Forecast reconciliation

A brief overview

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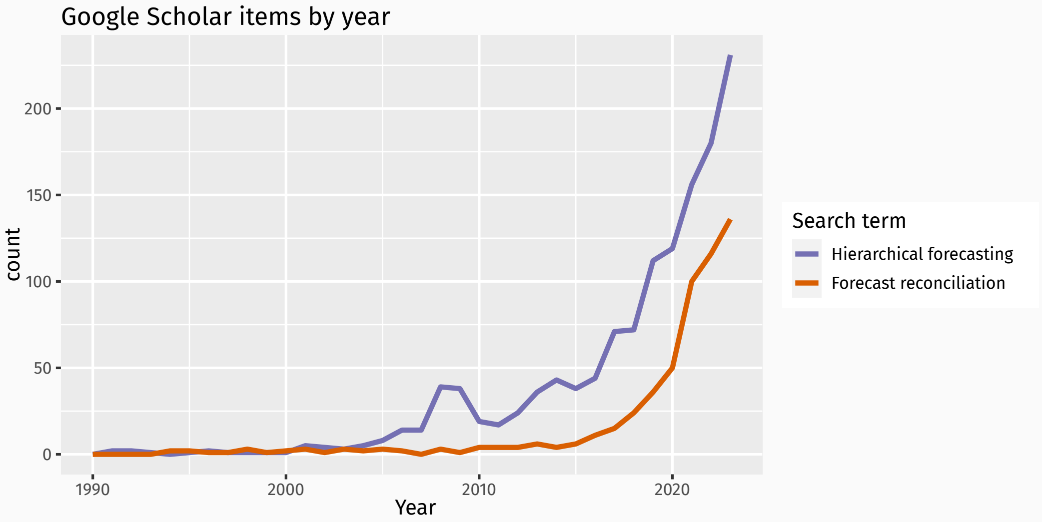
Forthcoming paper

- Athanasopoulos, Hyndman, Kourentzes, Panagiotelis (2024). "Forecast reconciliation: a review". In press. Preprint: robjhyndman.com/frreview.



Forecast reconciliation research

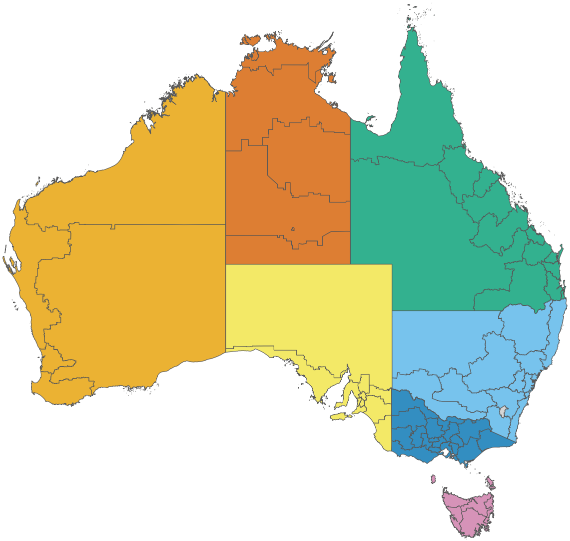


Outline

1. Australian tourism aggregations
2. Optimal forecast reconciliation
3. Probabilistic forecast reconciliation
4. Cross-temporal probabilistic forecast reconciliation

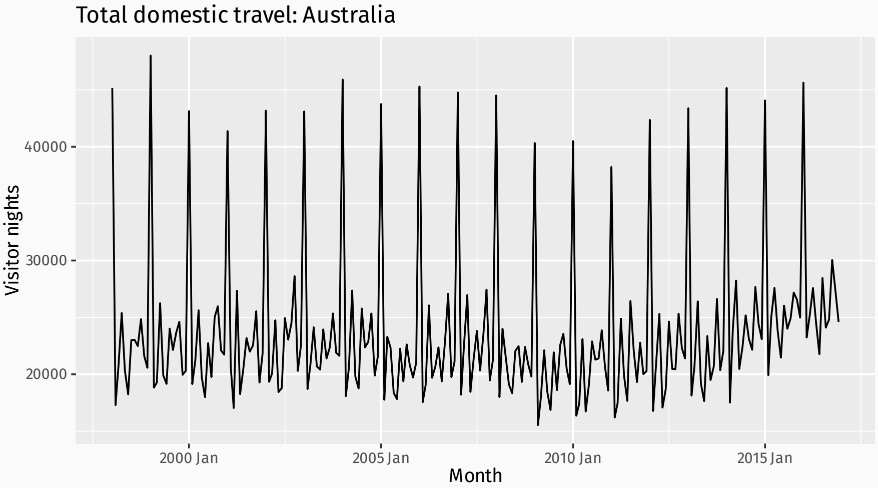
Outline

Australian tourism regions



- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+
- Geographical hierarchy:
  - 7 states
  - 27 zones
  - 75 regions
- Also disaggregated by purpose of travel (business, holiday, visiting, other)

Australian tourism data



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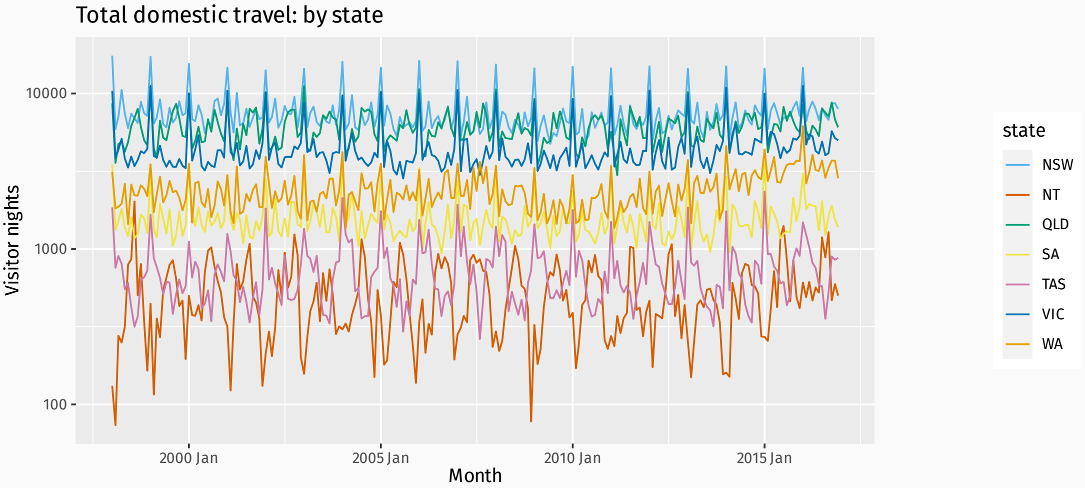

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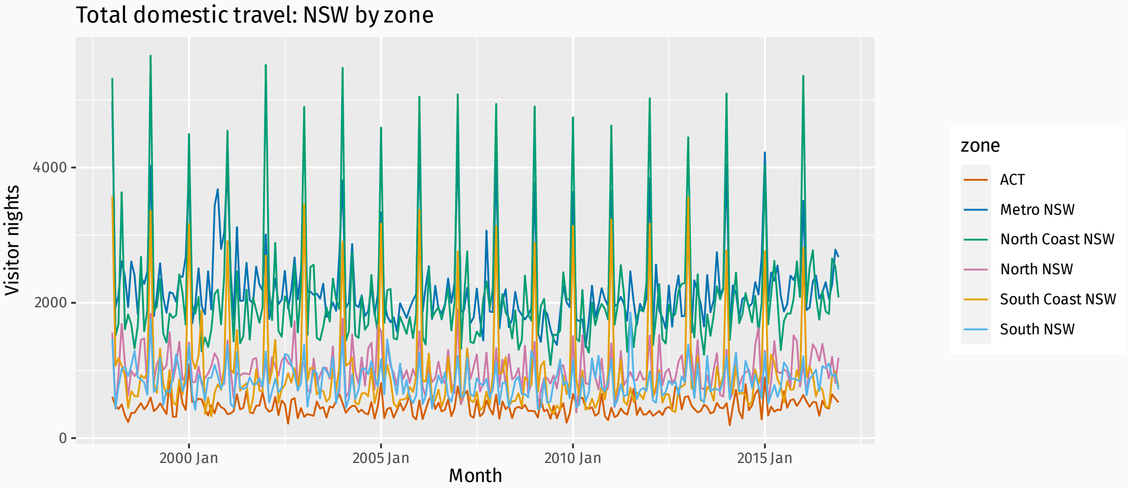

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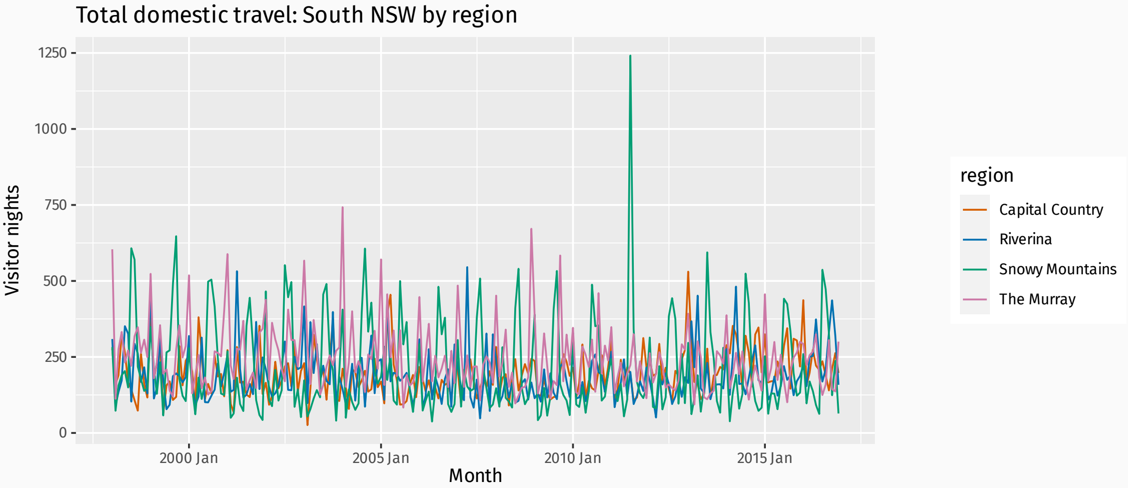

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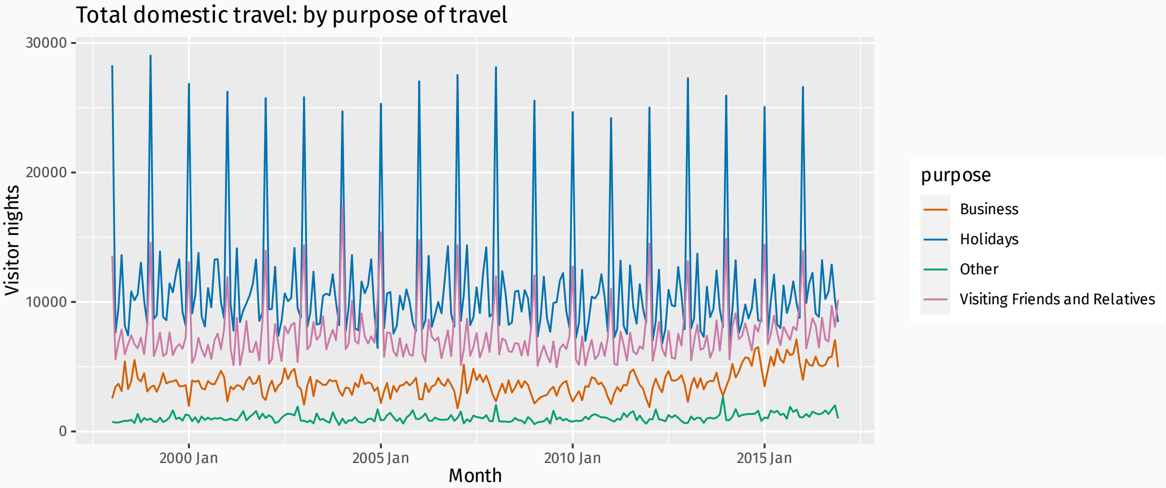

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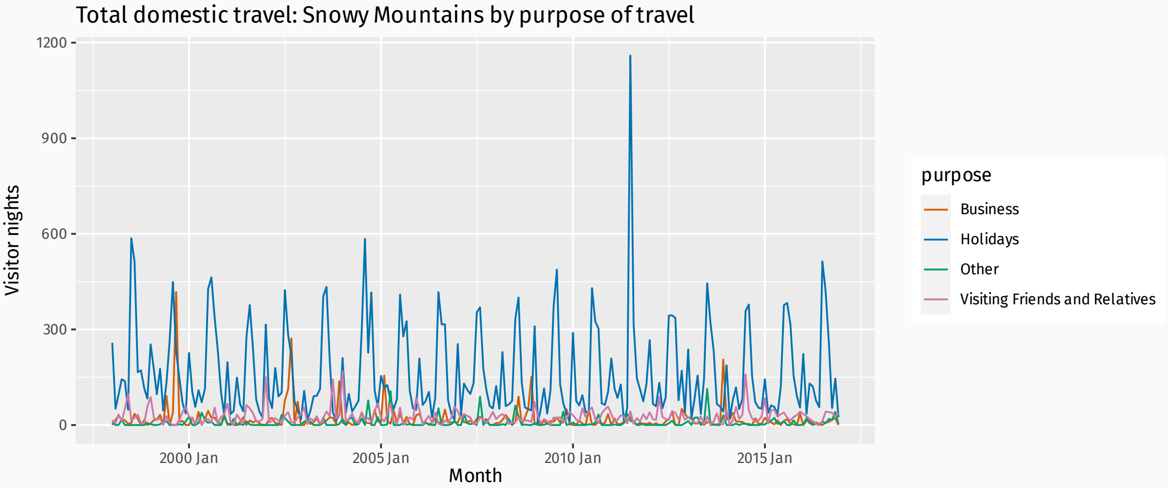
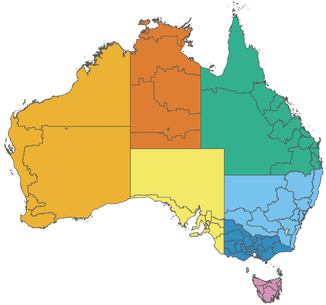


Australian tourism data

Geographical division



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Purpose of travel

Holiday, Visiting friends & relatives, Business, Other

- **Grouped ts**
  (geographical divisions × purpose of travel)

	AUS	States	Zones*	Regions	Tot
geographical	1	7	21	76	105
purpose	4	28	84	304	420
total	5	35	105	380	**525**

$n_a = 221$, $n_b = 304$, and $\mathbf{n = 525}$

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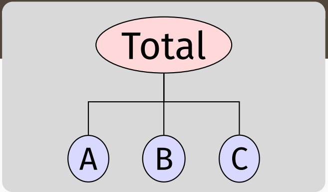
Linearly constrained time series



Almost all collections of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "structural matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Forecast reconciliation

What we want

- Forecasts of all series at all levels of aggregation.

Solution

- We model and forecast all series independently.
- We adjust (or "reconcile") the forecasts to make them coherent.

The coherent subspace

Coherent subspace

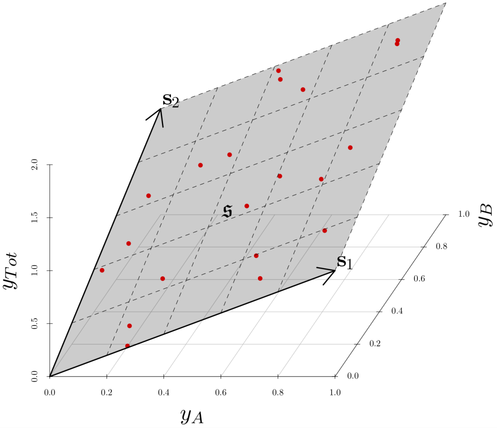
n_b-dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

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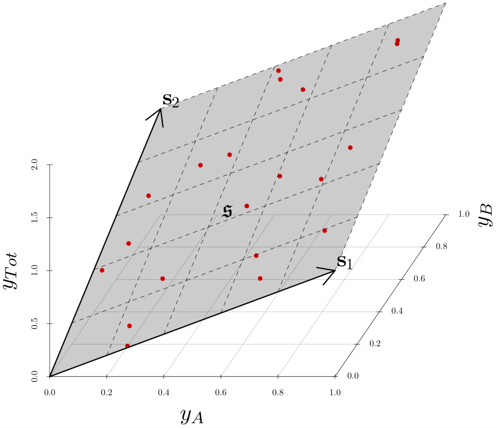
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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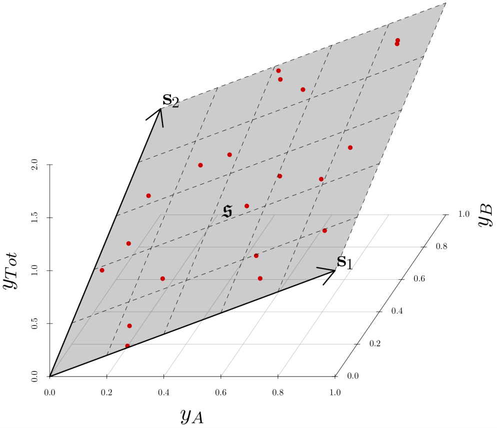

$y_{Tot} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let $\boldsymbol{M}$ be a projection matrix.
$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear projection reconciliation

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

- If $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$, then projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.
- Coherent base forecasts are unchanged since $\boldsymbol{M}\hat{\boldsymbol{y}} = \hat{\boldsymbol{y}}$
- If $\hat{\boldsymbol{y}}$ is unbiased, then $\tilde{\boldsymbol{y}}$ is also unbiased.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$ is the covariance matrix of the base forecast errors.
- $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]) = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$ is the covariance matrix of the reconciled forecast errors.
- How to choose the best Ψ?

Minimum trace reconciliation

Wickramasuriya et al (2019)

Minimum trace (MinT) reconciliation

If $\boldsymbol{M}$ is a projection, then trace of $\boldsymbol{V}_h$ is minimized when $\Psi = \boldsymbol{W}_h$, so that

$$\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$$

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{M}\,\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciled forecasts   Base forecasts

- $\boldsymbol{S}$ is the structural matrix, defining linear constraints.
- $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$
- Trace of $\boldsymbol{V}_h$ is sum of forecast variances.
- MinT is L_2 optimal amongst linear unbiased forecasts.
- Approximate $\boldsymbol{W}_h \approx k_h \boldsymbol{W}_1$, then use shrinkage estimator.

Key papers

- Di Fonzo, Girolimetto (2024). Forecast combination-based forecast reconciliation: Insights and extensions. *International J Forecasting*. In press.
- Hyndman, Ahmed, Athanasopoulos, Shang (2011). Optimal combination forecasts for hierarchical time series. *Computational Statistics & Data Analysis* **55**(9), 2579–2589.
- Hyndman, Lee, Wang (2016). Fast computation of reconciled forecasts for hierarchical and grouped time series. *Computational Statistics & Data Analysis* **97**, 16–32.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2021). Forecast reconciliation: A geometric view with new insights on bias correction. *International J Forecasting* **37**(1), 343–359.
- Wickramasuriya, Athanasopoulos, Hyndman (2019). Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization. *J American Statistical Association* **114**(526), 804–819.

Example: Australian tourism

```
tourism
```

```
# A tsibble: 69,312 x 6 [1M]
# Key:       state, zone, region, purpose [304]
      month state zone  region   purpose  visitors
      <mth> <chr> <chr> <chr>    <chr>       <dbl>
 1 1998 Jan NSW   ACT   Canberra Business     25.0
 2 1998 Feb NSW   ACT   Canberra Business    148.
 3 1998 Mar NSW   ACT   Canberra Business    111.
 4 1998 Apr NSW   ACT   Canberra Business     93.1
 5 1998 May NSW   ACT   Canberra Business     78.1
 6 1998 Jun NSW   ACT   Canberra Business     44.3
 7 1998 Jul NSW   ACT   Canberra Business    129.
 8 1998 Aug NSW   ACT   Canberra Business     71.3
 9 1998 Sep NSW   ACT   Canberra Business     77.7
10 1998 Oct NSW   ACT   Canberra Business    145.
# i 69,302 more rows
```

Example: Australian tourism

```
tourism_agg <- tourism |>
  aggregate_key((state/zone/region) * purpose, visitors = sum(visitors))
```

```
# A tsibble: 126,540 x 6 [1M]
# Key:       state, purpose, zone, region [555]
      month state        purpose      zone         region       visitors
      <mth> <chr*>       <chr*>       <chr*>       <chr*>          <dbl>
 1 1998 Jan <aggregated> <aggregated> <aggregated> <aggregated>   45151.
 2 1998 Feb <aggregated> <aggregated> <aggregated> <aggregated>   17295.
 3 1998 Mar <aggregated> <aggregated> <aggregated> <aggregated>   20725.
 4 1998 Apr <aggregated> <aggregated> <aggregated> <aggregated>   25389.
 5 1998 May <aggregated> <aggregated> <aggregated> <aggregated>   20330.
 6 1998 Jun <aggregated> <aggregated> <aggregated> <aggregated>   18238.
 7 1998 Jul <aggregated> <aggregated> <aggregated> <aggregated>   23005.
 8 1998 Aug <aggregated> <aggregated> <aggregated> <aggregated>   23033.
 9 1998 Sep <aggregated> <aggregated> <aggregated> <aggregated>   22483.
10 1998 Oct <aggregated> <aggregated> <aggregated> <aggregated>   24845.
# i 126,530 more rows
```

Example: Australian tourism

```r
fit <- tourism_agg |>
  filter(year(month) <= 2015) |>
  model(ets = ETS(visitors))
```

```
# A mable: 555 x 5
# Key:     state, purpose, zone, region [555]
   state  purpose  zone             region                ets
   <chr*> <chr*>   <chr*>           <chr*>            <model>
 1 NSW    Business ACT              Canberra     <ETS(M,N,M)>
 2 NSW    Business ACT              <aggregated> <ETS(M,N,M)>
 3 NSW    Business Metro NSW        Central Coast <ETS(A,N,N)>
 4 NSW    Business Metro NSW        Sydney       <ETS(A,N,A)>
 5 NSW    Business Metro NSW        <aggregated> <ETS(A,N,A)>
 6 NSW    Business North Coast NSW Hunter       <ETS(M,N,M)>
 7 NSW    Business North Coast NSW North Coast NSW <ETS(M,N,A)>
 8 NSW    Business North Coast NSW <aggregated> <ETS(M,N,M)>
 9 NSW    Business North NSW        Blue Mountains <ETS(A,N,N)>
10 NSW    Business North NSW        Central NSW <ETS(M,Ad,N)>
```

Example: Australian tourism

```r
fc <- fit |>
  reconcile(mint_s = min_trace(ets, method = "mint_shrink")) |>
  forecast(h = "2 years")
```

```
# A fable: 26,640 x 8 [1M]
# Key:     state, purpose, zone, region, .model [1,110]
   state   purpose  zone    region   .model    month      visitors .mean
   <chr*>  <chr*>   <chr*>  <chr*>   <chr>     <mth>        <dist> <dbl>
 1 NSW     Business ACT     Canberra ets    2016 Jan   N(64, 1190)  63.9
 2 NSW     Business ACT     Canberra ets    2016 Feb N(112, 3709) 112.
 3 NSW     Business ACT     Canberra ets    2016 Mar N(170, 8814) 170.
 4 NSW     Business ACT     Canberra ets    2016 Apr N(124, 4776) 124.
 5 NSW     Business ACT     Canberra ets    2016 May N(122, 4700) 122.
 6 NSW     Business ACT     Canberra ets    2016 Jun N(133, 5730) 133.
 7 NSW     Business ACT     Canberra ets    2016 Jul N(173, 9860) 173.
 8 NSW     Business ACT     Canberra ets    2016 Aug N(138, 6385) 138.
 9 NSW     Business ACT     Canberra ets    2016 Sep N(159, 8657) 159.
10 NSW     Business ACT     Canberra ets    2016 Oct N(162, 9164) 162.
```

Example: Australian tourism

```
fc |>
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>
  group_by(.model) |>
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>
  arrange(rmsse)
```

```
# A tibble: 2 x 3
  .model   mase rmsse
  <chr>   <dbl> <dbl>
1 mint_s  0.891 0.853
2 ets     0.891 0.868
```

- Overall, MinT reconciliation forecasts are better than the base ETS forecasts.

Example: Australian tourism

```r
fc |>
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>
  group_by(.model, level) |>
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>
  arrange(level, rmsse)
```

```
# A tibble: 10 x 4
# Groups:   .model [2]
   .model level     mase rmsse
   <chr>  <fct>    <dbl> <dbl>
 1 ets    National 0.806 0.755
 2 mint_s National 0.862 0.886
 3 mint_s State    0.894 0.898
 4 ets    State    0.921 0.919
 5 mint_s Zone     0.883 0.854
 6 ets    Zone     0.936 0.935
 7 mint_s Region   0.833 0.807
 8 ets    Region   0.866 0.858
 9 mint_s Purpose  0.933 0.887
10 ets    Purpose  0.967 0.932
```

- Overall, MinT reconciliation forecasts are better than the base ETS forecasts.
- MinT reconciliation forecasts are better than the base ETS forecasts at all levels except national.

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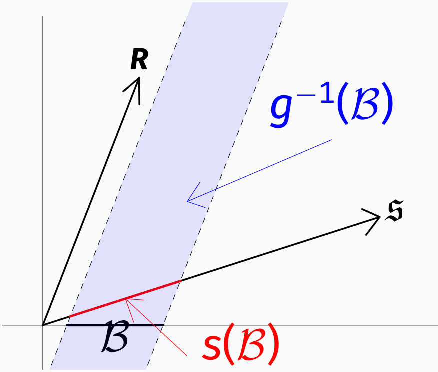
Coherent probabilistic forecasts

Coherent probabilistic forecasts

A probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is coherent with the bottom probability triple $(\chi^m, \mathscr{F}_{\chi^m}, \nu)$, if

$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\chi^m}$$

- Random draws from coherent distribution must lie on $\mathfrak{s}$.
- The probability of points not on $\mathfrak{s}$ is zero.
- The reconciled distribution is a transformation of the base forecast distribution that is coherent on $\mathfrak{s}$.



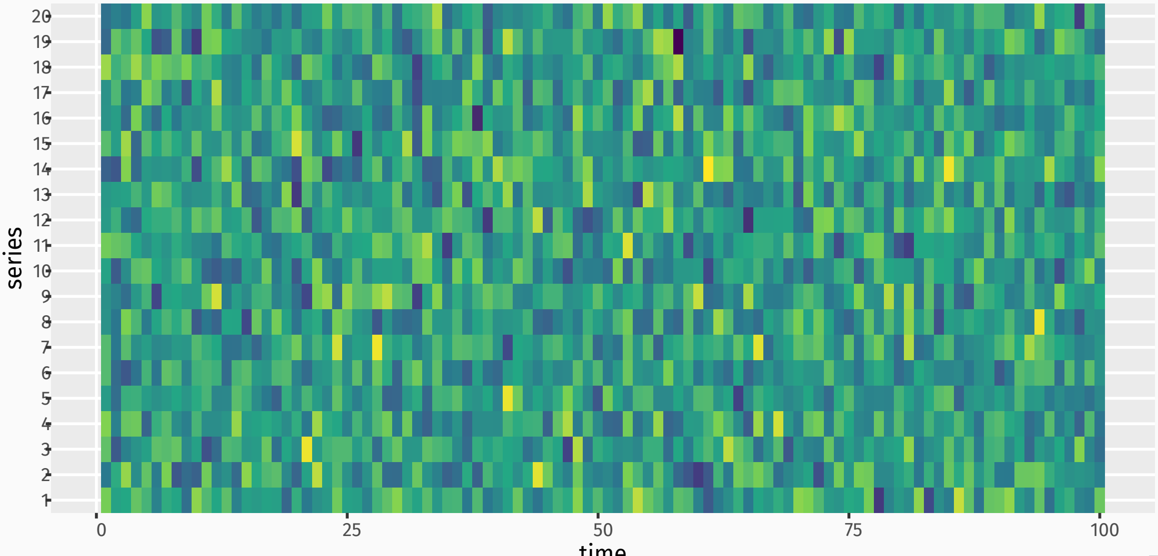
$$\psi = s \circ g$$

Simulation from a reconciled distribution

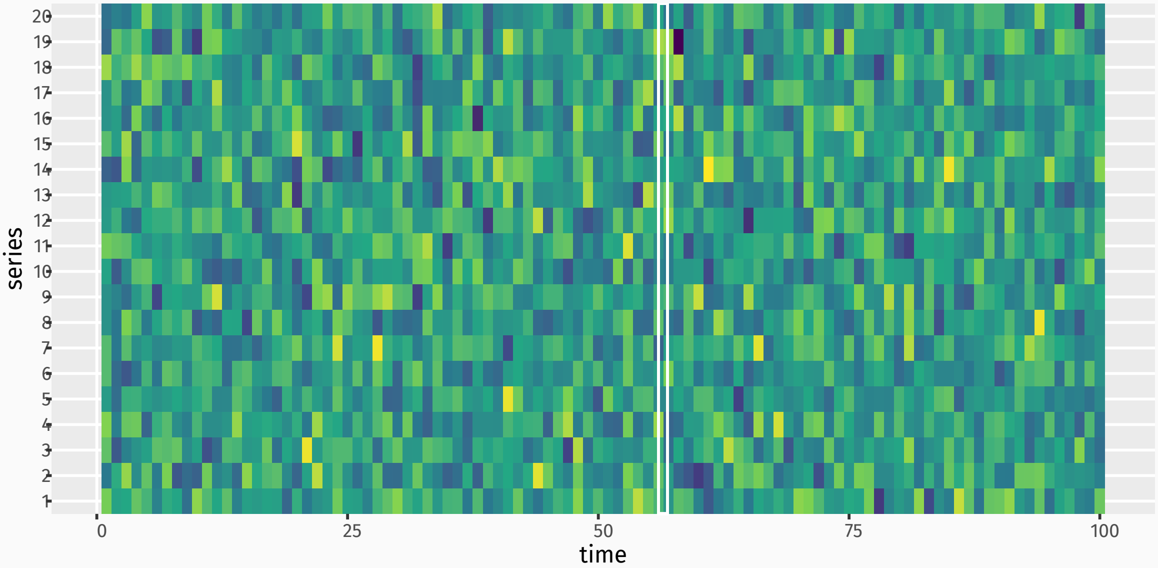
Suppose that $(\hat{\boldsymbol{y}}^{[1]},\dots,\hat{\boldsymbol{y}}^{[L]})$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $(\tilde{\boldsymbol{y}}^{[1]},\dots,\tilde{\boldsymbol{y}}^{[L]})$ where $\tilde{\boldsymbol{y}}^{[\ell]} := \psi(\hat{\boldsymbol{y}}^{[\ell]})$ for $\ell = 1,\dots,L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

- Simulate future sample paths for each series, by simulating from each model using a multivariate bootstrap of the residuals (to preserve cross-correlations).
- Reconcile the sample paths.
- The reconciled sample paths are a sample from the reconciled distribution.

Simulation from a reconciled distribution



Simulation from a reconciled distribution

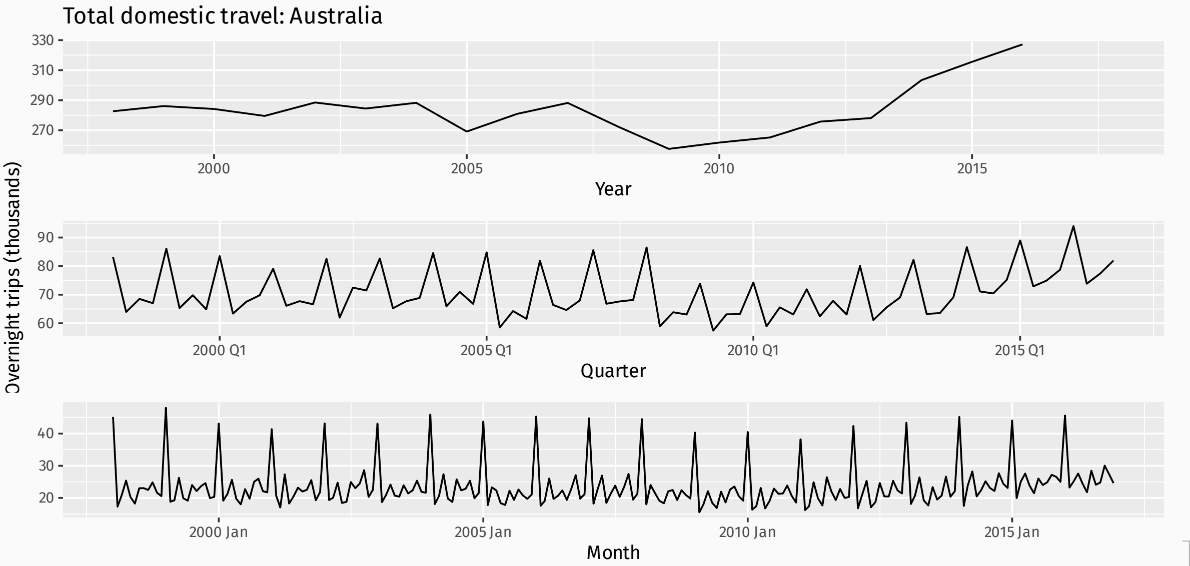


Key papers

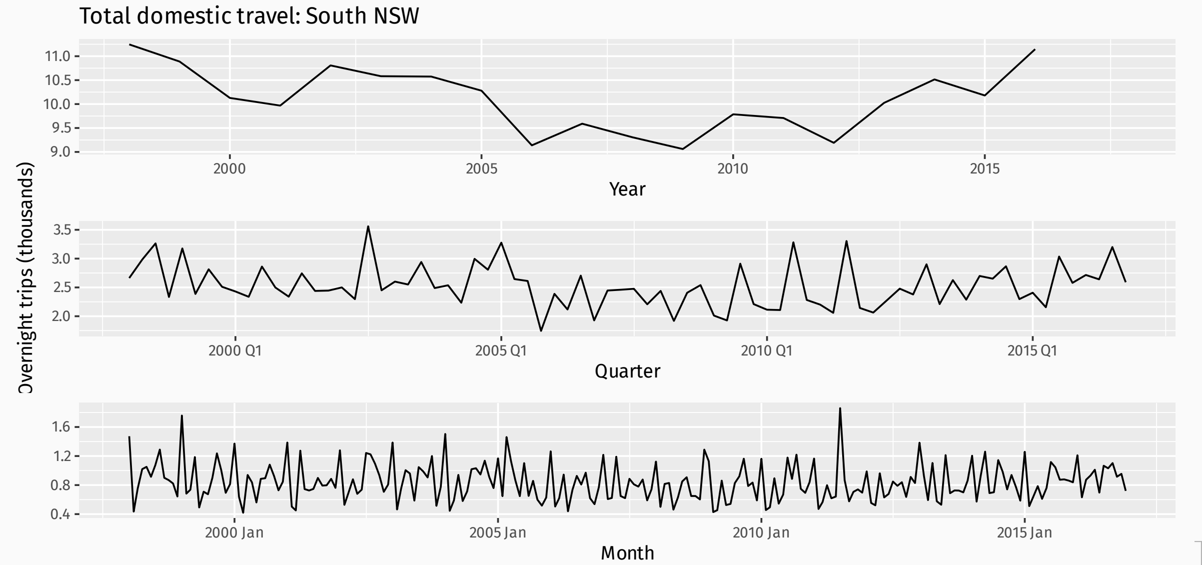
- Corani, Azzimonti, Rubattu (2024). Probabilistic reconciliation of count time series. *International J Forecasting*. In press.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2023). Probabilistic forecast reconciliation: properties, evaluation and score optimisation. *European J Operational Research* **306**(2), 693–706.

Outline

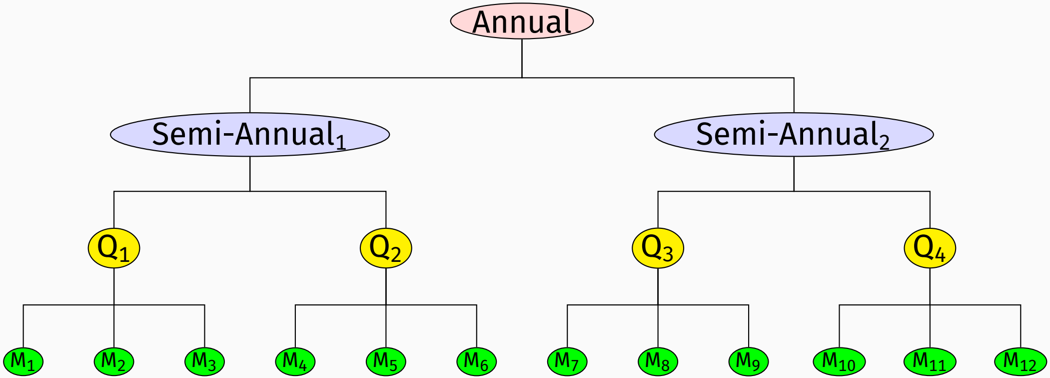
Temporal aggregations



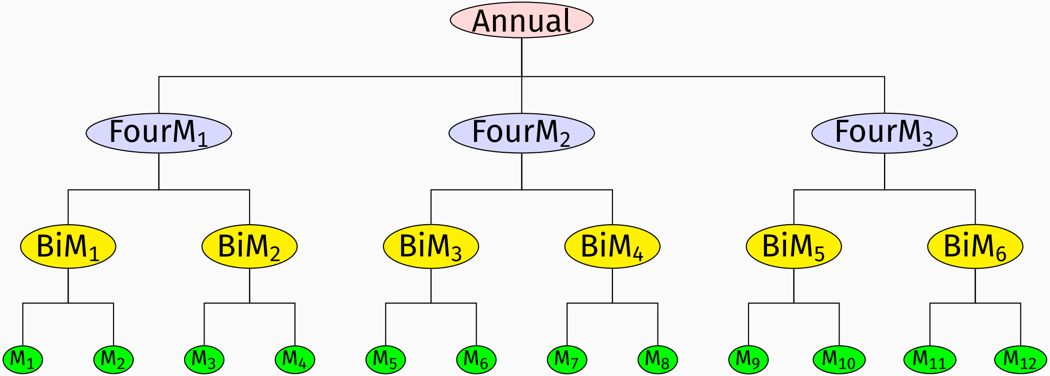
Temporal aggregations



Temporal aggregations: monthly data

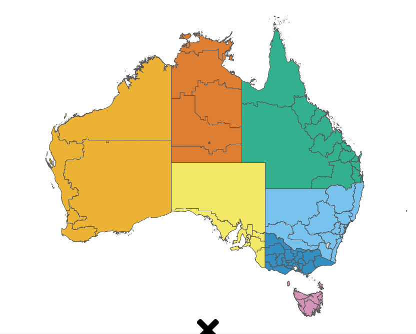


Temporal aggregations: monthly data



Monthly Australian Tourism Demand

Geographical division



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- **Temporal framework**, frequencies:
  - Monthly
  - Bi-Monthly
  - Quarterly
  - Four-Monthly
  - Semi-Annual
  - Annual

Monthly Australian Tourism Demand

- Monthly data: January 1998 to December 2016
- Time series cross-validation; initial training set 10 years.
- One-month increase in each training set
- For each training set, compute temporally aggregated series for $k \in \{1, 2, 3, 4, 6, 12\}$, and produce forecasts up to $h_2 = 6$, $h_3 = 4$, $h_4 = 3$, $h_6 = 2$ and $h_{12} = 1$ steps ahead.
- Automatic ETS forecasts on log-transformed data

Monthly Australian tourism data – CRPS skill scores

Reconciliation using different covariance matrix ($\boldsymbol{W}_h$) estimates

	Worse than benchmark $\forall k \in \{12, 6, 4, 3, 2, 1\}$	**Best** $k = 1$
base	1.000	1.000
ct(bu)	1.321	1.077
ct(shr_{cs}, bu_{te})	1.057	0.976
ct($wlsv_{te}, bu_{cs}$)	1.062	0.976
oct(ols)	0.989	0.982
oct($struc$)	0.982	0.970
oct($wlsv$)	0.987	0.952
oct($bdshr$)	0.975	**0.949**
oct$_h$($hbshr$)	0.989	0.982
oct$_h$($bshr$)	0.994	0.988
oct$_h$($hshr$)	**0.969**	0.953
oct$_h$(shr)	1.007	1.000

Key papers

- Athanasopoulos, Hyndman, Kourentzes, Petropoulos (2017). Forecasting with temporal hierarchies. *European J Operational Research* **262**(1), 60–74.
- Di Fonzo, Girolimetto (2023). Cross-temporal forecast reconciliation: Optimal combination method and heuristic alternatives. *International J Forecasting* **39**(1), 39–57.
- Girolimetto, Athanasopoulos, Di Fonzo, Hyndman (2024). Cross-temporal probabilistic forecast reconciliation. *International J Forecasting*. In press.

Forecast reconciliation software

Package	Language	Cross-sectional	Temporal	Cross-temporal	Probabilistic
`hts`	R	✓			
`thief`	R		✓		
`fable`	R	✓			✓
`FoReco`	R	✓	✓	✓	✓
`pyhts`	Python	✓	✓		
`hierarchicalforecast`	Python	✓			✓

- `hts`, `thief`, and `FoReco` use `ts` objects
- `fable` uses `tsibble` objects
- `fable` has plans to implement temporal and cross-temporal reconciliation

More information

- robjhyndman.com/frreview
- robjhyndman.com/frslides
- aus.social/@robjhyndman
- @robjhyndman
- rob.hyndman@monash.edu

References

- Athanasopoulos, Hyndman, Kourentzes, Panagiotelis (2024). "Forecast reconciliation: a review". In press. Preprint: robjhyndman.com/frreview.
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- Di Fonzo, Girolimetto (2024). Forecast combination-based forecast reconciliation: Insights and extensions. *International J Forecasting*. In press.

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- Girolimetto, Athanasopoulos, Di Fonzo, Hyndman (2024). Cross-temporal probabilistic forecast reconciliation. *International J Forecasting*. In press.
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