MONASH University

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Functional time series

with applications in demography

1. Tools for functional time series analysis

Mortality rates

Fertility rates

Outline

1 **Functional time series**

2 Functional principal components

3 Data visualization

4 References

Functional time series

$$y_t(x_i) = g_\lambda(z_t(x_i)) = \begin{cases} \log[z_t(x_i)] & \text{if } \lambda = 0; \\ \lambda^{-1}\left[z_t^\lambda(x_i) - 1\right] & \text{otherwise.} \end{cases}$$
$$= s_t(x_i) + \sigma_t(x_i)\varepsilon_{t,i}$$

- $z_t(x_i)$ is observed data for age x_i in year t, $i = 1, \dots, N, \quad t = 1, \dots, T$.
- λ chosen so that $\varepsilon_{t,i} \sim \text{NID}(0, 1)$.
- We assume $s_t(x)$ is a smooth function of x.
- We need to estimate $s_t(x)$ from the data for $x_1 < x < x_N$.
- We want to forecast **whole curve** $z_t(x)$ for $t = T + 1, \dots, T + h$.

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Smoothing functional time series

$$y_t(x_i) = g_\lambda(z_t(x_i)) = \begin{cases} \log(z_t(x_i)) & \text{if } \lambda = 0; \\ \lambda^{-1}\left(z_t^\lambda(x_i) - 1\right) & \text{otherwise.} \end{cases}$$
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- Estimate $s_t(x)$ using penalized regression spline with a large number of knots.
- For mortality data, use $\lambda = 0$ and constrain $s_t(x)$ to be monotonic for $x > 50$.
- For fertility data, use $\lambda = 0.4$ and constrain $s_t(x)$ to be concave.
- Fit is weighted with $w_t(x_i) = \sigma_t^{-2}(x_i)$.

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$D_t(x_i)$ = number of deaths at age x_i in year t.

$E_t(x_i)$ = total population aged x_i on June 30 in year t.

$m_t(x_i) = D_t(x_i)/E_t(x_i)$ = observed mortality rate.

$\mu_t(x_i)$ = "true" mortality rate.

$$D_t(x_i) \sim \text{Poisson}(E_t(x_i)\mu_t(x_i))$$

$\text{E}[m_t(x_i)] = \mu_t(x)$ and $\text{V}[m_t(x_i)] = \mu_t(x_i)E_t^{-1}(x_i)$.

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Taylor series approx

$$\text{V}[g_\lambda(X)] = \sigma_X^2[g'_\lambda(\mu_X)]^2$$
$$\text{V}[\log(X)] = \sigma_X^2/\mu_X^2$$

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$$\begin{aligned}\sigma^2(x_i) &= \text{V}(\log[m_t(x_i)]) \\ &\approx \left[\mu_t(x_i)E_t^{-1}(x_i)\right]\mu_t(x)^{-2} \\ &= \mu_t(x_i)^{-1}E_t(x_i)^{-1}\end{aligned}$$

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Fertility

$B_t(x_i)$ = number of births to women aged x_i in year t.

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$$\begin{aligned}
\sigma^2(x_i) &= \text{V}(g_\lambda[f_t(x_i)]) \\
&= \left[\mu_t(x_i)E_t^{-1}(x_i)\right]\mu_t(x)^{2\lambda-2} \\
&= \mu_t(x_i)^{2\lambda-1}E_t(x_i)^{-1} \\
&\approx f_t(x_i)^{2\lambda-1}E_t(x_i)^{-1}
\end{aligned}$$

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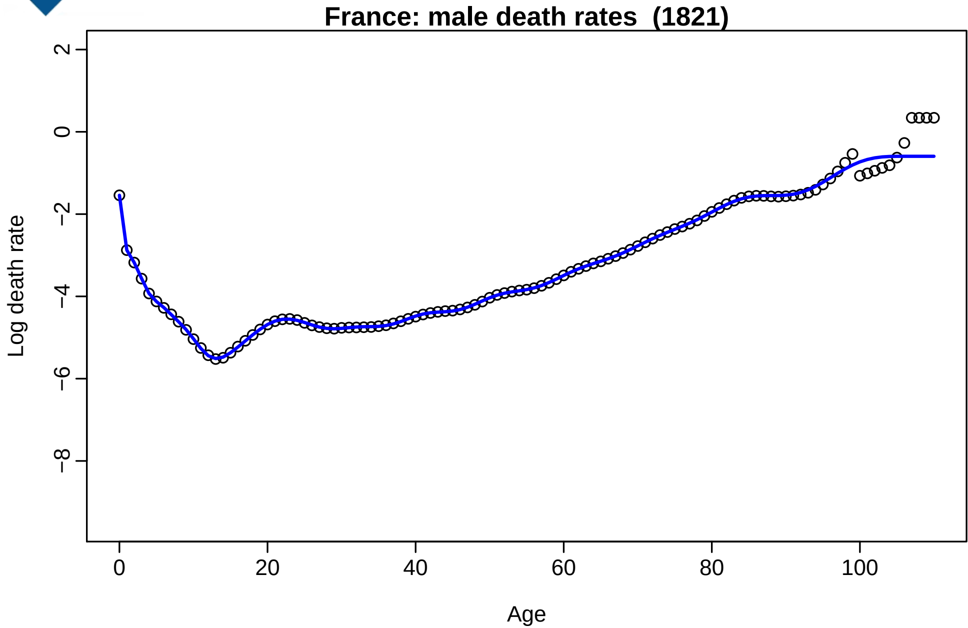

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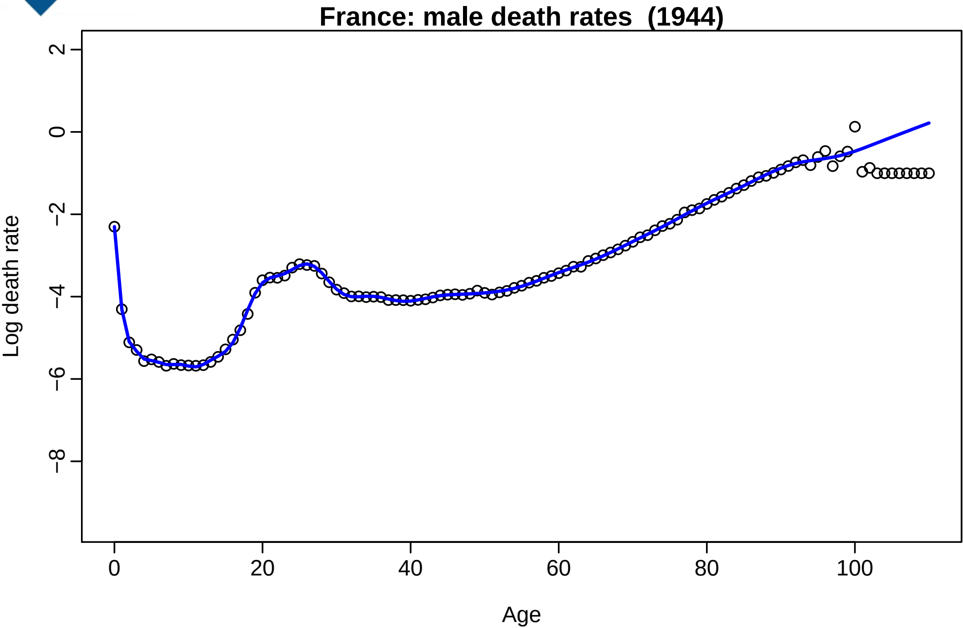

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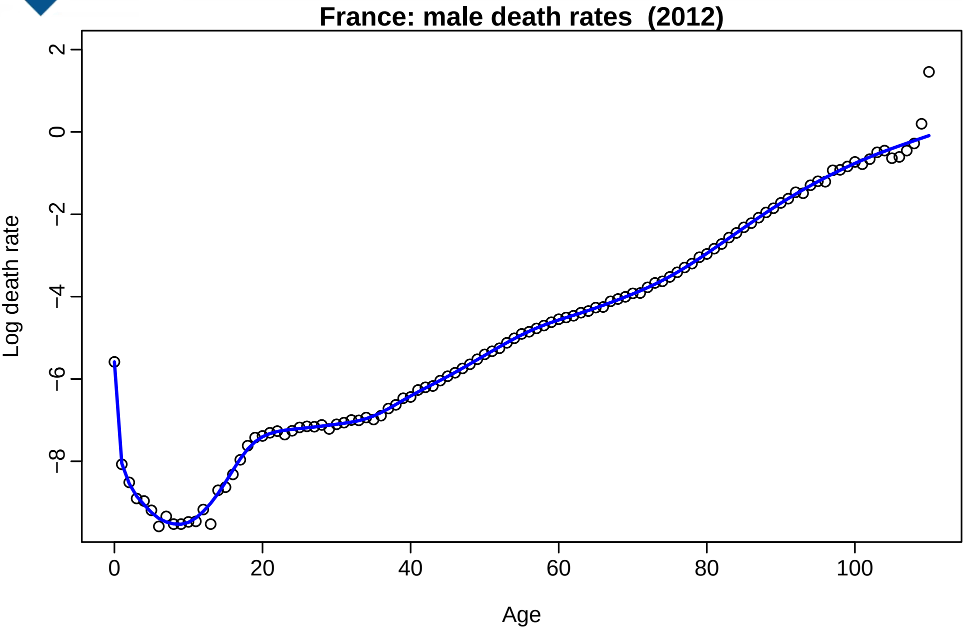

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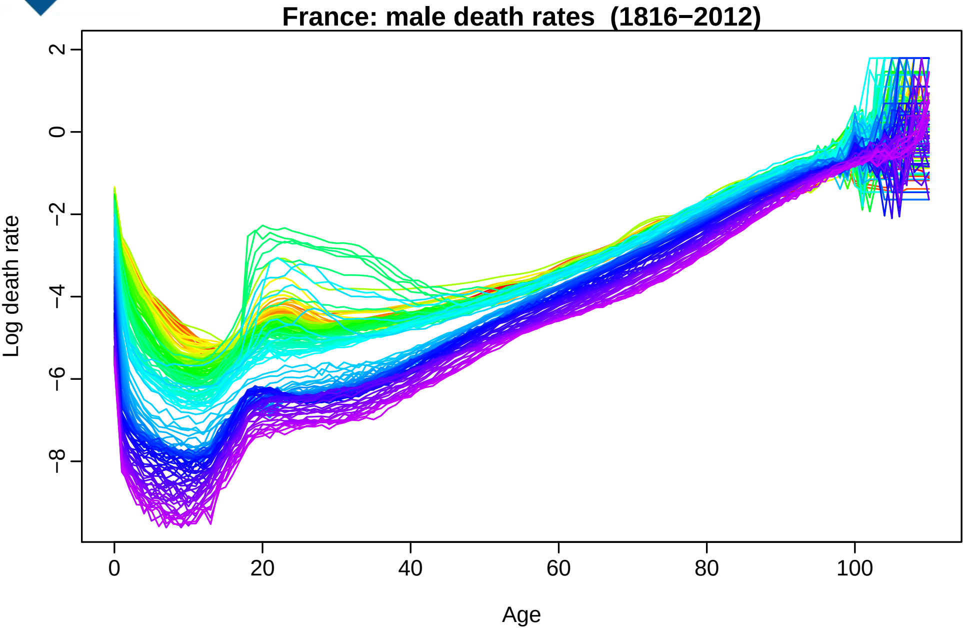

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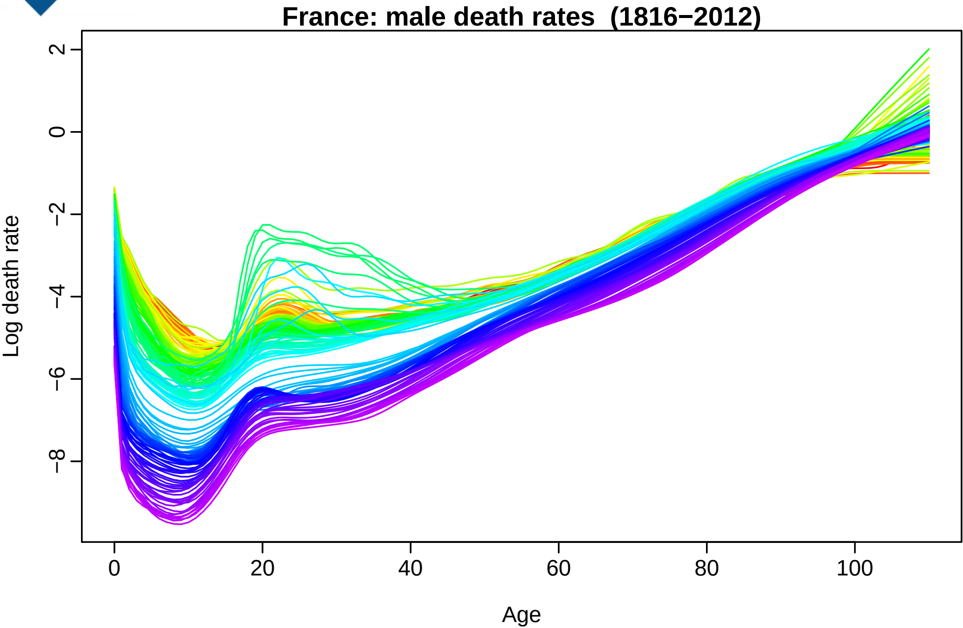

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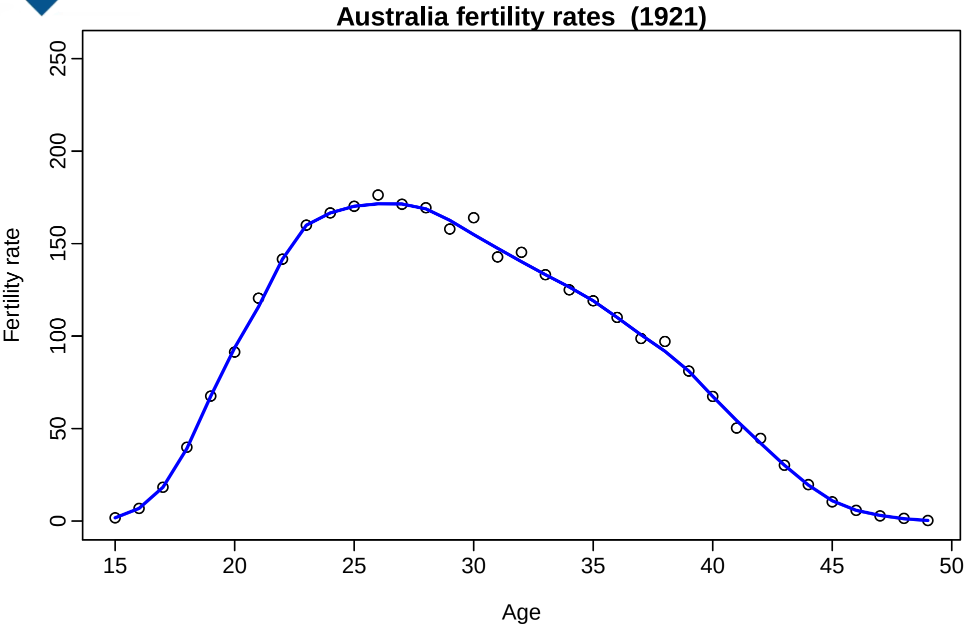

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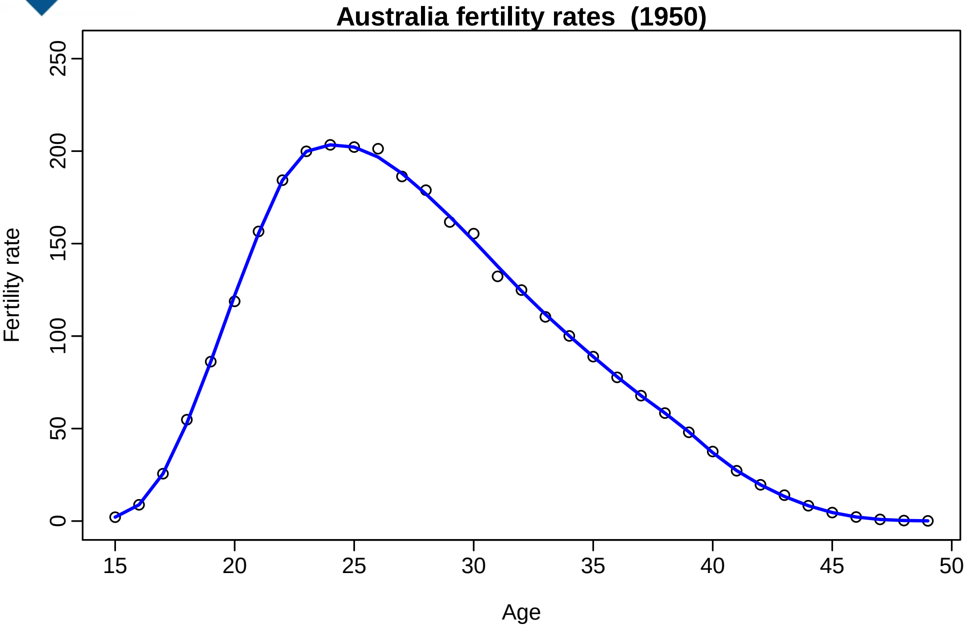

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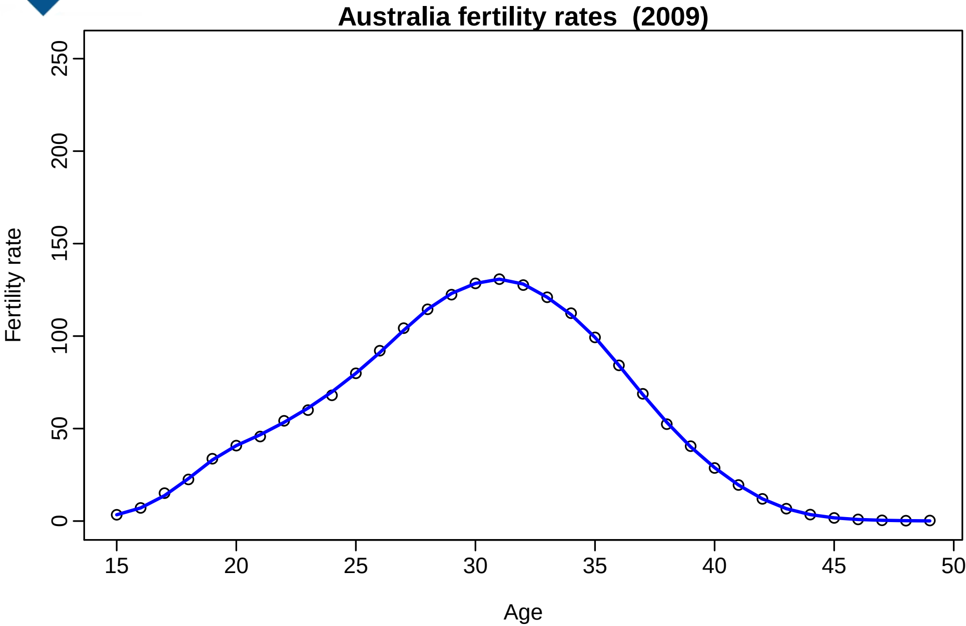

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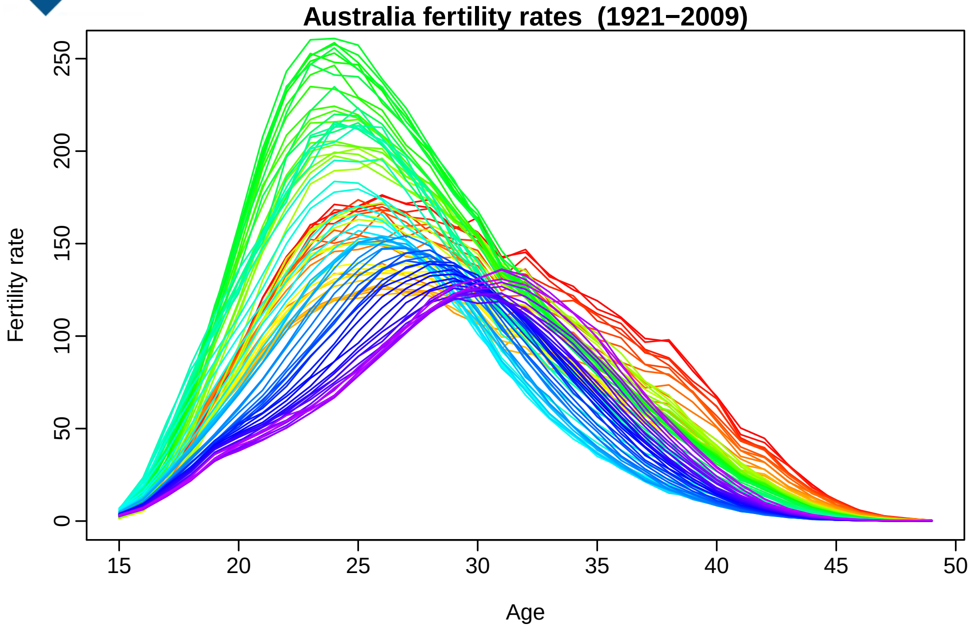

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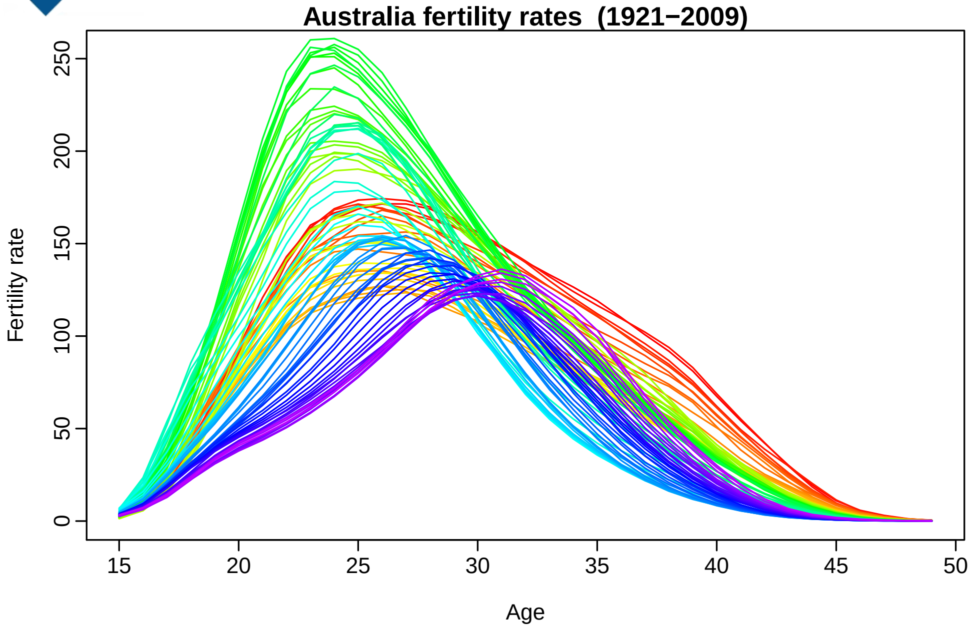

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2 Functional principal components

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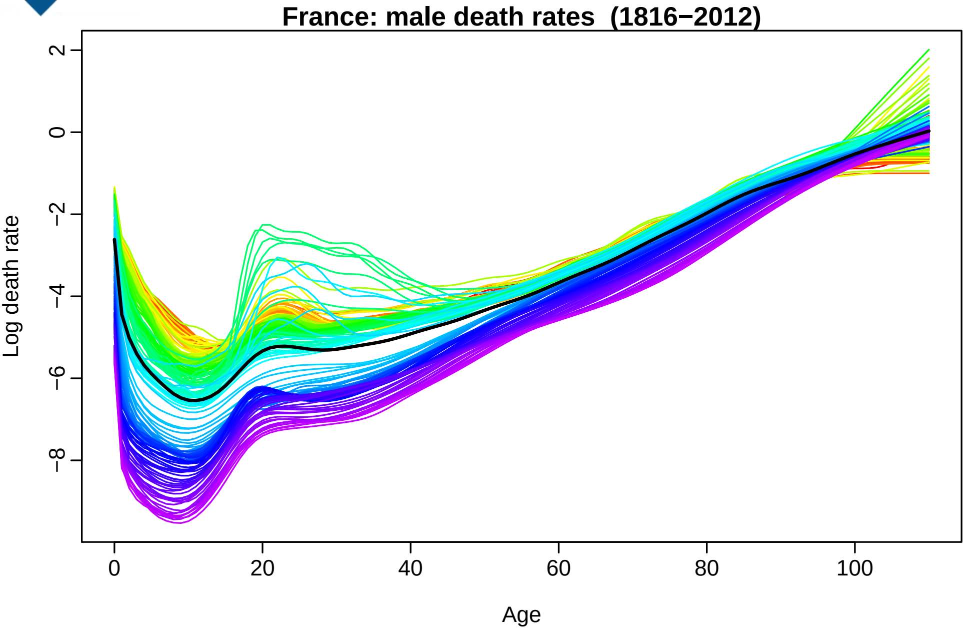
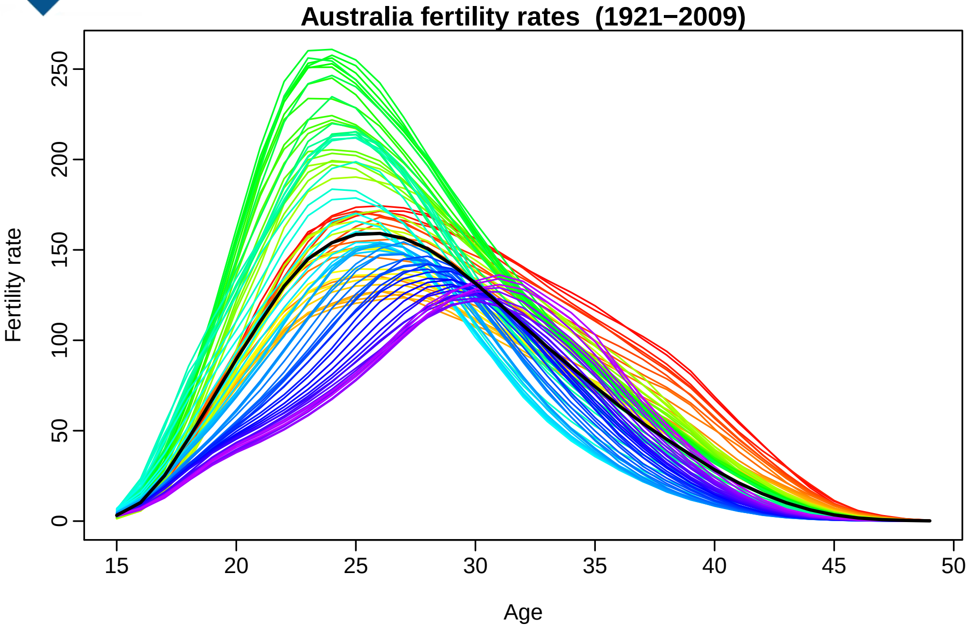
Functional principal components

$$y_t(x_i) = s_t(x_i) + \sigma_t(x_i)\varepsilon_{t,i},$$
$$s_t(x) = \mu(x) + \sum_{k=1}^{T-1} \beta_{t,k}\,\phi_k(x)$$

1. Estimate smooth functions $s_t(x)$ using weighted penalized regression splines.
2. Compute $\mu(x)$ as $\bar{s}(x)$ across years.
3. Compute $\beta_{t,k}$ and $\phi_k(x)$ using functional principal components.

$\mu(x)$


France: male death rates (1816−2012)
Log death rate
2
0
−2
−4
−6
−8
0
20
40
60
80
100
Age


Australia fertility rates (1921−2009)
Fertility rate
0
50
100
150
200
250
15
20
25
30
35
40
45
50
Age

Functional principal components

(Ramsay and Silverman, 1997,2002).

- In FDA, each principal component is specified by a weight function $\phi_k(x)$.
- The PC scores for each year are given by

$$\beta_{k,t} = \int \phi_k(x) \left[\hat{s}_t(x) - \bar{s}(x)\right] dx$$

- The aim is to:
  1. Find the weight function $\phi_1(x)$ that maximizes the variance of $\beta_{1,t}$ subject to the constraint $\int \phi_1^2(x)dx = 1$.
  2. Find the weight function $\phi_2(x)$ that maximizes the variance of $\beta_{2,t}$ such that $\int \phi_2^2(x)dx = 1$ and $\int \phi_1(x)\phi_2(x)dx = 0$.
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Functional principal components

The optimal basis functions

Approximate $s_t(x)$ using

$$s_t(x) = \bar{s}(x) + \sum_{k=0}^{K} \beta_{t,k}\phi_k(x) + r_t(x)$$

The basis function $\phi_k(x)$ which minimizes $\text{MISE} = \frac{1}{T}\sum_{t=1}^{T}\int r_t^2 dx$ is the kth principal component (computed recursively).

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$$s_t(x) = \bar{s}(x) + \sum_{k=0}^{K} \beta_{t,k}\phi_k(x) + r_t(x)$$

The basis function $\phi_k(x)$ which minimizes

$\text{MISE} = \dfrac{1}{T}\displaystyle\sum_{t=1}^{T} \int r_t^2 dx$ is the kth principal component

(computed recursively).

Functional principal components

Computationally equivalent approach

- Let $s_t^*(x) = s_t(x) - \bar{s}(x)$.
- Discretize $s_t^*(x)$ on a dense grid of q equally spaced points.
- Denote discretized $s_t^*(x)$ as $T \times q$ matrix $\boldsymbol{G}$.
- SVD of $\boldsymbol{G} = \Phi\Lambda\Psi'$ where $\phi_k(x)$ is kth column of Φ.
- $\beta_{t,k}$ is (t,k)th element of $\boldsymbol{G}\Phi$.
- The basis functions are orthogonal.
- This means the coefficients series are also uncorrelated with each other. i.e., $\text{Corr}(\hat{\beta}_{t,i}, \hat{\beta}_{t,j}) = 0$ for $i \neq j$. However, $\text{Corr}(\hat{\beta}_{t,i}, \hat{\beta}_{s,j}) \neq 0$ in general for $t \neq s$ and $i \neq j$.

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Functional principal components

Eigenvector approach

- Let $\boldsymbol{V} = (T-1)^{-1}\boldsymbol{G}'\boldsymbol{G}$ be $m \times m$ sample covariance matrix of $\boldsymbol{G}$.
- Let $\Phi_K = [\phi_1, \dots, \phi_K]$ consist of the first K eigenvectors of $\boldsymbol{V}$ where $K \leq T-1$. The (i,j)th element of Φ_K is $\phi_i(x_j^*)$.
- Robust versions possible using robust "covariance" estimation.

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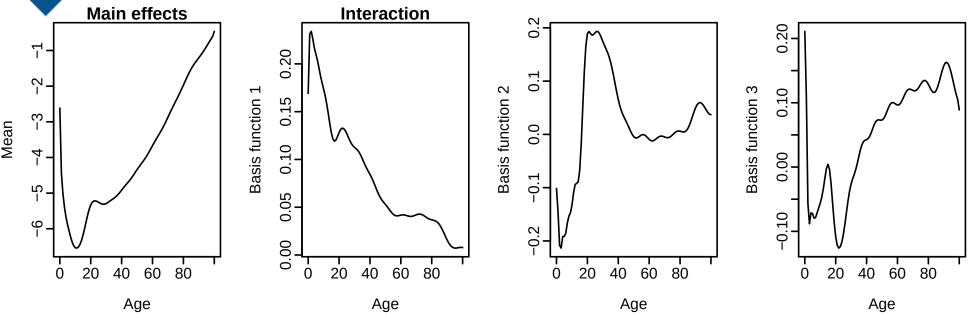
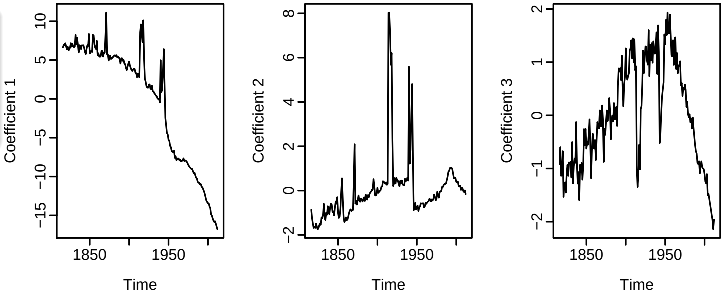


French male mortality



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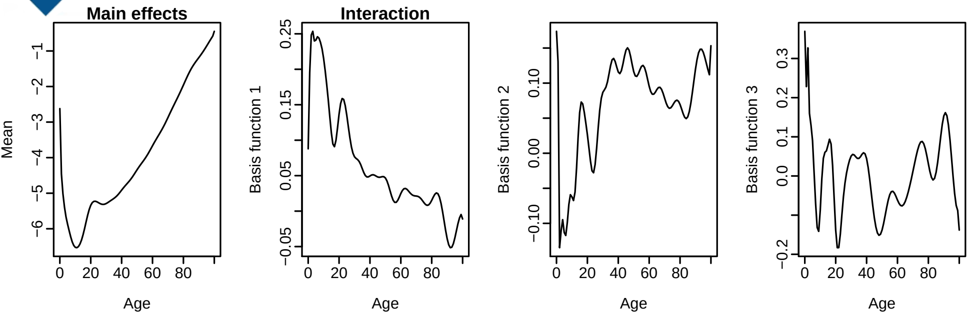
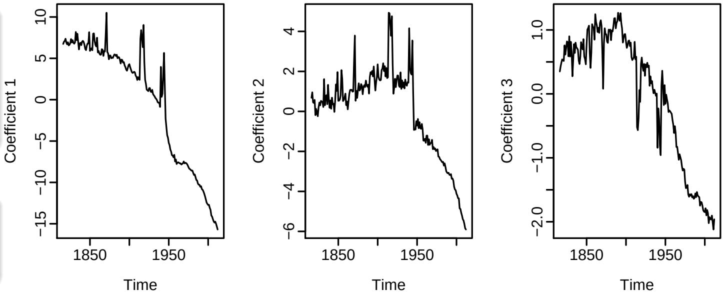


French male mortality

Robust PCA



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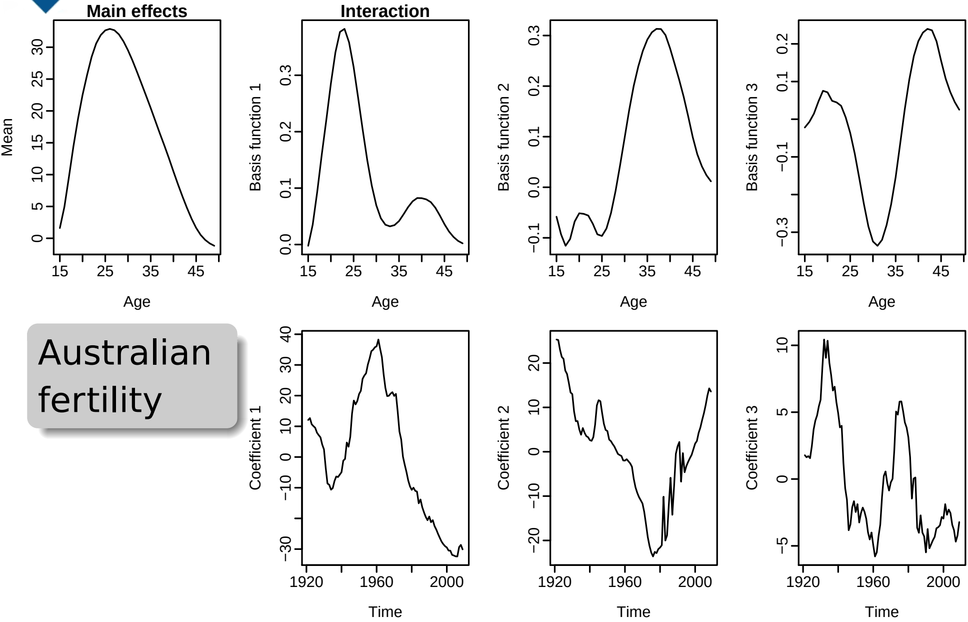

Australian fertility

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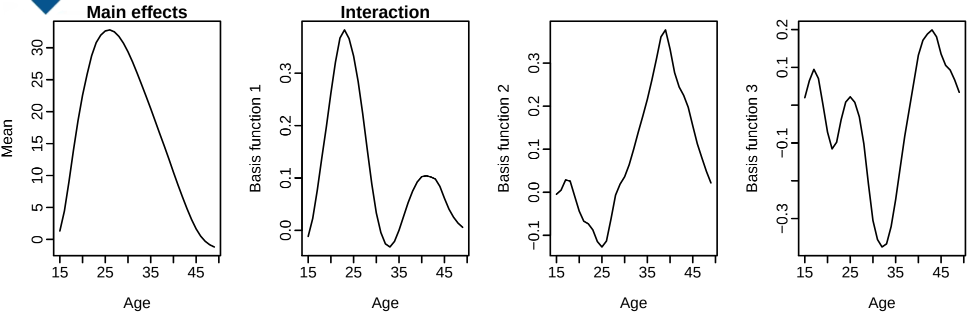

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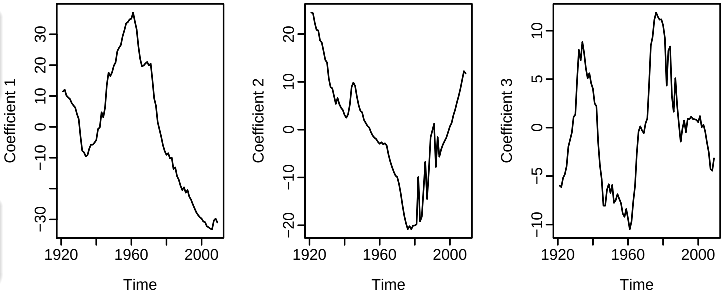

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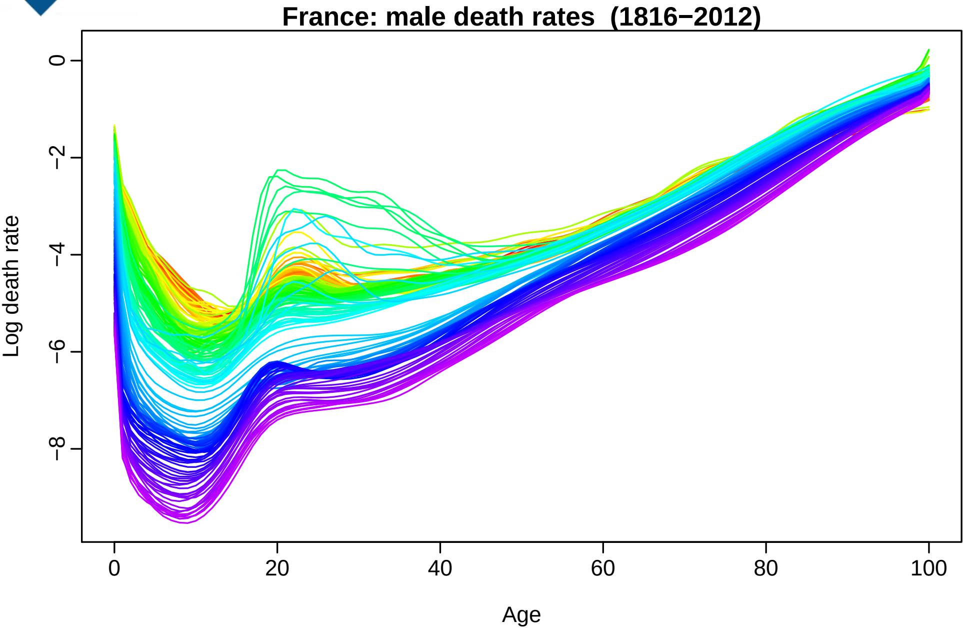
1 Functional time series

2 Functional principal components

3 Data visualization

4 References

French male mortality rates



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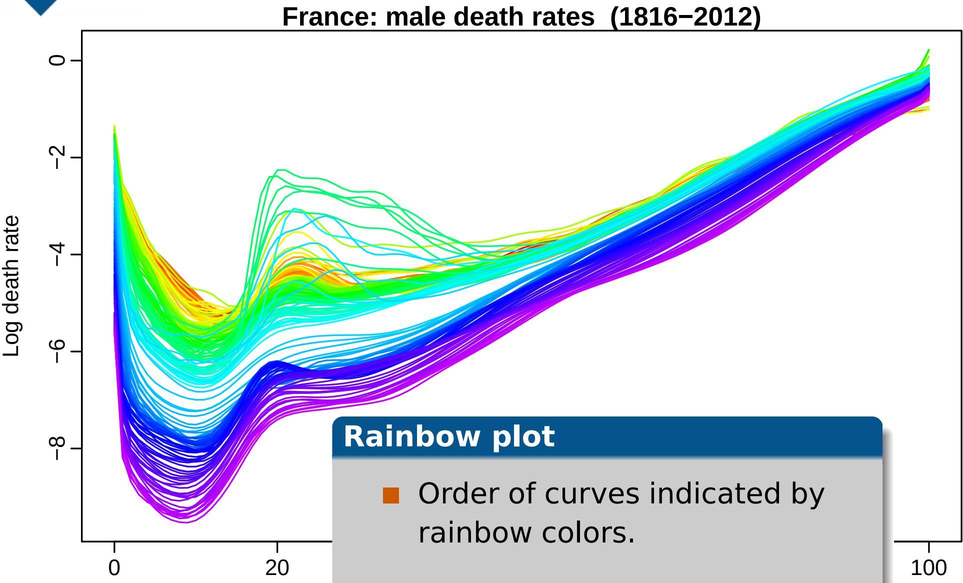

Rainbow plot

- Order of curves indicated by rainbow colors.
- Other orderings are possible.

Order by functional depth

Febrero, Galeano and Gonzalez-Manteiga (2007) proposed:

$$o_t = \int D(y_t(x))\,dx$$

where $D(y_t(x))$ is a univariate depth measure for each x.

- o_t provides an ordering of curves by “functional depth”.
- Problem: may not detect shape outliers.

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Bivariate functional depth

Alternative: Apply bivariate depth measures to first two PC scores.

Plot $\beta_{t,2}$ vs $\beta_{t,1}$

- Each point in scatterplot represents one curve.
- Outliers show up in bivariate score space.
- Curves can be ordered by bivariate depth.

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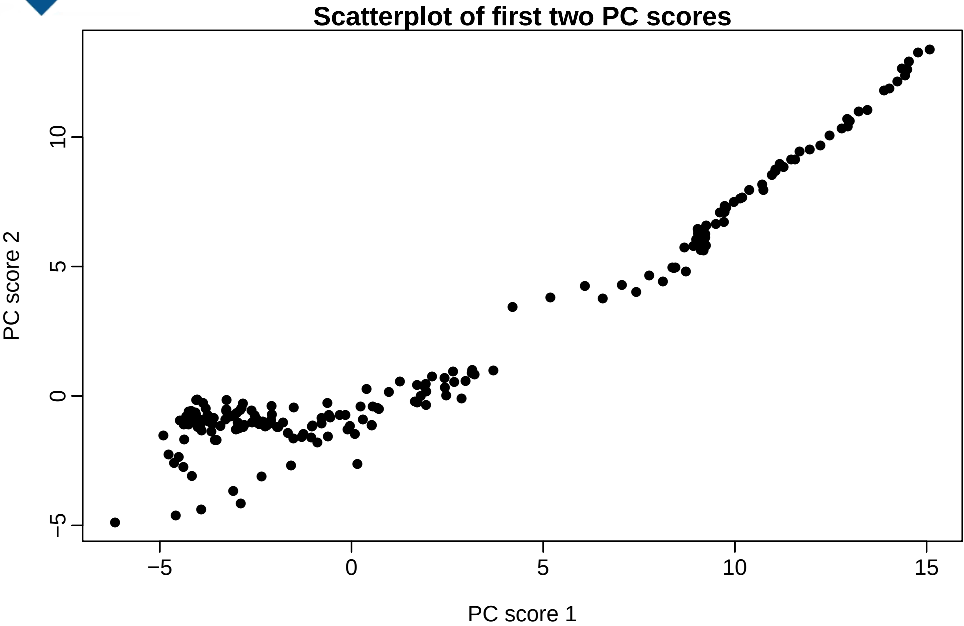

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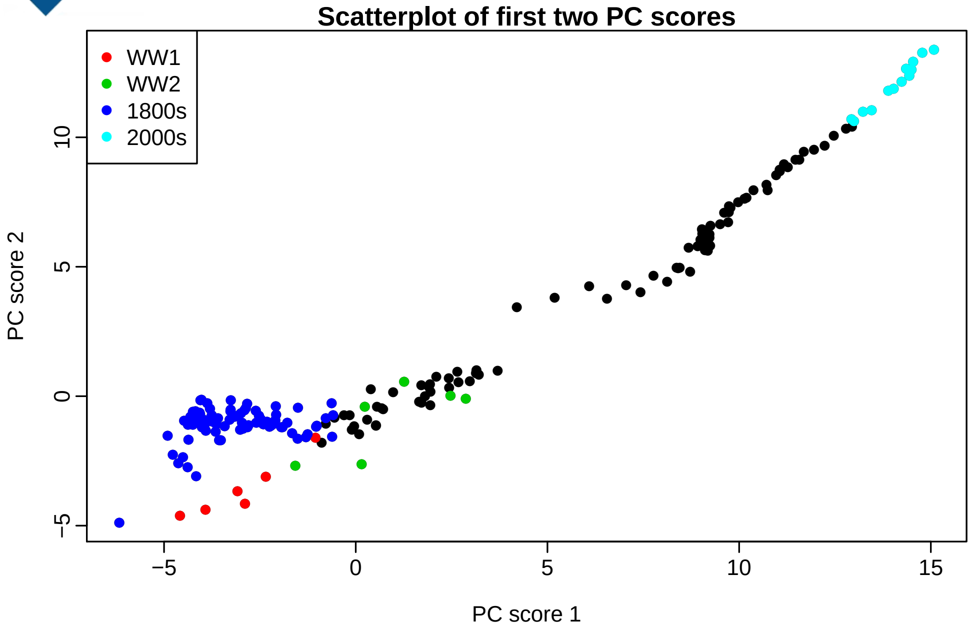

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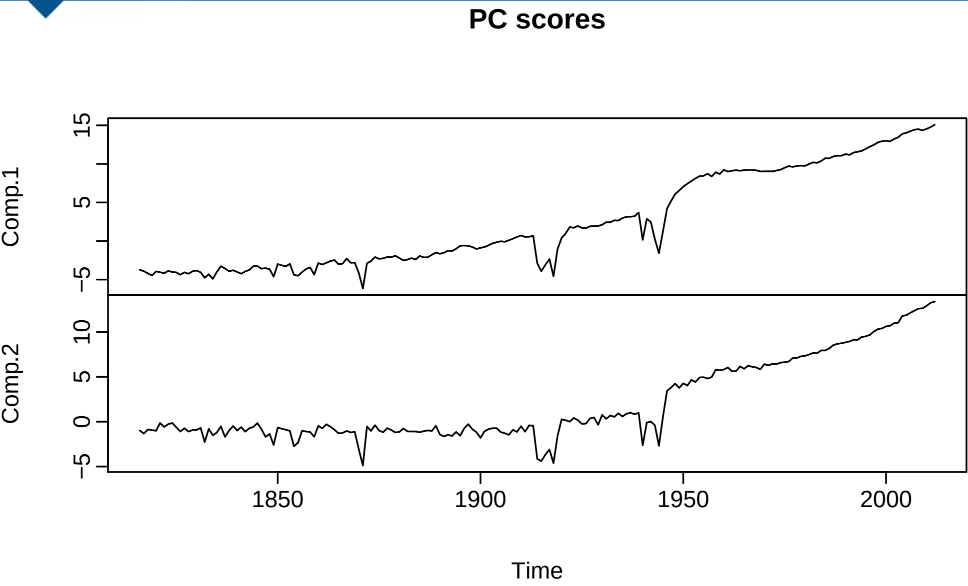

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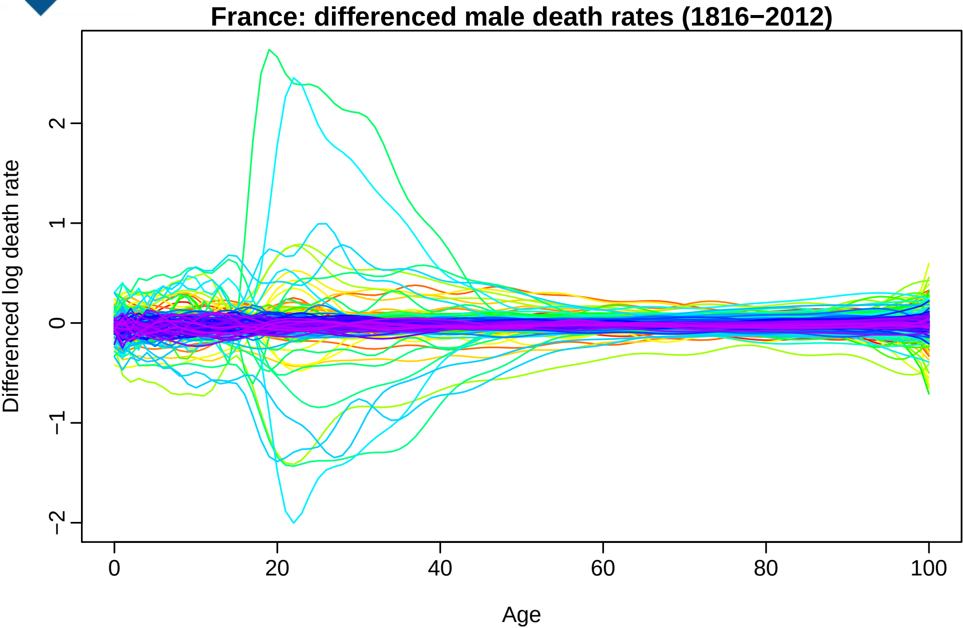

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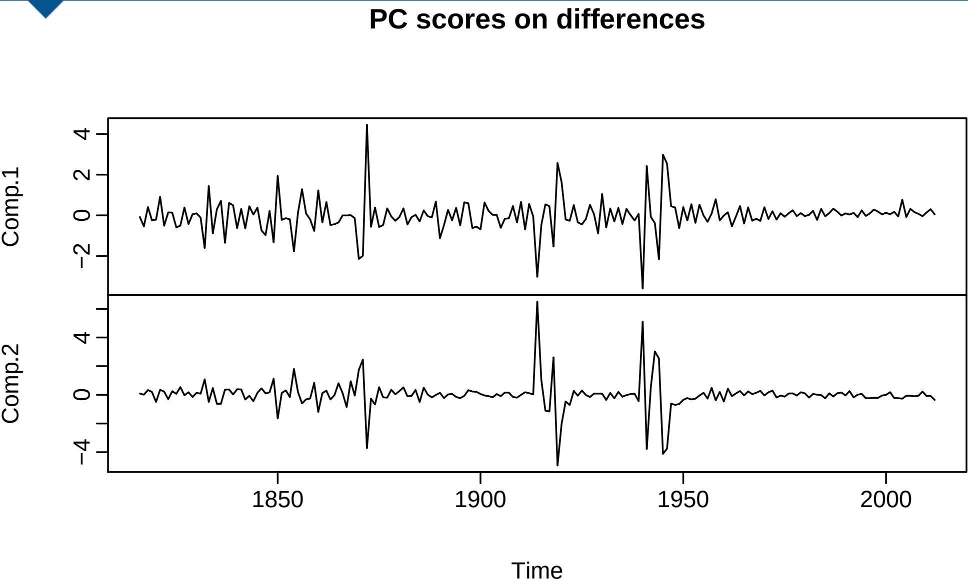

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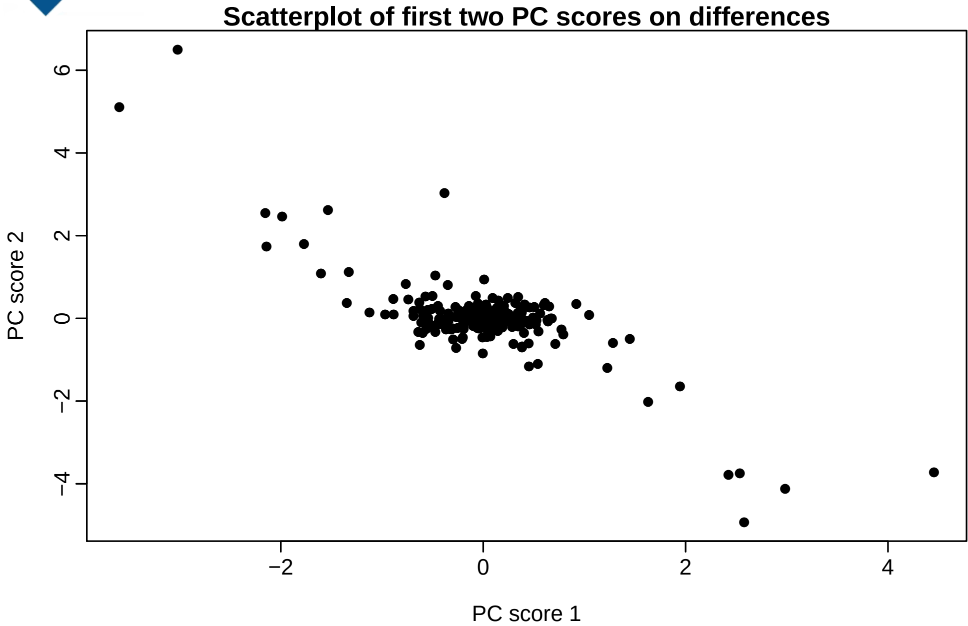

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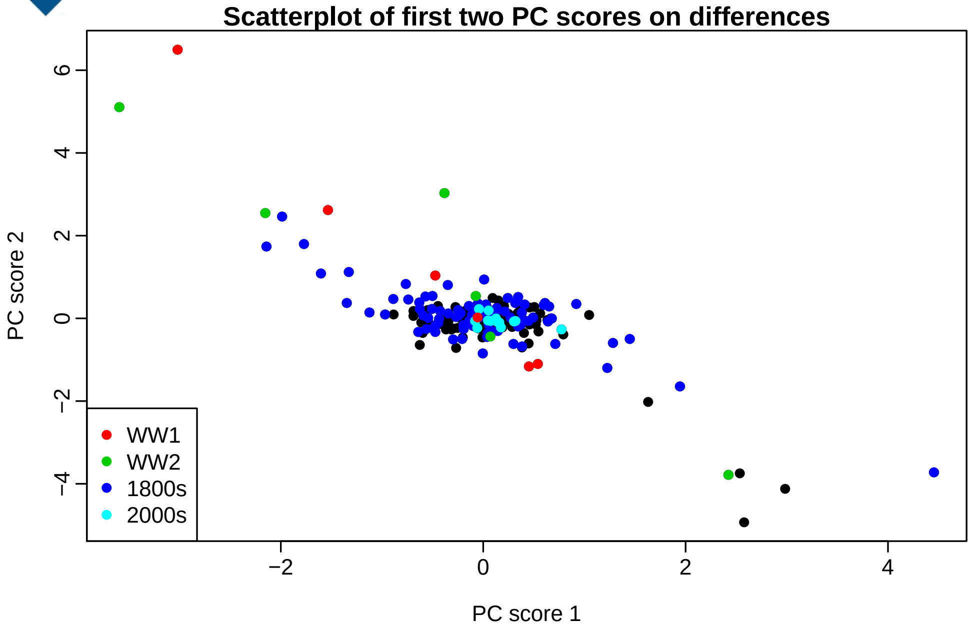

Halfspace location depth

The halfspace depth of a point q:

(Due to Hotelling, 1929; Tukey, 1975)

- For each closed halfspace that contains q, count number of observations not in halfspace. The minimum over all halfspaces is the depth of that point.
- The median is the point with maximum depth (not generally unique).
- Any point outside convex hull of the data has depth zero.

Halfspace location depth

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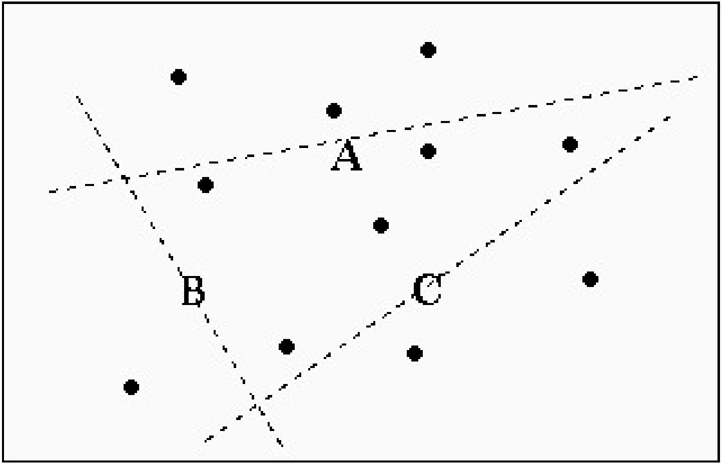

Fig.5 : Depth of certain points with respect to a data set.
depth(A) = 3 , depth(B) = 1 , depth(C) = 2.

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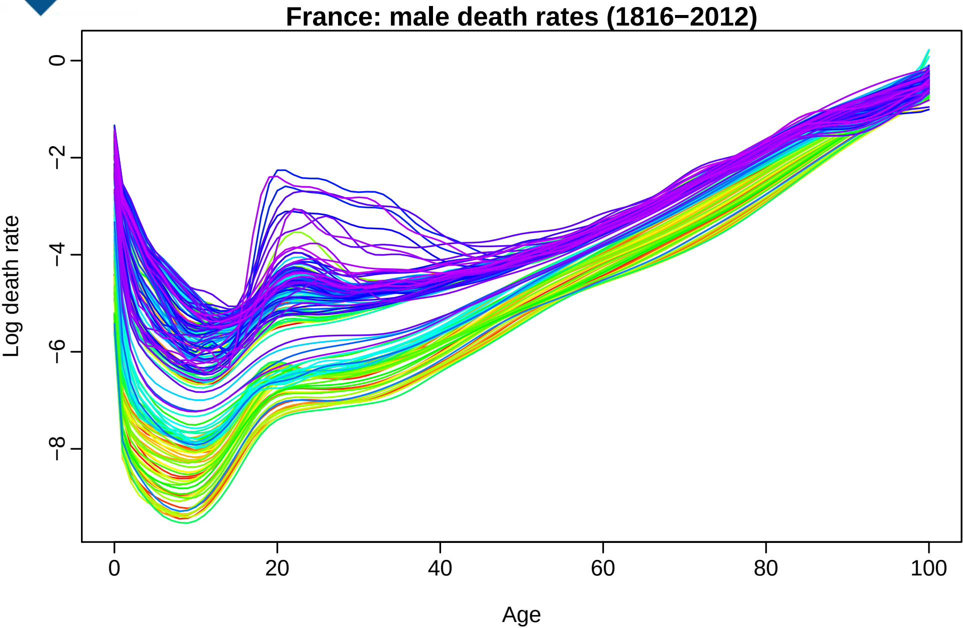
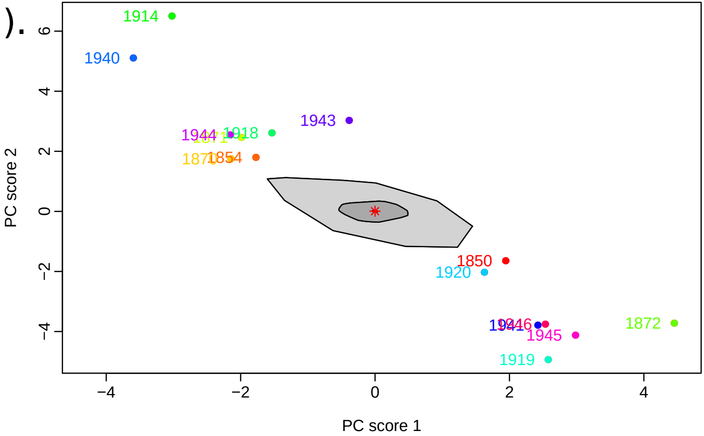
Ordering by halfspace depth



Bivariate bagplot

Due to Rousseeuw, Ruts & Tukey (Am.Stat. 1999).

- Rank points by halfspace location depth.
- Display median, 50% convex hull and outer convex hull (with 99% coverage if bivariate normal).



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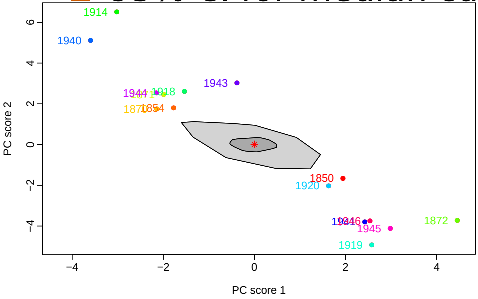
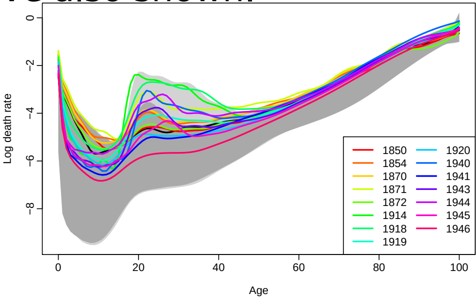
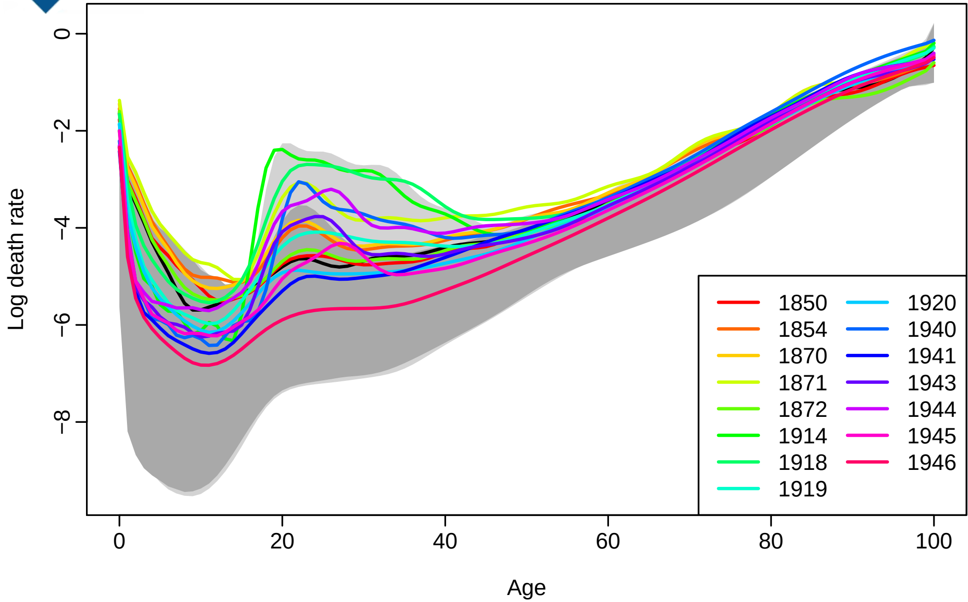
- Boundaries contain all curves inside bags.
- 95% CI for median curve also shown.





Functional bagplot



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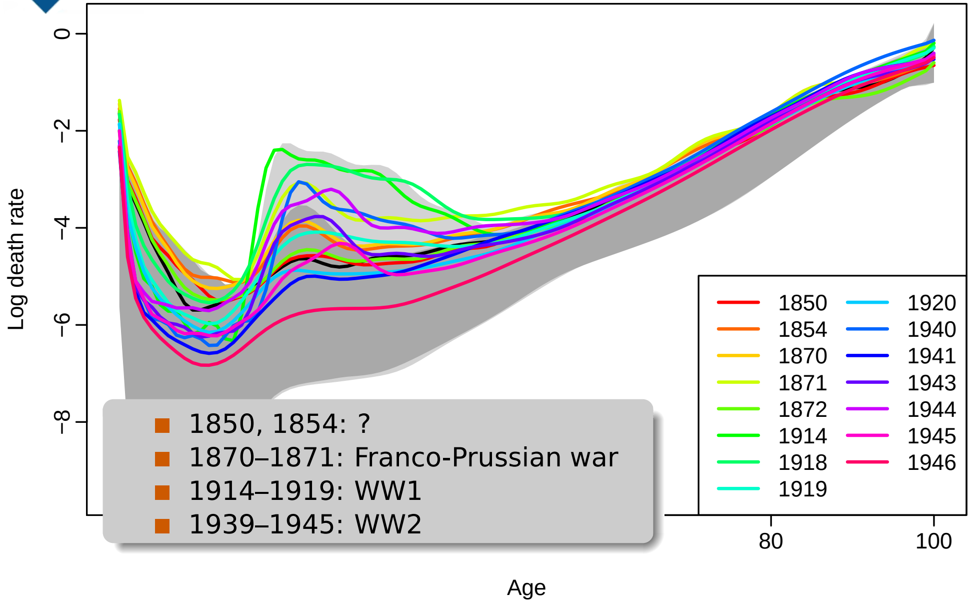
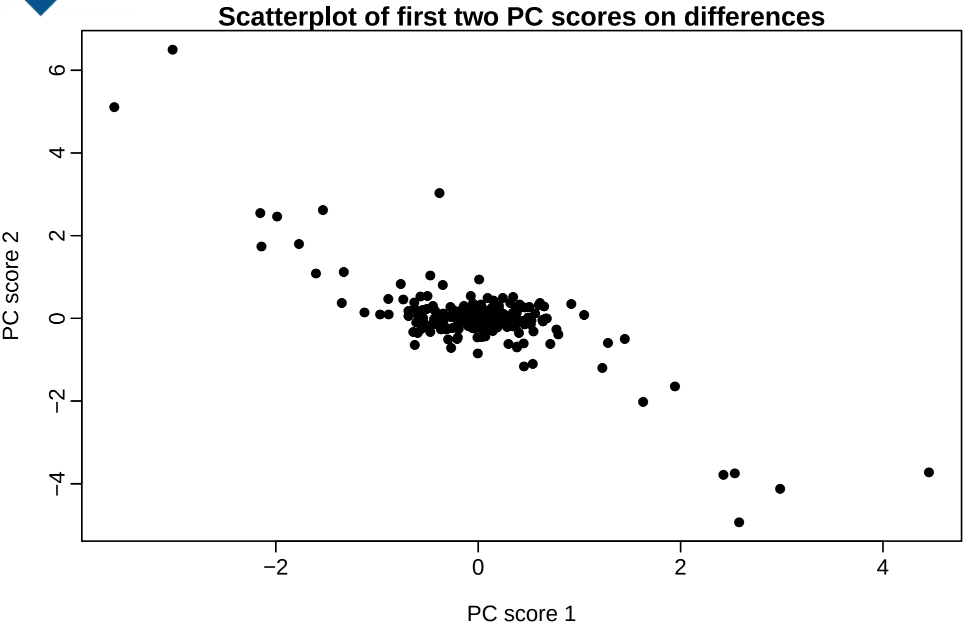


- 1850, 1854: ?
- 1870–1871: Franco-Prussian war
- 1914–1919: WW1
- 1939–1945: WW2

Kernel density estimate



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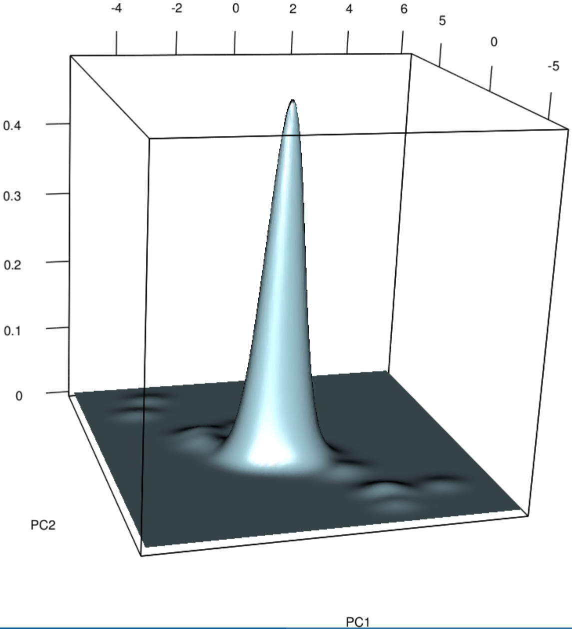
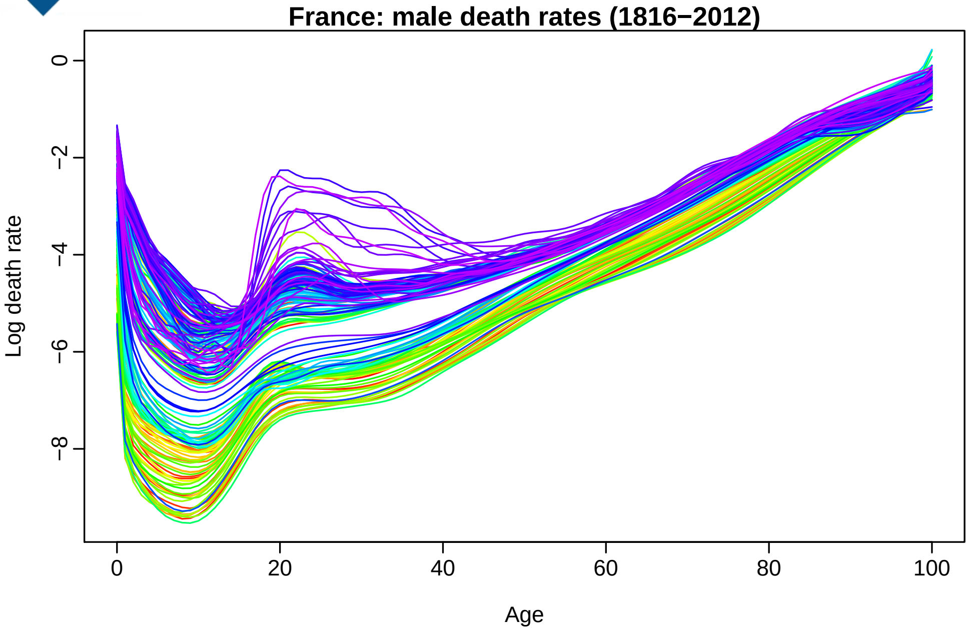
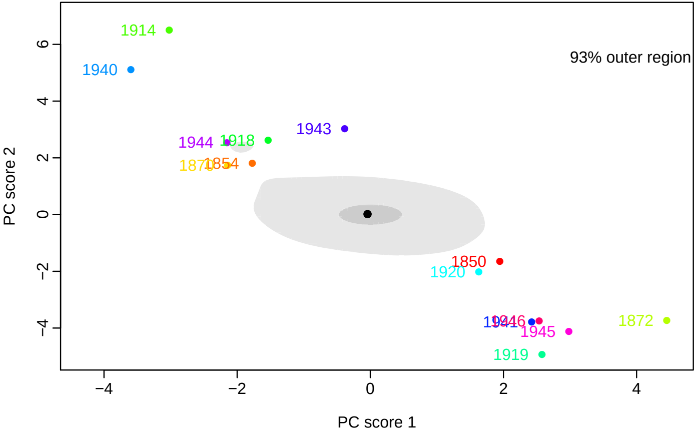


Ordering by bivariate density



Functional HDR boxplot

- Bivariate HDR boxplot due to Hyndman (1996).
- Rank points by value of kernel density estimate.
- Display mode, 50% and (usually) 99% highest density regions (HDRs) and mode.



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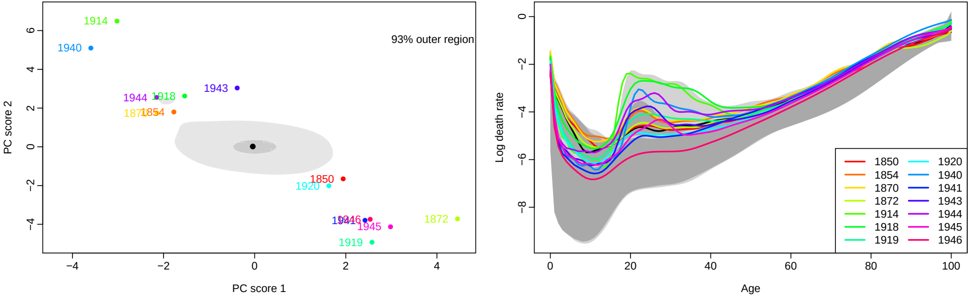

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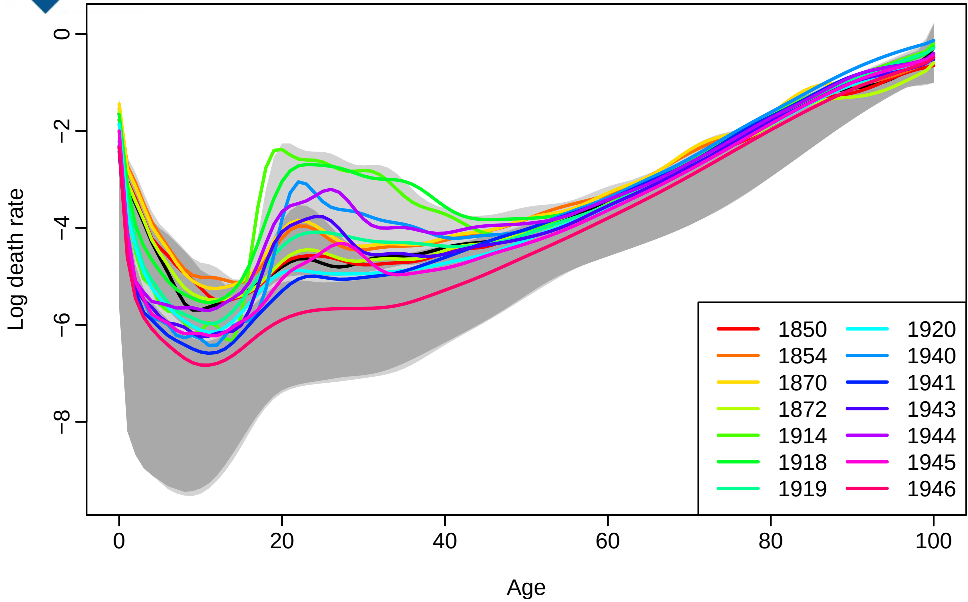

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2 Functional principal components

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Selected references

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