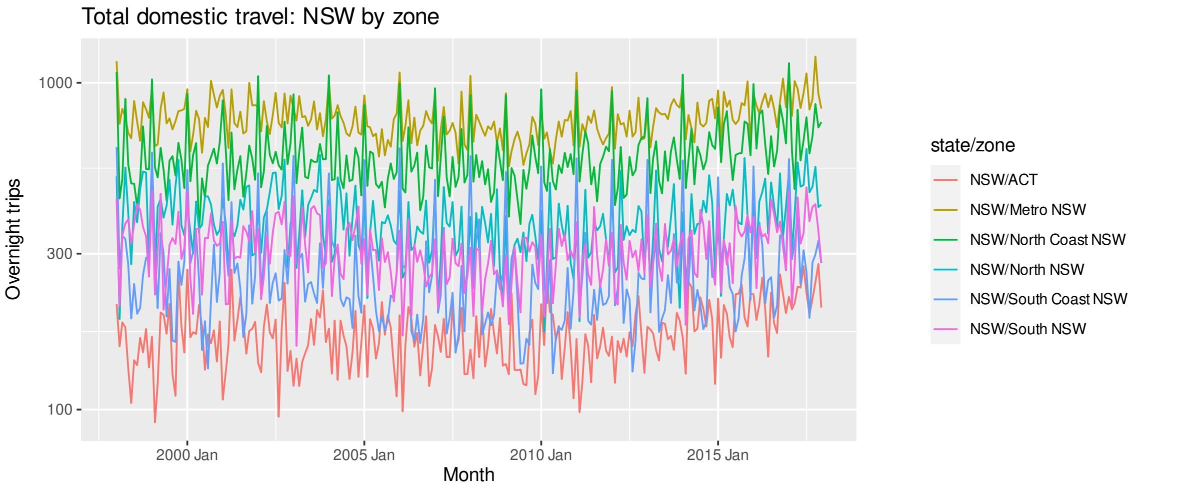

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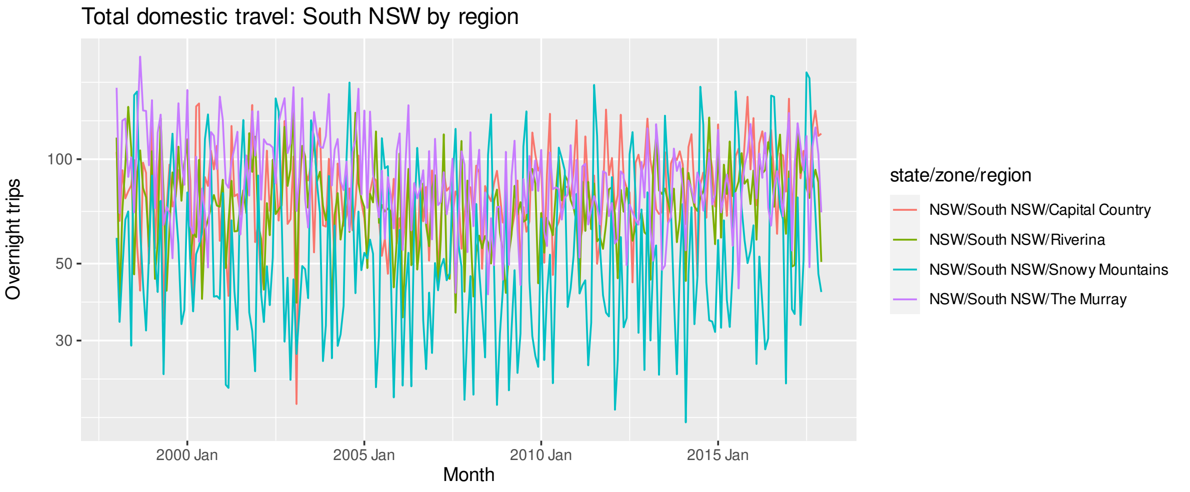
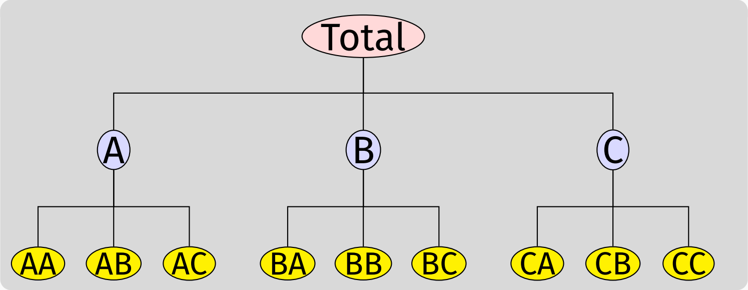


Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



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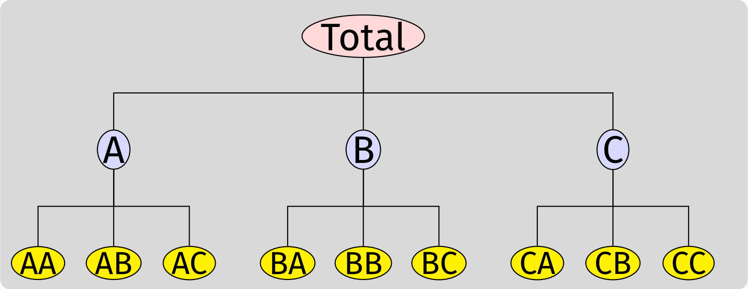
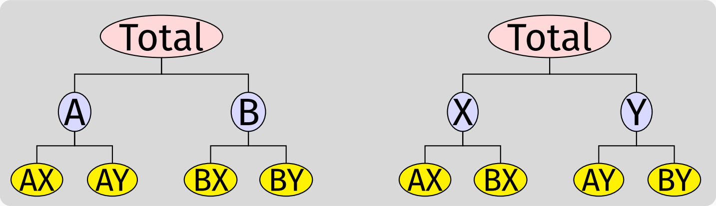


Examples

- Tourism by state and region
- Retail sales by product groups, sub groups, and SKUs

Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



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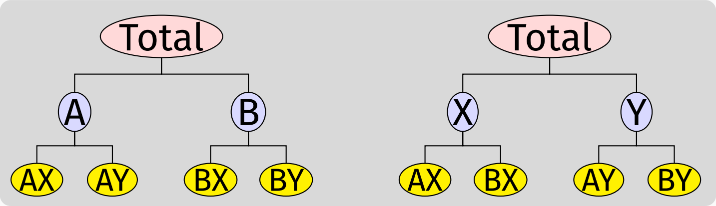
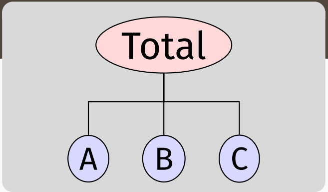


Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Hierarchical and grouped time series



Almost all collections of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "structural matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

The forecast reconciliation problem

- We want "coherent" forecasts of all series — i.e., the forecasts should obey the same constraints as the data.
- We model and forecast all series independently. The resulting forecasts will almost certainly be incoherent.
- We "reconcile" the forecasts to make them coherent.

Notation

- $\boldsymbol{y}_t$ = vector of data for all series at time t.
- $\hat{\boldsymbol{y}}_{t+h|t}$ = vector of incoherent "base" forecasts for time $t+h$ using data to time t.
- $\tilde{\boldsymbol{y}}_{t+h|t}$ = vector of coherent "reconciled" forecasts for time $t+h$ using data to time t.

Outline

1. Hierarchical time series notation
2. Optimal linear forecast reconciliation
3. Probabilistic forecast reconciliation
4. Cross-temporal probabilistic forecast reconciliation
5. Final comments

The coherent subspace

Coherent subspace

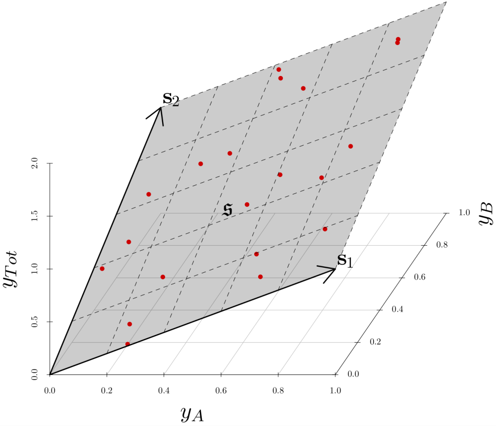
n_b-dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

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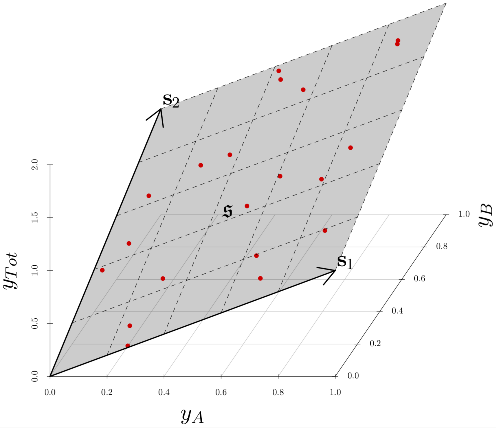
$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



$y_{Tot} = y_A + y_B$

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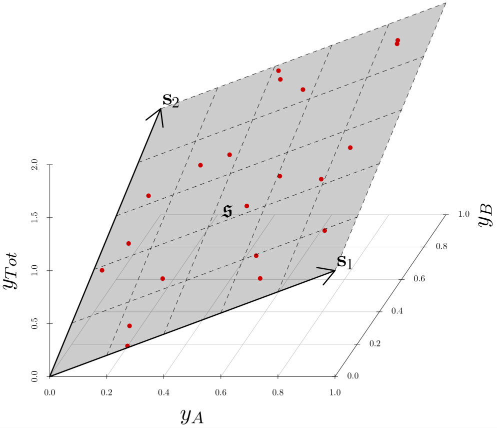

$y_{Tot} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \chi^n \rightarrow \mathfrak{s}$.
$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t}) = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

- If $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$, then projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.
- Coherent base forecasts are unchanged since $\boldsymbol{M}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}$
- If $\hat{\boldsymbol{y}}$ is unbiased, then $\tilde{\boldsymbol{y}}$ is also unbiased.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$ is the covariance matrix of the base forecast errors.
- $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]) = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$ is the covariance matrix of the reconciled forecast errors.
- How to choose the best Ψ?

Minimum trace reconciliation

Wickramasuriya et al (2019)

Minimum trace (MinT) reconciliation

If $\boldsymbol{M}$ is a projection, then trace of $\boldsymbol{V}_h$ is minimized when $\Psi = \boldsymbol{W}_h$, so that

$$\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$$

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\,\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciled forecasts　　Base forecasts

- Trace of $\boldsymbol{V}_h$ is sum of forecast variances.
- MinT is L_2 optimal amongst linear unbiased forecasts.
- How to estimate $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$?

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciliation method	$\boldsymbol{G}$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS(var)	$(\boldsymbol{S}'\Lambda_v\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_v$
WLS(struct)	$(\boldsymbol{S}'\Lambda_s\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_s$
MinT(sample)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}$
MinT(shrink)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\boldsymbol{W}_h = k_h\boldsymbol{W}_1$.

- $\Lambda_v = \text{diag}(\boldsymbol{W}_1)^{-1}$
- $\Lambda_s = \text{diag}(\boldsymbol{S}\mathbf{1})^{-1}$
- $\hat{\boldsymbol{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_{\text{sam}}) + (1-\tau)\hat{\boldsymbol{W}}_{\text{sam}}$ where τ selected optimally.
- Still need a good estimate of $\boldsymbol{W}_h$ for forecast variance.

Example: Australian tourism

```r
tourism
```

```
# A tsibble: 18,000 x 5 [1M]
# Key:       state, zone, region [75]
      month state zone      region visitors
      <mth> <chr> <chr>     <chr>     <dbl>
 1 1998 Jan NSW   Metro NSW Sydney     926.
 2 1998 Feb NSW   Metro NSW Sydney     647.
 3 1998 Mar NSW   Metro NSW Sydney     716.
 4 1998 Apr NSW   Metro NSW Sydney     621.
 5 1998 May NSW   Metro NSW Sydney     598.
 6 1998 Jun NSW   Metro NSW Sydney     601.
 7 1998 Jul NSW   Metro NSW Sydney     720.
 8 1998 Aug NSW   Metro NSW Sydney     645.
 9 1998 Sep NSW   Metro NSW Sydney     633.
10 1998 Oct NSW   Metro NSW Sydney     771.
# i 17,990 more rows
```

Example: Australian tourism

```
tourism_agg <- tourism |>
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
# A tsibble: 26,400 x 5 [1M]
# Key:       state, zone, region [110]
      month state        zone         region       visitors
      <mth> <chr*>       <chr*>       <chr*>          <dbl>
 1 1998 Jan <aggregated> <aggregated> <aggregated>   10376.
 2 1998 Feb <aggregated> <aggregated> <aggregated>    5746.
 3 1998 Mar <aggregated> <aggregated> <aggregated>    7129.
 4 1998 Apr <aggregated> <aggregated> <aggregated>    7939.
 5 1998 May <aggregated> <aggregated> <aggregated>    6552.
 6 1998 Jun <aggregated> <aggregated> <aggregated>    5969.
 7 1998 Jul <aggregated> <aggregated> <aggregated>    7041.
 8 1998 Aug <aggregated> <aggregated> <aggregated>    6382.
 9 1998 Sep <aggregated> <aggregated> <aggregated>    6907.
10 1998 Oct <aggregated> <aggregated> <aggregated>    7744.
# i 26,390 more rows
```

Example: Australian tourism

```r
fit <- tourism_agg |>
  filter(year(month) <= 2015) |>
  model(ets = ETS(visitors))
```

```
# A mable: 110 x 4
# Key:     state, zone, region [110]
   state  zone            region                     ets
   <chr*> <chr*>          <chr*>                 <model>
 1 NSW    ACT             Canberra          <ETS(M,N,A)>
 2 NSW    ACT             <aggregated>      <ETS(M,N,A)>
 3 NSW    Metro NSW       Central Coast     <ETS(M,N,M)>
 4 NSW    Metro NSW       Sydney            <ETS(M,N,A)>
 5 NSW    Metro NSW       <aggregated>      <ETS(M,N,A)>
 6 NSW    North Coast NSW Hunter            <ETS(M,N,M)>
 7 NSW    North Coast NSW North Coast NSW   <ETS(M,N,M)>
 8 NSW    North Coast NSW <aggregated>      <ETS(M,N,M)>
 9 NSW    North NSW       Blue Mountains    <ETS(M,N,A)>
10 NSW    North NSW       Central NSW       <ETS(M,N,M)>
```

Example: Australian tourism

```r
fc <- fit |>
  reconcile(
    ols = min_trace(ets, method = "ols"),
    wlsv = min_trace(ets, method = "wls_var"),
    wlss = min_trace(ets, method = "wls_struct"),
    # mint_c = min_trace(ets, method="mint_cov"),
    mint_s = min_trace(ets, method = "mint_shrink"),
  ) |>
  forecast(h = "2 years")
```

```
# A fable: 13,200 x 7 [1M]
# Key:     state, zone, region, .model [550]
   state  zone   region   .model    month      visitors .mean
   <chr*> <chr*> <chr*>   <chr>     <mth>        <dist> <dbl>
 1 NSW    ACT    Canberra ets    2016 Jan N(202, 1437)  202.
 2 NSW    ACT    Canberra ets    2016 Feb  N(160, 912)  160.
 3 NSW    ACT    Canberra ets    2016 Mar N(204, 1489)  204.
 4 NSW    ACT    Canberra ets    2016 Apr N(207, 1527)  207.
```

Example: Australian tourism

```r
fc |>
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>
  group_by(.model) |>
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>
  arrange(rmsse)
```

```
# A tibble: 5 x 3
  .model  mase rmsse
  <chr>  <dbl> <dbl>
1 ols    0.930 0.926
2 wlss   0.949 0.948
3 mint_s 0.953 0.954
4 wlsv   0.964 0.965
5 ets    0.968 0.968
```

Example: Australian tourism

```r
fc |>
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3 mint_s 0.953 0.954
4 wlsv   0.964 0.965
5 ets    0.968 0.968
```

- Overall, every reconciliation method is better than the base ETS forecasts.

Example: Australian tourism

```
# A tibble: 20 x 4
# Groups:   .model [5]
   .model level     mase rmsse
   <chr>  <fct>    <dbl> <dbl>
 1 ets    National 1.44  1.27
 2 ols    National 1.46  1.29
 3 wlss   National 1.61  1.43
 4 mint_s National 1.64  1.45
 5 wlsv   National 1.69  1.49
 6 ols    State    1.07  1.08
 7 ets    State    1.10  1.11
 8 wlss   State    1.13  1.14
 9 mint_s State    1.15  1.15
10 wlsv   State    1.18  1.17
11 ols    Zone     0.954 0.948
12 wlss   Zone     0.987 0.980
13 mint_s Zone     0.995 0.988
14 ets    Zone     1.01  0.999
15 wlsv   Zone     1.01  1.00
16 ols    Region   0.901 0.895
17 wlss   Region   0.910 0.907
18 mint_s Region   0.911 0.911
19 wlsv   Region   0.917 0.919
20 ets    Region   0.935 0.938
```

Example: Australian tourism

```
# A tibble: 20 x 4
# Groups:   .model [5]
   .model level     mase rmsse
   <chr>  <fct>    <dbl> <dbl>
 1 ets    National 1.44  1.27
 2 ols    National 1.46  1.29
 3 wlss   National 1.61  1.43
 4 mint_s National 1.64  1.45
 5 wlsv   National 1.69  1.49
 6 ols    State    1.07  1.08
 7 ets    State    1.10  1.11
 8 wlss   State    1.13  1.14
 9 mint_s State    1.15  1.15
10 wlsv   State    1.18  1.17
11 ols    Zone     0.954 0.948
12 wlss   Zone     0.987 0.980
13 mint_s Zone     0.995 0.988
14 ets    Zone     1.01  0.999
15 wlsv   Zone     1.01  1.00
16 ols    Region   0.901 0.895
17 wlss   Region   0.910 0.907
18 mint_s Region   0.911 0.911
19 wlsv   Region   0.917 0.919
20 ets    Region   0.935 0.938
```

- OLS is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

Outline

The coherent subspace

Coherent subspace

m-dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

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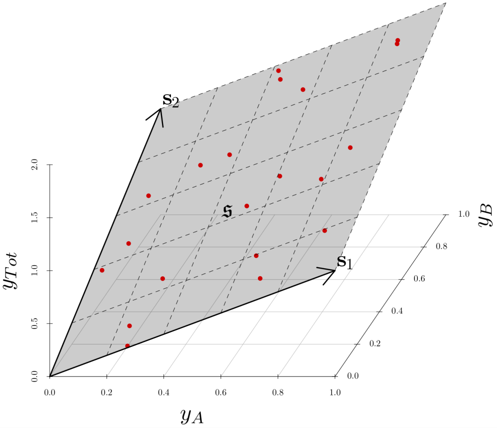

$y_{Tot} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

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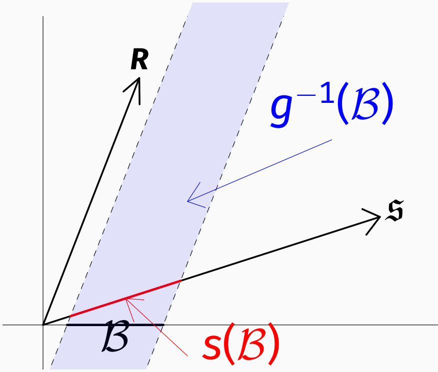
Coherent probabilistic forecasts

Coherent probabilistic forecasts

A probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is coherent with the bottom probability triple $(\chi^m, \mathscr{F}_{\chi^m}, \nu)$, if

$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\chi^m}$$

- Random draws from coherent distribution must lie on $\mathfrak{s}$.
- The probability of points not on $\mathfrak{s}$ is zero.
- The reconciled distribution is a transformation of the base forecast distribution that is coherent on $\mathfrak{s}$.



$$\psi = s \circ g$$

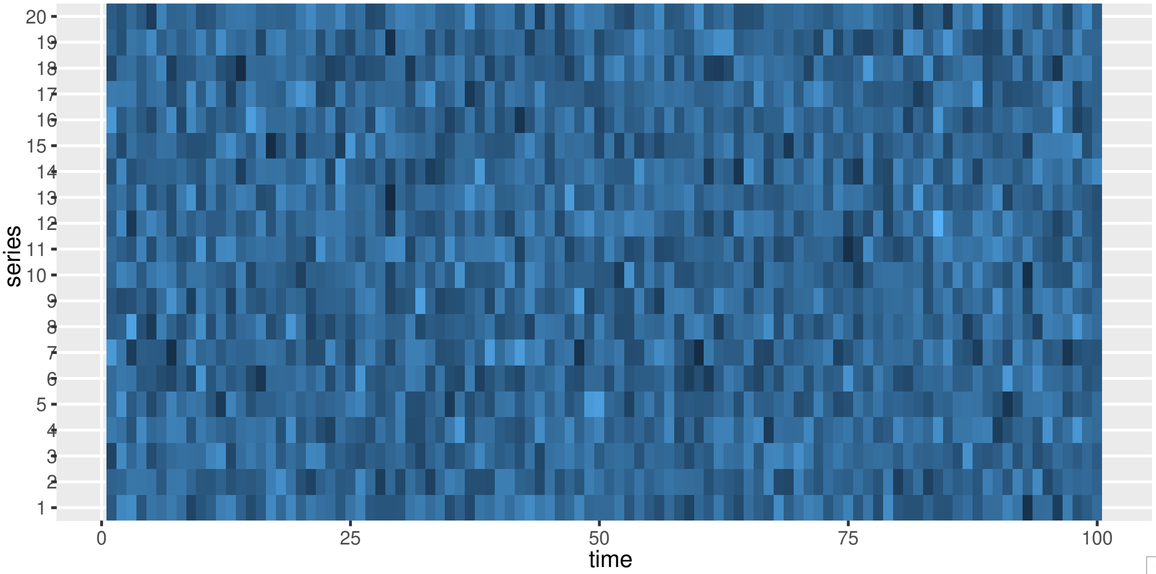
Panagiotelis et al. (2023) and Corani, Azzimonti, and Rubattu (2024)

Simulation from a reconciled distribution

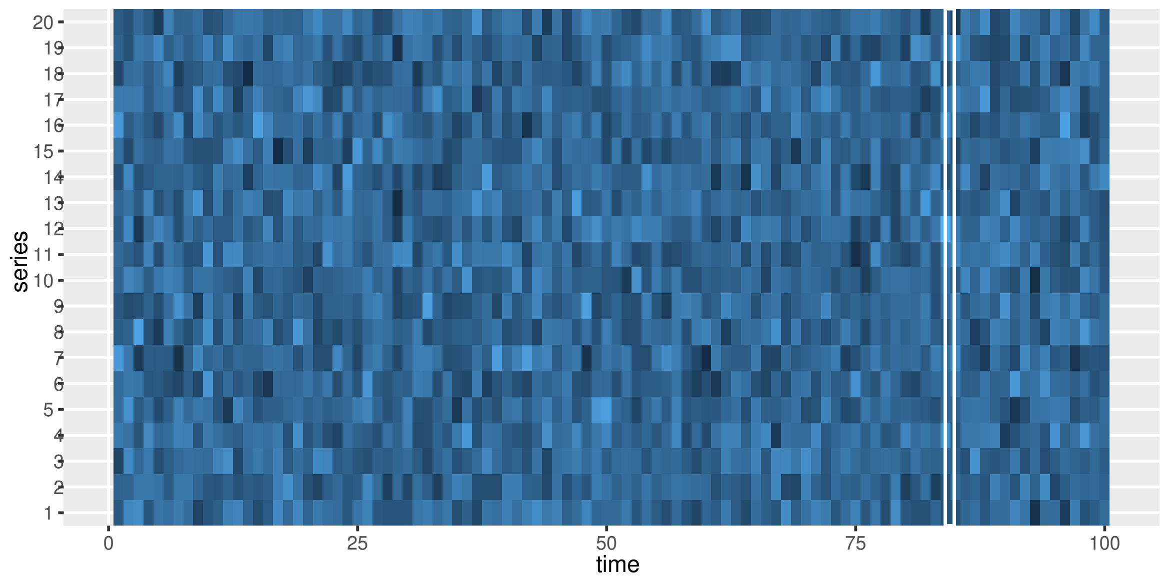
Suppose that $(\hat{\boldsymbol{y}}^{[1]},\dots,\hat{\boldsymbol{y}}^{[L]})$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $(\tilde{\boldsymbol{y}}^{[1]},\dots,\tilde{\boldsymbol{y}}^{[L]})$ where $\tilde{\boldsymbol{y}}^{[\ell]} := \psi(\hat{\boldsymbol{y}}^{[\ell]})$ for $\ell = 1,\dots,L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

- Simulate future sample paths for each series, by simulating from each model using a multivariate bootstrap of the residuals (to preserve cross-correlations).
- Reconcile the sample paths.
- The reconciled sample paths are a sample from the reconciled distribution.

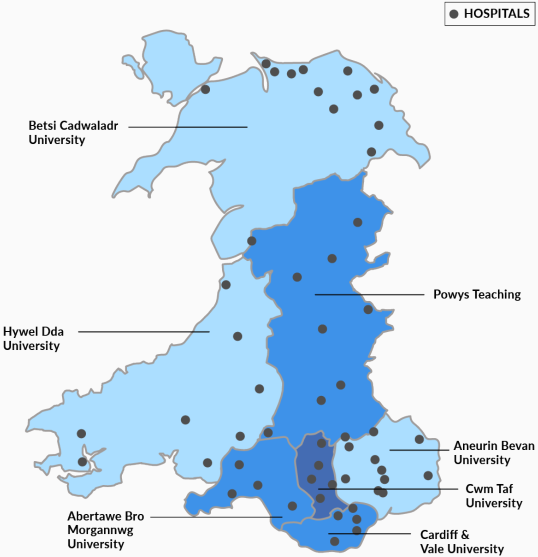
Simulation from a reconciled distribution



Simulation from a reconciled distribution



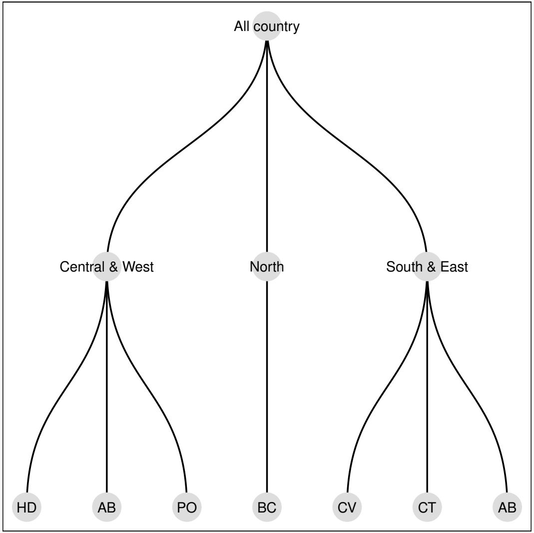
Wales Health Board Areas



Data

- Daily number of attended incidents:
  1 October 2015 – 31 July 201
  Disaggregated by: * control area (x3) * health board (x7) priority (x3) * nature of incidents (x35) * 2,142,000 rows observations from 1,53 time series.

Data structure



*

Priority

Red
Amber
Green

*

Nature of incident

Chest pain
Stroke
Breathing problem
...
Abdominal pain

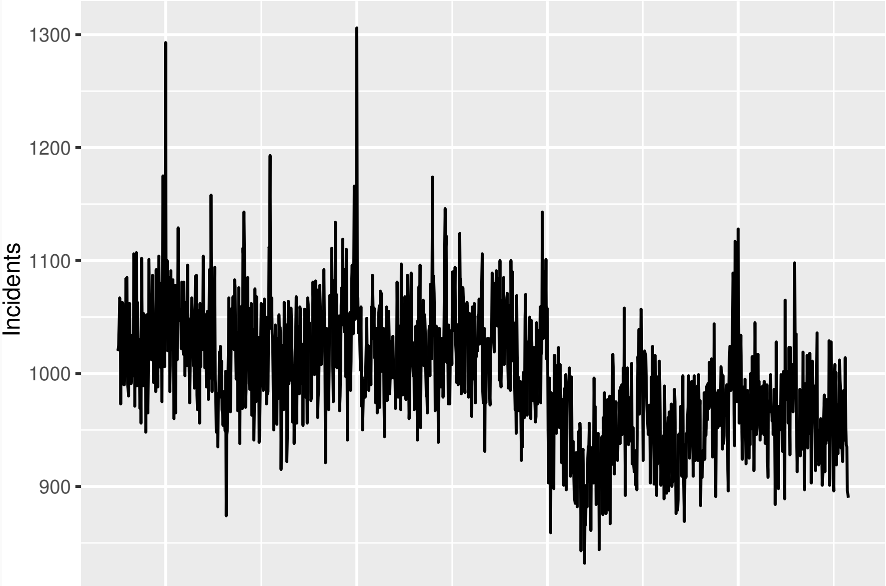
Data

```
# A tsibble: 2,142,000 x 6 [1D]
# Key:       region, category, nature, lhb [1,530]
   date       region       category     nature       lhb          incident
   <date>     <chr*>       <chr*>       <chr*>       <chr*>          <dbl>
 1 2015-10-01 <aggregated> <aggregated> <aggregated> <aggregated>     1020
 2 2015-10-02 <aggregated> <aggregated> <aggregated> <aggregated>     1021
 3 2015-10-03 <aggregated> <aggregated> <aggregated> <aggregated>     1025
 4 2015-10-04 <aggregated> <aggregated> <aggregated> <aggregated>     1043
 5 2015-10-05 <aggregated> <aggregated> <aggregated> <aggregated>     1067
 6 2015-10-06 <aggregated> <aggregated> <aggregated> <aggregated>     1063
 7 2015-10-07 <aggregated> <aggregated> <aggregated> <aggregated>      973
 8 2015-10-08 <aggregated> <aggregated> <aggregated> <aggregated>     1057
 9 2015-10-09 <aggregated> <aggregated> <aggregated> <aggregated>     1026
10 2015-10-10 <aggregated> <aggregated> <aggregated> <aggregated>     1063
# i 2,141,990 more rows
```

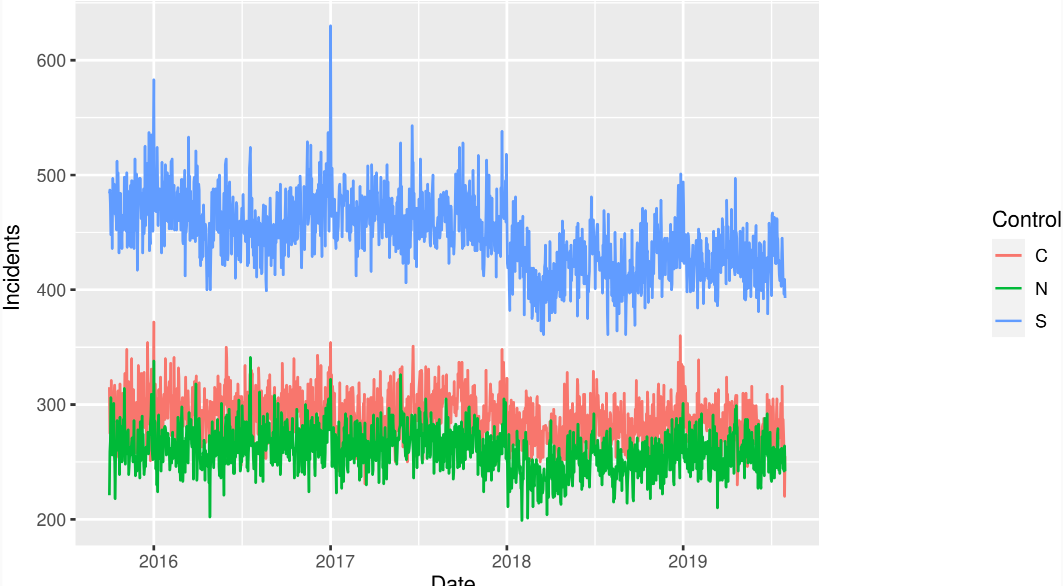
Data

```
# A tsibble: 2,142,000 x 6 [1D]
# Key:       region, category, nature, lhb [1,530]
   date       region category nature    lhb          incident
   <date>     <chr*> <chr*>   <chr*>    <chr*>          <dbl>
 1 2015-10-01 C      Amber    ABDOMINAL HD                  0
 2 2015-10-01 C      Amber    ABDOMINAL PO                  0
 3 2015-10-01 C      Amber    ABDOMINAL SB                  0
 4 2015-10-01 C      Amber    ABDOMINAL <aggregated>        0
 5 2015-10-01 C      Amber    ALLERGIES HD                  0
 6 2015-10-01 C      Amber    ALLERGIES PO                  1
 7 2015-10-01 C      Amber    ALLERGIES SB                  0
 8 2015-10-01 C      Amber    ALLERGIES <aggregated>        1
 9 2015-10-01 C      Amber    ANIMALBIT HD                  0
10 2015-10-01 C      Amber    ANIMALBIT PO                  0
# i 2,141,990 more rows
```

Aggregated daily incidents

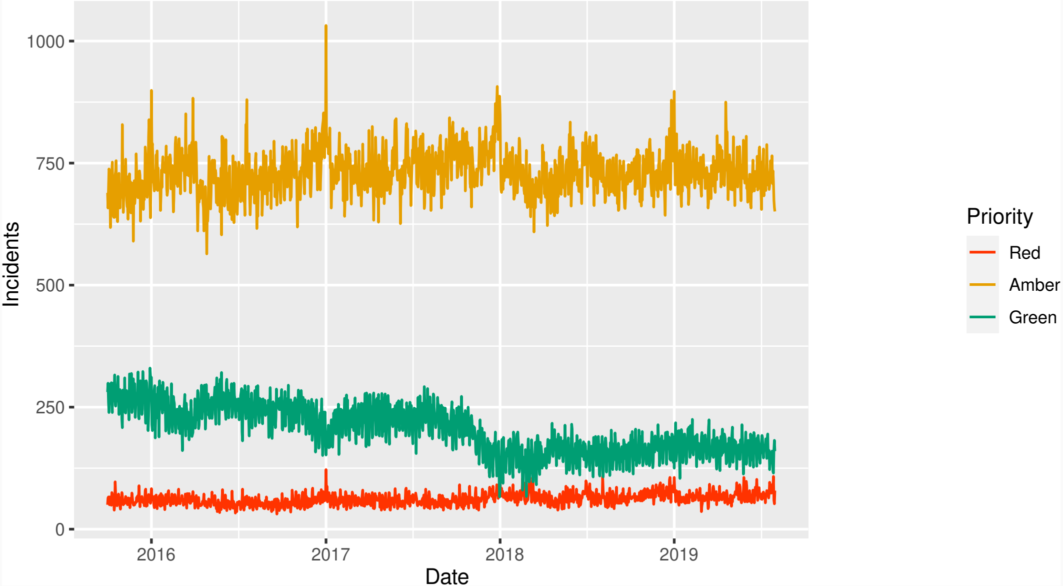


Daily incidents by control area

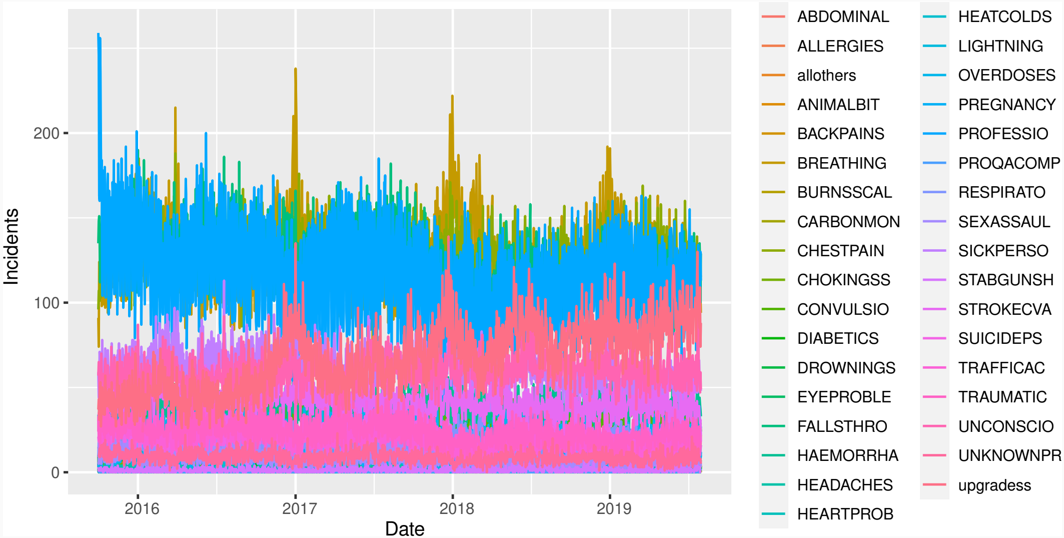


Data incidents by health board

Data incidents by priority



Data incidents by nature of incident



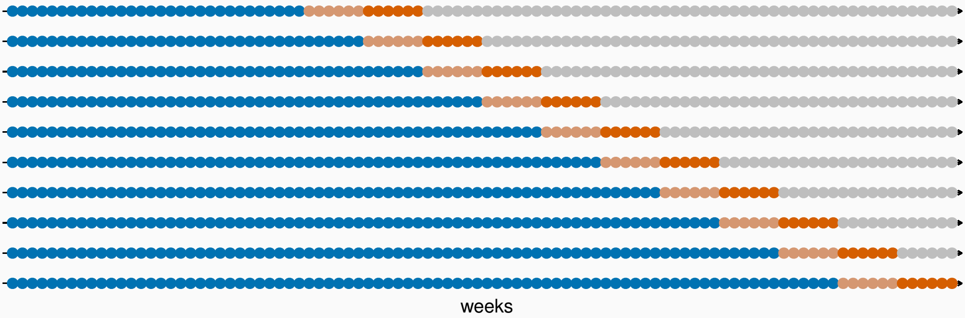
Data incidents by nature of incident

Forecasting methods

1. **Naïve**: Empirical distribution of past daily attended incidents.
2. **ETS**: Exponential Smoothing State Space models.
3. **GLM**: Poission Regression with spline trend, day of the week, annual Fourier seasonality, public holidays, school holidays, Christmas Day, New Year's Day.
4. **TSGLM**: Poisson Regression with same covariates plus three autoregressive terms.
5. **Ensemble**: Mixture distribution of 1–4.

Performance evaluation

- Ten-fold time series cross-validation
- Forecast horizon of 1–84 days
- Each training set contains an additional 42 days.
- Forecasts at 43–84 days correspond to planning horizon.



Forecast accuracy: 43–84 days ahead

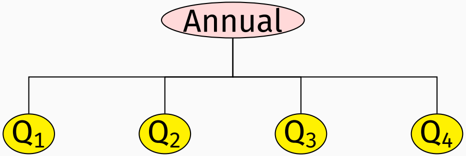
		CRPS			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	30.387	10.882	5.500	0.302
Base	ETS	14.309	6.074	3.476	0.244
Base	GLM	15.396	6.253	3.576	0.244
Base	TSGLM	15.316	6.227	3.575	0.245
Base	Ensemble	12.978	**5.727**	3.430	0.243
MinT	Naïve	30.368	10.902	5.498	0.313
MinT	ETS	13.515	5.967	3.547	**0.243**
MinT	GLM	13.839	5.917	3.453	0.246
MinT	TSGLM	14.000	5.947	3.455	0.248
MinT	Ensemble	**12.585**	5.728	**3.426**	0.247

Forecast accuracy

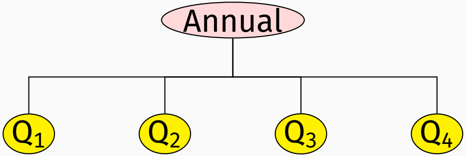
- Ensemble mixture distributions give better forecasts than any component methods.
- Forecast reconciliation improves forecast accuracy, even when some component methods are quite poor.
- The ensemble without the Naïve method was worse.
- Forecast reconciliation allows coordinated planning and resource allocation.

Outline

Temporal reconciliation: quarterly data

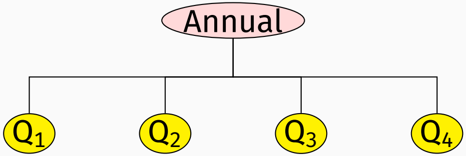


Temporal reconciliation: quarterly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: quarterly data

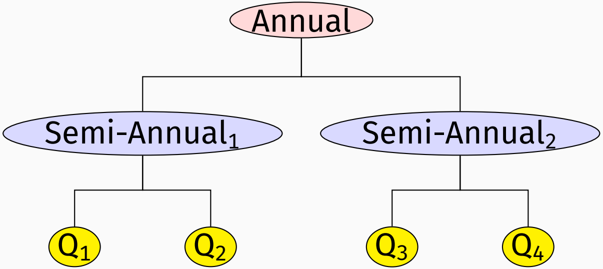


$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

τ = index of largest temporal aggregation level.

Temporal reconciliation: quarterly data

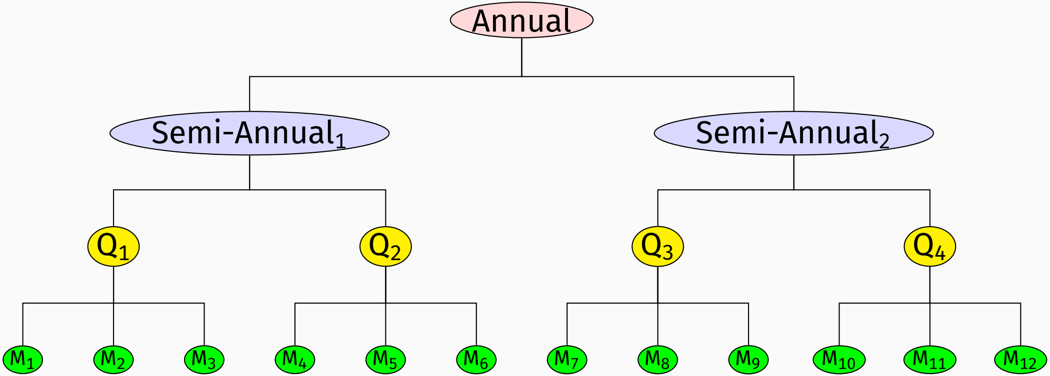


- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[2]} \\ x_{\tau,2}^{[2]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

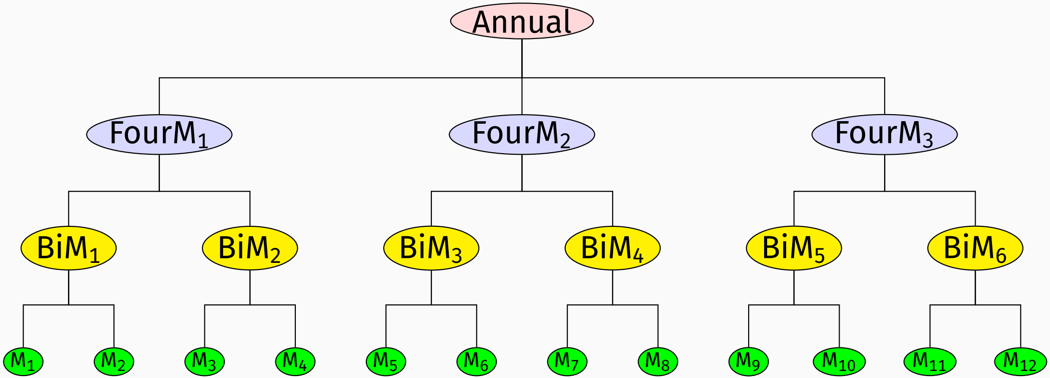
τ = index of largest temporal aggregation level.

Temporal reconciliation: monthly data



- ➥ Forecast series at each available frequency.
- ➥ Optimally combine forecasts within the same year.

Temporal reconciliation: monthly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: monthly data

$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[12]} \\ \boldsymbol{x}_\tau^{[6]} \\ \boldsymbol{x}_\tau^{[4]} \\ \boldsymbol{x}_\tau^{[3]} \\ \boldsymbol{x}_\tau^{[2]} \\ \boldsymbol{x}_\tau^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & & & & & & \\ & & & & & \boldsymbol{I}_{12} & & & & & & \end{bmatrix}$$

Temporal reconciliation

For a time series $y_1, \dots, y_T$, observed at frequency m:

$$x_j^{[k]} = \sum_{t=(j-1)k+1}^{jk} y_t \qquad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in \mathcal{K} = \{k_1, \dots, k_p\}$ denote the p factors of m in ascending order, where $k_1 = 1$ and $k_p = m$
- $x_j^{[1]} = y_t$
- A single unique hierarchy is only possible when there are no coprime pairs in $\mathcal{K}$.
- $M_k = m/k$ is seasonal period of aggregated series.

Temporal reconciliation

$$\boldsymbol{x}_\tau = \boldsymbol{S}\boldsymbol{x}_\tau^{[1]}, \qquad \boldsymbol{S} = \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{I} \end{bmatrix}$$

where

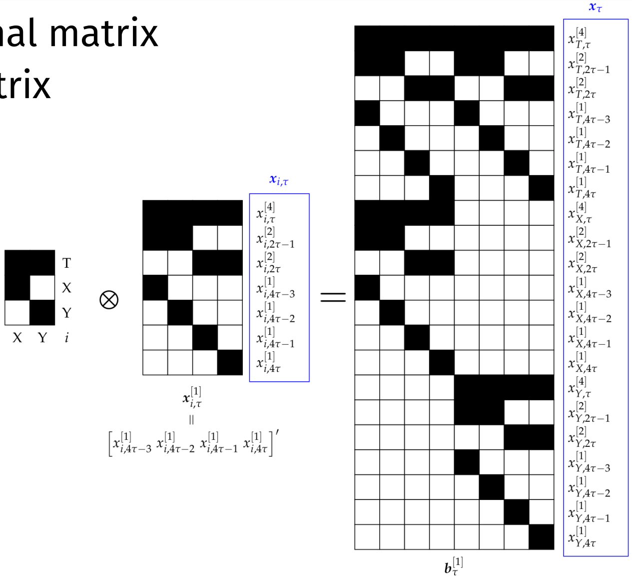
$$\boldsymbol{x}_\tau = \begin{bmatrix} x_\tau^{[k_p]} \\ \boldsymbol{x}_\tau^{[k_{p-1}]} \\ \vdots \\ \boldsymbol{x}_\tau^{[k_1]} \end{bmatrix} \qquad \boldsymbol{x}_\tau^{[k]} = \begin{bmatrix} x_{M_k(\tau-1)+1}^{[k]} \\ x_{M_k(\tau-1)+2}^{[k]} \\ \vdots \\ x_{M_k\tau}^{[k]} \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{1}'_m \\ \boldsymbol{I}_{m/k_{p-1}} \otimes \boldsymbol{1}'_{k_{p-1}} \\ \vdots \\ \boldsymbol{I}_{m/k_2} \otimes \boldsymbol{1}'_{k_2} \end{bmatrix}$$

τ is time index for most aggregated series,

$k \in \mathcal{K} = \{k_1, \dots, k_p\}, \quad k_1 = 1, \quad k_p = m, \quad \tau = 1, \dots, T/m.$

Cross-temporal forecast reconciliation

- $\boldsymbol{S}_{cs}$ = structural cross-sectional matrix
- $\boldsymbol{S}_{te}$ = structural temporal matrix
- $\boldsymbol{S}_{ct} = \boldsymbol{S}_{cs} \otimes \boldsymbol{S}_{te}$
- $\boldsymbol{x}_{\tau} = \boldsymbol{S}_{ct}\boldsymbol{b}_{\tau}$
- $\boldsymbol{b}_{\tau} = \begin{bmatrix} \boldsymbol{x}^{[1]}_{1,\tau} \\ \vdots \\ \boldsymbol{x}^{[1]}_{n,\tau} \end{bmatrix}.$

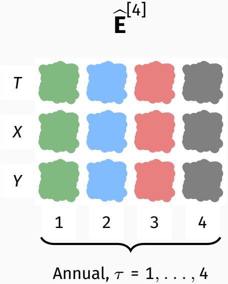


Cross-temporal probabilistic forecast reconciliation

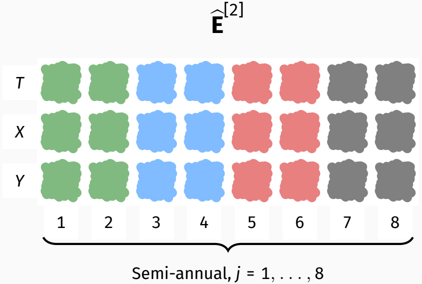
Nonparametric bootstrap

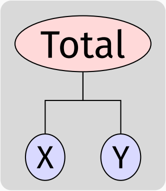
- Simulate future sample paths from all models using bootstrapped residuals, then reconcile them to obtain coherent sample paths.
- Need to generate samples that preserve cross-temporal relationships.
- Draw residual samples of all series at same time from most temporally aggregated level.
- Residuals for other levels obtained using the corresponding time indices.

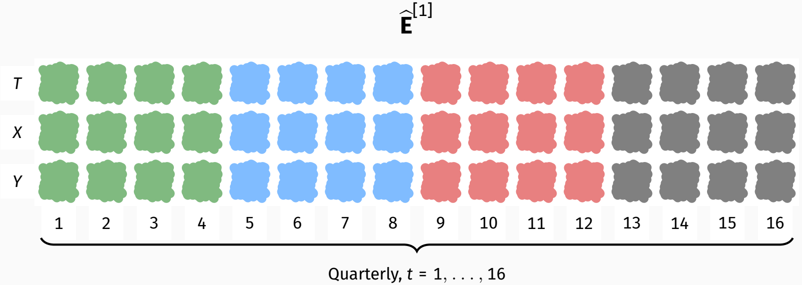
Cross-temporal probabilistic forecast reconciliation



Data

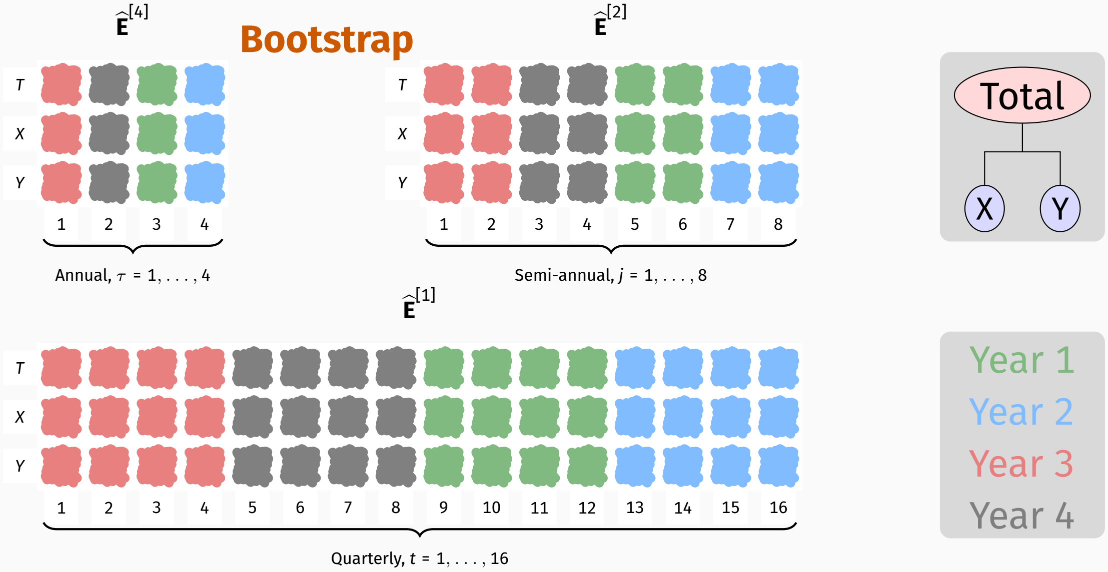






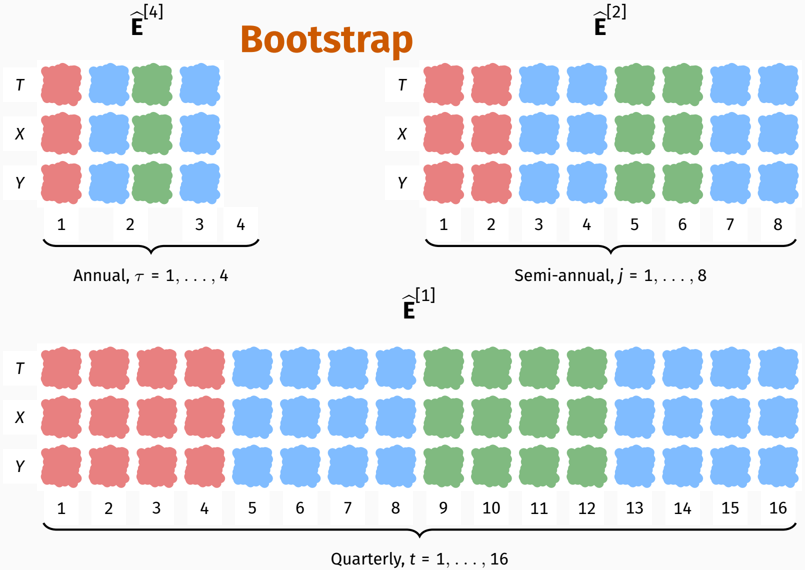
Year 1
Year 2
Year 3
Year 4

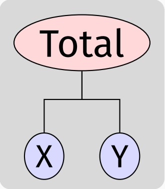
Cross-temporal probabilistic forecast reconciliation



Cross-temporal probabilistic forecast reconciliation

Bootstrap

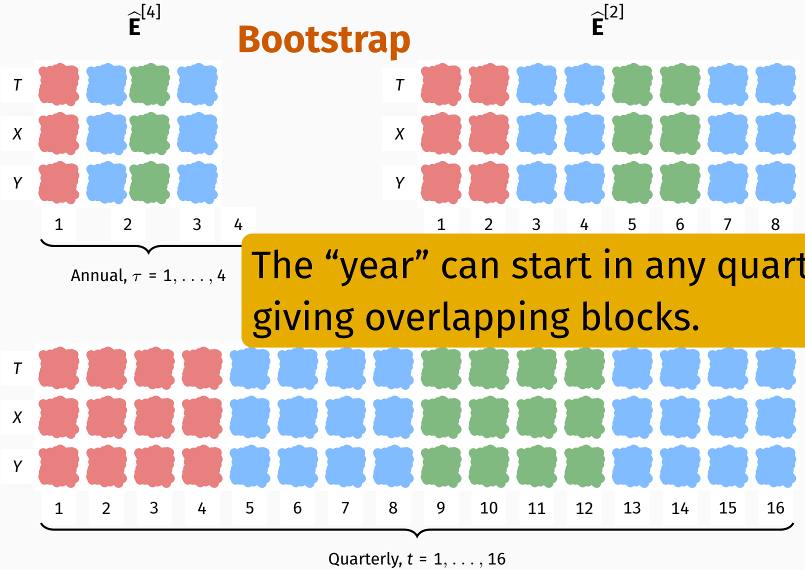




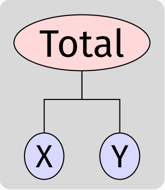
Year 1
Year 2
Year 3
Year 4

Cross-temporal probabilistic forecast reconciliation

Bootstrap



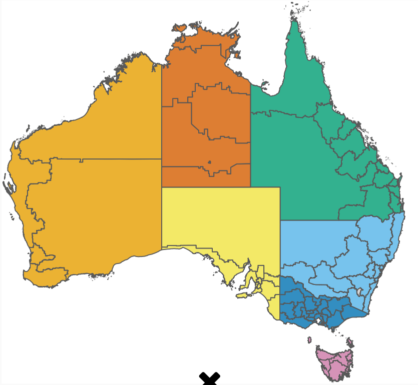
The “year” can start in any quarter, giving overlapping blocks.



Year 1
Year 2
Year 3
Year 4

Monthly Australian Tourism Demand

Geographical division



×

Purpose of travel

Holiday, Visiting friends & relatives, Business, Other

- **Grouped ts**

  (geographical divisions × purpose of travel)

	AUS	States	Zones*	Regions	Tot
geographical	1	7	21	76	105
purpose	4	28	84	304	420
total	5	35	105	380	**525**

$n_a = 221$, $n_b = 304$, and $\mathbf{n = 525}$

- **Temporal framework**, frequencies:
  - Monthly
  - Bi-Monthly
  - Quarterly
  - Four-Monthly
  - Semi-Annual
  - Annual

Monthly Australian Tourism Demand

- Monthly data: January 1998 to December 2016
- Time series cross-validation; initial training set 10 years.
- One-month increase in each training set
- For each training set, compute temporally aggregated series for $k \in \{1, 2, 3, 4, 6, 12\}$, and produce forecasts up to $h_2 = 6$, $h_3 = 4$, $h_4 = 3$, $h_6 = 2$ and $h_{12} = 1$ steps ahead.
- Automatic ETS forecasts on log-transformed data

Monthly Australian Tourism Demand

Reconciliation approaches

- Cross-temporal **bottom-up** and **partly bottom-up**

$\text{ct}(bu)$	$\text{ct}(shr_{cs}, bu_{te})$	$\text{ct}(wlsv_{te}, bu_{cs})$

- Optimal forecast reconciliation with **one-step residuals**

$\text{oct}(ols)$	$\text{oct}(struc)$	$\text{oct}(wlsv)$	$\text{oct}(bdshr)$

- Optimal forecast reconciliation with **multi-step residuals**

$\text{oct}_h(hbshr)$	$\text{oct}_h(bshr)$	$\text{oct}_h(hshr)$	$\text{oct}_h(shr)$

Monthly Australian tourism data – CRPS skill scores

Worse than benchmark **Best**

	$\forall k \in \{12, 6, 4, 3, 2, 1\}$	$k = 1$
base	1.000	1.000
ct(bu)	1.321	1.077
ct(shr_{cs}, bu_{te})	1.057	0.976
ct($wlsv_{te}, bu_{cs}$)	1.062	0.976
oct(ols)	0.989	0.982
oct($struc$)	0.982	0.970
oct($wlsv$)	0.987	0.952
oct($bdshr$)	0.975	**0.949**
oct$_h$($hbshr$)	0.989	0.982
oct$_h$($bshr$)	0.994	0.988
oct$_h$($hshr$)	**0.969**	0.953
oct$_h$(shr)	1.007	1.000

Outline

Software

Package	Language	Cross-sectional	Temporal	Cross-temporal	Probabilistic
`hts`	R	✓			
`thief`	R		✓		
`fable`	R	✓			✓
`FoReco`	R	✓	✓	✓	✓
`pyhts`	Python	✓	✓		
`hierarchicalforecast`	Python	✓			✓

- `hts`, `thief`, and `FoReco` use `ts` objects
- `fable` uses `tsibble` objects
- `fable` has plans to implement temporal and cross-temporal reconciliation

Thanks!

More information

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