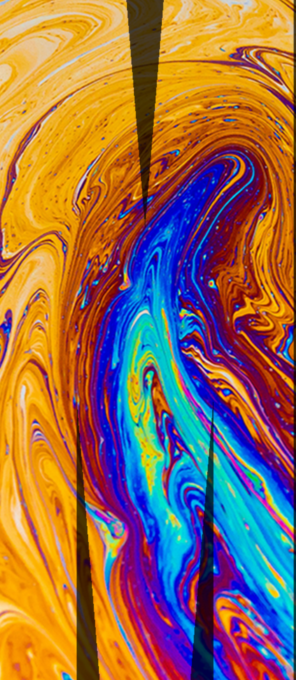

The geometry of forecast reconciliation

Rob J Hyndman

28 August 2020

Outline

1. Hierarchical and grouped time series
2. Forecast reconciliation using projections
3. Probabilistic forecast reconciliation
4. Evaluating probabilistic forecasts

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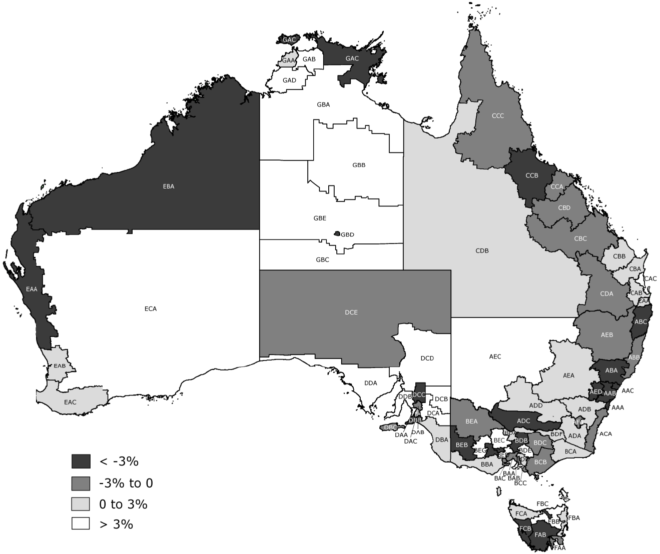
Australian tourism



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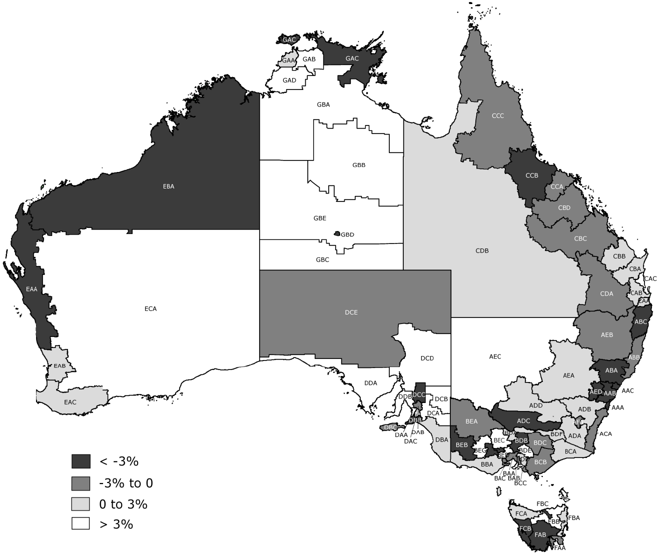
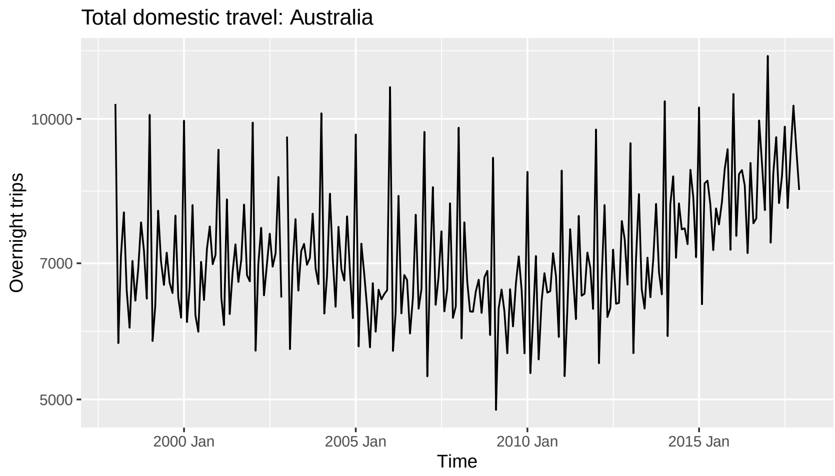


- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - 7 states
  - 27 zones
  - 75 regions

Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney     926.
##  2 1998 Feb NSW   Metro NSW Sydney     647.
##  3 1998 Mar NSW   Metro NSW Sydney     716.
##  4 1998 Apr NSW   Metro NSW Sydney     621.
##  5 1998 May NSW   Metro NSW Sydney     598.
##  6 1998 Jun NSW   Metro NSW Sydney     601.
##  7 1998 Jul NSW   Metro NSW Sydney     720.
##  8 1998 Aug NSW   Metro NSW Sydney     645.
##  9 1998 Sep NSW   Metro NSW Sydney     633.
## 10 1998 Oct NSW   Metro NSW Sydney     771.
## # ... with 17,990 more rows
```

Australian tourism data



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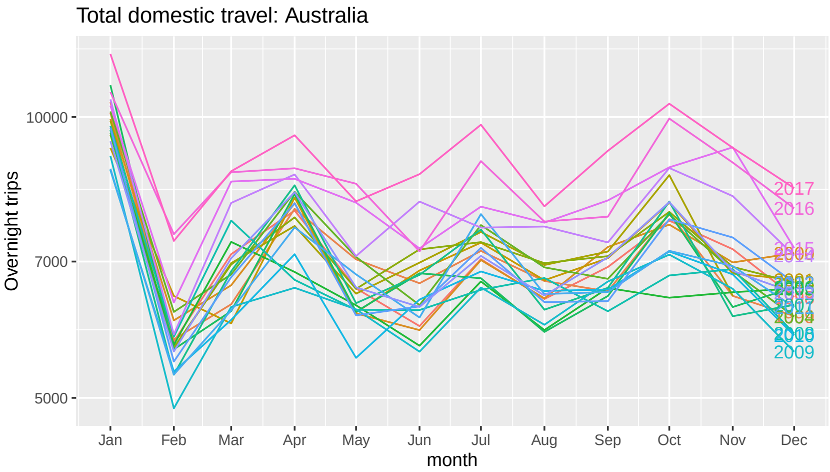

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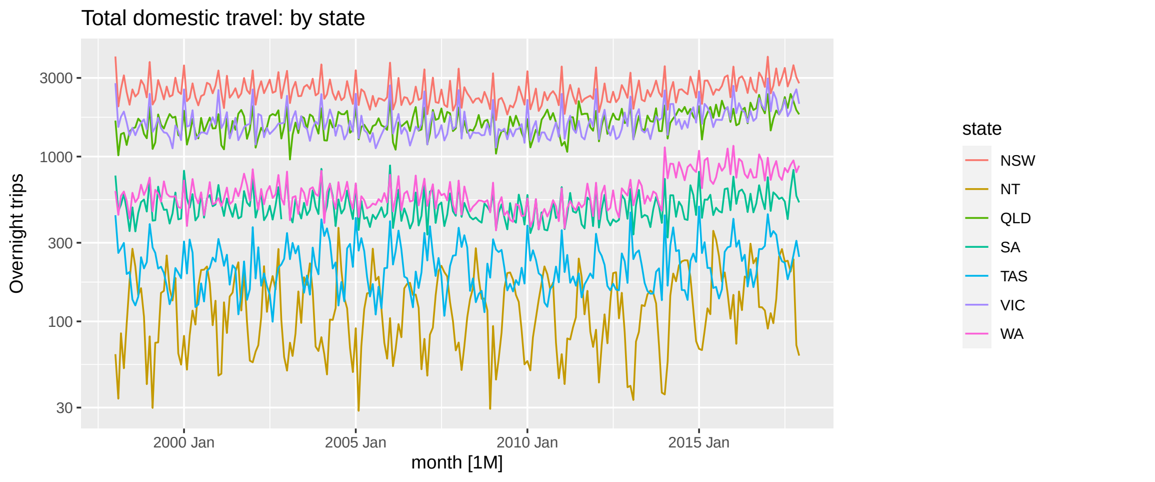

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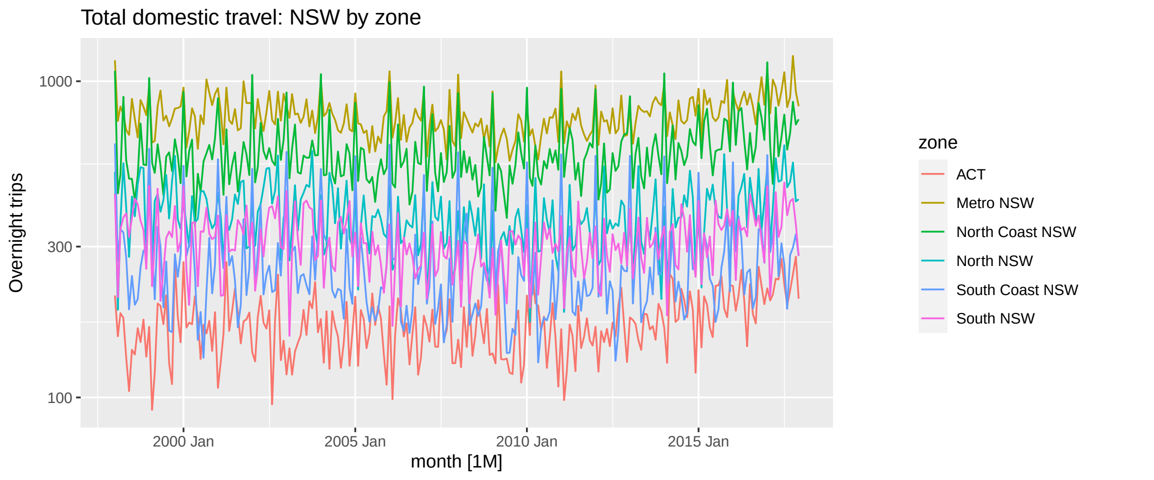

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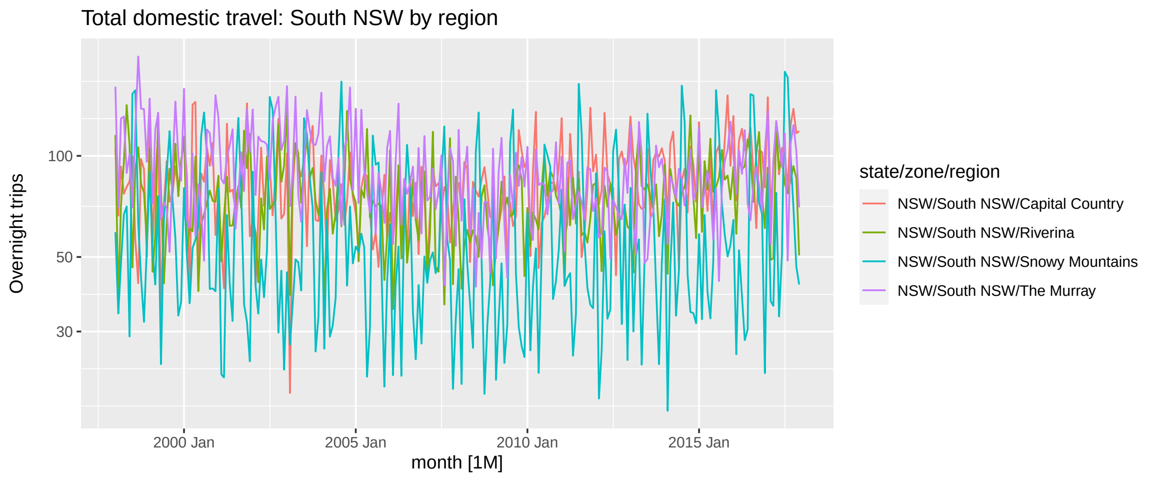
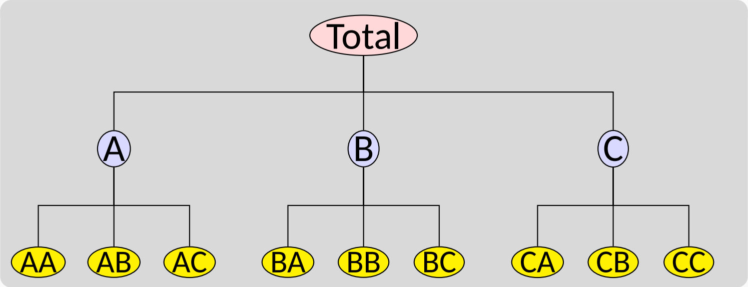


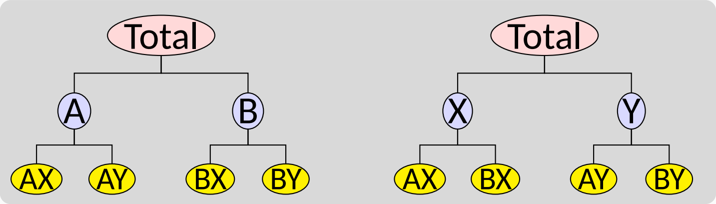
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



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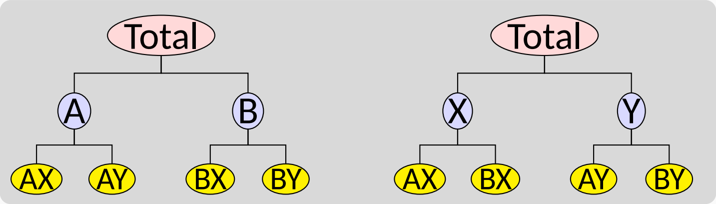

Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Outline

The problem

How to produce **coherent** forecasts at all nodes?

The problem

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Old approaches (pre 2009)

- Bottom-up forecasting
- Top-down forecasting
- Middle-out forecasting

The problem

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Old approaches (pre 2009)

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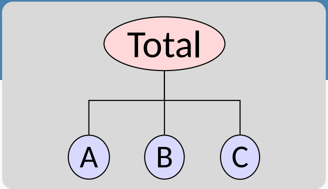
Forecast reconcilation approach

1. Forecast all series at all levels of aggregation using an automatic forecasting algorithm. (e.g., `ETS`, `ARIMA`, ...)
2. Reconcile the resulting forecasts so they are coherent using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).

Key forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 *CSDA*) Optimal combination forecasts for hierarchical time series.
- Athanasopoulos, Ahmed, Hyndman (2009 *IJF*) Hierarchical forecasts for Australian domestic tourism.
- Hyndman, Lee, Wang (2016 *CSDA*) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 *JASA*) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 *IJF*) Forecast reconciliation: A geometric view with new insights on bias correction.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020) Probabilistic forecast reconciliation: properties, evaluation and score optimisation.

Hierarchical and grouped time series



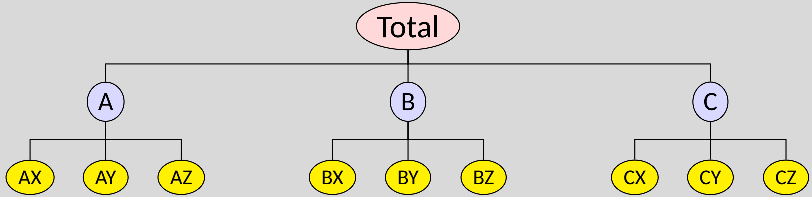
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- y_t = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Hierarchical time series



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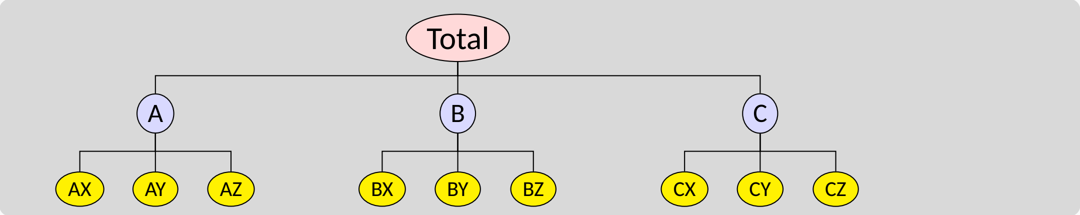
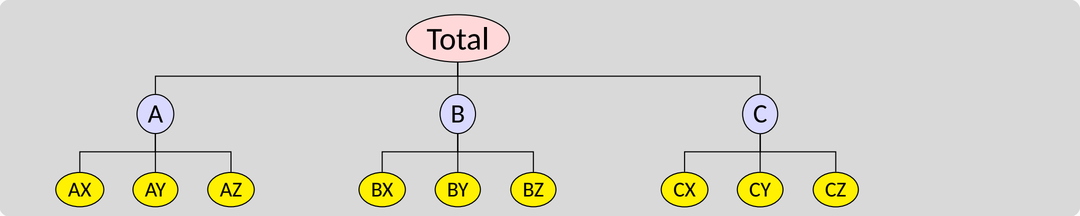


$$
\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}
$$

Hierarchical time series



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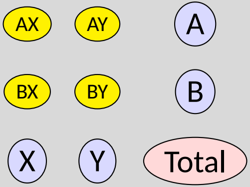

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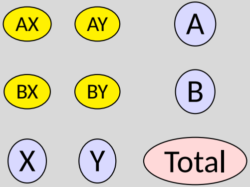

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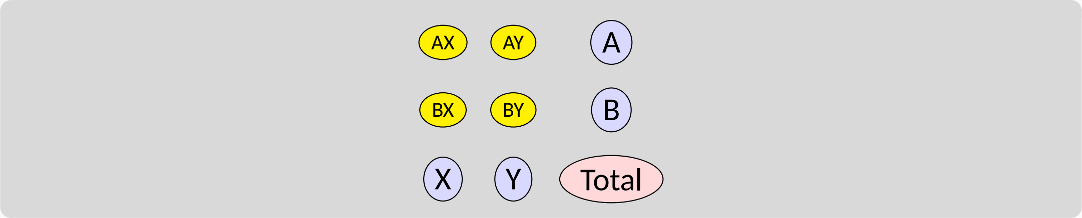

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Definitions

Coherent subspace

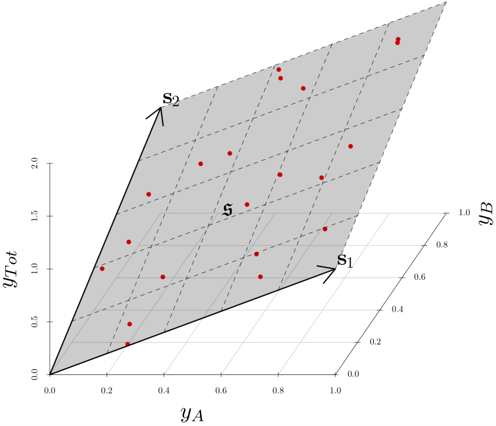
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

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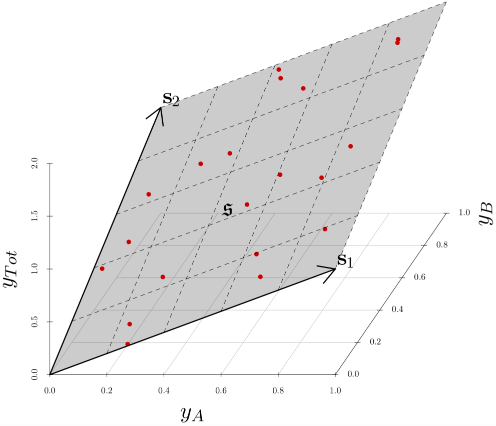
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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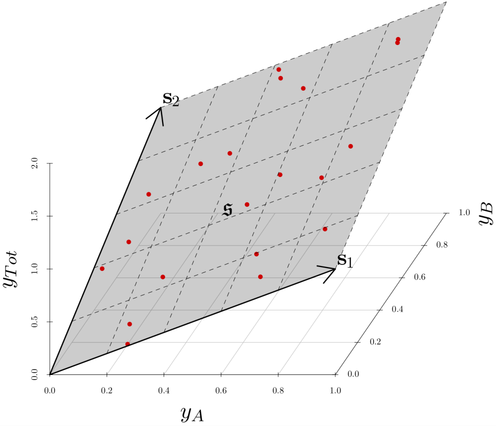

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Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$.

$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear reconciliation

If ψ is a linear function and **G** is some matrix, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{G}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.
- e.g., OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$

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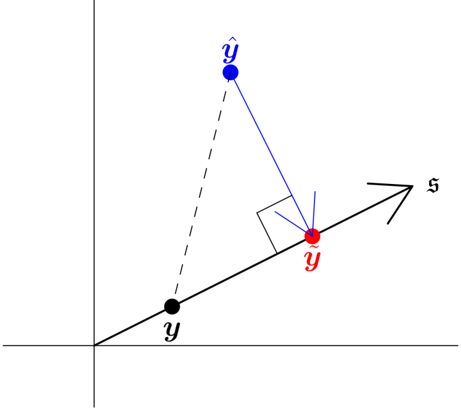

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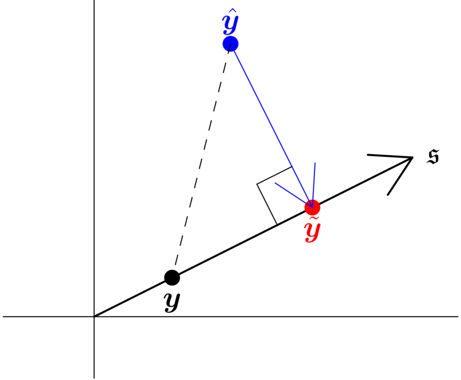

Projections

Suppose $\boldsymbol{S}\boldsymbol{G}$ is a projection onto $\mathfrak{s}$, then

- Coherent base forecasts are unchanged.
- Unbiased base forecasts remain unbiased.

- Orthogonal projections lead to smallest possible adjustments of base forecasts.

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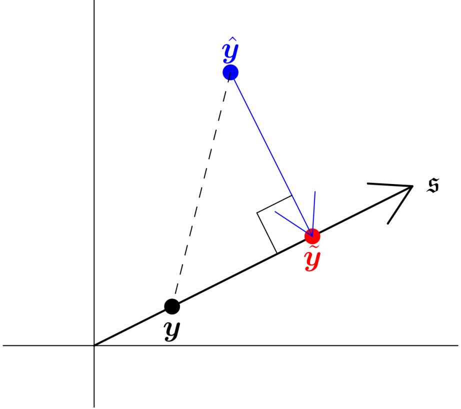
Distance reducing property

If $\boldsymbol{S}\boldsymbol{G}$ is an orthogonal projection onto $\mathfrak{s}$ then:

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\| \le \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|.$$



- Distance reduction holds for any realisation and any forecast.
- Other measures of forecast accuracy may be worse.
- Not necessarily the optimal reconciliation.

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

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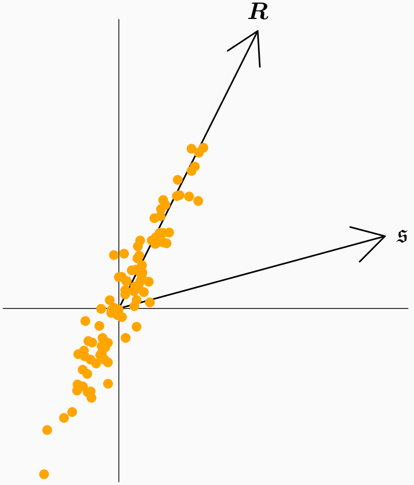
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Orange: in-sample errors



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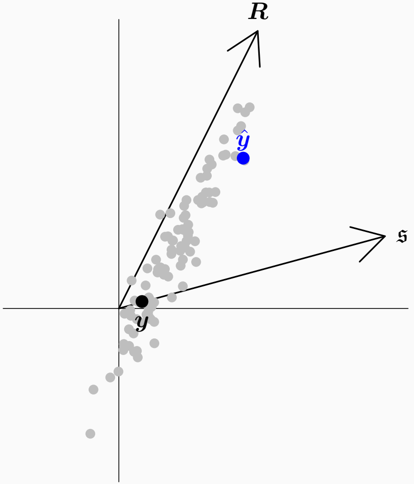

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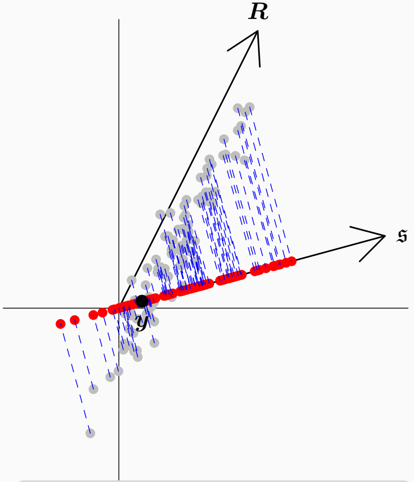

Orthogonal projection

Linear projections

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Variance

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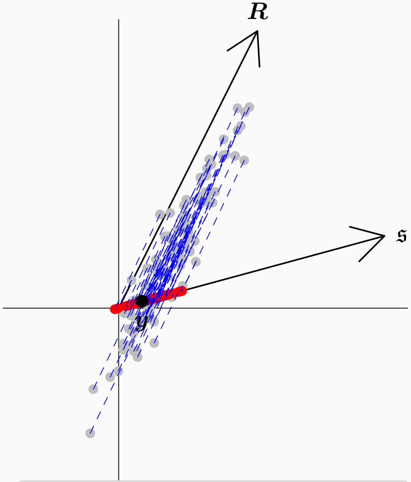
- Grey: potential base forecasts
- Red: reconciled forecsts



Oblique projection

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciliation method	$\boldsymbol{G}$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS	$(\boldsymbol{S}'\boldsymbol{\Lambda}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{\Lambda}$
MinT(Sample)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}$
MinT(Shrink)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\boldsymbol{W}_h = k_h\boldsymbol{W}_1$.

- $\boldsymbol{\Lambda}$ is diagonal matrix
- $\hat{\boldsymbol{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_{\text{sam}}) + (1-\tau)\hat{\boldsymbol{W}}_{\text{sam}}$ where $\tau = \dfrac{\sum_{i\neq j}\hat{\text{Var}}(\hat{\sigma}_{ij})}{\sum_{i\neq j}\hat{\sigma}_{ij}^2}$ and σ_{ij} denotes the (i,j)th element of $\hat{\boldsymbol{W}}_{\text{sam}}$.

Outline

Coherent probabilistic forecasts

Coherent probabilistic forecasts

Given the triple $(\mathbb{R}^m, \mathscr{F}_{\mathbb{R}^m}, \nu)$, a coherent probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is such that

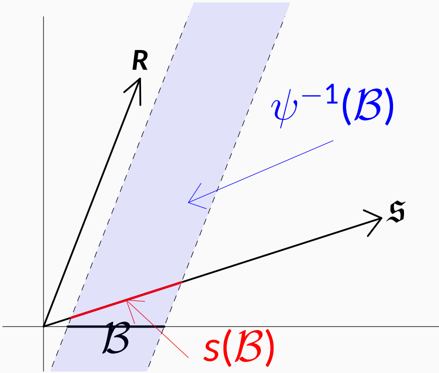
$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\mathbb{R}^m}.$$

Probabilistic forecast reconciliation

The reconciled probability measure of $\hat{\nu}$ wrt $\psi(.)$ is such that

$$\tilde{\nu}(\mathcal{B}) = \hat{\nu}(\psi^{-1}(\mathcal{B})) \qquad \forall \mathcal{B} \in \mathscr{F}_{\mathfrak{s}}\,,$$

where $\psi^{-1}(\mathcal{B}) := \{\mathbf{y} \in \mathbb{R}^n : \psi(\mathbf{y}) \in \mathcal{B}\}$ is the pre-image of $\mathcal{B}$, that is the set of all points in $\mathbb{R}^n$ that $\psi(.)$ maps to a point in $\mathcal{B}$.



Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_{\boldsymbol{b}}(\boldsymbol{b}) = |\boldsymbol{G}^*| \int \hat{f}(\boldsymbol{G}^- \boldsymbol{b} + \boldsymbol{G}_\perp \boldsymbol{a}) d\boldsymbol{a}$$

- $\hat{f}$ is density of incoherent base probabilistic forecast
- $\boldsymbol{G}^-$ is $n \times m$ generalised inverse of $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}^- = \boldsymbol{I}$
- $\boldsymbol{G}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}_\perp = \boldsymbol{0}$
- $\boldsymbol{G}^* = \left(\boldsymbol{G}^- \vdots \boldsymbol{G}_\perp\right)$, and $\boldsymbol{b}$ and $\boldsymbol{a}$ are obtained via

  the change of variables $\boldsymbol{y} = \boldsymbol{G}^* \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix} \boldsymbol{S}^- \\ \boldsymbol{S}'_\perp \end{pmatrix}$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n - m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

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- $\boldsymbol{S}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Gaussian reconciliation

If the incoherent base forecasts are $\text{N}(\hat{\mu}, \hat{\Sigma})$,
then the reconciled density is $\text{N}(\boldsymbol{SG}\hat{\mu}, \boldsymbol{SG}\hat{\Sigma}\boldsymbol{G}'\boldsymbol{S}')$.

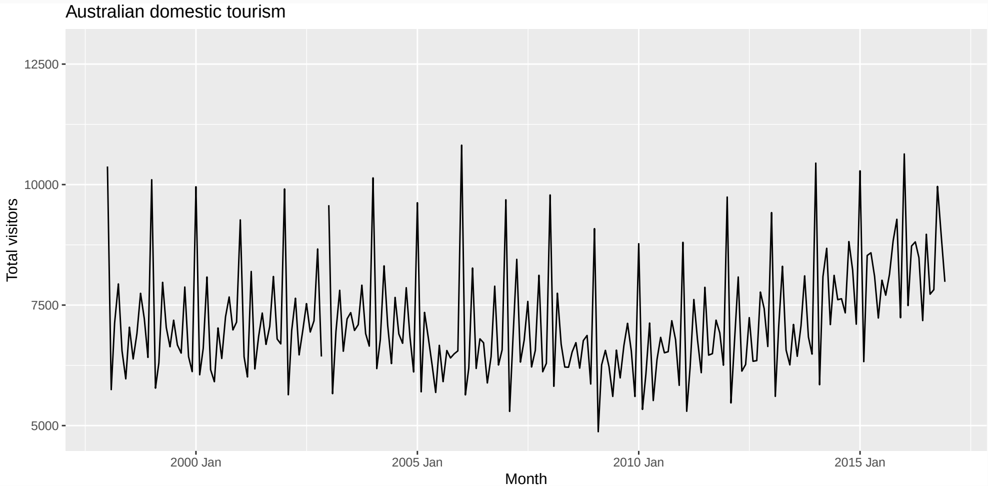
Simulation from a reconciled distribution

Suppose that $\left(\hat{\boldsymbol{y}}^{[1]}, \dots, \hat{\boldsymbol{y}}^{[L]}\right)$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $\left(\tilde{\boldsymbol{y}}^{[1]}, \dots, \tilde{\boldsymbol{y}}^{[L]}\right)$ where $\tilde{\boldsymbol{y}}^{[\ell]} := \psi(\hat{\boldsymbol{y}}^{[\ell]})$ for $\ell = 1, \dots, L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

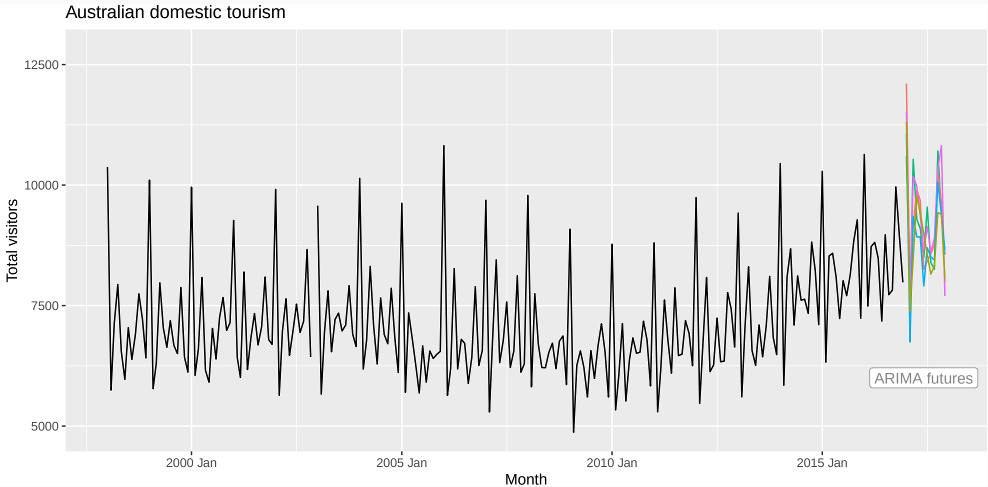
- So reconciling sample paths from incoherent distributions works.

Outline

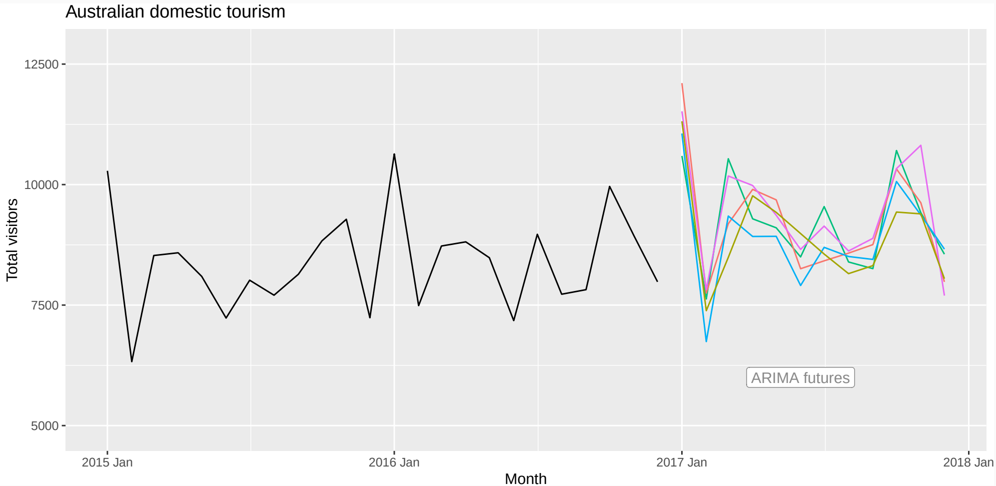
Evaluating probabilistic forecasts



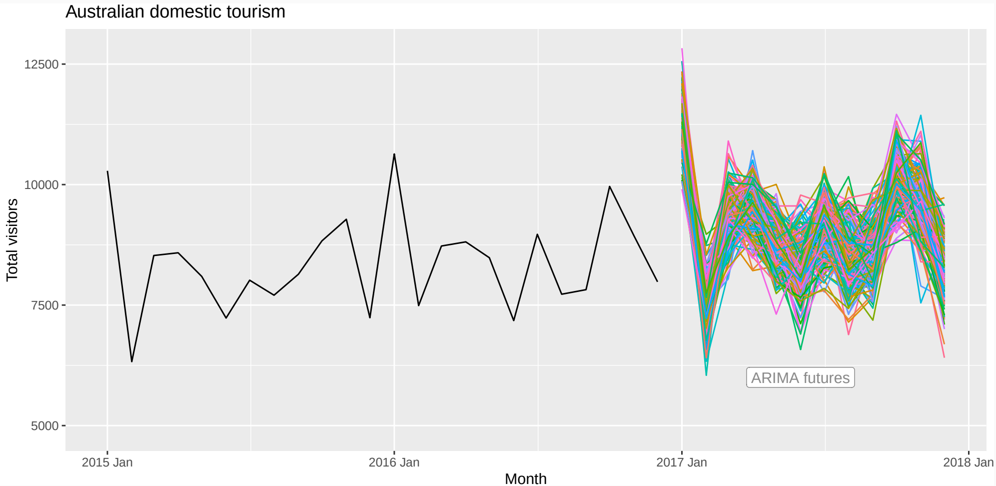
Evaluating probabilistic forecasts



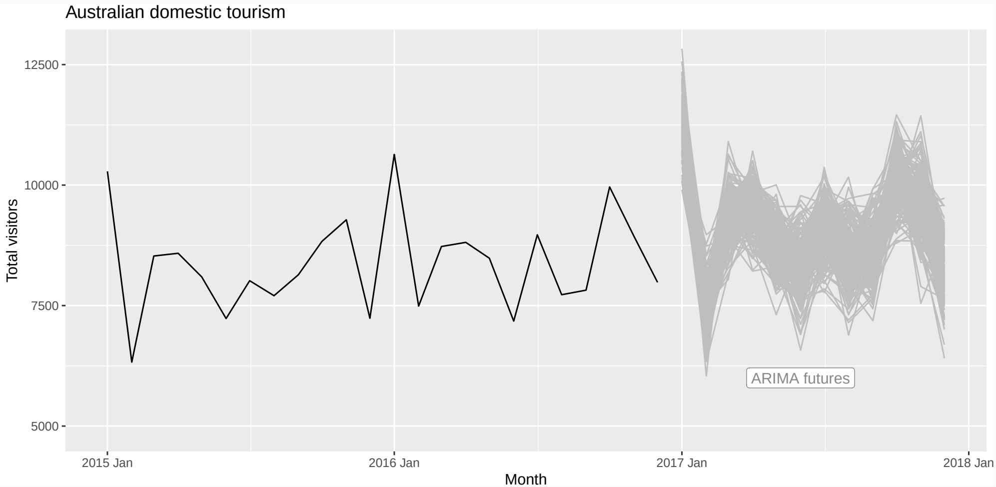
Evaluating probabilistic forecasts



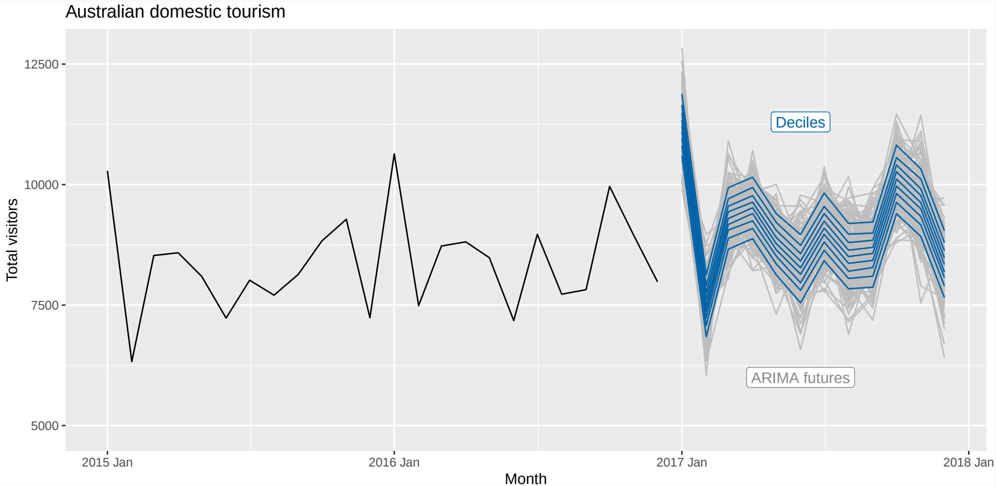
Evaluating probabilistic forecasts



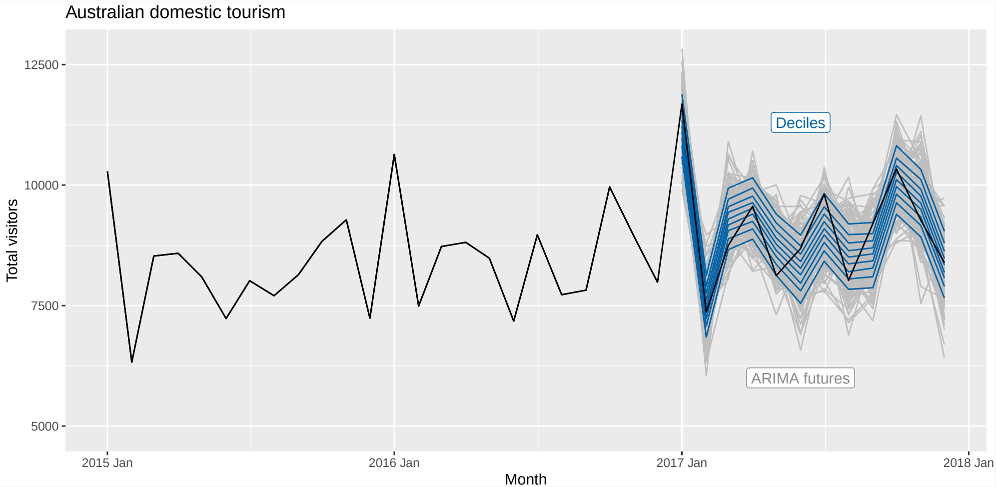
Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



Evaluating probabilistic forecasts

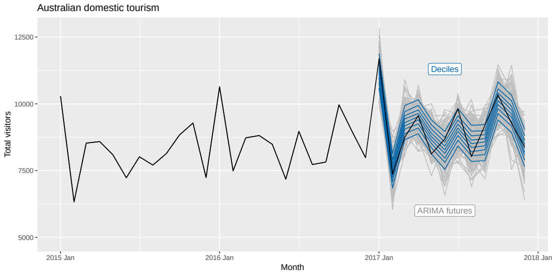
$f_{p,t}$ = quantile forecast with prob. p at time t.

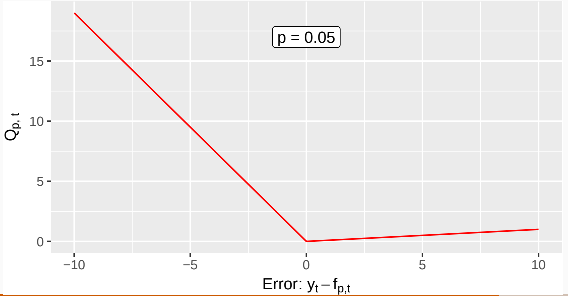
y_t = observation at time t

Quantile score

$$Q_{p,t} = \begin{cases} 2(1-p)|y_t - f_{p,t}|, & \text{if } y_t < f_{p,t} \\ 2p|y_t - f_{p,t}|, & \text{if } y_t \geq f_{p,t} \end{cases}$$

- Low $Q_{p,t}$ is good
- Multiplier of 2 often omitted, but useful for interpretation
- $Q_{p,t}$ like absolute error, weighted to account for likely exceedance
- Average $Q_{p,t}$ over p = CRPS (Continuous Rank Probability Score)





Evaluating probabilistic forecasts

Continuous Rank Probability Score

- For univariate forecasts
- Average $Q_{p,t}$ over p.
- E

Energy score

- For multivariate forecasts
- $\text{ES}(P, \boldsymbol{\omega}) = \text{E}_P||\boldsymbol{y} - \boldsymbol{\omega}||^\alpha - \frac{1}{2}\text{E}_P||\boldsymbol{y} - \boldsymbol{y}^*||^\alpha, \;\; \alpha \in (0, 2]$
- Approximate via simulations

Other scoring rules:

- Scoring rule: $K(P, y)$

To add from probabilistic paper

- Score optimal reconciliation
- Electricity example