

Forecast reconciliation

A brief overview

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MONASH University

Outline

- 1 Hierarchical time series notation
- 2 Optimal linear forecast reconciliation
- 3 Probabilistic forecast reconciliation
- 4 Cross-temporal probabilistic forecast reconciliation
- 5 Final comments

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Labour market participation

Australia & New Zealand Standard Classification of Occupations

- 8 major groups
 - ▶ 43 sub-major groups
 - ★ 97 minor groups
 - 359 unit groups
 - 1023 occupations

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Example: statistician

2 Professionals

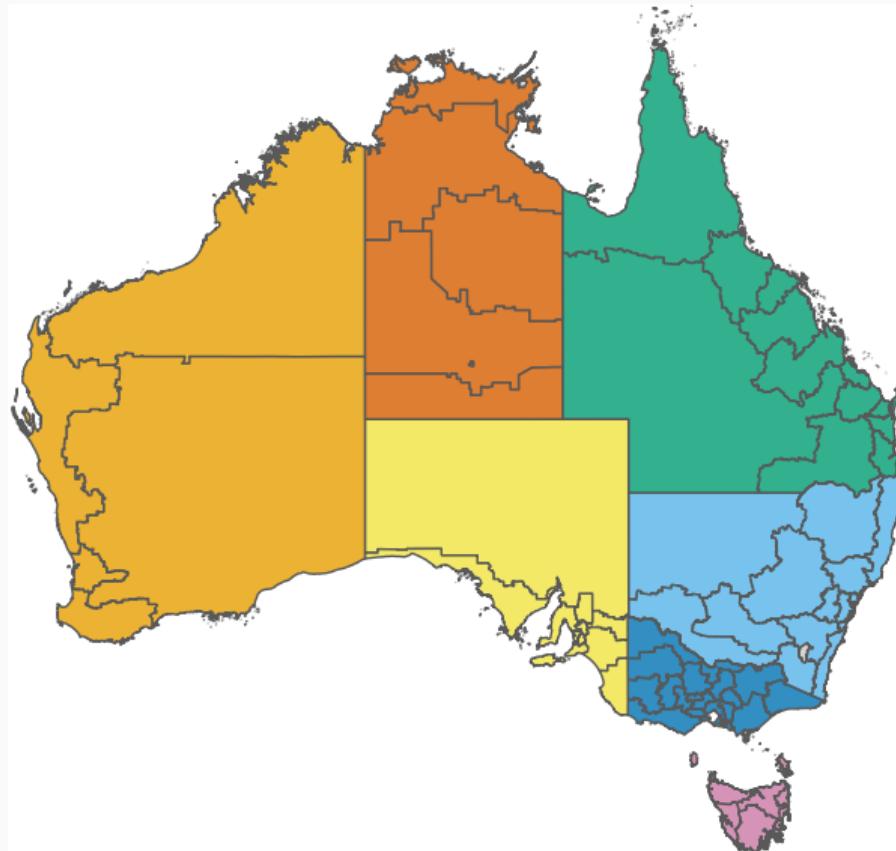
22 Business, Human Resource and Marketing Professionals

224 Information and Organisation Professionals

2241 Actuaries, Mathematicians and Statisticians

224113 Statistician

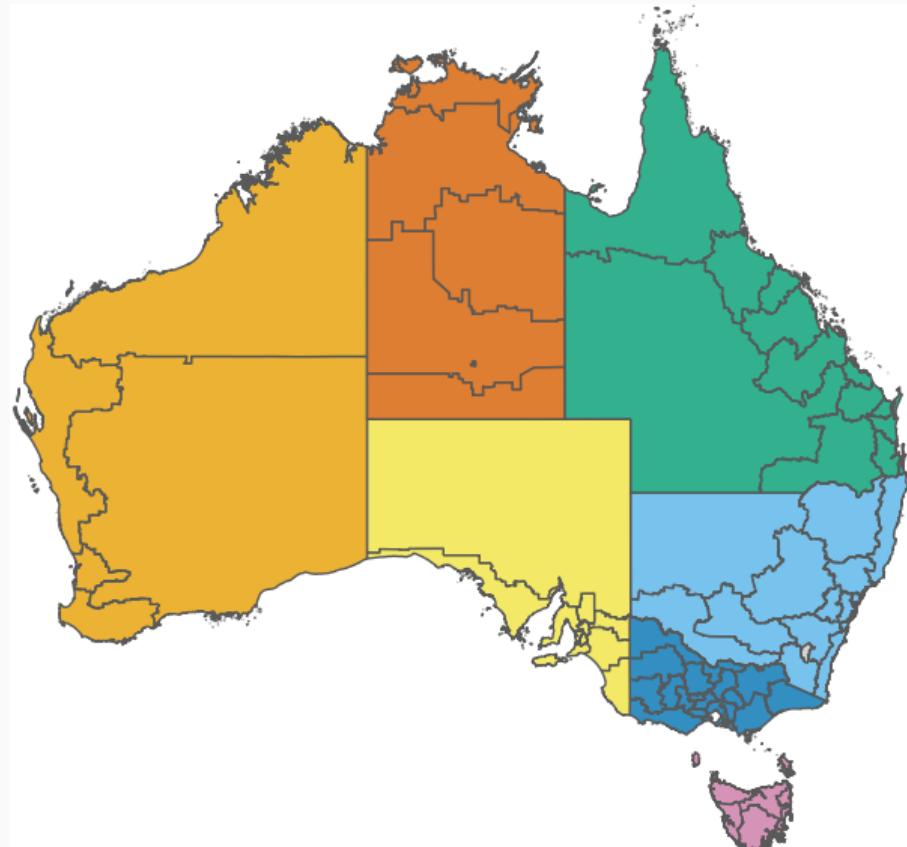
Australian tourism regions



State

- Australian Capital Territory
- New South Wales
- Northern Territory
- Queensland
- South Australia
- Tasmania
- Victoria
- Western Australia

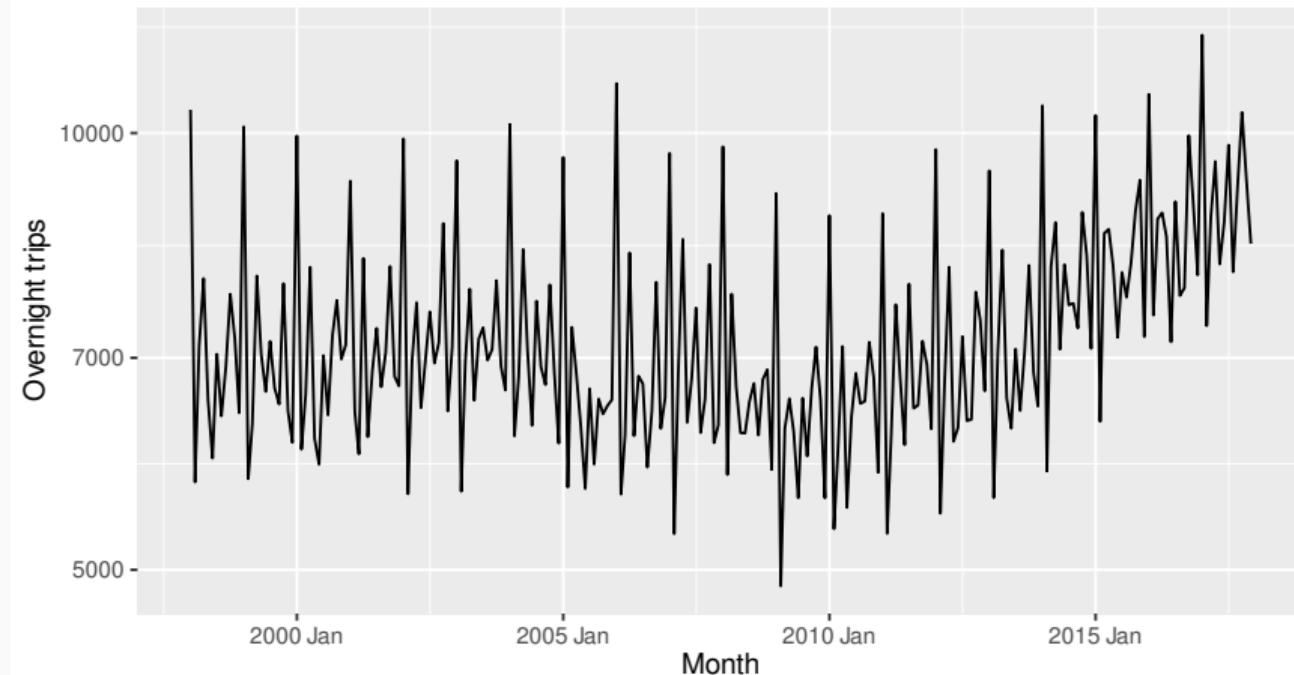
Australian tourism regions



- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
 - ▶ 7 states
 - ▶ 27 zones
 - ▶ 75 regions

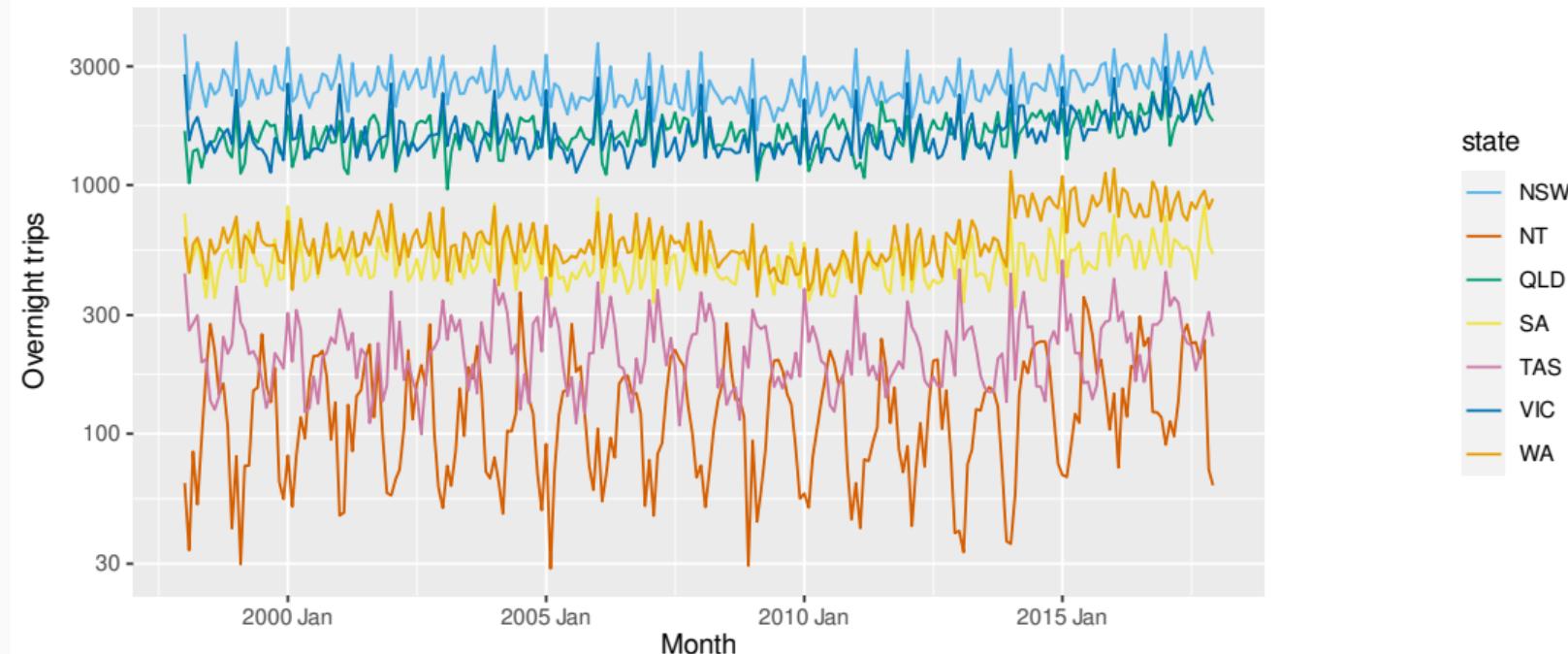
Australian tourism data

Total domestic travel: Australia



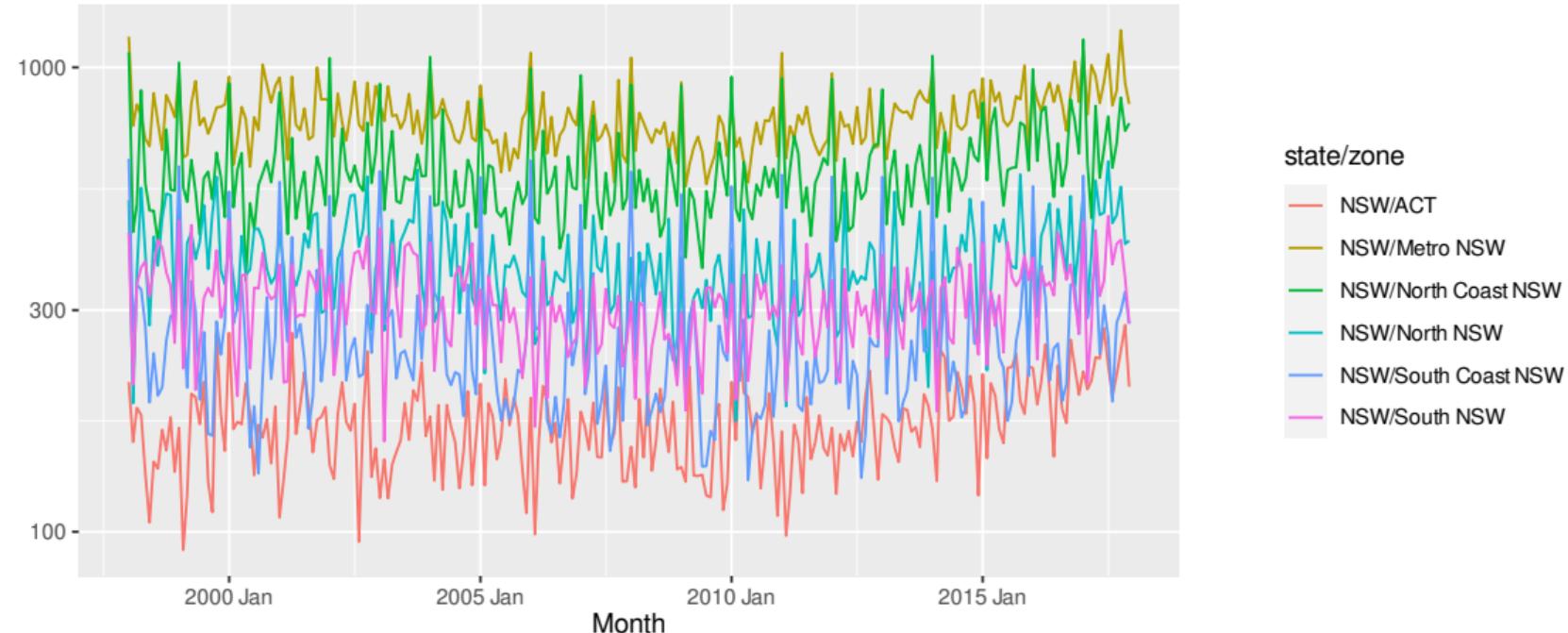
Australian tourism data

Total domestic travel: by state



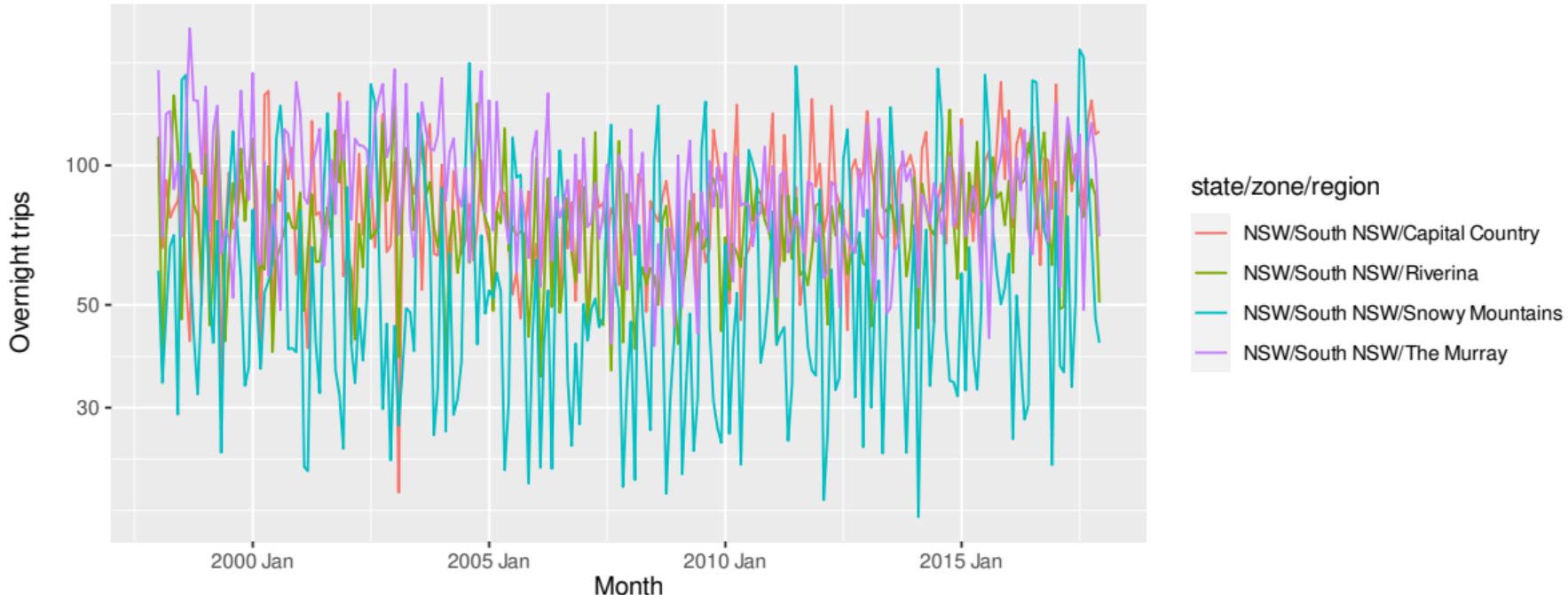
Australian tourism data

Total domestic travel: NSW by zone



Australian tourism data

Total domestic travel: South NSW by region



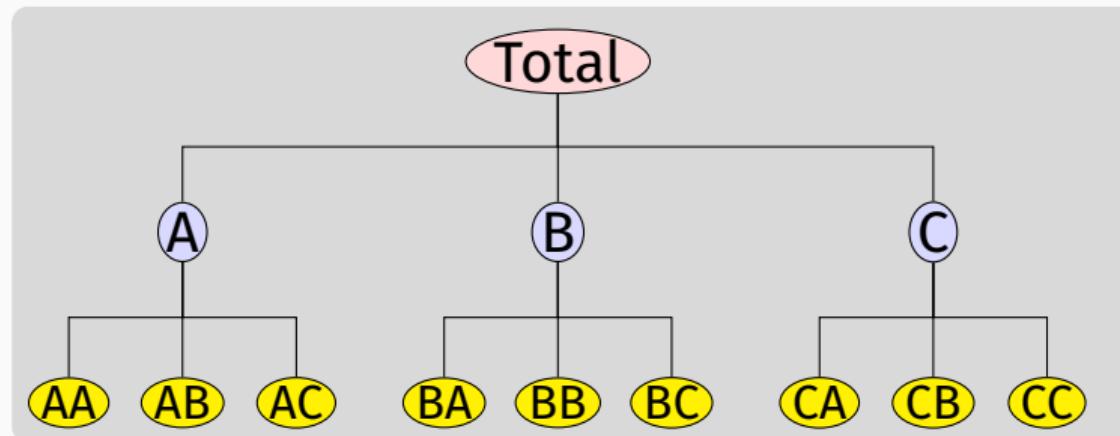
Spectacle sales

- Monthly UK sales data from 2000 – 2014
- Provided by a large spectacle manufacturer
- Split by brand (26), gender (3), price range (6), materials (4), and stores (600)
- About 1 million bottom-level series



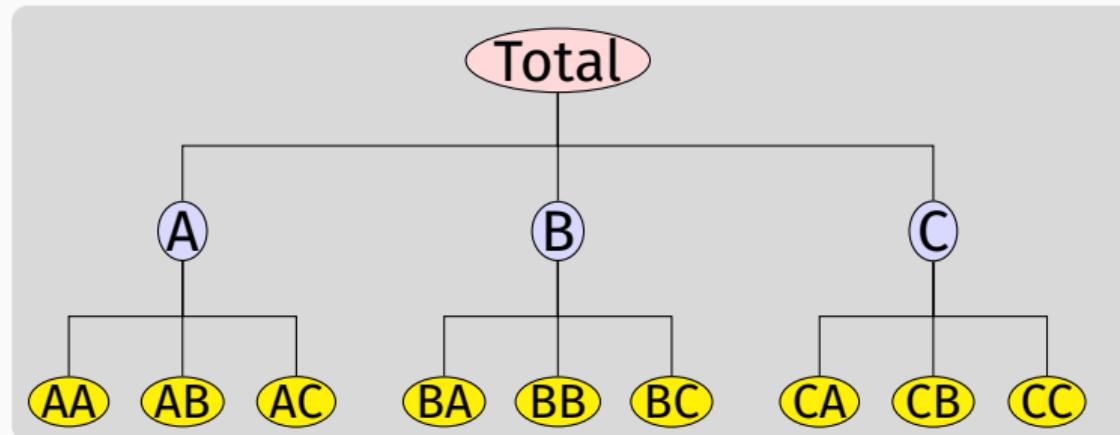
Hierarchical time series

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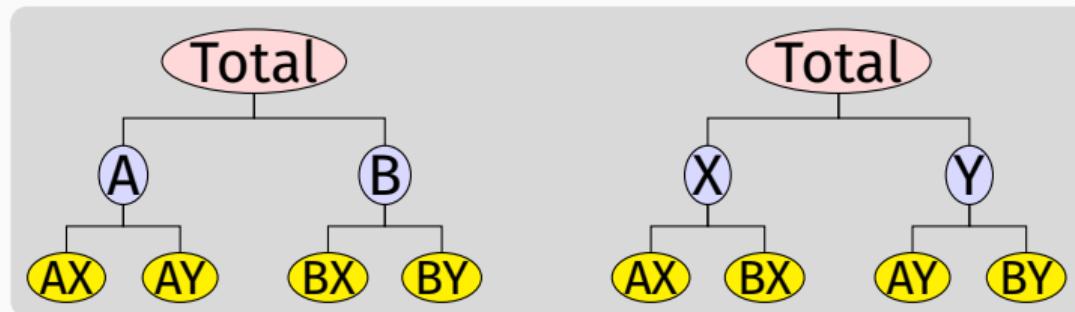


Examples

- Tourism by state and region
- Retail sales by product groups, sub groups, and SKUs

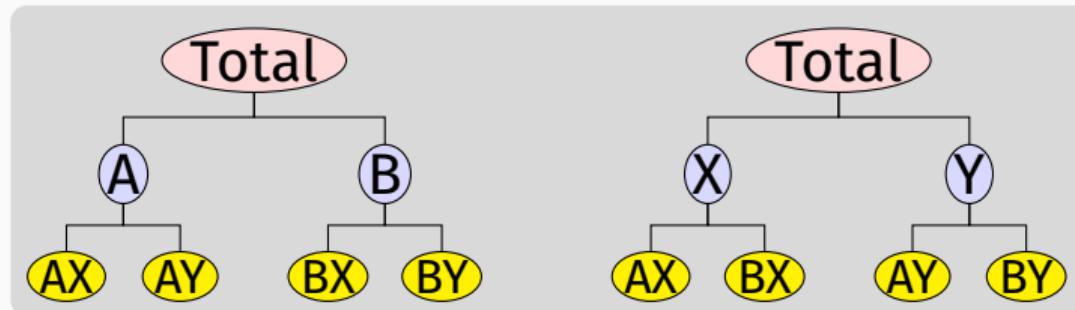
Grouped time series

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Examples

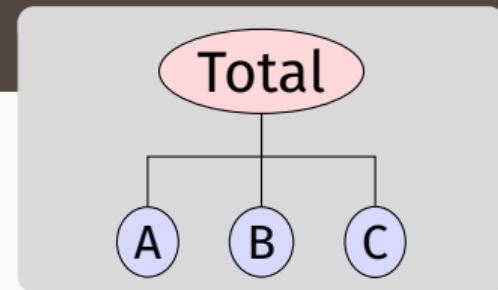
- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Hierarchical and grouped time series

Almost all collections of time series with linear constraints can be written as

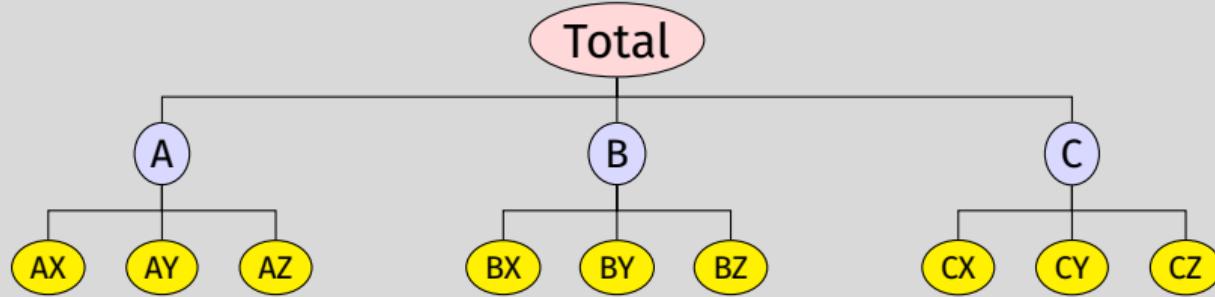
$$\mathbf{y}_t = \mathbf{S} \mathbf{b}_t$$

- \mathbf{y}_t = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t .
- $y_{X,t}$ = value of series X at time t .
- \mathbf{b}_t = vector of most disaggregated series at time t
- \mathbf{S} = “summing matrix” containing the linear constraints.

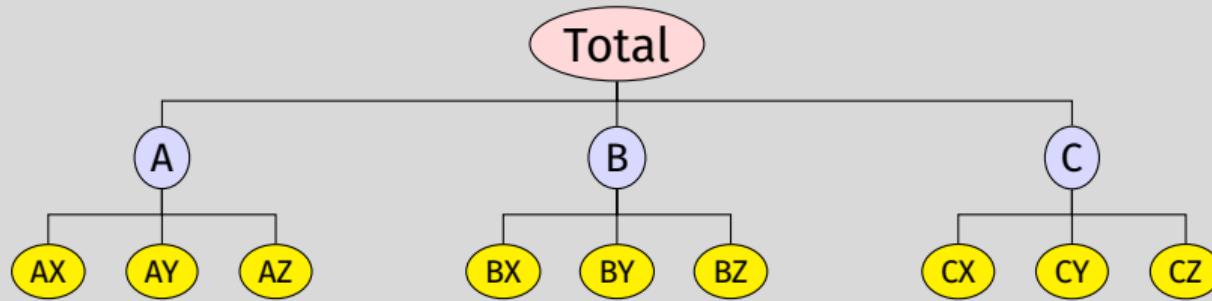


$$\begin{aligned}\mathbf{y}_t &= \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t}\end{aligned}$$

Hierarchical time series

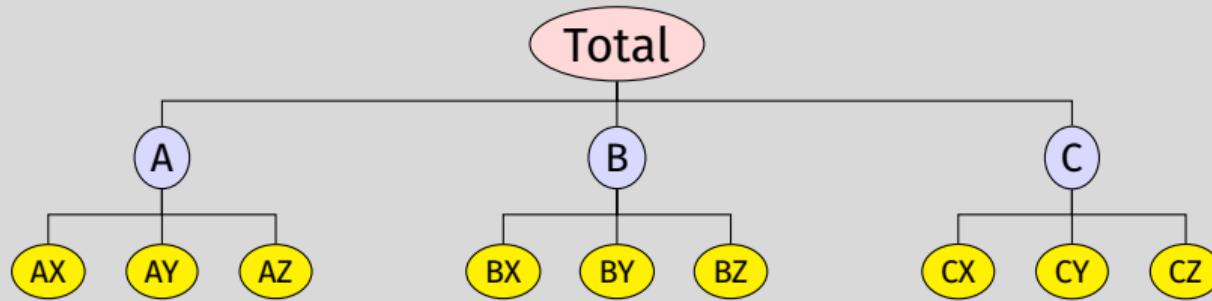


Hierarchical time series



$$\mathbf{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

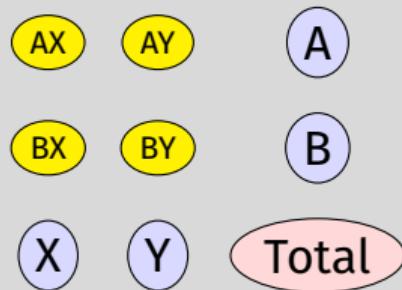
Hierarchical time series



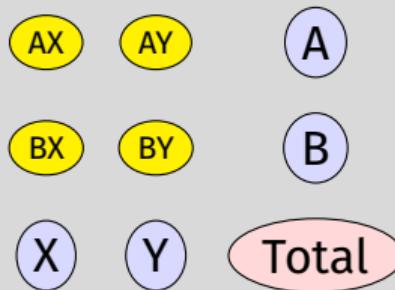
$$\mathbf{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

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Grouped data

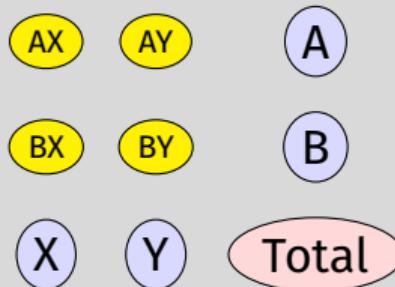


Grouped data



$$\mathbf{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}$$

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$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

The hierarchical forecasting problem

- We want forecasts at all levels of aggregation.
- If we model and forecast each series independently, the forecasts will almost certainly not add up.
- We need to impose constraints on the forecasts to ensure they are “coherent”.
- We need to do this in a way that is computationally efficient.

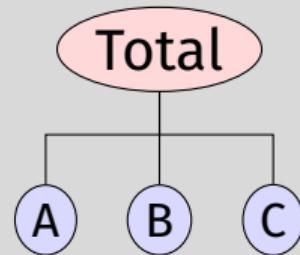
Constraint notation

Aggregation matrix

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

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$$\begin{pmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{pmatrix} = \begin{pmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{pmatrix} \mathbf{b}_t$$



Constraint matrix

$$\mathbf{C}\mathbf{y}_t = \mathbf{0}$$

$$\begin{aligned} \text{where } \mathbf{C} &= [1 \ -1 \ -1 \ -1] \\ &= [\mathbf{I}_{n_a} \ -\mathbf{A}] \end{aligned}$$

Constraint notation

Aggregation matrix A

$$\mathbf{y}_t = \begin{bmatrix} \mathbf{a}_t \\ \mathbf{b}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I}_{n_b} \end{bmatrix} \mathbf{b}_t = \mathbf{S}\mathbf{b}_t$$

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Constraint matrix C

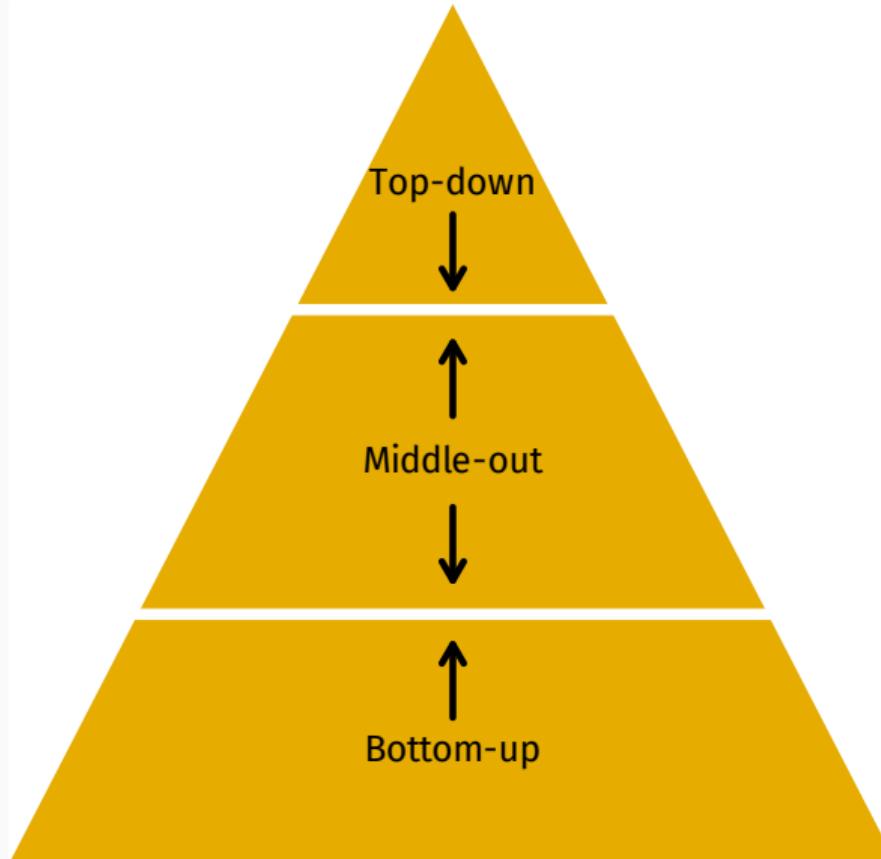
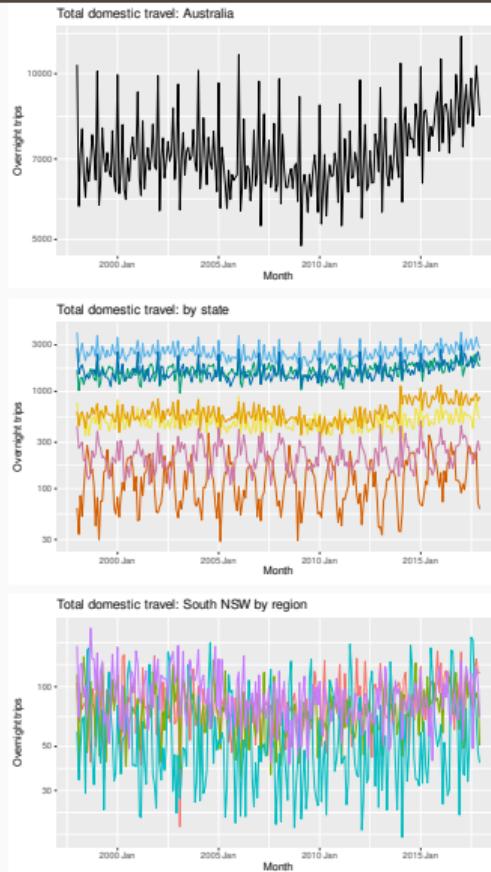
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- Constraint matrix approach more general & more parsimonious.
- $\mathbf{C} = [\mathbf{I}_{n_a} \quad -\mathbf{A}]$.
- \mathbf{S}, \mathbf{A} and \mathbf{C} may contain any real values (not just 0s and 1s).

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Hierarchical forecasting 20 years ago

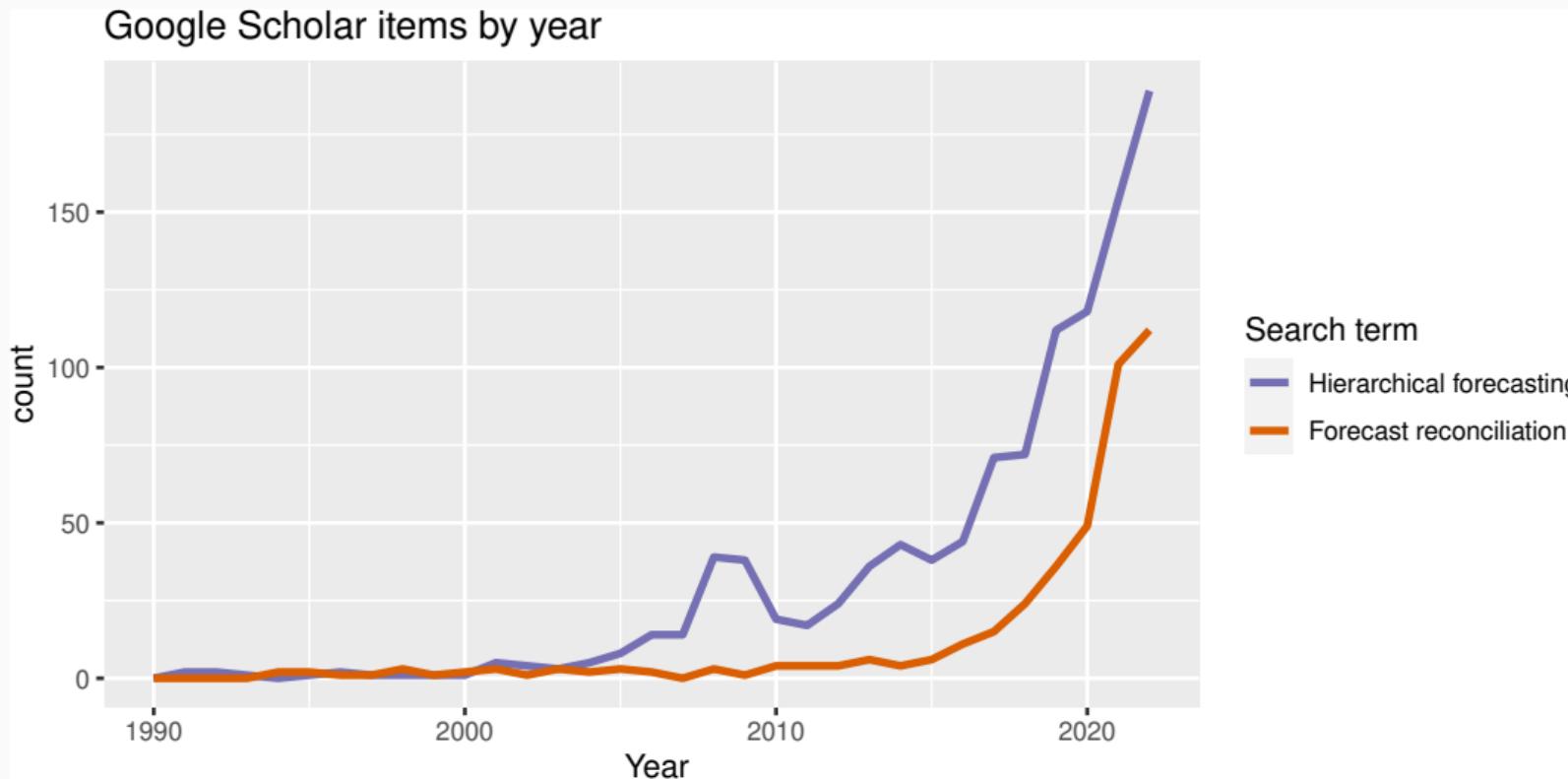


Early history of forecast reconciliation

History

- 2001:** Idea to use all available series to forecast Australia's labour market by occupation.
- 2004:** PhD student Roman Ahmed begins, co-supervised with George Athanasopoulos.
- 2006:** Presentation at ISF, Santander.
- 2007:** Pre-print of “Optimal combination forecasts for hierarchical time series”.
- 2009:** Application to Australian tourism published in IJF.
- 2010:** First version of hts package on CRAN.
- 2011:** “Optimal combination forecasts for hierarchical time series” appears in CSDA.
- 2019:** “Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization” appears in JASA.

Forecast reconciliation research



The coherent subspace

Coherent subspace

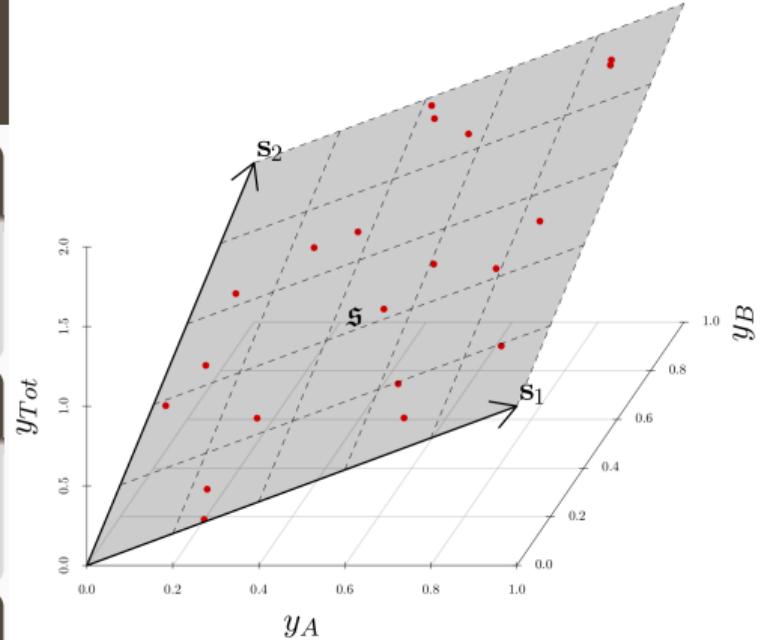
n_b -dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\mathbf{y} \in \mathfrak{s}$.

Hierarchical time series

An n -dimensional multivariate time series such that $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$ is *coherent* if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

The coherent subspace

Coherent subspace

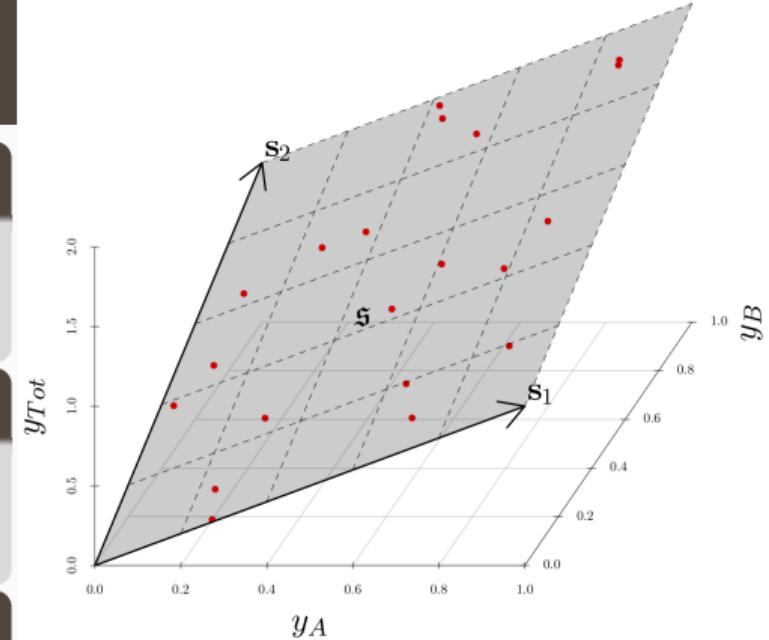
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$$y_{Tot} = y_A + y_B$$

Base forecasts

Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h -step forecasts.

The coherent subspace

Coherent subspace

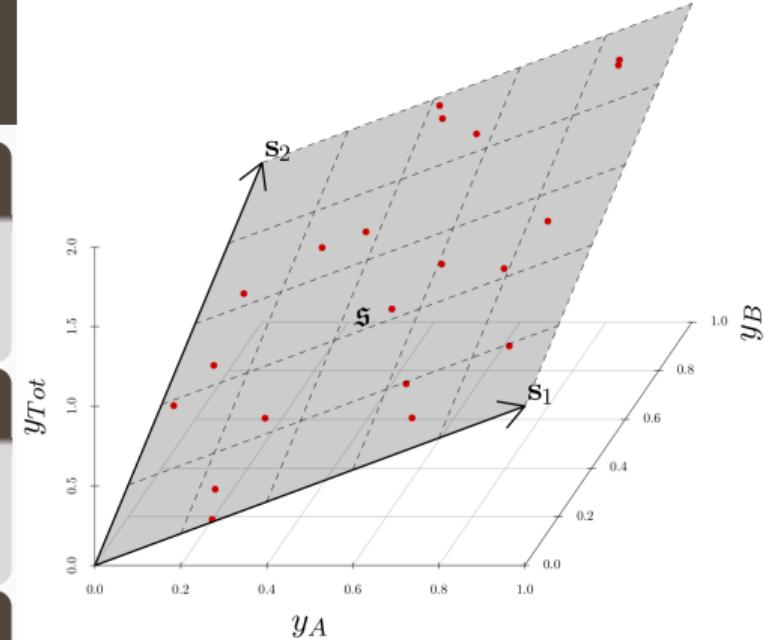
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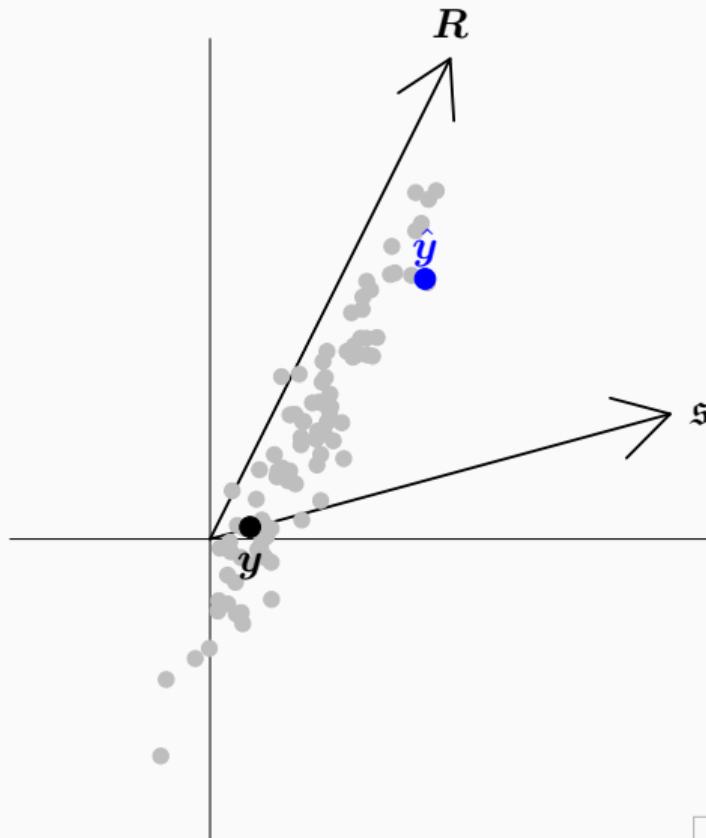
Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h -step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \chi^n \rightarrow \mathfrak{s}$.
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$ “reconciles” $\hat{\mathbf{y}}_{t+h|t}$.

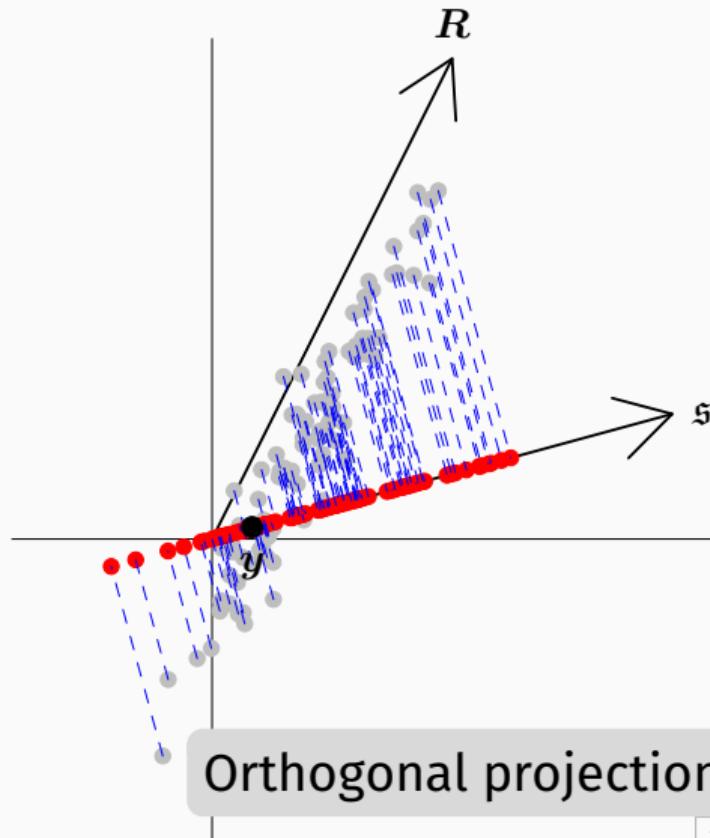
Linear projection reconciliation

- R is the most likely direction of deviations from \hat{s} .
- Grey: potential base forecasts



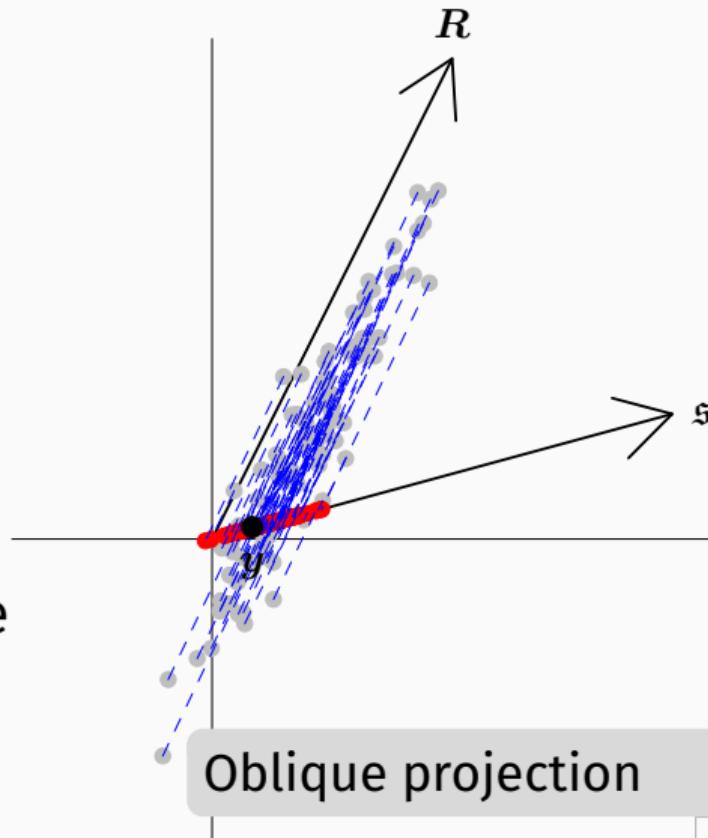
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- R is the most likely direction of deviations from \hat{s} .
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.



Linear projection reconciliation

- R is the most likely direction of deviations from \hat{s} .
- Grey: potential base forecasts
- Red: reconciled forecasts
- Orthogonal projections (i.e., OLS) lead to smallest possible adjustments of base forecasts.
- Oblique projections (i.e., MinT) give reconciled forecasts with smallest variance.



Linear projection reconciliation

$$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

- \mathbf{M} is a projection onto \mathfrak{s} if and only if $\mathbf{M}\mathbf{y} = \mathbf{y}$ for all $\mathbf{y} \in \mathfrak{s}$.
- Coherent base forecasts are unchanged since $\mathbf{M}\hat{\mathbf{y}} = \hat{\mathbf{y}}$
- If $\hat{\mathbf{y}}$ is unbiased, then $\tilde{\mathbf{y}}$ is also unbiased since

$$E(\tilde{\mathbf{y}}_{t+h|t}) = E(\mathbf{M}\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}E(\hat{\mathbf{y}}_{t+h|t}) = E(\hat{\mathbf{y}}_{t+h|t}),$$

and unbiased estimates must lie on \mathfrak{s} .

- If \mathbf{S} forms a basis set for \mathfrak{s} , then projections are of the form $\mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi$ where Ψ is a positive definite matrix.
- The projection is orthogonal if and only if $\mathbf{M}' = \mathbf{M}$. Equivalently, $\Psi = \mathbf{I}$.

Linear projection reconciliation

$$\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t}) = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

$$\text{where } \mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi = \mathbf{I} - \Psi\mathbf{C}'(\mathbf{C}\Psi\mathbf{C}')^{-1}\mathbf{C}$$

OLS:

$$\Psi = \mathbf{I}$$

WLS:

$$\Psi = \text{diagonal}$$

MinT:

$$\Psi = \mathbf{W}_h$$

- $\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T]$ is the covariance matrix of the base forecast errors.
- $\mathbf{V}_h = \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T] = \mathbf{M}\mathbf{W}_h\mathbf{M}'$ is the covariance matrix of the reconciled forecast errors.

Minimum trace reconciliation

Minimum trace (MinT) reconciliation

If \mathbf{M} is a projection, then trace of \mathbf{V}_h is minimized when $\Psi = \mathbf{W}_h$, so that

$$\mathbf{M} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1} = \mathbf{I} - \mathbf{W}_h\mathbf{C}'(\mathbf{C}\mathbf{W}_h\mathbf{C}')^{-1}\mathbf{C}$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{M} \hat{\mathbf{y}}_{T+h|T}$$

Reconciled forecasts

Base forecasts

- Trace of \mathbf{V}_h is sum of forecast variances.
- MinT is L_2 optimal amongst linear unbiased forecasts.
- How to estimate $\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_T]?$

Reconciliation method \mathbf{G}

OLS	$(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
WLS(var)	$(\mathbf{S}'\Lambda_v\mathbf{S})^{-1}\mathbf{S}'\Lambda_v$
WLS(struct)	$(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s$
MinT(sample)	$(\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}$
MinT(shrink)	$(\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\mathbf{W}_h = k_h \mathbf{W}_1$.

- $\Lambda_v = \text{diag}(\mathbf{W}_1)^{-1}$ ■ $\Lambda_s = \text{diag}(\mathbf{S}\mathbf{1})^{-1}$
- $\hat{\mathbf{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\mathbf{W}}_{\text{shr}}$ is shrinkage estimator $\tau \text{diag}(\hat{\mathbf{W}}_{\text{sam}}) + (1 - \tau)\hat{\mathbf{W}}_{\text{sam}}$ where τ selected optimally.
- Still need a good estimate of \mathbf{W}_h for forecast variance.

Example: Australian tourism

tourism

```
# A tsibble: 18,000 x 5 [1M]
# Key:      state, zone, region [75]
  month state zone      region visitors
  <mth> <chr> <chr>      <chr>     <dbl>
1 1998  Jan  NSW  Metro  NSW  Sydney    926.
2 1998  Feb  NSW  Metro  NSW  Sydney    647.
3 1998  Mar  NSW  Metro  NSW  Sydney    716.
4 1998  Apr  NSW  Metro  NSW  Sydney    621.
5 1998  May  NSW  Metro  NSW  Sydney    598.
6 1998  Jun  NSW  Metro  NSW  Sydney    601.
7 1998  Jul  NSW  Metro  NSW  Sydney    720.
8 1998  Aug  NSW  Metro  NSW  Sydney    645.
9 1998  Sep  NSW  Metro  NSW  Sydney    633.
10 1998  Oct  NSW  Metro  NSW  Sydney   771.
# i 17,990 more rows
```

Example: Australian tourism

```
tourism_agg <- tourism |>  
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
# A tsibble: 26,400 x 5 [1M]  
# Key:      state, zone, region [110]  
  month state       zone       region     visitors  
  <mth> <chr*>    <chr*>    <chr*>    <dbl>  
1 1998 Jan <aggregated> <aggregated> <aggregated> 10376.  
2 1998 Feb <aggregated> <aggregated> <aggregated> 5746.  
3 1998 Mar <aggregated> <aggregated> <aggregated> 7129.  
4 1998 Apr <aggregated> <aggregated> <aggregated> 7939.  
5 1998 May <aggregated> <aggregated> <aggregated> 6552.  
6 1998 Jun <aggregated> <aggregated> <aggregated> 5969.  
7 1998 Jul <aggregated> <aggregated> <aggregated> 7041.  
8 1998 Aug <aggregated> <aggregated> <aggregated> 6382.  
9 1998 Sep <aggregated> <aggregated> <aggregated> 6907.  
10 1998 Oct <aggregated> <aggregated> <aggregated> 7744.  
# i 26,390 more rows
```

Example: Australian tourism

```
fit <- tourism_agg |>  
  filter(year(month) <= 2015) |>  
  model(ets = ETS(visitors))
```

```
# A mable: 110 x 4  
# Key: state, zone, region [110]  
  
  state   zone           region          ets  
  <chr*> <chr*>        <chr*>        <model>  
1 NSW     ACT            Canberra       <ETS(M,N,A)>  
2 NSW     ACT            <aggregated> <ETS(M,N,A)>  
3 NSW     Metro NSW      Central Coast <ETS(M,N,M)>  
4 NSW     Metro NSW      Sydney         <ETS(M,N,A)>  
5 NSW     Metro NSW      <aggregated> <ETS(M,N,A)>  
6 NSW     North Coast NSW Hunter       <ETS(M,N,M)>  
7 NSW     North Coast NSW North Coast NSW <ETS(M,N,M)>  
8 NSW     North Coast NSW <aggregated> <ETS(M,N,M)>  
9 NSW     North NSW       Blue Mountains <ETS(M,N,A)>  
10 NSW    North NSW      Central NSW    <ETS(M,N,M)>
```

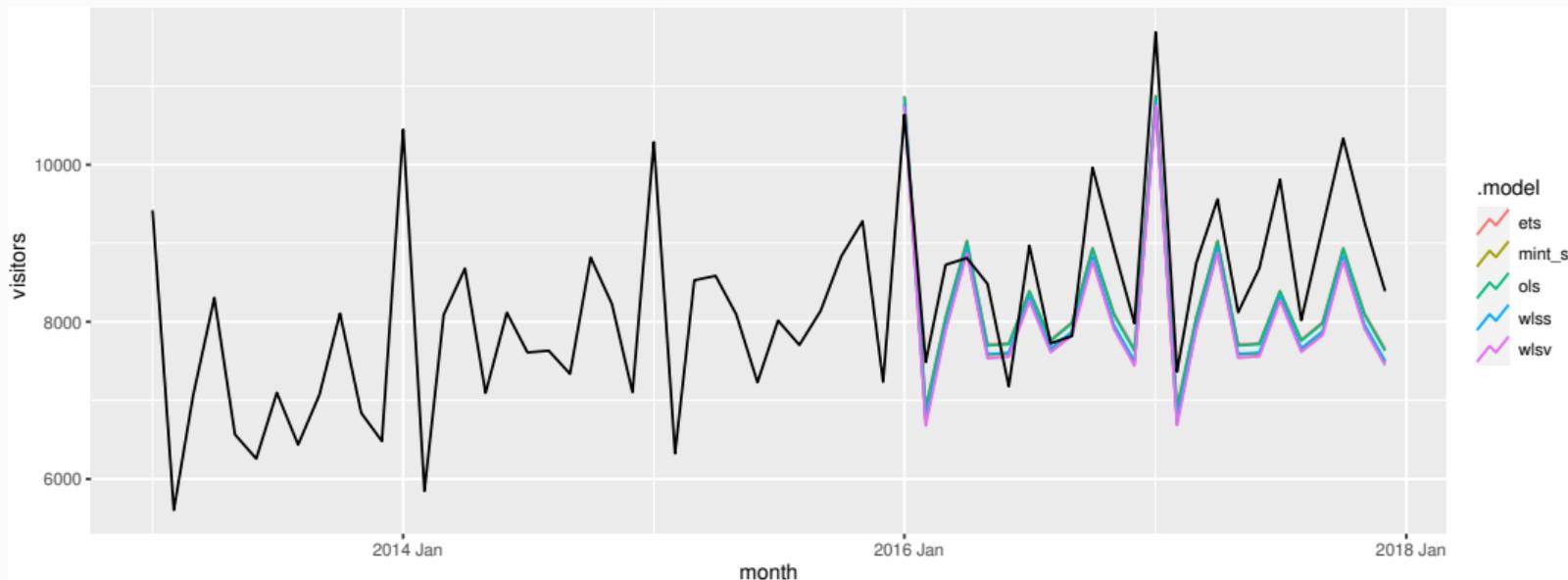
Example: Australian tourism

```
fc <- fit |>
  reconcile(
    ols = min_trace(ets, method = "ols"),
    wlsv = min_trace(ets, method = "wls_var"),
    wlss = min_trace(ets, method = "wls_struct"),
    # mint_c = min_trace(ets, method="mint_cov"),
    mint_s = min_trace(ets, method = "mint_shrink"),
  ) |>
  forecast(h = "2 years")
```

```
# A fable: 13,200 x 7 [1M]
# Key:      state, zone, region, .model [550]
  state   zone   region   .model   month   visitors .mean
  <chr*> <chr*> <chr*>   <chr>    <mth>     <dist> <dbl>
1 NSW     ACT     Canberra  ets     2016 Jan N(202, 1437) 202.
2 NSW     ACT     Canberra  ets     2016 Feb N(160, 912) 160.
3 NSW     ACT     Canberra  ets     2016 Mar N(204, 1489) 204.
4 NSW     ACT     Canberra  ets     2016 Apr N(207, 1527) 207.
```

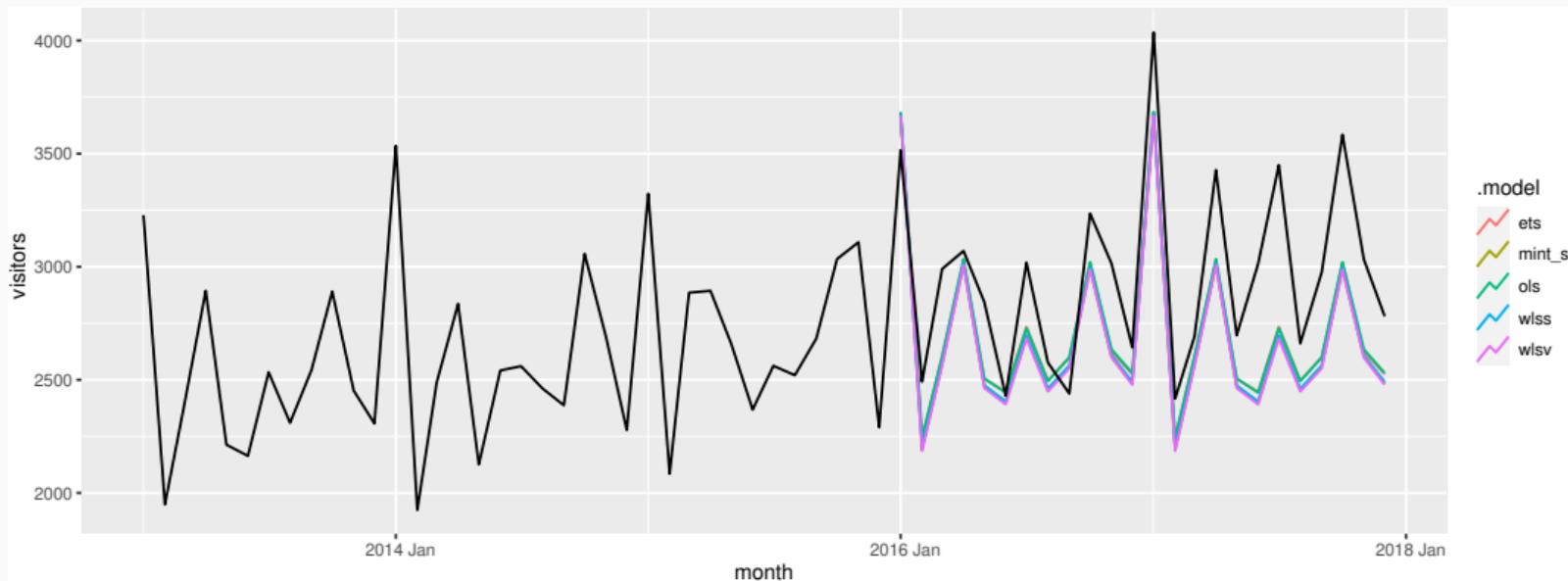
Example: Australian tourism

```
fc |>  
  filter(is_aggregated(state)) |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



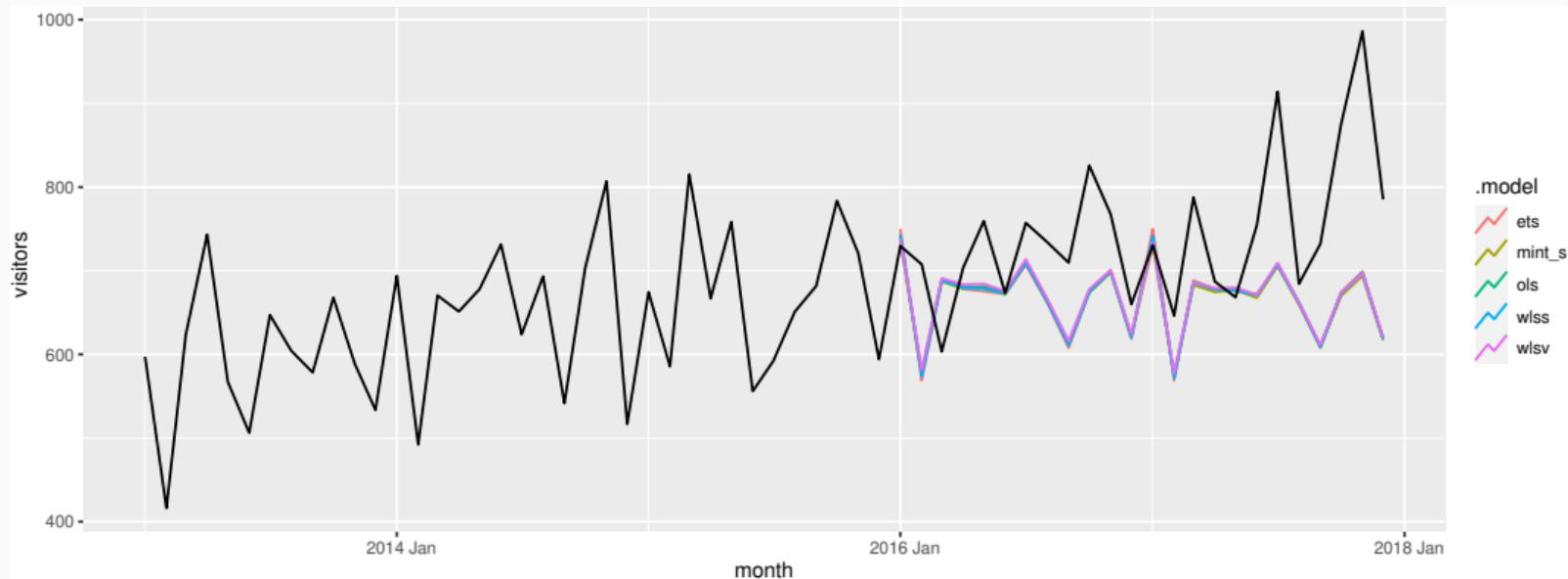
Example: Australian tourism

```
fc |>  
  filter(state == "NSW" & is_aggregated(zone)) |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



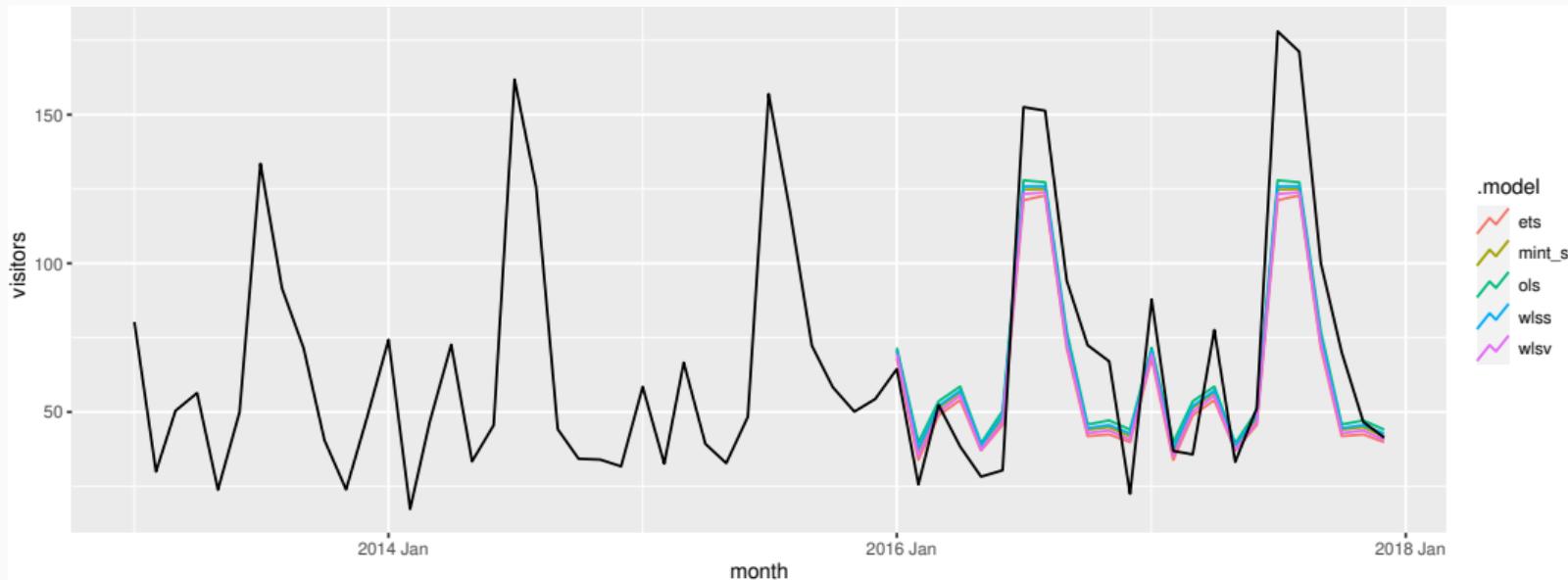
Example: Australian tourism

```
fc |>  
  filter(region == "Melbourne") |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



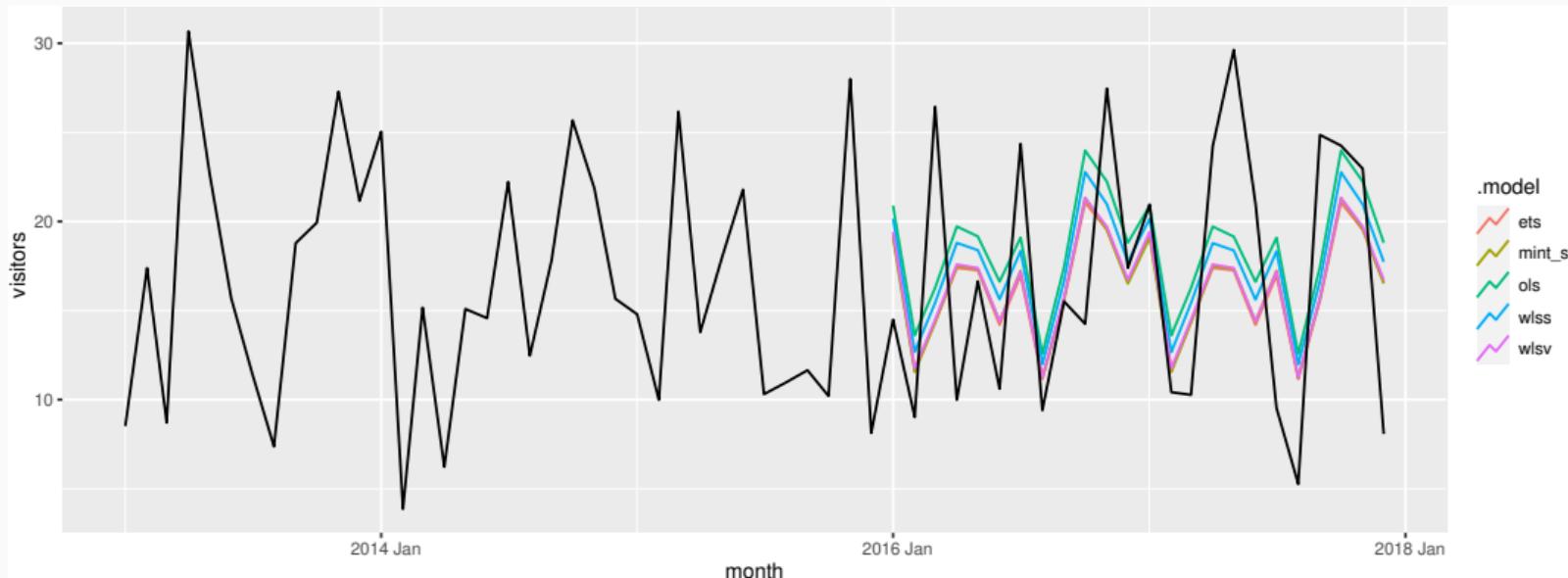
Example: Australian tourism

```
fc |>  
  filter(region == "Snowy Mountains") |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Example: Australian tourism

```
fc |>  
  filter(region == "Barossa") |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Performance evaluation

$$\text{MASE} = \text{mean}(|q_j|)$$

$$q_j = \frac{e_j}{\frac{1}{T-m} \sum_{t=m+1}^T |y_t - y_{t-m}|}$$

- y_t = observation for period t
- e_j = forecast error for forecast horizon j
- T = size of training set
- $m = 12$

Performance evaluation

$$\text{RMSSE} = \sqrt{\text{mean}(q_j^2)}$$

$$q_j^2 = \frac{e_j^2}{\frac{1}{T-m} \sum_{t=m+1}^T (y_t - y_{t-m})^2}$$

- y_t = observation for period t
- e_j = forecast error for forecast horizon j
- T = size of training set
- $m = 12$

Example: Australian tourism

```
fc |>  
accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE))
```

```
# A tibble: 550 x 7  
  .model state zone          region      .type  mase  rmsse  
  <chr>   <chr> <chr>        <chr>      <chr> <dbl> <dbl>  
1 ets     NSW    ACT         Canberra    Test   0.866 0.835  
2 ets     NSW    ACT         <aggregated> Test   0.866 0.835  
3 ets     NSW    Metro NSW Central Coast Test   0.777 0.747  
4 ets     NSW    Metro NSW Sydney      Test   1.20   1.16  
5 ets     NSW    Metro NSW <aggregated> Test   1.18   1.18  
6 ets     NSW    North Coast NSW Hunter    Test   1.33   1.21  
7 ets     NSW    North Coast NSW North Coast NSW Test   0.845 0.884  
8 ets     NSW    North Coast NSW <aggregated> Test   1.11   1.02  
9 ets     NSW    North NSW       Blue Mountains Test   1.02   1.02  
10 ets    NSW    North NSW      Central NSW   Test   1.30   1.18  
# i 540 more rows
```

Example: Australian tourism

```
fc |>  
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>  
  group_by(.model) |>  
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>  
  arrange(rmsse)
```

```
# A tibble: 5 x 3  
  .model    mase   rmsse  
  <chr>    <dbl>  <dbl>  
1 ols      0.930  0.926  
2 wlss     0.949  0.948  
3 mint_s   0.953  0.954  
4 wlsv     0.964  0.965  
5 ets      0.968  0.968
```

Example: Australian tourism

```
fc |>  
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>  
  group_by(.model) |>  
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>  
  arrange(rmsse)
```

```
# A tibble: 5 x 3  
  .model    mase   rmsse  
  <chr>    <dbl>  <dbl>  
1 ols      0.930  0.926  
2 wlss     0.949  0.948  
3 mint_s   0.953  0.954  
4 wlsv     0.964  0.965  
5 ets      0.968  0.968
```

■ Overall, every reconciliation method is better than the base ETS forecasts.

Example: Australian tourism

```
# A tibble: 20 x 4
# Groups:   .model [5]
  .model level     mase rmsse
  <chr>  <fct>    <dbl> <dbl>
1 ets    National  1.44  1.27
2 ols    National  1.46  1.29
3 wlss   National  1.61  1.43
4 mint_s National  1.64  1.45
5 wlsv   National  1.69  1.49
6 ols    State     1.07  1.08
7 ets    State     1.10  1.11
8 wlss   State     1.13  1.14
9 mint_s State     1.15  1.15
10 wlsv  State     1.18  1.17
11 ols    Zone      0.954 0.948
12 wlss   Zone      0.987 0.980
13 mint_s Zone      0.995 0.988
14 ets    Zone      1.01  0.999
15 wlsv  Zone      1.01  1.00
16 ols    Region    0.901 0.895
17 wlss   Region    0.910 0.907
18 mint_s Region    0.911 0.911
19 wlsv  Region    0.917 0.919
20 ets    Region    0.935 0.938
```

Example: Australian tourism

```
# A tibble: 20 x 4
# Groups:   .model [5]
  .model level      mase rmsse
  <chr>  <fct>     <dbl> <dbl>
1 ets    National  1.44  1.27
2 ols    National  1.46  1.29
3 wlss   National  1.61  1.43
4 mint_s National  1.64  1.45
5 wlsv   National  1.69  1.49
6 ols    State     1.07  1.08
7 ets    State     1.10  1.11
8 wlss   State     1.13  1.14
9 mint_s State     1.15  1.15
10 wlsv   State    1.18  1.17
11 ols    Zone      0.954 0.948
12 wlss   Zone      0.987 0.980
13 mint_s Zone      0.995 0.988
14 ets    Zone      1.01  0.999
15 wlsv   Zone      1.01  1.00
16 ols    Region    0.901 0.895
17 wlss   Region    0.910 0.907
18 mint_s Region    0.911 0.911
19 wlsv   Region    0.917 0.919
20 ets    Region    0.935 0.938
```

- OLS is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

Distance reducing property

Let $\|\mathbf{u}\|_{\Psi} = \mathbf{u}'\Psi\mathbf{u}$. Then

$$\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}\|_{\Psi} \leq \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|_{\Psi}$$

- Ψ -projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure.*
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.

$$\begin{aligned}\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h}\|_2^2 &= \|\mathbf{M}(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h})\|_2^2 \\ &\leq \|\mathbf{M}\|_2^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h}\|_2^2 \\ &= \sigma_{\max}^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h}\|_2^2\end{aligned}$$

- σ_{\max} is the largest eigenvalue of \mathbf{M}
- $\sigma_{\max} \geq 1$ as \mathbf{M} is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

$$\begin{aligned} & \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]'[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]\right) \end{aligned}$$

Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left(\mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left(\mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Solve via quadratic programming:

$$\min_{\mathbf{b}} [\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - 2\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}] \quad \text{s.t. } \mathbf{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Non-negative forecasts

$$\begin{aligned} \min_{\mathbf{G}_h} \text{tr} & \left(\mathbb{E}[\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}]' [\mathbf{y}_{t+h} - \mathbf{S}\mathbf{G}_h \hat{\mathbf{y}}_{t+h|t}] \right) \\ \text{such that } & \mathbf{b}_{t+h|t} = \mathbf{G}_h \hat{\mathbf{y}}_{t+h|t} \geq 0 \end{aligned}$$

Solve via quadratic programming:

$$\min_{\mathbf{b}} [\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - 2\mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}] \quad \text{s.t. } \mathbf{b} \geq 0$$

(Wickramasuriya, Turlach, and Hyndman, 2020)

Set-negative-to-zero heuristic solution

- Negative reconciled forecasts at bottom level set to zero
- Remaining forecasts computed via aggregation
(Di Fonzo and Girolimetto, 2023c)

Reconciliation and regularization

Mishchenko, Montgomery, and Vaggi (2019):
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{y}}_t} \sum_{t=1}^T \|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_2 + \lambda \sum_{t=1}^T \|\hat{\mathbf{y}}_t - \mathbf{s}_t \hat{\mathbf{b}}_t\|_2$$

Reconciliation and regularization

Mishchenko, Montgomery, and Vaggi (2019):
Optimize all forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{y}}_t} \sum_{t=1}^T \|\mathbf{y}_t - \hat{\mathbf{y}}_t\|_2 + \lambda \sum_{t=1}^T \|\hat{\mathbf{y}}_t - \mathbf{s}_t \hat{\mathbf{b}}_t\|_2$$

Shiratori, Kobayashi, and Takano (2020):
Optimize bottom level forecasts with an incoherence penalty

$$\min_{\hat{\mathbf{b}}_t} \sum_{t=1}^T \|\mathbf{b}_t - \hat{\mathbf{b}}_t\|_2 + \sum_{t=1}^T \|\Lambda(\mathbf{a}_t - \mathbf{A}_t \hat{\mathbf{b}}_t)\|_2$$

Outline

- 1 Hierarchical time series notation
- 2 Optimal linear forecast reconciliation
- 3 Probabilistic forecast reconciliation
- 4 Cross-temporal probabilistic forecast reconciliation
- 5 Final comments

The coherent subspace

Coherent subspace

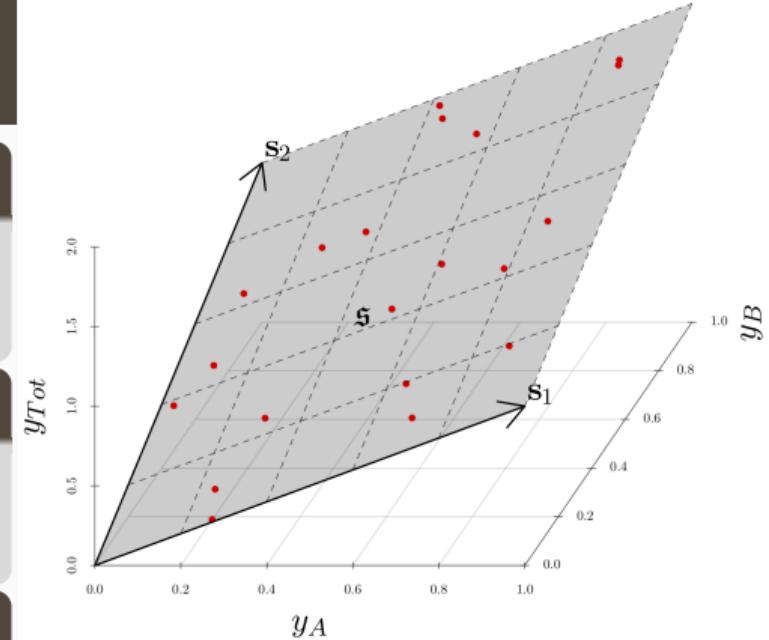
m -dimensional linear subspace $\mathfrak{s} \subset \chi^n$ for which linear constraints hold for all $\mathbf{y} \in \mathfrak{s}$.

Hierarchical time series

An n -dimensional multivariate time series such that $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$ is *coherent* if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

Base forecasts

Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h -step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \chi^n \rightarrow \mathfrak{s}$.
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$ “reconciles” $\hat{\mathbf{y}}_{t+h|t}$.

Coherent probabilistic forecasts

Coherent probabilistic forecasts

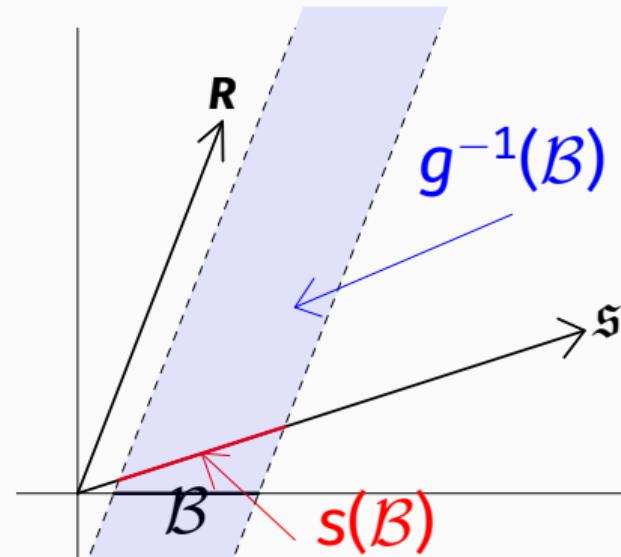
A probability triple $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \check{\nu})$ is coherent with the bottom probability triple $(\chi^m, \mathcal{F}_{\chi^m}, \nu)$, if

$$\check{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathcal{F}_{\chi^m}$$

i.e., probability of any point not on \mathfrak{s} is zero.

Probabilistic forecast reconciliation

Let $(\chi^n, \mathcal{F}_{\chi^n}, \hat{\nu})$ be the base forecast. Then the reconciled probability distribution $\check{\nu}$ is a transformation of $\hat{\nu}$ that is coherent on $\mathcal{F}_{\mathfrak{s}}$.



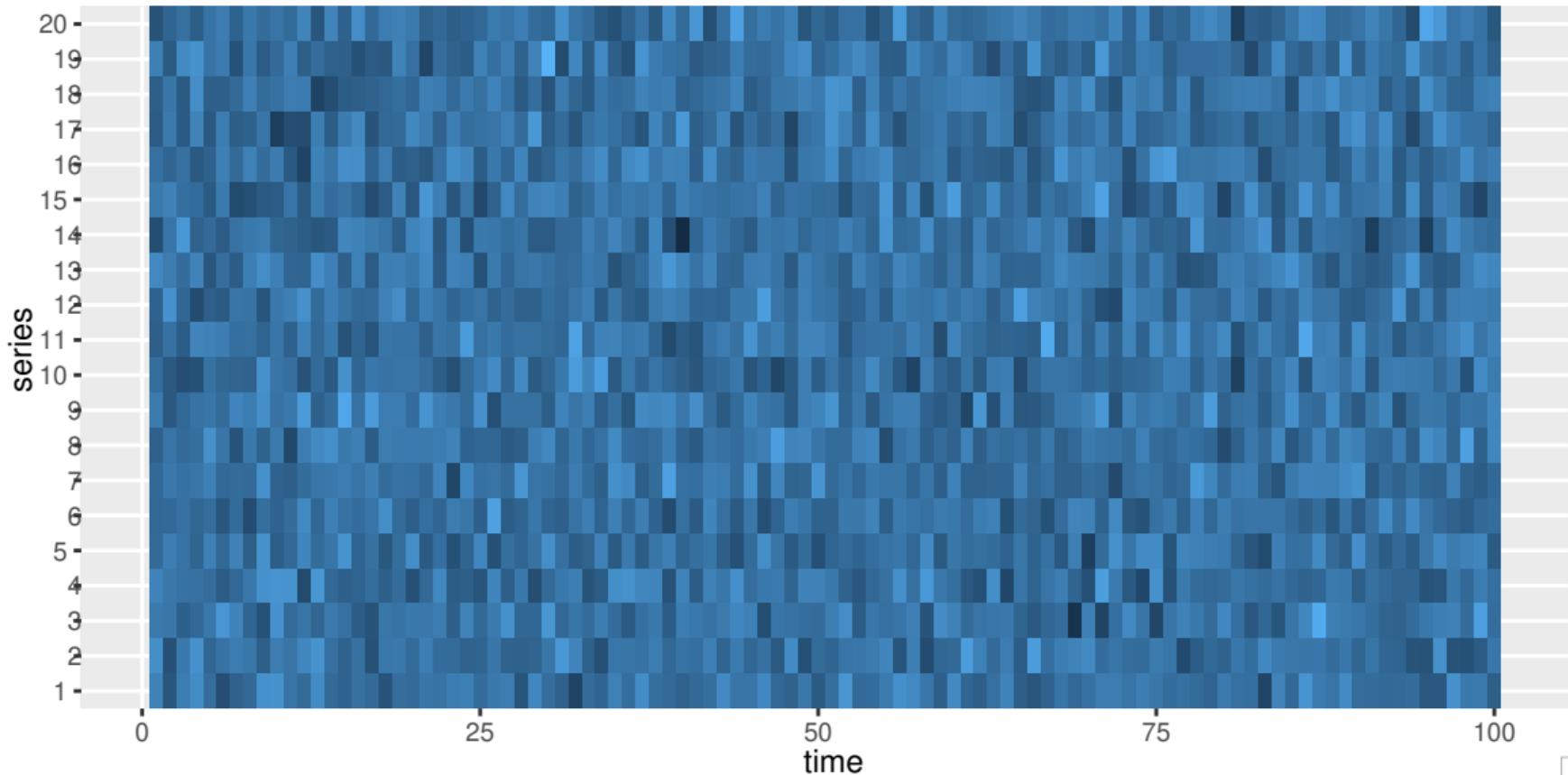
$$\psi = s \circ g$$

Simulation from a reconciled distribution

Suppose that $(\hat{\mathbf{y}}^{[1]}, \dots, \hat{\mathbf{y}}^{[L]})$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $(\tilde{\mathbf{y}}^{[1]}, \dots, \tilde{\mathbf{y}}^{[L]})$ where $\tilde{\mathbf{y}}^{[\ell]} := \psi(\hat{\mathbf{y}}^{[\ell]})$ for $\ell = 1, \dots, L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

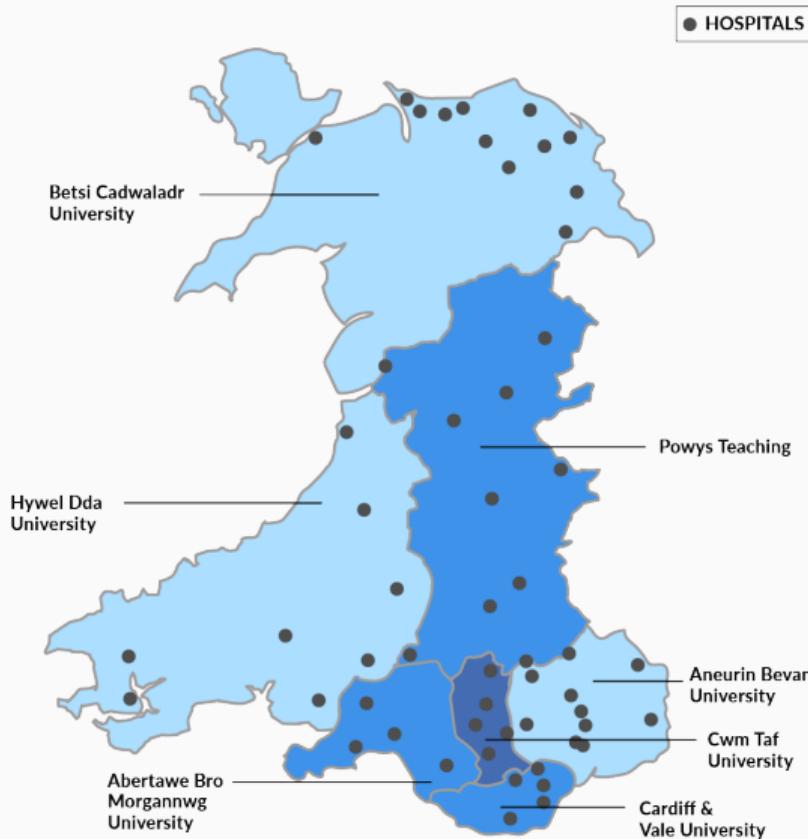
- Simulate future sample paths for each series, by simulating from each model using a multivariate bootstrap of the residuals (to preserve cross-correlations).
- Reconcile the sample paths.
- The reconciled sample paths are a sample from the reconciled distribution.

Simulation from a reconciled distribution



Simulation from a reconciled distribution

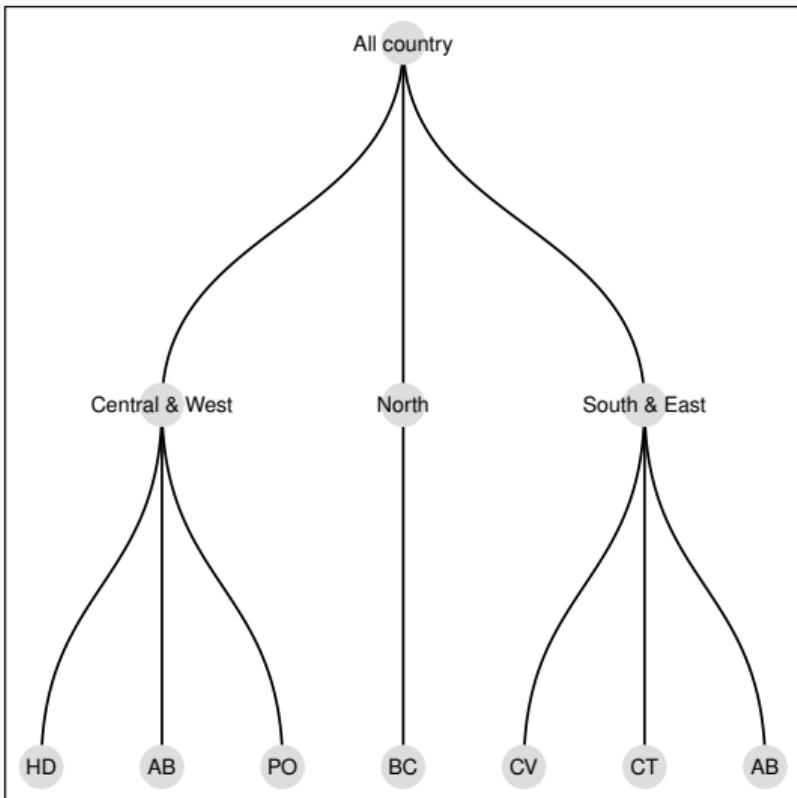
Wales Health Board Areas



Data

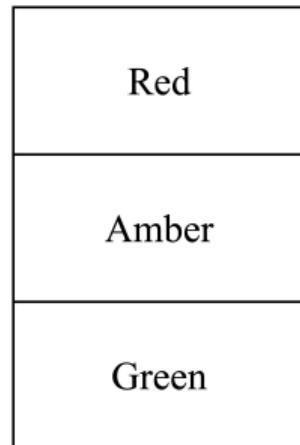
- Daily number of attended incidents:
1 October 2015 – 31 July 2019
- Disaggregated by:
 - ▶ control area
 - ▶ health board
 - ▶ priority
 - ▶ nature of incidents
- 2,142,000 rows observations from 1,530 time series.

Data structure



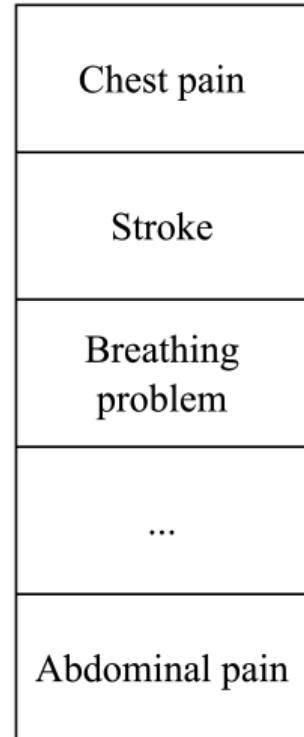
*

Priority



*

Nature of incident



Data structure

Level	Number of series
All country	1
Control	3
Health board	7
Priority	3
Priority * Control	9
Priority * Health board	21
Nature of incident	35
Nature of incident * Control	105
Nature of incident * Health board	245
Priority * Nature of incident	104
Control * Priority * Nature of incident	306
Control * Health board * Priority * Nature of incident (Bottom level)	691
Total	1530

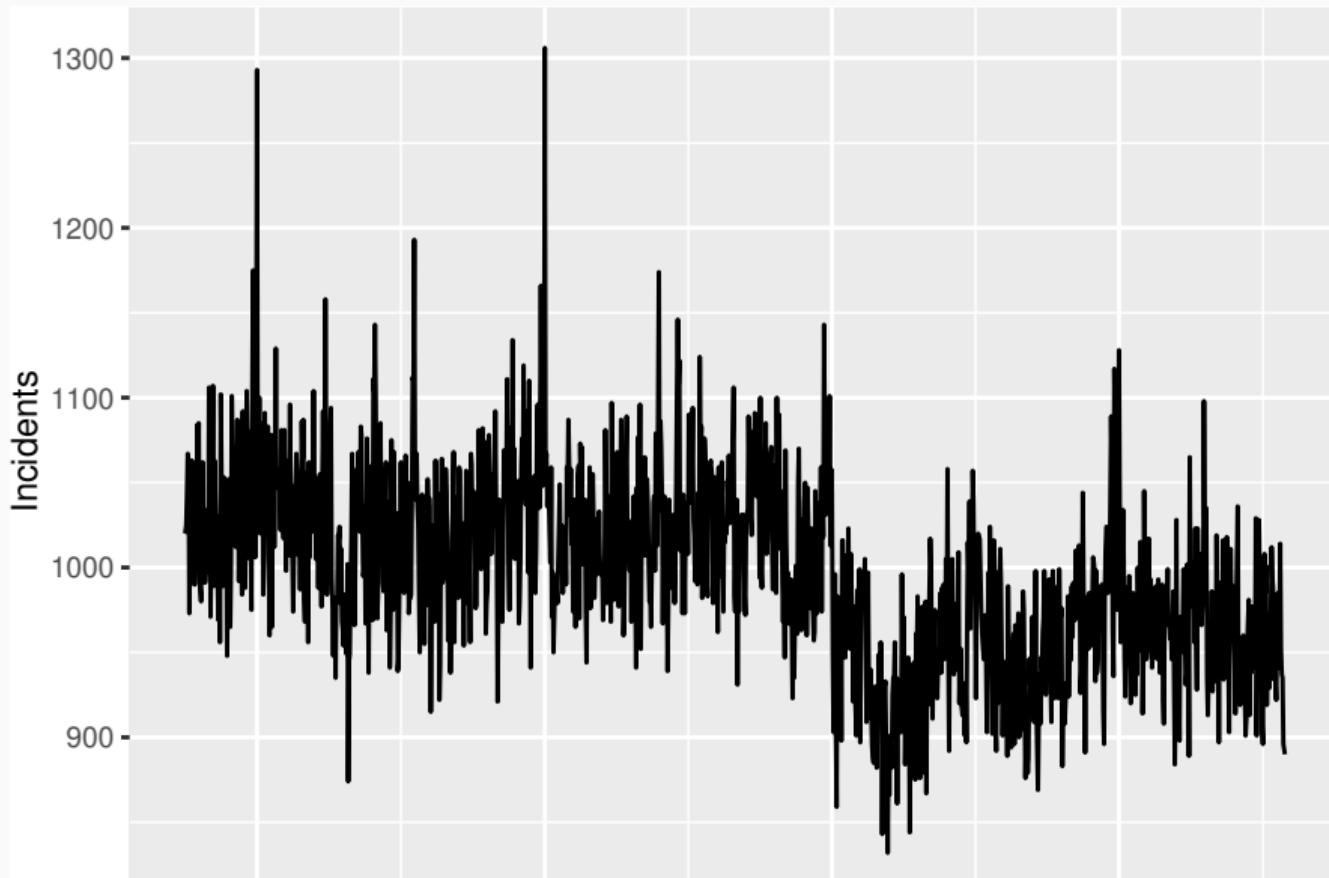
Data

```
# A tsibble: 2,142,000 x 6 [1D]
# Key:      region, category, nature, lhb [1,530]
  date      region     category     nature      lhb      incident
  <date>    <chr*>     <chr*>     <chr*>     <chr*>     <dbl>
1 2015-10-01 <aggregated> <aggregated> <aggregated> <aggregated> 1020
2 2015-10-02 <aggregated> <aggregated> <aggregated> <aggregated> 1021
3 2015-10-03 <aggregated> <aggregated> <aggregated> <aggregated> 1025
4 2015-10-04 <aggregated> <aggregated> <aggregated> <aggregated> 1043
5 2015-10-05 <aggregated> <aggregated> <aggregated> <aggregated> 1067
6 2015-10-06 <aggregated> <aggregated> <aggregated> <aggregated> 1063
7 2015-10-07 <aggregated> <aggregated> <aggregated> <aggregated> 973
8 2015-10-08 <aggregated> <aggregated> <aggregated> <aggregated> 1057
9 2015-10-09 <aggregated> <aggregated> <aggregated> <aggregated> 1026
10 2015-10-10 <aggregated> <aggregated> <aggregated> <aggregated> 1063
# i 2,141,990 more rows
```

Data

```
# A tsibble: 2,142,000 x 6 [1D]
# Key:      region, category, nature, lhb [1,530]
  date      region category nature    lhb      incident
  <date>    <chr*>  <chr*>   <chr*>  <chr*>    <dbl>
1 2015-10-01 C       Amber   ABDOMINAL HD        0
2 2015-10-01 C       Amber   ABDOMINAL PO        0
3 2015-10-01 C       Amber   ABDOMINAL SB        0
4 2015-10-01 C       Amber   ABDOMINAL <aggregated> 0
5 2015-10-01 C       Amber   ALLERGIES HD        0
6 2015-10-01 C       Amber   ALLERGIES PO        1
7 2015-10-01 C       Amber   ALLERGIES SB        0
8 2015-10-01 C       Amber   ALLERGIES <aggregated> 1
9 2015-10-01 C       Amber   ANIMALBIT HD        0
10 2015-10-01 C      Amber   ANIMALBIT PO       0
# i 2,141,990 more rows
```

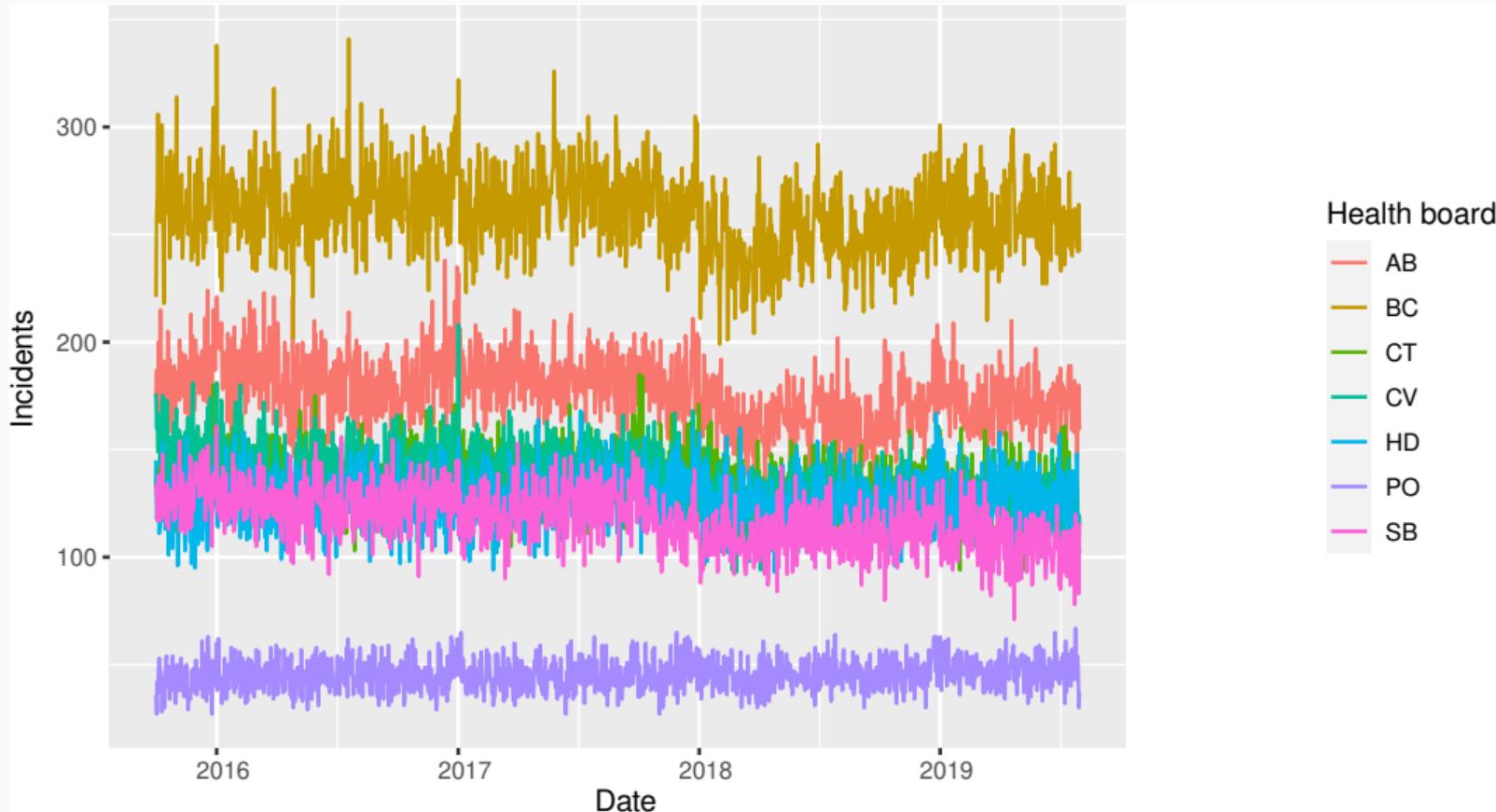
Aggregated daily incidents



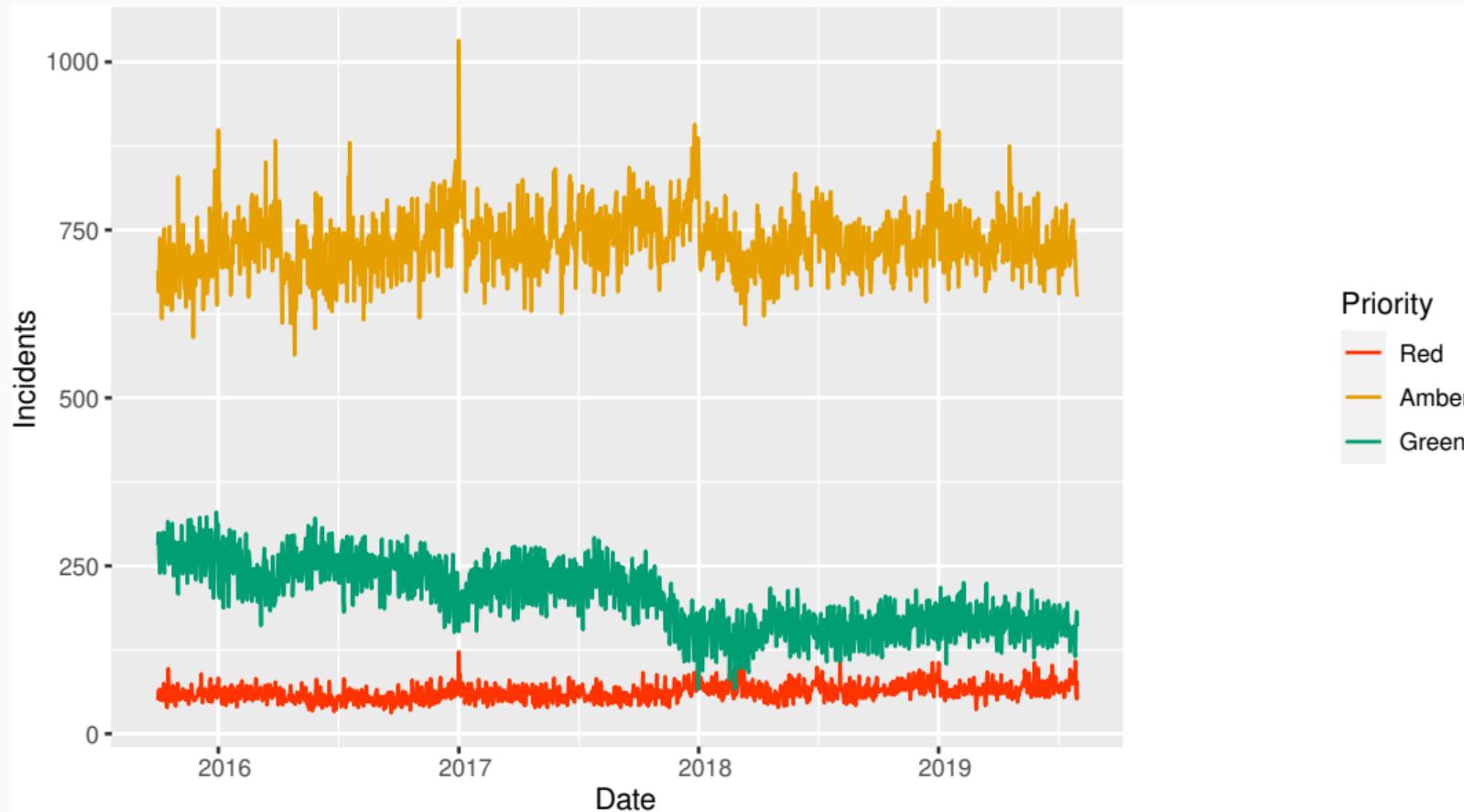
Daily incidents by control area



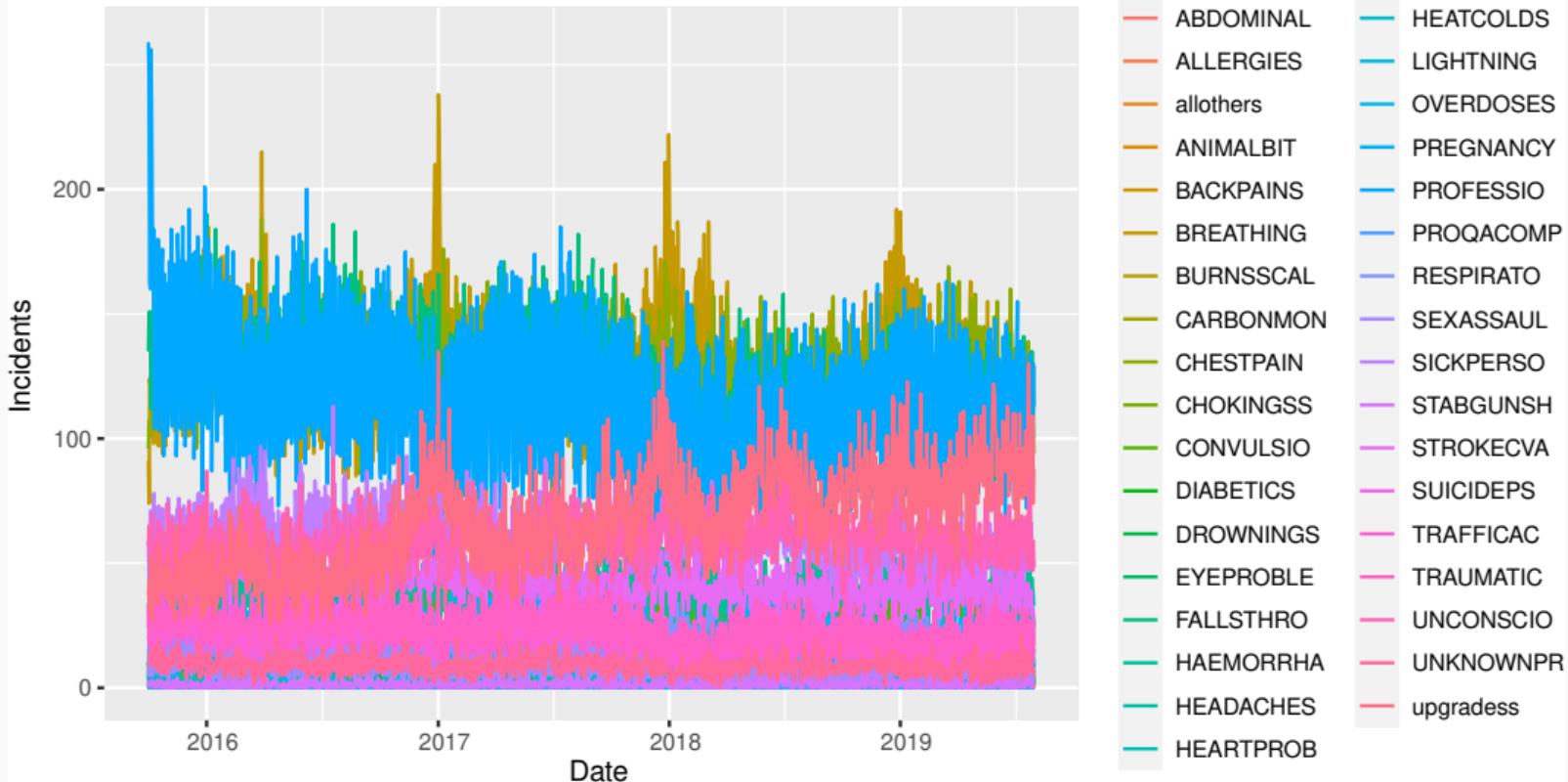
Data incidents by health board



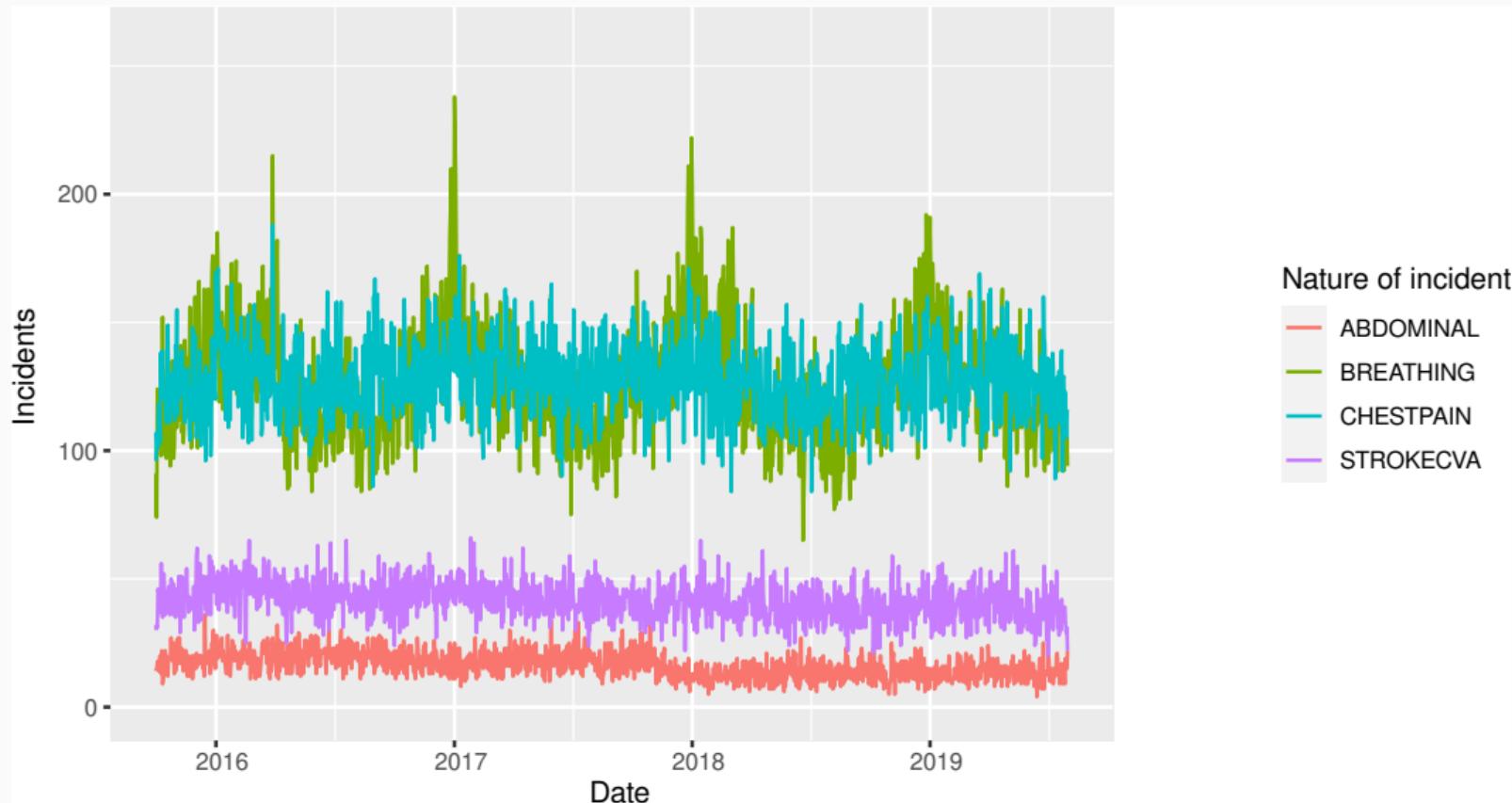
Data incidents by priority



Data incidents by nature of incident



Data incidents by nature of incident

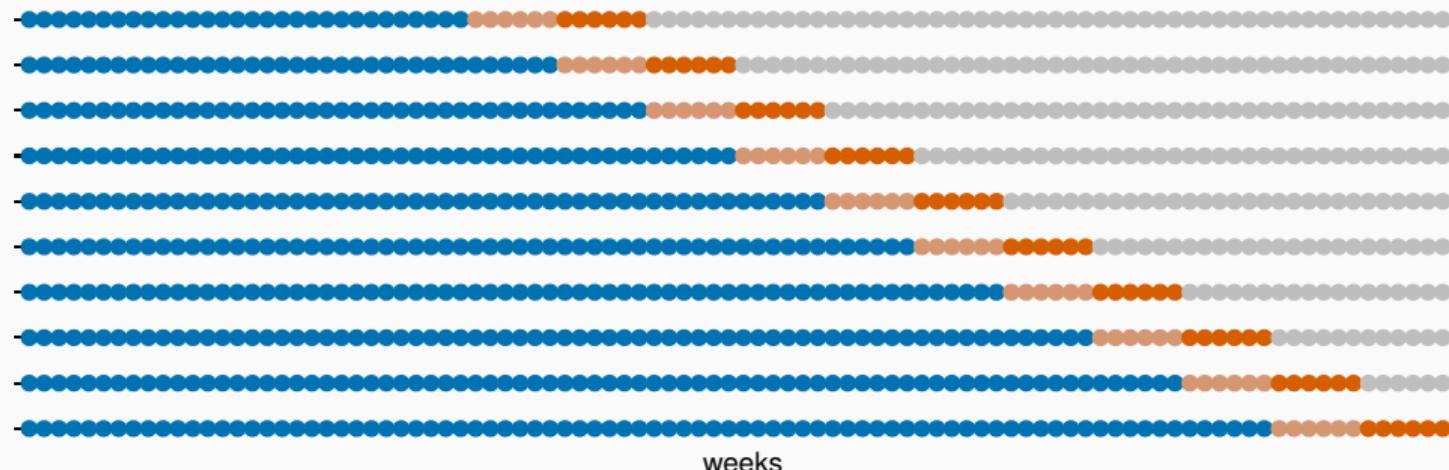


Forecasting methods

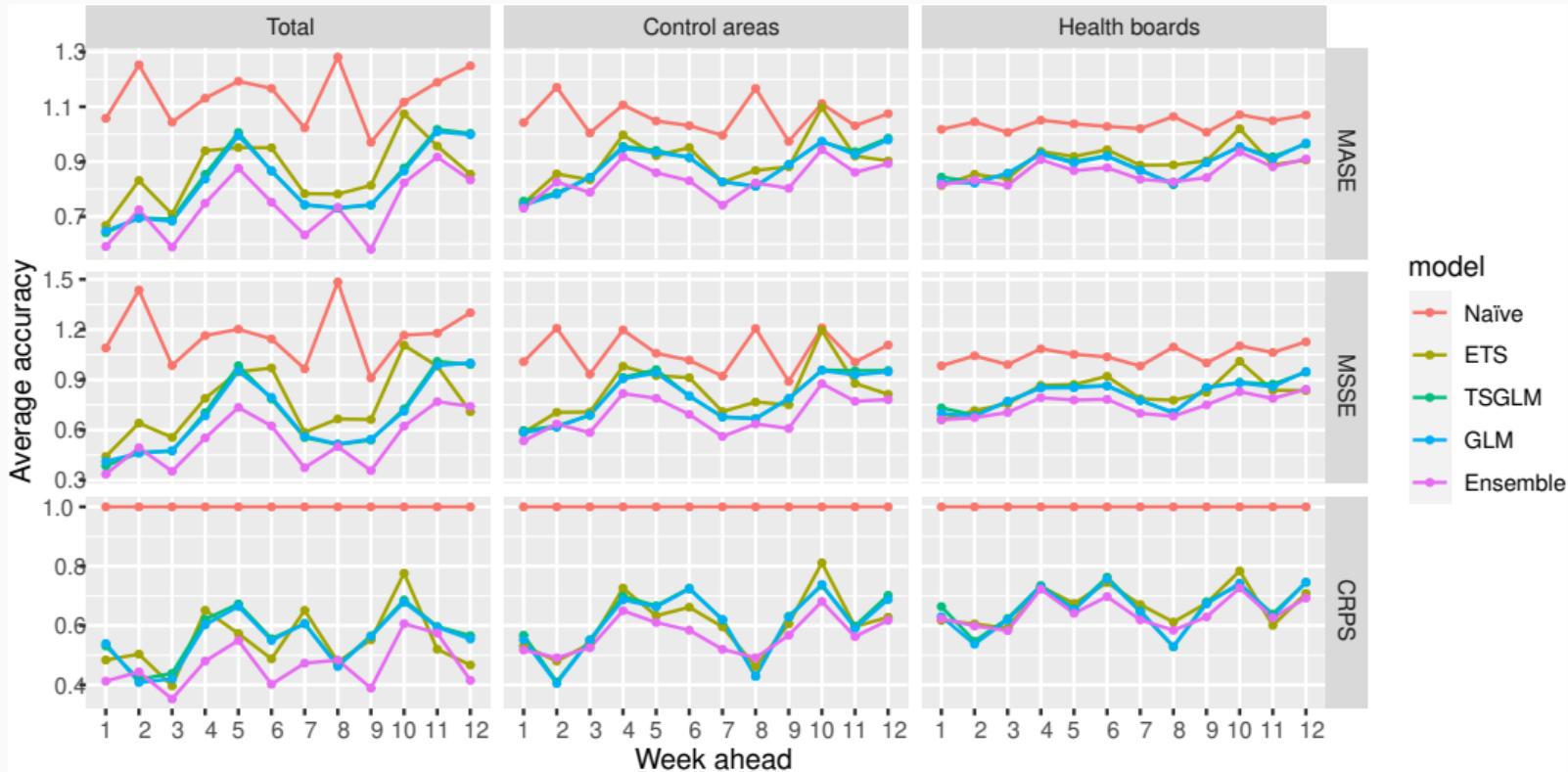
- 1 **Naïve:** Empirical distribution of past daily attended incidents.
- 2 **ETS:** Exponential Smoothing State Space models.
- 3 **GLM:** Poission Regression with spline trend, day of the week, annual Fourier seasonality, public holidays, school holidays, Christmas Day, New Year's Day.
- 4 **TSGLM:** Poisson Regression with same covariates plus three autoregressive terms.
- 5 **Ensemble:** Mixture distribution of 1–4.

Performance evaluation

- Ten-fold time series cross-validation
- Forecast horizon of 1–84 days
- Each training set contains an additional 42 days.
- Forecasts at 43–84 days correspond to planning horizon.



Forecast accuracy



Forecast accuracy: 43–84 days ahead

Method	Model	MSSE			
		Total	Control areas	Health boards	Bottom
Base	Naïve	1.169	1.056	1.062	1.031
Base	ETS	0.979	0.875	0.816	0.975
Base	GLM	0.813	0.897	0.875	1.009
Base	TSGLM	0.822	0.901	0.875	1.050
Base	Ensemble	0.599	0.729	0.774	0.993
MinT	Naïve	1.168	1.057	1.062	2.095
MinT	ETS	0.785	0.852	0.845	0.994
MinT	GLM	0.720	0.827	0.837	1.803
MinT	TSGLM	0.722	0.833	0.839	1.851
MinT	Ensemble	0.560	0.706	0.765	1.557

Forecast accuracy: 43–84 days ahead

		MASE			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	1.139	1.059	1.047	1.019
Base	ETS	0.963	0.930	0.899	1.038
Base	GLM	0.910	0.940	0.923	1.002
Base	TSGLM	0.911	0.939	0.924	1.005
Base	Ensemble	0.782	0.856	0.876	1.008
MinT	Naïve	1.138	1.059	1.047	2.651
MinT	ETS	0.877	0.916	0.915	1.289
MinT	GLM	0.848	0.901	0.902	2.493
MinT	TSGLM	0.852	0.903	0.903	2.513
MinT	Ensemble	0.753	0.844	0.872	2.260

Forecast accuracy: 43–84 days ahead

		CRPS			
Method	Model	Total	Control areas	Health boards	Bottom
Base	Naïve	30.387	10.882	5.500	0.302
Base	ETS	14.309	6.074	3.476	0.244
Base	GLM	15.396	6.253	3.576	0.244
Base	TSGLM	15.316	6.227	3.575	0.245
Base	Ensemble	12.978	5.727	3.430	0.243
MinT	Naïve	30.368	10.902	5.498	0.313
MinT	ETS	13.515	5.967	3.547	0.243
MinT	GLM	13.839	5.917	3.453	0.246
MinT	TSGLM	14.000	5.947	3.455	0.248
MinT	Ensemble	12.585	5.728	3.426	0.247

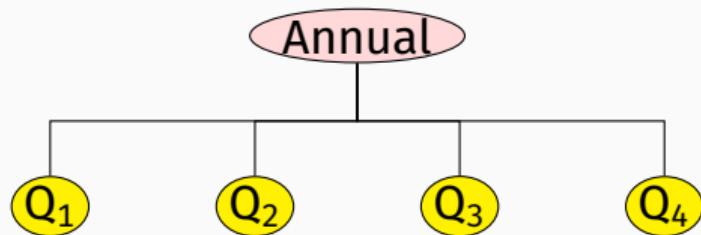
Forecast accuracy

- Ensemble mixture distributions give better forecasts than any component methods.
- Forecast reconciliation improves forecast accuracy, even when some component methods are quite poor.
- The ensemble without the Naïve method was worse.
- Forecast reconciliation allows coordinated planning and resource allocation.

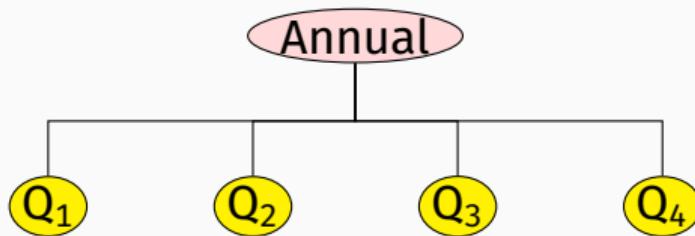
Outline

- 1 Hierarchical time series notation
- 2 Optimal linear forecast reconciliation
- 3 Probabilistic forecast reconciliation
- 4 Cross-temporal probabilistic forecast reconciliation
- 5 Final comments

Temporal reconciliation: quarterly data

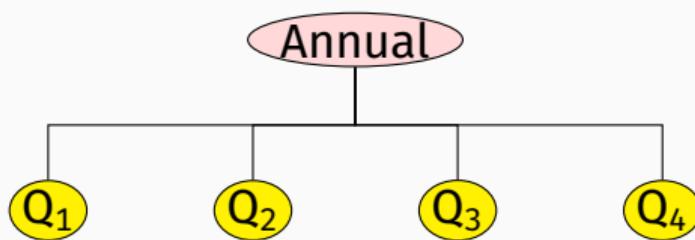


Temporal reconciliation: quarterly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: quarterly data

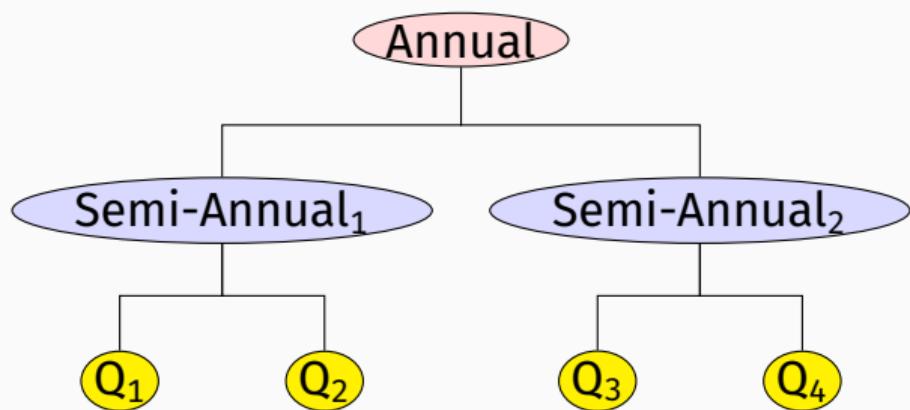


$$\mathbf{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

τ = index of largest temporal aggregation level.

Temporal reconciliation: quarterly data

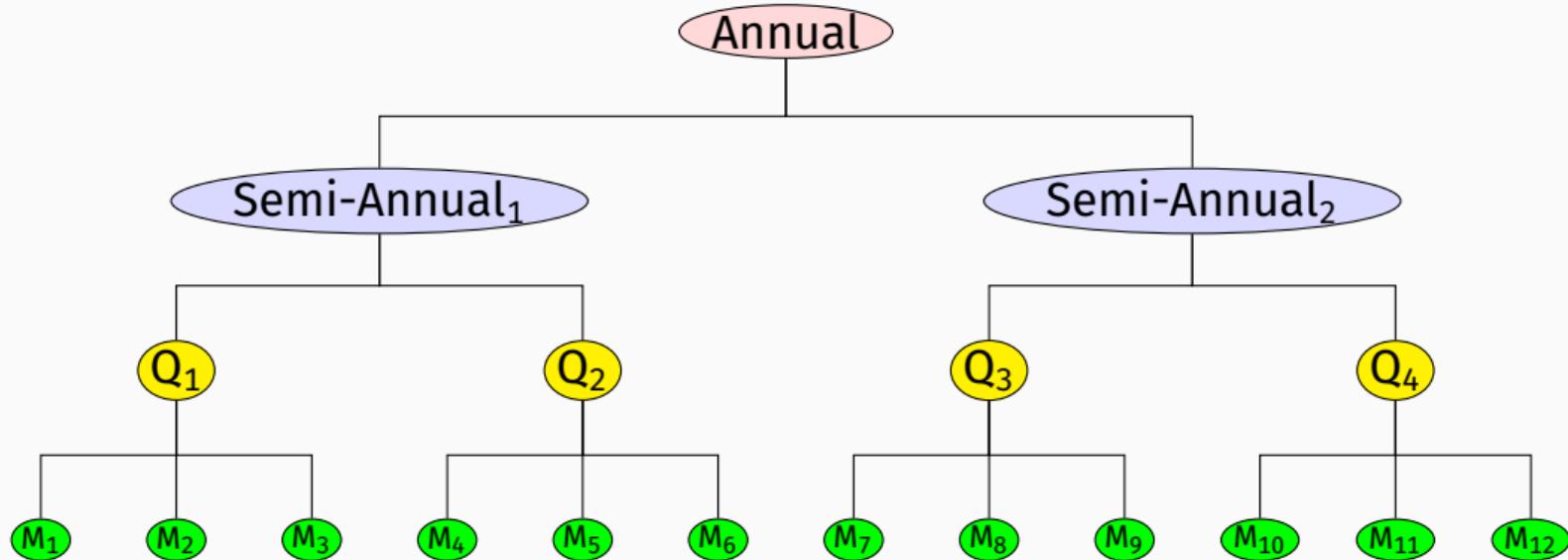


- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

$$\mathbf{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[2]} \\ x_{\tau,2}^{[2]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix}$$
$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

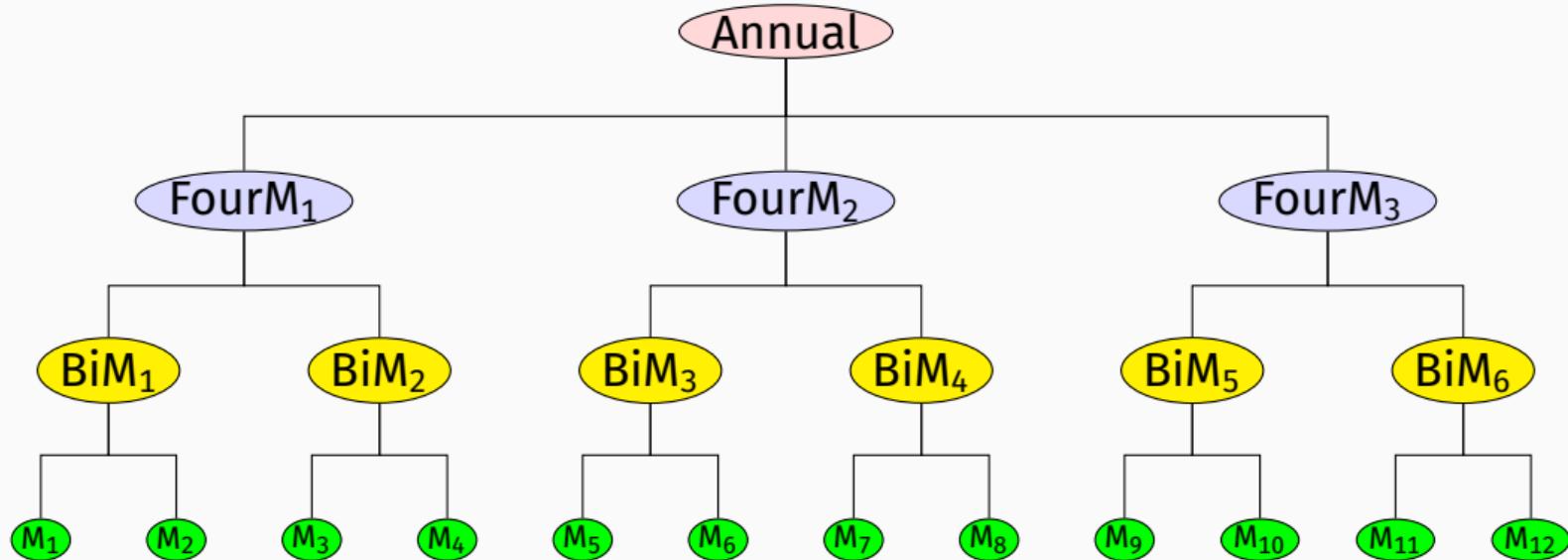
τ = index of largest temporal aggregation level.

Temporal reconciliation: monthly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: monthly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: monthly data

$$\mathbf{y}_\tau = \begin{bmatrix} \mathbf{x}_\tau^{[12]} \\ \mathbf{x}_\tau^{[6]} \\ \mathbf{x}_\tau^{[4]} \\ \mathbf{x}_\tau^{[3]} \\ \mathbf{x}_\tau^{[2]} \\ \mathbf{x}_\tau^{[1]} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \mathbf{I}_{12}$$

Temporal reconciliation

For a time series y_1, \dots, y_T , observed at frequency m :

$$x_j^{[k]} = \sum_{t=(j-1)k+1}^{jk} y_t \quad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in \mathcal{K} = \{k_1, \dots, k_p\}$ denote the p factors of m in ascending order, where $k_1 = 1$ and $k_p = m$
- $x_j^{[1]} = y_t$
- A single unique hierarchy is only possible when there are no coprime pairs in \mathcal{K} .
- $M_k = m/k$ is seasonal period of aggregated series.

Temporal reconciliation

$$\mathbf{x}_\tau = \mathbf{S} \mathbf{x}_\tau^{[1]}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix}$$

where

$$\mathbf{x}_\tau = \begin{bmatrix} \mathbf{x}_\tau^{[k_p]} \\ \mathbf{x}_\tau^{[k_{p-1}]} \\ \vdots \\ \mathbf{x}_\tau^{[k_1]} \end{bmatrix}, \quad \mathbf{x}_\tau^{[k]} = \begin{bmatrix} \mathbf{x}_{M_k(\tau-1)+1}^{[k]} \\ \mathbf{x}_{M_k(\tau-1)+2}^{[k]} \\ \vdots \\ \mathbf{x}_{M_k\tau}^{[k]} \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} \mathbf{1}'_m \\ \mathbf{I}_{m/k_{p-1}} \otimes \mathbf{1}'_{k_{p-1}} \\ \vdots \\ \mathbf{I}_{m/k_2} \otimes \mathbf{1}'_{k_2} \end{bmatrix}$$

τ is time index for most aggregated series,

$$k \in \mathcal{K} = \{k_1, \dots, k_p\}, \quad k_1 = 1, \quad k_p = m, \quad \tau = 1, \dots, T/m.$$

Cross-temporal forecast reconciliation

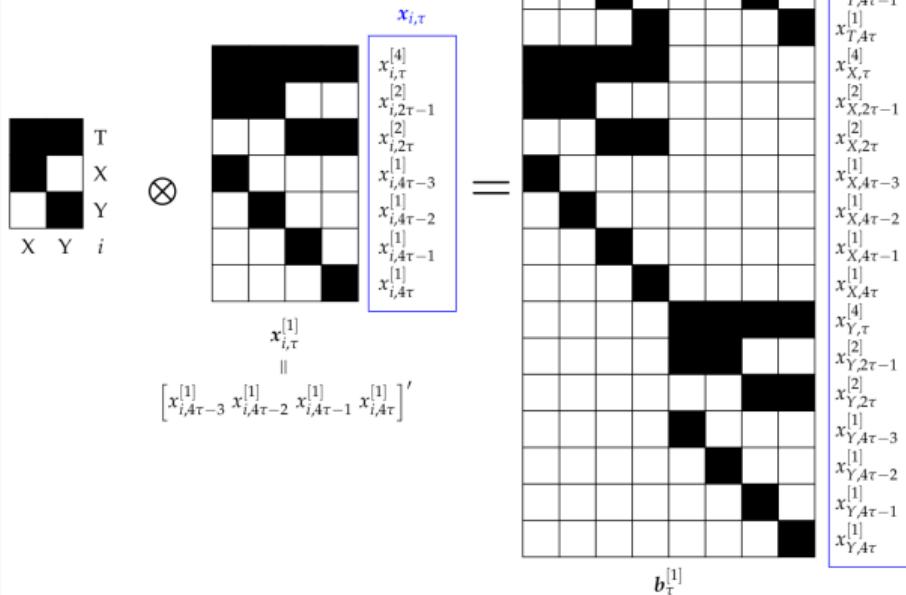
Structural matrix approach

- \mathbf{S}_{cs} = structural cross-sectional matrix
- \mathbf{S}_{te} = structural temporal matrix
- $\mathbf{S}_{ct} = \mathbf{S}_{cs} \otimes \mathbf{S}_{te}$

$$\mathbf{x}_\tau = \mathbf{S}_{ct} \mathbf{b}_\tau, \quad \text{where} \quad \mathbf{b}_\tau = \begin{bmatrix} \mathbf{x}_{1,\tau}^{[1]} \\ \vdots \\ \mathbf{x}_{n,\tau}^{[1]} \end{bmatrix}.$$

Cross-temporal forecast reconciliation

$$\mathbf{S}_{cs} \otimes \mathbf{S}_{te} = \mathbf{S}_{ct}$$



Cross-temporal forecast reconciliation

Constraint matrix approach

- \mathbf{C}_{cs} = cross-sectional constraint matrix
- \mathbf{C}_{te} = temporal constraint matrix

$$\mathbf{C}_{ct}\mathbf{x}_\tau = \mathbf{0} \quad \text{where} \quad \mathbf{C}_{ct} = \begin{bmatrix} (\mathbf{0}_{(n_a m \times nk^*)} \ I_m \otimes \mathbf{C}_{cs}) \mathbf{P}' \\ \ I_n \otimes \mathbf{C}_{te} \end{bmatrix}$$

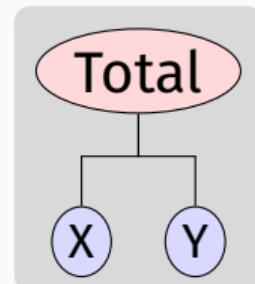
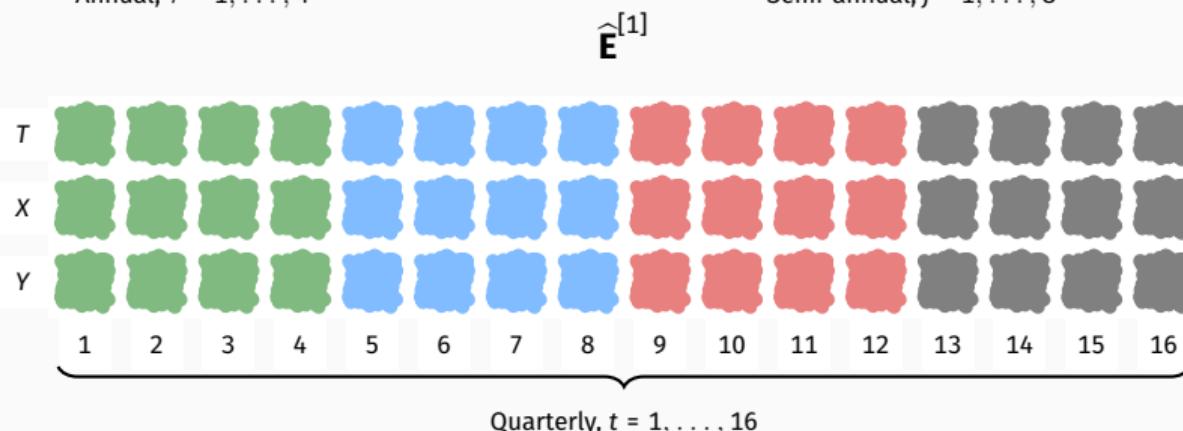
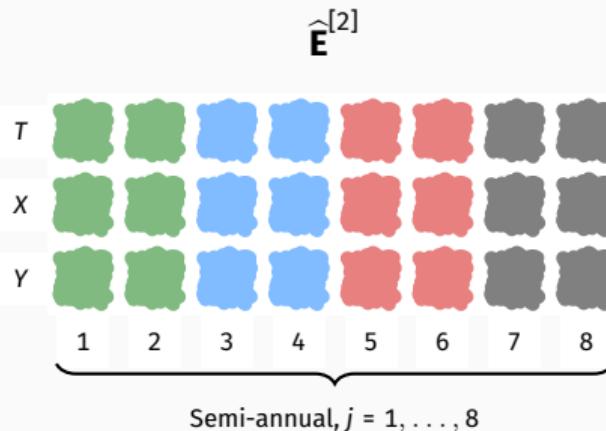
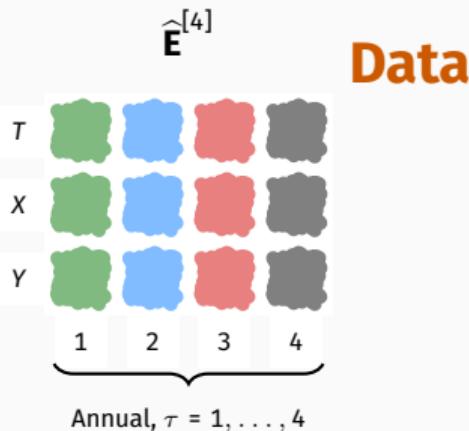
- $k^* = \sum_{k \in \mathcal{K} \setminus \{k_1\}} \frac{m}{k}$
- \mathbf{P} = the permutation matrix such that $\mathbf{P}\text{vec}(\mathbf{Y}_\tau) = \text{vec}(\mathbf{Y}'_\tau)$.

Cross-temporal probabilistic forecast reconciliation

Nonparametric bootstrap

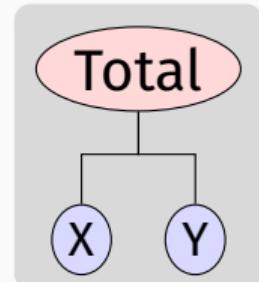
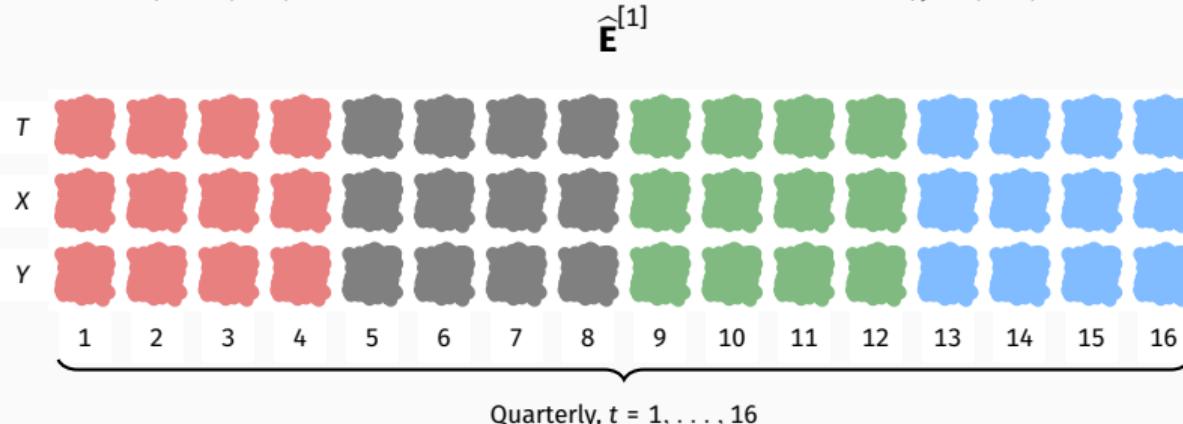
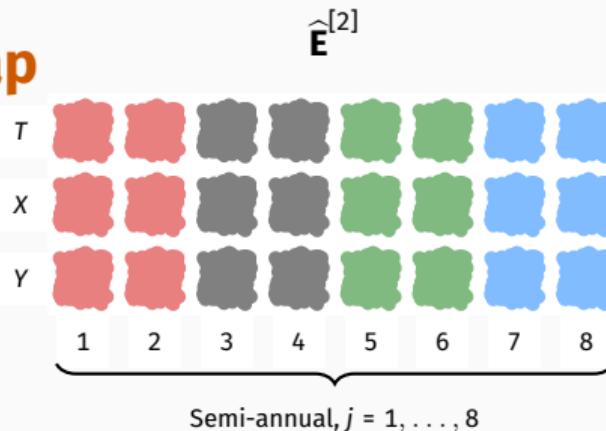
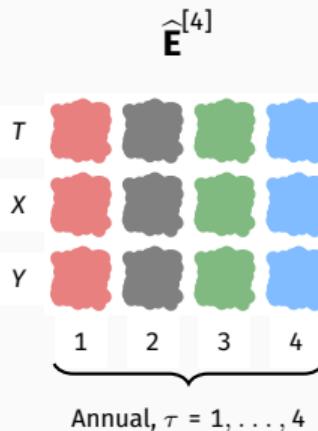
- Simulate future sample paths from all models using bootstrapped residuals, then reconcile them to obtain coherent sample paths.
- Need to generate samples that preserve cross-temporal relationships.
- Draw residual samples of all series at same time from most temporally aggregated level.
- Residuals for other levels obtained using the corresponding time indices.

Cross-temporal probabilistic forecast reconciliation



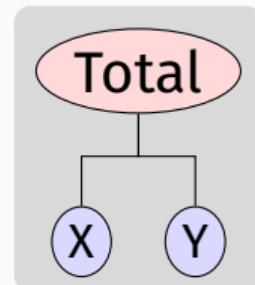
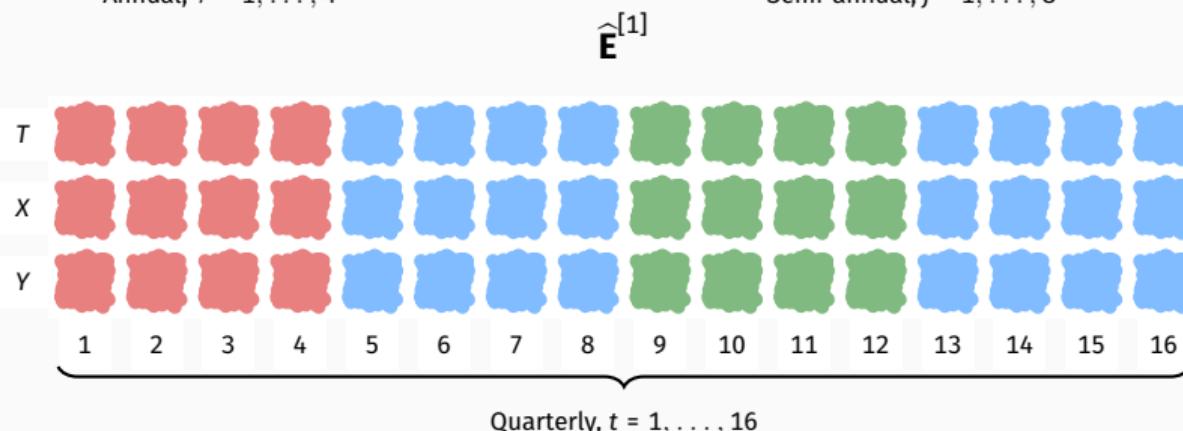
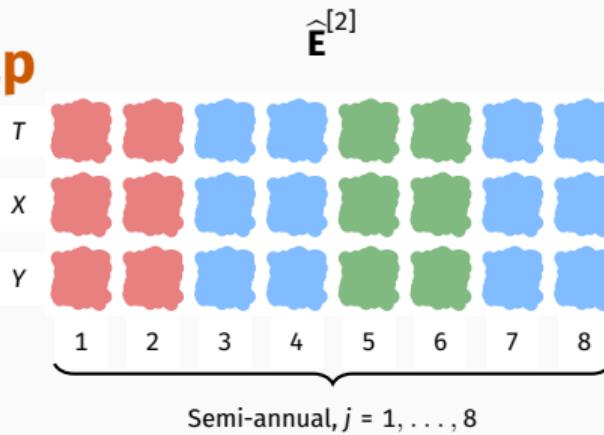
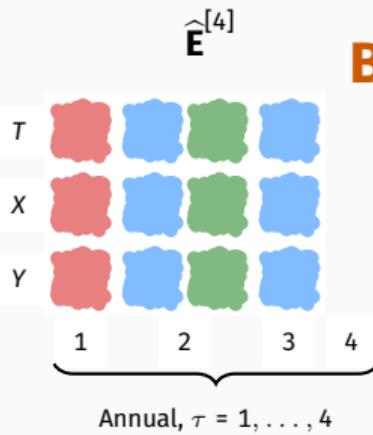
Year 1
Year 2
Year 3
Year 4

Cross-temporal probabilistic forecast reconciliation



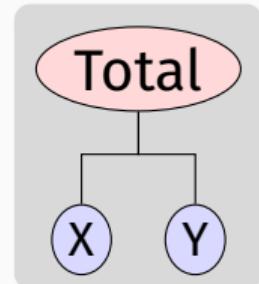
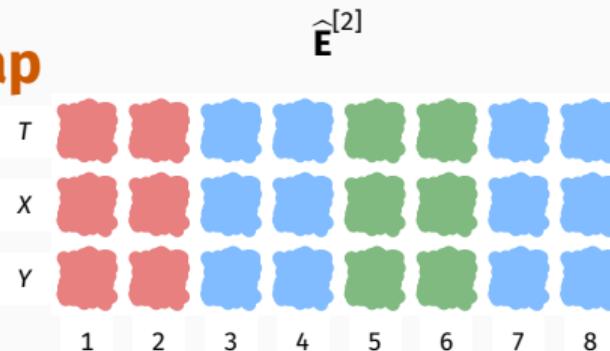
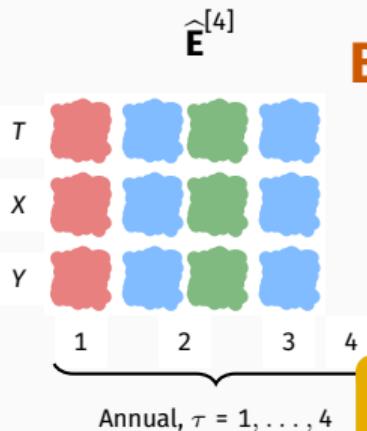
Year 1
Year 2
Year 3
Year 4

Cross-temporal probabilistic forecast reconciliation

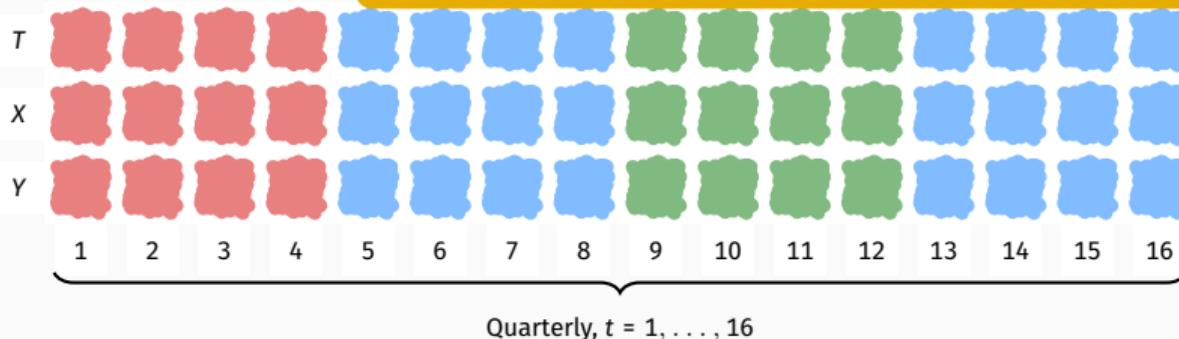


Year 1
Year 2
Year 3
Year 4

Cross-temporal probabilistic forecast reconciliation



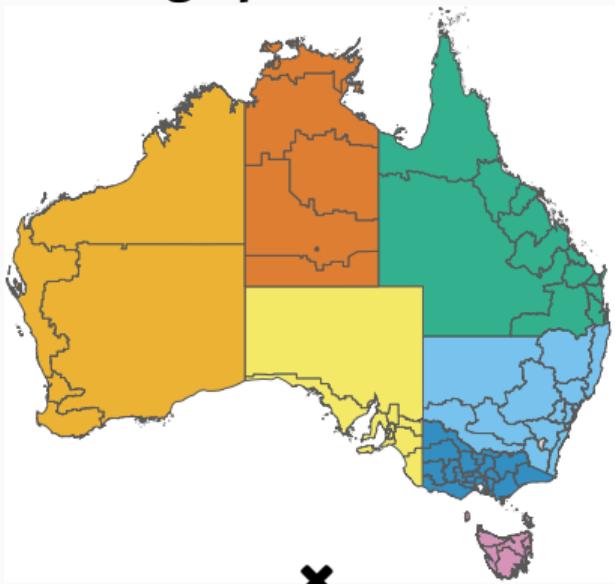
The “year” can start in any quarter, giving overlapping blocks.



Year 1
Year 2
Year 3
Year 4

Monthly Australian Tourism Demand

Geographical division



Purpose of travel

Holiday, Visiting friends & relatives, Business, Other

Grouped ts

(geographical divisions × purpose of travel)

	AUS	States	Zones*	Regions	Tot
geographical	1	7	21	76	105
purpose	4	28	84	304	420
total	5	35	105	380	525

$$n_a = 221, n_b = 304, \text{ and } n = 525$$

Temporal framework, frequencies:

- ▶ Monthly
- ▶ Bi-Monthly
- ▶ Quarterly
- ▶ Four-Monthly
- ▶ Semi-Annual
- ▶ Annual

Monthly Australian Tourism Demand

- Monthly data: January 1998 to December 2016
- Time series cross-validation; initial training set 10 years.
- One-month increase in each training set
- For each training set, compute temporally aggregated series for $k \in \{1, 2, 3, 4, 6, 12\}$, and produce forecasts up to $h_2 = 6$, $h_3 = 4$, $h_4 = 3$, $h_6 = 2$ and $h_{12} = 1$ steps ahead.
- Automatic ETS forecasts on log-transformed data

Monthly Australian Tourism Demand

Reconciliation approaches

- Cross-temporal **bottom-up** and **partly bottom-up**

$ct(bu)$ | $ct(shr_{cs}, bu_{te})$ | $ct(wlsv_{te}, bu_{cs})$

- Optimal forecast reconciliation with **one-step residuals**

$oct(ols)$ | $oct(struc)$ | $oct(wlsv)$ | $oct(bdshr)$

- Optimal forecast reconciliation with **multi-step residuals**

$oct_h(hbshr)$ | $oct_h(bshr)$ | $oct_h(hshr)$ | $oct_h(shr)$

Monthly Australian tourism data – CRPS skill scores

	Worse than benchmark	Best
	$\forall k \in \{12, 6, 4, 3, 2, 1\}$	$k = 1$
base	1.000	1.000
ct(bu)	1.321	1.077
ct(shr _{cs} , bu _{te})	1.057	0.976
ct(wlsv _{te} , bu _{cs})	1.062	0.976
oct(ols)	0.989	0.982
oct(struc)	0.982	0.970
oct(wlsv)	0.987	0.952
oct(bdshr)	0.975	0.949
oct _h (hbshr)	0.989	0.982
oct _h (bshr)	0.994	0.988
oct _h (hshr)	0.969	0.953
oct _h (shr)	1.007	1.000

Outline

- 1 Hierarchical time series notation
- 2 Optimal linear forecast reconciliation
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- 5 Final comments

Software

Package	Language	Cross-sectional	Temporal	Cross-temporal	Probabilistic
hts	R	✓			
thief	R		✓		
fable	R	✓			✓
FoReco	R	✓	✓	✓	✓
pyhts	Python	✓	✓		
hierarchicalforecast	Python	✓			✓

- hts, thief, and FoReco use ts objects
- fable uses tsibble objects
- fable has plans to implement temporal and cross-temporal reconciliation

Thanks!



More information

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 [@robjhyndman](https://github.com/robjhyndman)

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