The geometry of forecast reconciliation

Rob J Hyndman

CUHK: 16 September 2021

Outline

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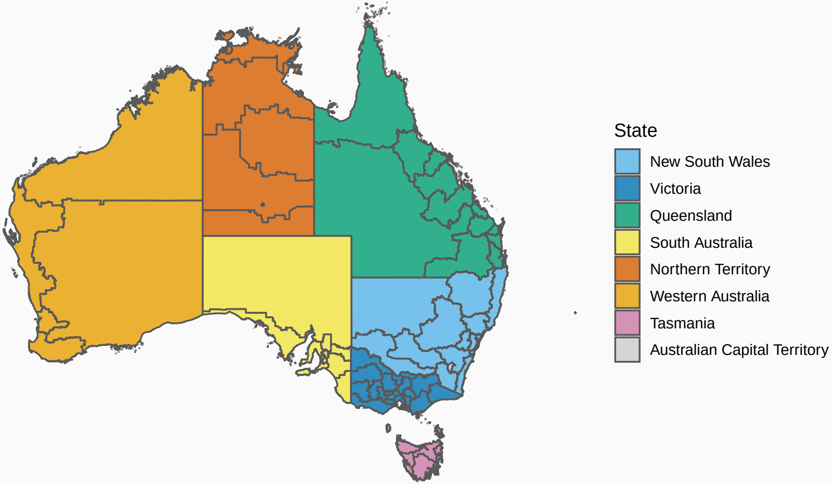
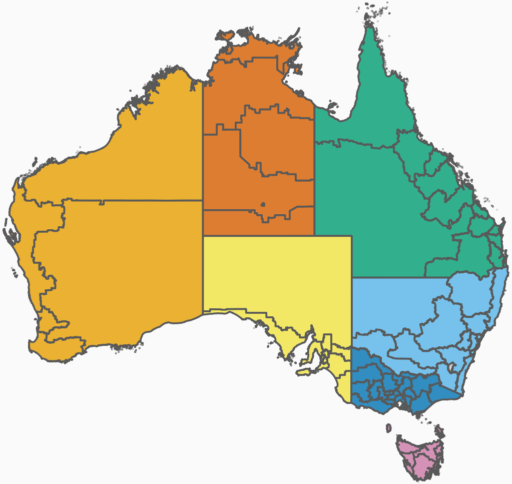
Australian tourism regions



Australian tourism regions

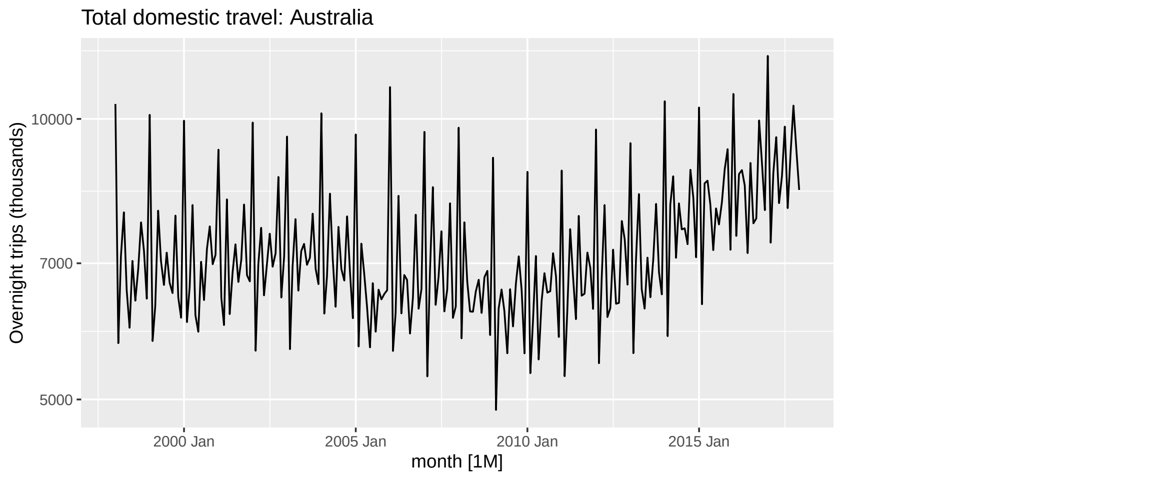


- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - 7 states
  - 27 zones
  - 75 regions

Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney     926.
##  2 1998 Feb NSW   Metro NSW Sydney     647.
##  3 1998 Mar NSW   Metro NSW Sydney     716.
##  4 1998 Apr NSW   Metro NSW Sydney     621.
##  5 1998 May NSW   Metro NSW Sydney     598.
##  6 1998 Jun NSW   Metro NSW Sydney     601.
##  7 1998 Jul NSW   Metro NSW Sydney     720.
##  8 1998 Aug NSW   Metro NSW Sydney     645.
##  9 1998 Sep NSW   Metro NSW Sydney     633.
## 10 1998 Oct NSW   Metro NSW Sydney     771.
## # ... with 17,990 more rows
```

Australian tourism data



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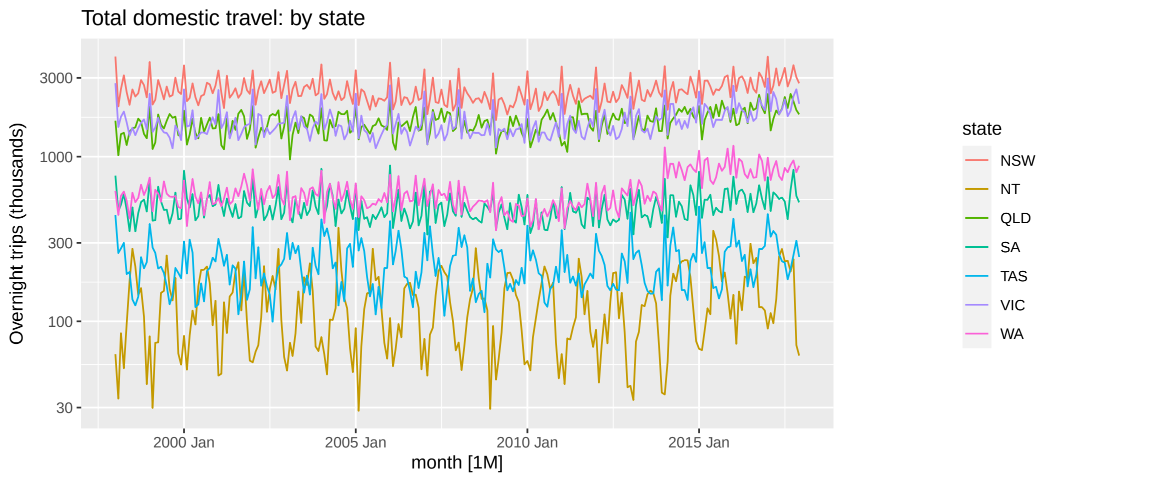

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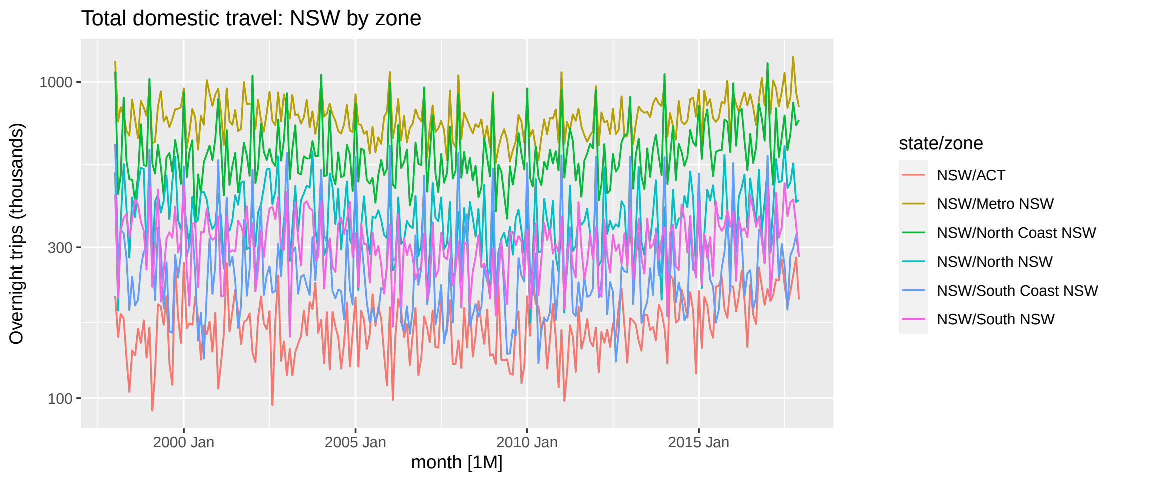

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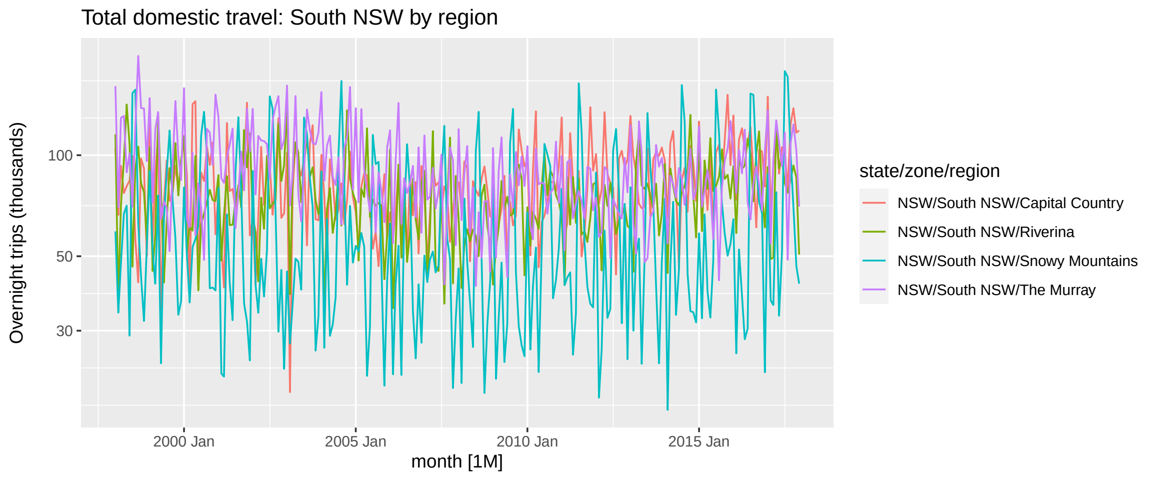
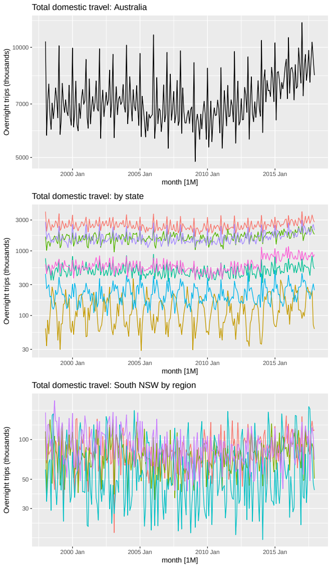
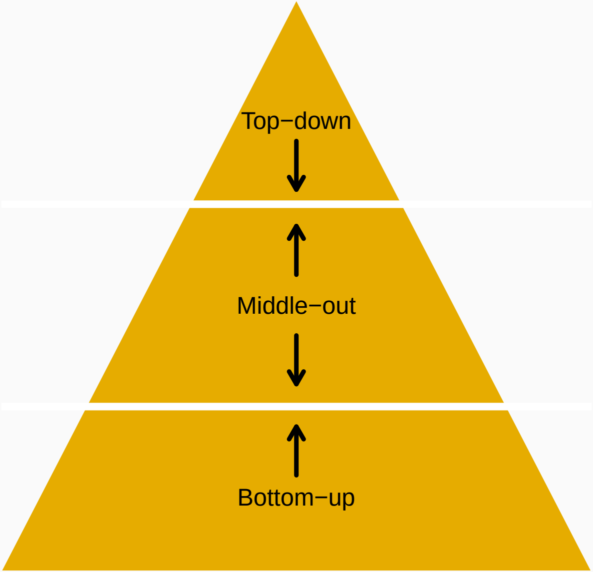


Hierarchical forecasting 20 years ago

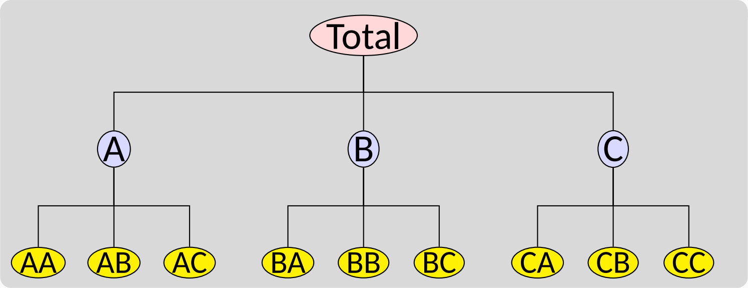




Outline

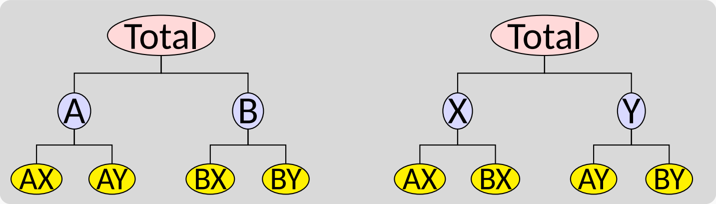
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



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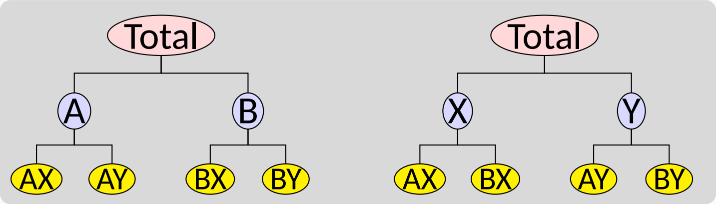

Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

The problem

How to produce **coherent** forecasts at all nodes?

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Old approaches (pre 2009)

- Bottom-up forecasting
- Top-down forecasting
- Middle-out forecasting

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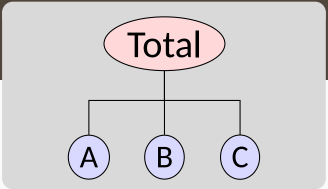
Forecast reconcilation approach

1. Forecast all series at all levels of aggregation using an automatic forecasting algorithm. (e.g., `ETS`, `ARIMA`, ...)
2. Reconcile the resulting forecasts so they are coherent using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).

Key forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 *CSDA*) Optimal combination forecasts for hierarchical time series.
- Athanasopoulos, Ahmed, Hyndman (2009 *IJF*) Hierarchical forecasts for Australian domestic tourism.
- Hyndman, Lee, Wang (2016 *CSDA*) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 *JASA*) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 *IJF*) Forecast reconciliation: A geometric view with new insights on bias correction.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020) Probabilistic forecast reconciliation: properties, evaluation and score optimisation.

Hierarchical and grouped time series



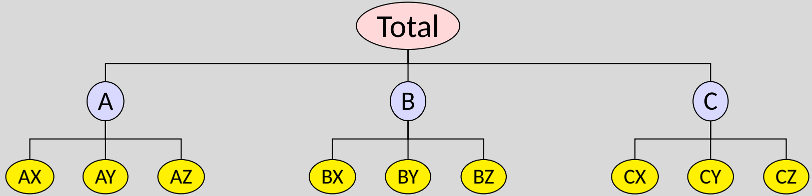
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- y_t = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Hierarchical time series



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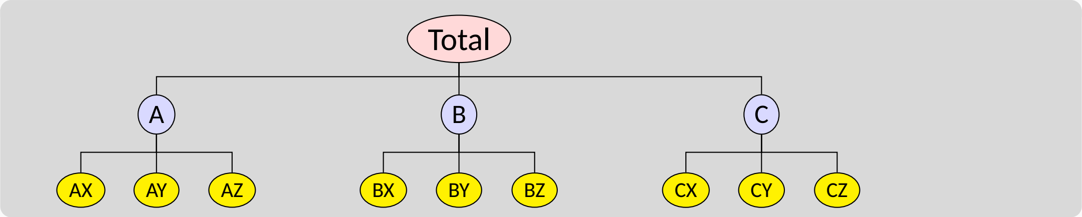

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

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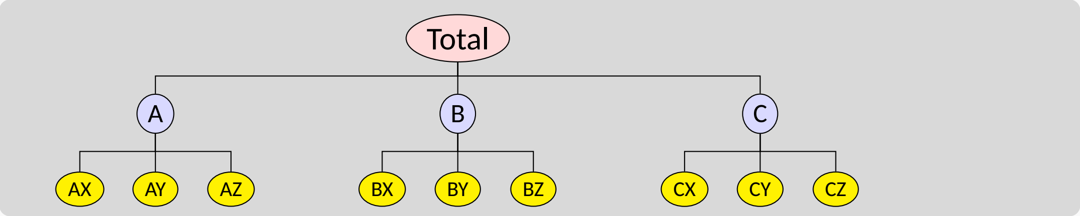

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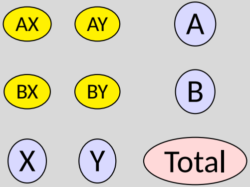

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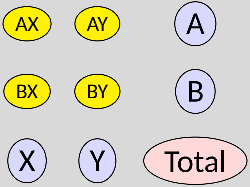

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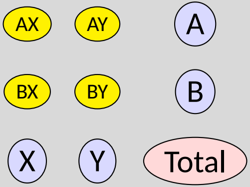

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Definitions

Coherent subspace

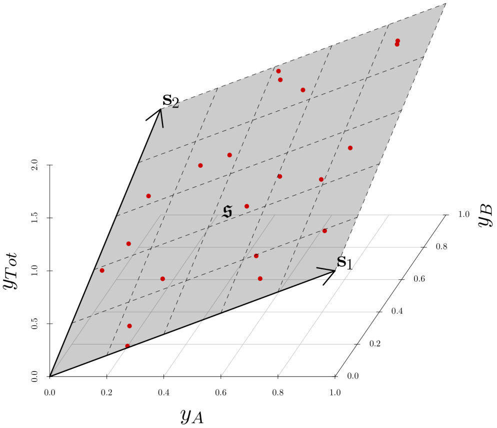
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$$y_{Tot} = y_A + y_B$$

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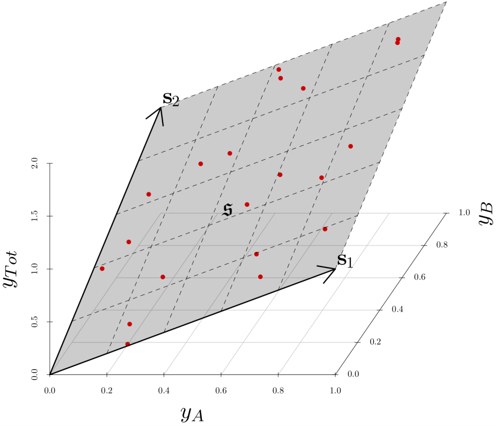
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



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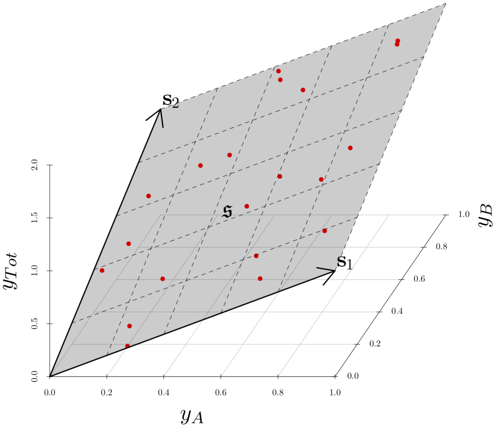

$$y_{Tot} = y_A + y_B$$

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Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$.
$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear reconciliation

If ψ is a linear function and **G** is some matrix, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{G}$ combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.
- e.g., OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$

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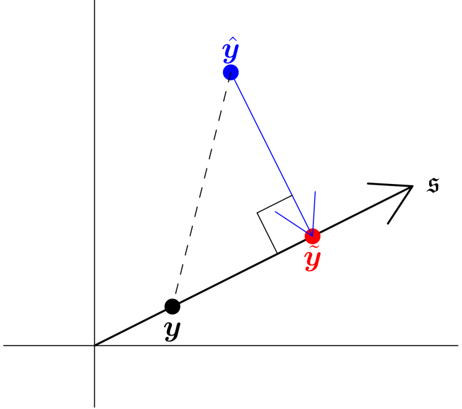

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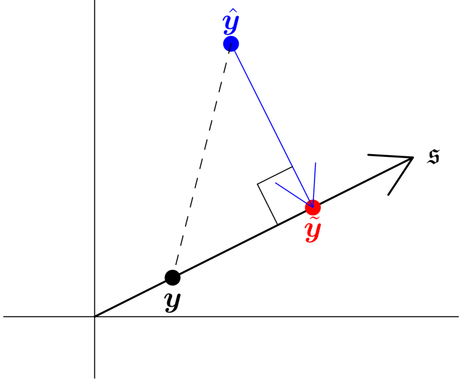

Projections

Suppose $\boldsymbol{SG}$ is a projection onto $\mathfrak{s}$, then

- Coherent base forecasts are unchanged.
- Unbiased base forecasts remain unbiased.

- Orthogonal projections lead to smallest possible adjustments of base forecasts.

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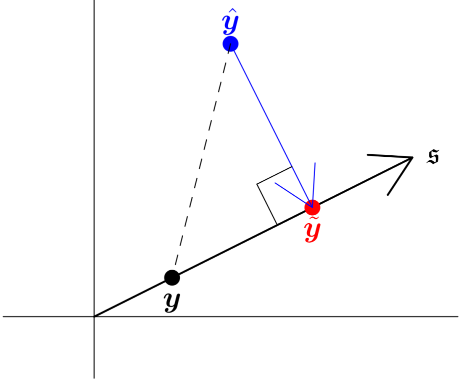
Distance reducing property

If $\boldsymbol{SG}$ is an orthogonal projection onto $\mathfrak{s}$ then:

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\| \leq \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|.$$



- Distance reduction holds for any realisation and any forecast.
- Other measures of forecast accuracy may be worse.
- Not necessarily the optimal reconciliation.

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

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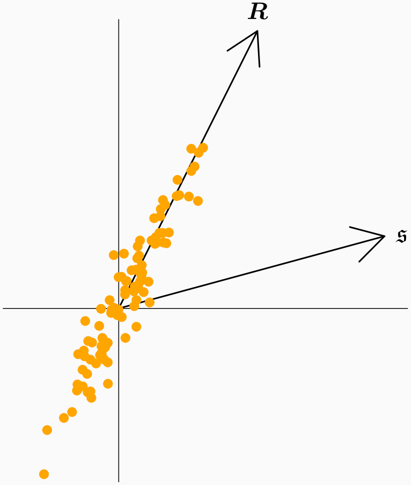
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Orange: in-sample errors



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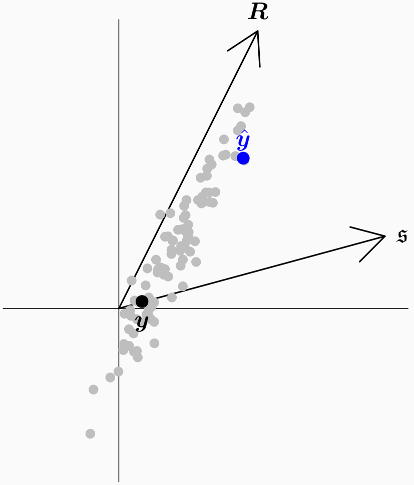
- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts



Linear projections

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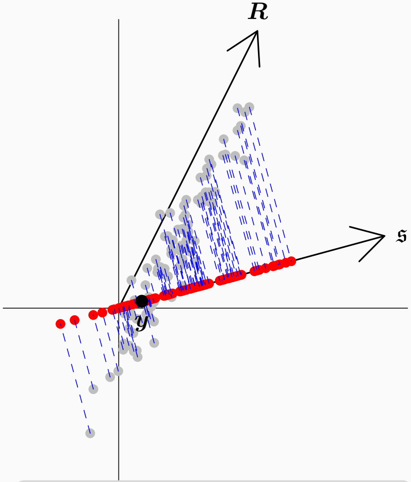

Orthogonal projection

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

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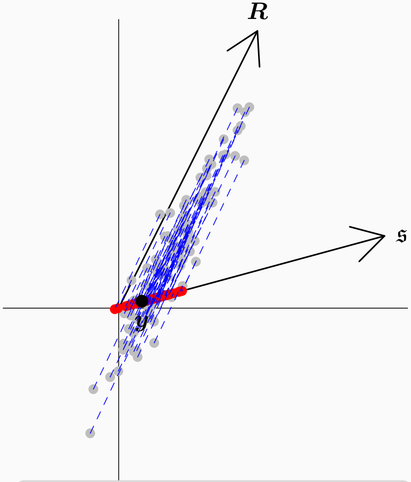
- Red: reconciled forecsts



Oblique projection

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciliation method	$\boldsymbol{G}$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS	$(\boldsymbol{S}'\boldsymbol{\Lambda}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{\Lambda}$
MinT(Sample)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}$
MinT(Shrink)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\boldsymbol{W}_h = k_h\boldsymbol{W}_1$.

- $\boldsymbol{\Lambda}$ is diagonal matrix
- $\hat{\boldsymbol{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_{\text{sam}}) + (1-\tau)\hat{\boldsymbol{W}}_{\text{sam}}$ where $\tau = \dfrac{\sum_{i\neq j}\hat{\text{Var}}(\hat{\sigma}_{ij})}{\sum_{i\neq j}\hat{\sigma}_{ij}^2}$ and σ_{ij} denotes the (i,j)th element of $\hat{\boldsymbol{W}}_{\text{sam}}$.

Outline

1 Hierarchical forecasting

2 Point forecast reconciliation

3 Example: Australian tourism

4 Probabilistic forecast reconciliation

5 Evaluating probabilistic forecasts

6 Example: Australian electricity generation

Example: Australian tourism

```
tourism
```

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney     926.
##  2 1998 Feb NSW   Metro NSW Sydney     647.
##  3 1998 Mar NSW   Metro NSW Sydney     716.
##  4 1998 Apr NSW   Metro NSW Sydney     621.
##  5 1998 May NSW   Metro NSW Sydney     598.
##  6 1998 Jun NSW   Metro NSW Sydney     601.
##  7 1998 Jul NSW   Metro NSW Sydney     720.
##  8 1998 Aug NSW   Metro NSW Sydney     645.
##  9 1998 Sep NSW   Metro NSW Sydney     633.
## 10 1998 Oct NSW   Metro NSW Sydney     771.
```

Example: Australian tourism

```
tourism_agg <- tourism %>%
  aggregate_key(state/zone/region, visitors = sum(visitors))
```

```
## # A tsibble: 26,400 x 5 [1M]
## # Key:       state, zone, region [110]
##       month state        zone         region       visitors
##       <mth> <chr*>       <chr*>       <chr*>          <dbl>
##  1 1998 Jan <aggregated> <aggregated> <aggregated>   10376.
##  2 1998 Feb <aggregated> <aggregated> <aggregated>    5746.
##  3 1998 Mar <aggregated> <aggregated> <aggregated>    7129.
##  4 1998 Apr <aggregated> <aggregated> <aggregated>    7939.
##  5 1998 May <aggregated> <aggregated> <aggregated>    6552.
##  6 1998 Jun <aggregated> <aggregated> <aggregated>    5969.
##  7 1998 Jul <aggregated> <aggregated> <aggregated>    7041.
##  8 1998 Aug <aggregated> <aggregated> <aggregated>    6382.
##  9 1998 Sep <aggregated> <aggregated> <aggregated>    6907.
```

Example: Australian tourism

```
fit <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(ets = ETS(visitors))
```

```
## # A mable: 110 x 4
## # Key:     state, zone, region [110]
##    state  zone             region                  ets
##    <chr*> <chr*>           <chr*>              <model>
##  1 NSW    ACT              Canberra         <ETS(M,N,A)>
##  2 NSW    ACT              <aggregated>     <ETS(M,N,A)>
##  3 NSW    Metro NSW        Central Coast    <ETS(M,N,M)>
##  4 NSW    Metro NSW        Sydney           <ETS(M,N,A)>
##  5 NSW    Metro NSW        <aggregated>     <ETS(M,N,A)>
##  6 NSW    North Coast NSW Hunter            <ETS(M,N,M)>
##  7 NSW    North Coast NSW North Coast NSW <ETS(M,N,M)>
##  8 NSW    North Coast NSW <aggregated>     <ETS(M,N,M)>
```

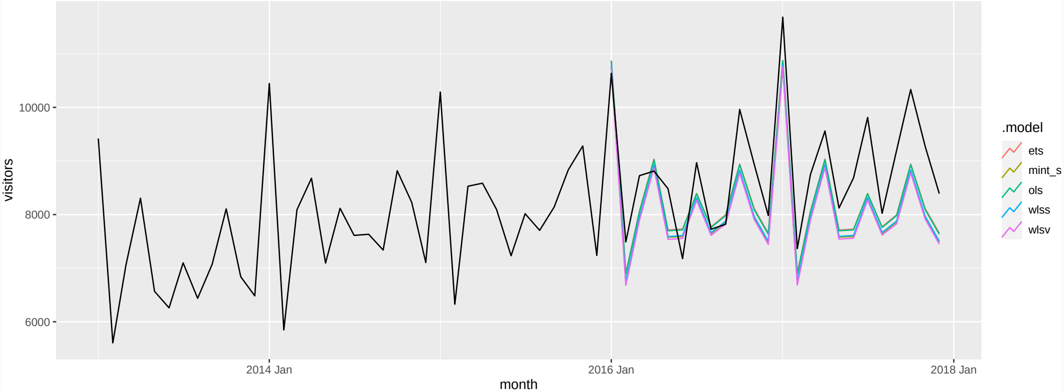
Example: Australian tourism

```r
fc <- fit %>%
  reconcile(
    ols = min_trace(ets, method="ols"),
    wlsv = min_trace(ets, method="wls_var"),
    wlss = min_trace(ets, method="wls_struct"),
    #mint_c = min_trace(ets, method="mint_cov"),
    mint_s = min_trace(ets, method="mint_shrink"),
  ) %>%
  forecast(h = "2 years")
```

```
## # A fable: 5,280 x 7 [1M]
## # Key:     state, zone, region, .model [220]
##    state  zone   region   .model    month      visitors .mean
##    <chr*> <chr*> <chr*>   <chr>     <mth>        <dist> <dbl>
##  1 NSW    ACT    Canberra ets    2016 Jan N(202, 1437)  202.
##  2 NSW    ACT    Canberra ets    2016 Feb  N(160, 912)  160.
```

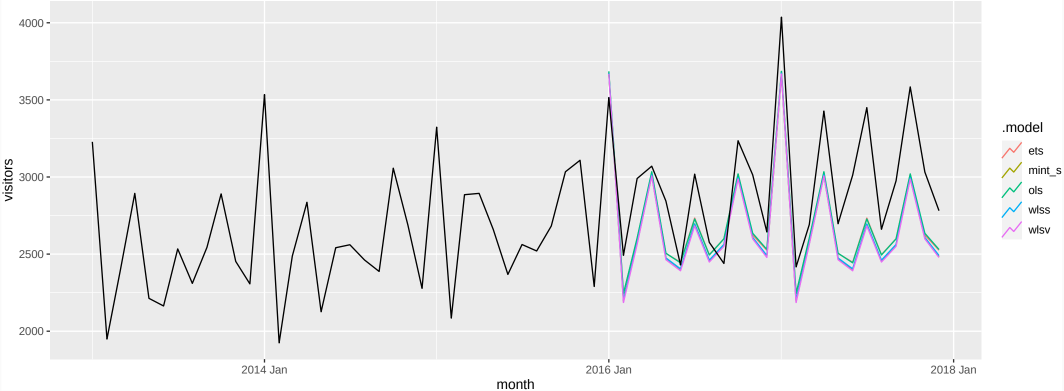
Example: Australian tourism

```
fc %>%
  filter(is_aggregated(state)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



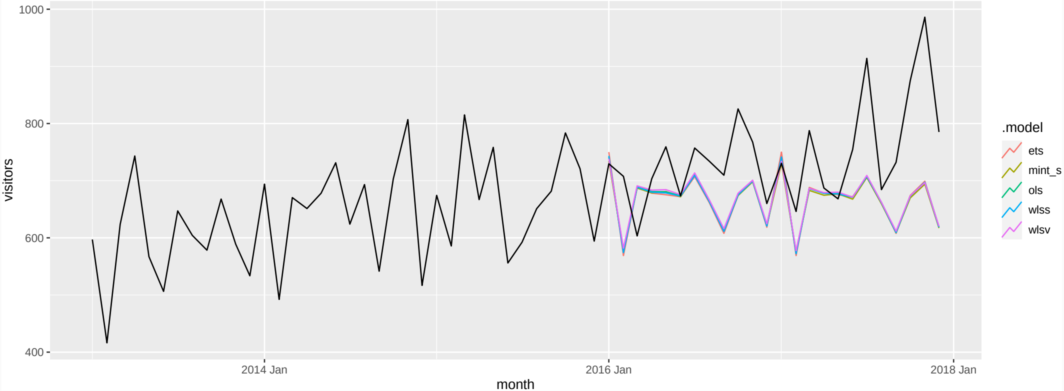
Example: Australian tourism

```r
fc %>%
  filter(state == "NSW" & is_aggregated(zone)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



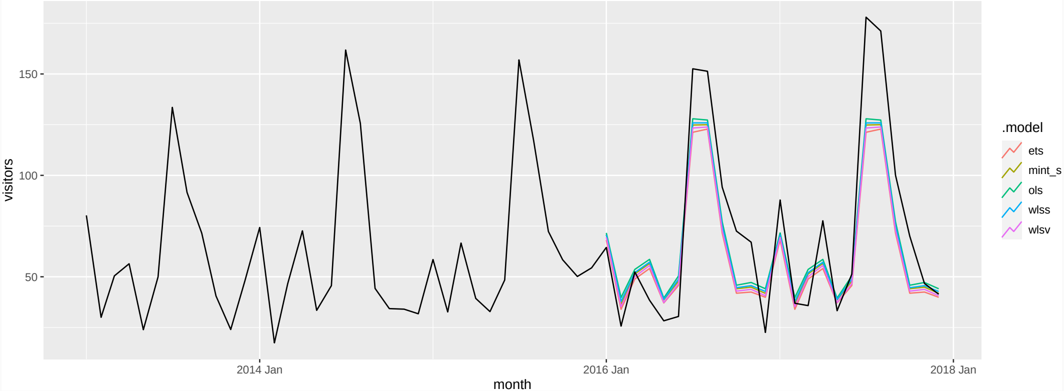
Example: Australian tourism

```r
fc %>%
  filter(region == "Melbourne") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



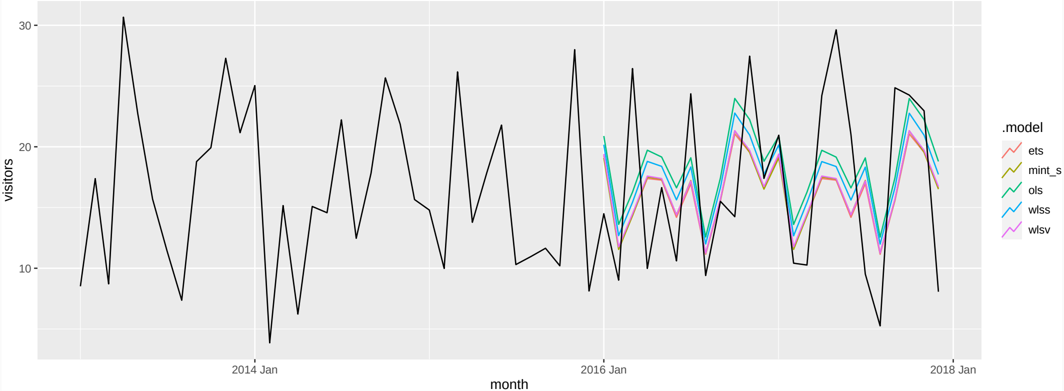
Example: Australian tourism

```
fc %>%
  filter(region == "Snowy Mountains") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Example: Australian tourism

```
fc %>%
  filter(region == "Barossa") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Example: Australian tourism

```
fc %>%
  accuracy(data = tourism_agg,
           measures = list(rmsse = RMSSE))
```

```
## # A tibble: 550 x 6
##    .model state  zone            region          .type rmsse
##    <chr>  <chr*> <chr*>          <chr*>          <chr> <dbl>
##  1 ets    NSW    ACT             Canberra        Test  0.835
##  2 ets    NSW    ACT             <aggregated>    Test  0.835
##  3 ets    NSW    Metro NSW       Central Coast   Test  0.747
##  4 ets    NSW    Metro NSW       Sydney          Test  1.16
##  5 ets    NSW    Metro NSW       <aggregated>    Test  1.18
##  6 ets    NSW    North Coast NSW Hunter          Test  1.21
##  7 ets    NSW    North Coast NSW North Coast NSW Test  0.884
##  8 ets    NSW    North Coast NSW <aggregated>    Test  1.02
##  9 ets    NSW    North NSW       Blue Mountains  Test  1.02
```

Example: Australian tourism

```
fc %>%
  accuracy(tourism_agg,
           measures = list(mase = MASE, rmsse = RMSSE)) %>%
  group_by(.model) %>%
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) %>%
  arrange(rmsse)
```

```
## # A tibble: 5 x 3
##   .model  mase rmsse
##   <chr>  <dbl> <dbl>
## 1 ols    0.930 0.926
## 2 wlss   0.949 0.948
## 3 mint_s 0.953 0.954
## 4 wlsv   0.964 0.965
## 5 ets    0.968 0.968
```

Example: Australian tourism

```
fc %>%
  accuracy(tourism_agg,
           measures = list(mase = MASE, rmsse = RMSSE)) %>%
  group_by(.model) %>%
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) %>%
  arrange(rmsse)
```

```
## # A tibble: 5 x 3
##   .model  mase rmsse
##   <chr>  <dbl> <dbl>
## 1 ols    0.930 0.926
## 2 wlss   0.949 0.948
## 3 mint_s 0.953 0.954
## 4 wlsv   0.964 0.965
## 5 ets    0.968 0.968
```

- Overall, every reconciliation method is better than the base ETS forecasts.

Example: Australian tourism

```
## # A tibble: 20 x 4
## # Groups:   .model [5]
##    .model level     mase rmsse
##    <chr>  <fct>    <dbl> <dbl>
##  1 ets    National 1.44  1.27
##  2 ols    National 1.46  1.29
##  3 wlss   National 1.61  1.43
##  4 mint_s National 1.64  1.45
##  5 wlsv   National 1.69  1.49
##  6 ols    State    1.07  1.08
##  7 ets    State    1.10  1.11
##  8 wlss   State    1.13  1.14
##  9 mint_s State    1.15  1.15
## 10 wlsv   State    1.18  1.17
## 11 ols    Zone     0.954 0.948
## 12 wlss   Zone     0.987 0.980
## 13 mint_s Zone     0.995 0.988
## 14 ets    Zone     1.01  0.999
## 15 wlsv   Zone     1.01  1.00
## 16 ols    Region   0.901 0.895
## 17 wlss   Region   0.910 0.907
## 18 mint_s Region   0.911 0.911
## 19 wlsv   Region   0.917 0.919
## 20 ets    Region   0.935 0.938
```

Example: Australian tourism

```
## # A tibble: 20 x 4
## # Groups:   .model [5]
##    .model level     mase rmsse
##    <chr>  <fct>    <dbl> <dbl>
##  1 ets    National 1.44  1.27
##  2 ols    National 1.46  1.29
##  3 wlss   National 1.61  1.43
##  4 mint_s National 1.64  1.45
##  5 wlsv   National 1.69  1.49
##  6 ols    State    1.07  1.08
##  7 ets    State    1.10  1.11
##  8 wlss   State    1.13  1.14
##  9 mint_s State    1.15  1.15
## 10 wlsv   State    1.18  1.17
## 11 ols    Zone     0.954 0.948
## 12 wlss   Zone     0.987 0.980
## 13 mint_s Zone     0.995 0.988
## 14 ets    Zone     1.01  0.999
## 15 wlsv   Zone     1.01  1.00
## 16 ols    Region   0.901 0.895
## 17 wlss   Region   0.910 0.907
## 18 mint_s Region   0.911 0.911
## 19 wlsv   Region   0.917 0.919
## 20 ets    Region   0.935 0.938
```

- OLS is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

Outline

Coherent probabilistic forecasts

Coherent probabilistic forecasts

Given the triple $(\mathbb{R}^m, \mathscr{F}_{\mathbb{R}^m}, \nu)$, a coherent probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is such that

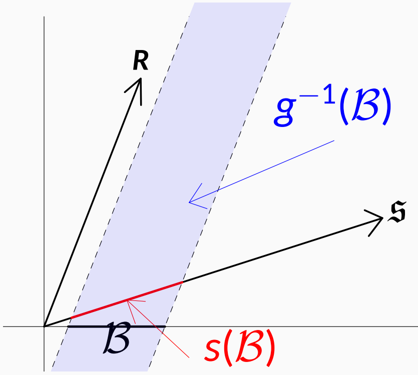
$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\mathbb{R}^m}.$$

Probabilistic forecast reconciliation

The reconciled probability measure of $\hat{\nu}$ wrt $\psi(.)$ is such that

$$\tilde{\nu}(\mathcal{B}) = \hat{\nu}(\psi^{-1}(\mathcal{B})) \qquad \forall \mathcal{B} \in \mathscr{F}_{\mathfrak{s}}\,,$$

where $\psi^{-1}(\mathcal{B}) := \{\boldsymbol{y} \in \mathbb{R}^n : \psi(\boldsymbol{y}) \in \mathcal{B}\}$ is the pre-image of $\mathcal{B}$, that is the set of all points in $\mathbb{R}^n$ that $\psi(.)$ maps to a point in $\mathcal{B}$.



Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_{\boldsymbol{b}}(\boldsymbol{b}) = |\boldsymbol{G}^*| \int \hat{f}(\boldsymbol{G}^- \boldsymbol{b} + \boldsymbol{G}_\perp \boldsymbol{a}) d\boldsymbol{a}$$

- $\hat{f}$ is density of incoherent base probabilistic forecast
- $\boldsymbol{G}^-$ is $n \times m$ generalised inverse of $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}^- = \boldsymbol{I}$
- $\boldsymbol{G}_\perp$ is $n \times (n - m)$ orthogonal complement to $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}_\perp = \boldsymbol{0}$
- $\boldsymbol{G}^* = \left(\boldsymbol{G}^- \vdots \boldsymbol{G}_\perp\right)$, and $\boldsymbol{b}$ and $\boldsymbol{a}$ are obtained via

  the change of variables $\boldsymbol{y} = \boldsymbol{G}^* \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix}\boldsymbol{S}^{-\prime} & \boldsymbol{S}_\perp\end{pmatrix}'$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}_\perp'\boldsymbol{S} = \boldsymbol{0}$.

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix}\boldsymbol{S}^{-\prime} & \boldsymbol{S}_\perp\end{pmatrix}'$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n - m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Gaussian reconciliation

If the incoherent base forecasts are $\text{N}(\hat{\mu}, \hat{\Sigma})$, then the reconciled density is $\text{N}(\boldsymbol{S}\boldsymbol{G}\hat{\mu}, \boldsymbol{S}\boldsymbol{G}\hat{\Sigma}\boldsymbol{G}'\boldsymbol{S}')$.

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix}\boldsymbol{S}^{-\prime} & \boldsymbol{S}_\perp\end{pmatrix}'$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n - m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}_\perp'\boldsymbol{S} = \boldsymbol{0}$.

Gaussian reconciliation

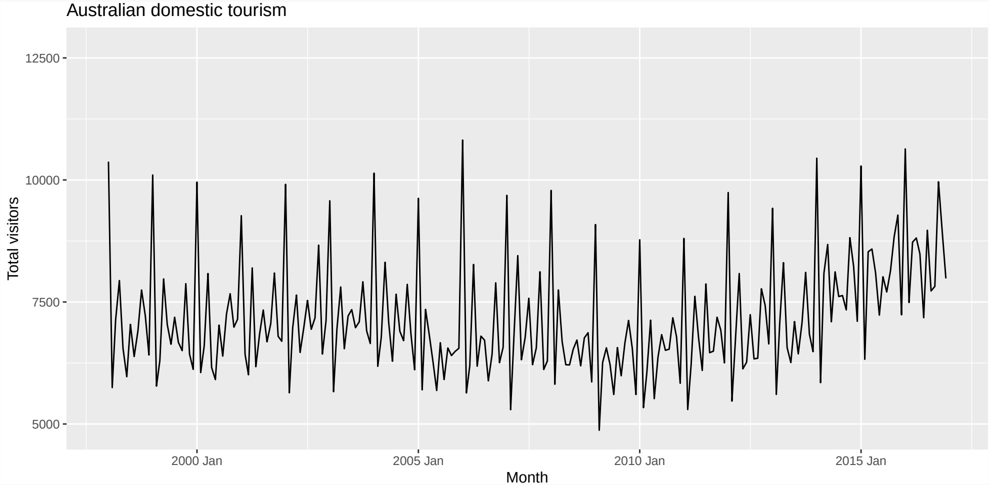
If the incoherent base forecasts are $\text{N}(\hat{\mu}, \hat{\Sigma})$, then the reconciled density is $\text{N}(\boldsymbol{S}\boldsymbol{G}\hat{\mu}, \boldsymbol{S}\boldsymbol{G}\hat{\Sigma}\boldsymbol{G}'\boldsymbol{S}')$.

Bootstrap reconciliation

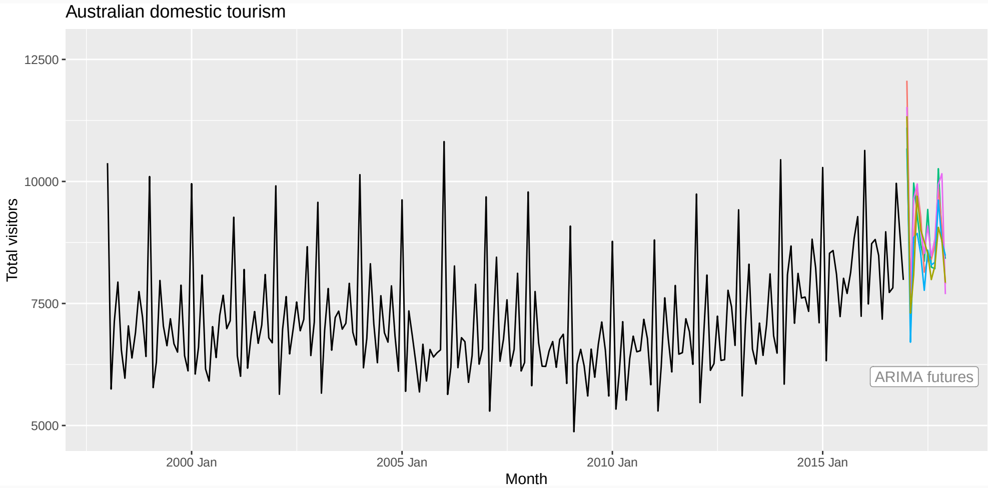
Reconciling sample paths from incoherent distributions works.

Outline

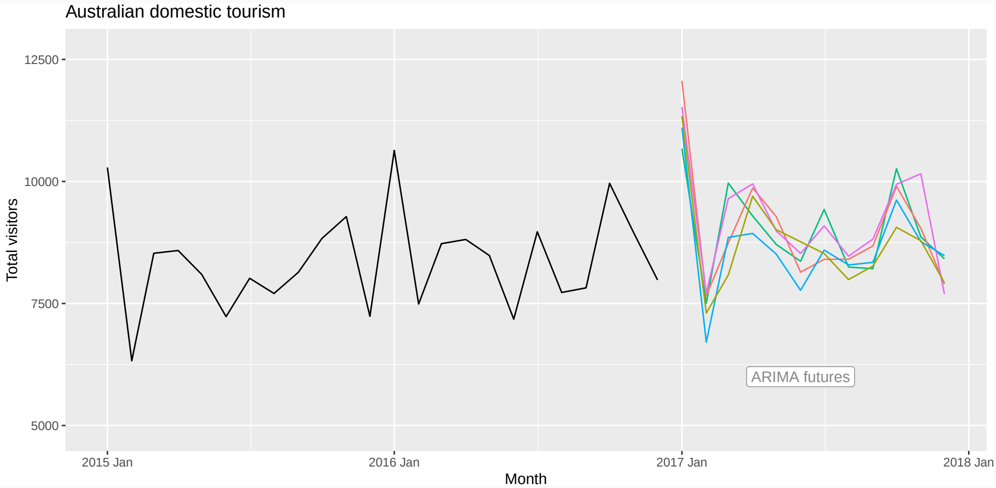
Evaluating probabilistic forecasts



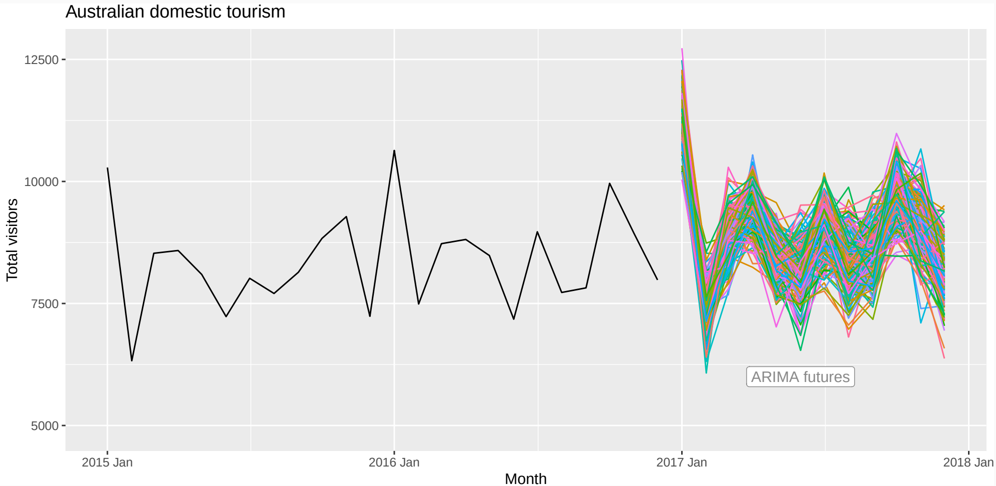
Evaluating probabilistic forecasts



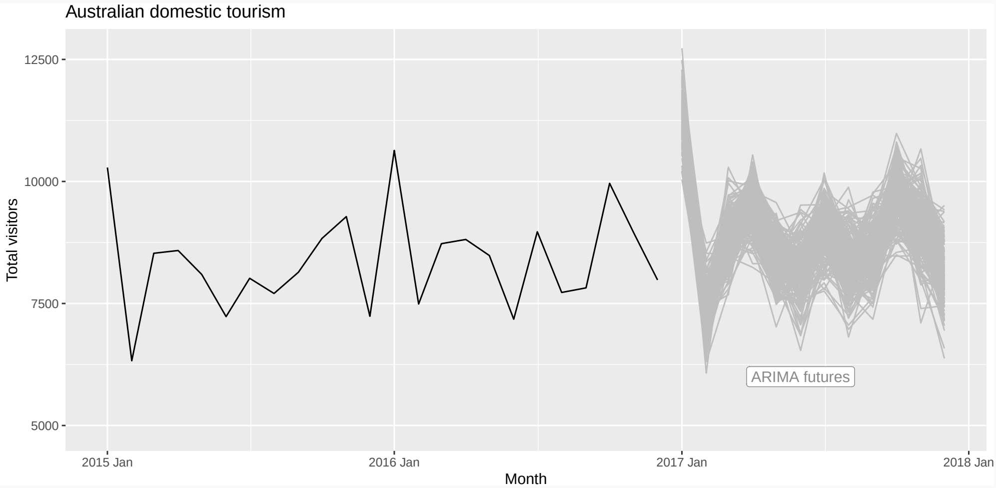
Evaluating probabilistic forecasts



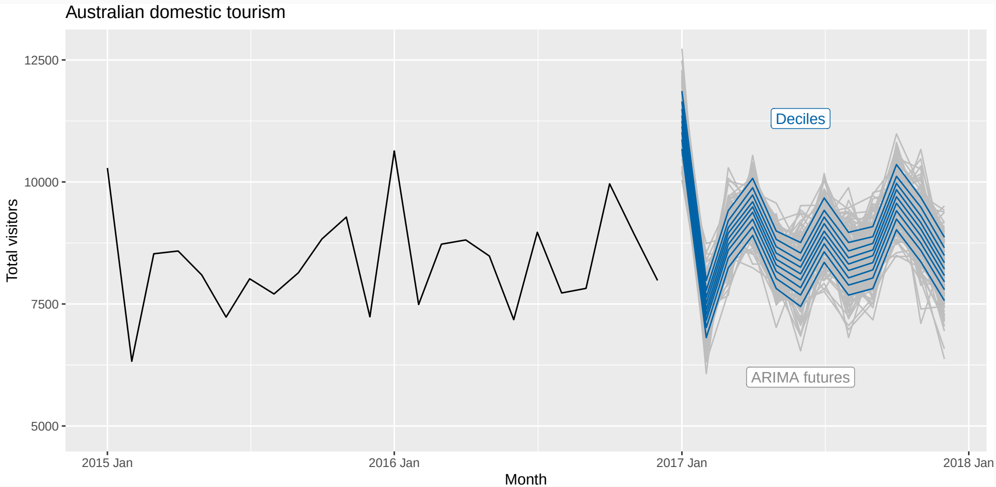
Evaluating probabilistic forecasts



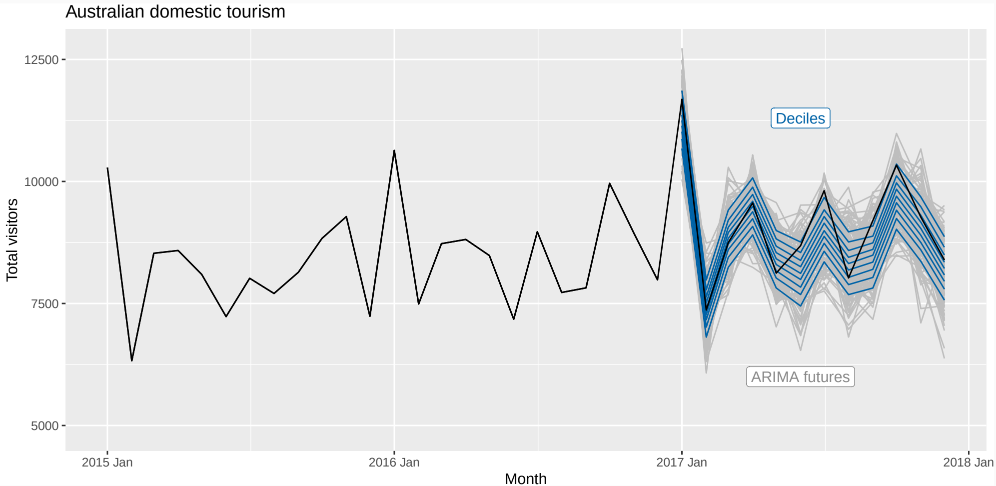
Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



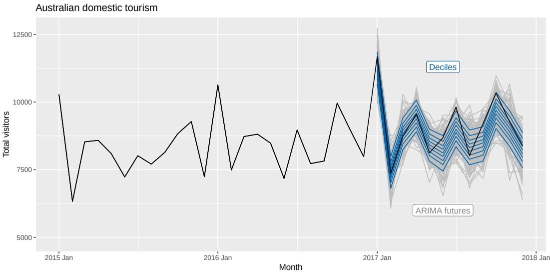
Evaluating probabilistic forecasts



Evaluating probabilistic forecasts

$q_{p,t}$ = quantile forecast with prob. p at time t.

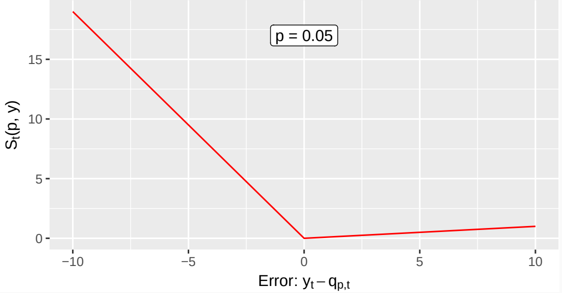
y_t = observation at time t



Quantile score

$$S_t(p, y) = \begin{cases} 2(1-p)|y_t - q_{p,t}|, & \text{if } y_t < q_{p,t} \\ 2p|y_t - q_{p,t}|, & \text{if } y_t \geq q_{p,t} \end{cases}$$

- Low S_t is good
- Multiplier of 2 often omitted, but useful for interpretation
- S_t like absolute error, weighted to account for likely exceedance
- Average $S_t(p, y)$ over p = CRPS (Continuous Rank Probability Score)



Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecsts)

Forecast distribution F_t and observation y_t.

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t)dp = \text{E}_F|Y - y_t| - \frac{1}{2}\text{E}_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t.
- Optimal when F_t is true distribution (i.e., it is a proper score)

Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecsts)

Forecast distribution F_t and observation y_t.

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t)dp = \text{E}_F|Y - y_t| - \frac{1}{2}\text{E}_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t.
- Optimal when F_t is true distribution (i.e., it is a proper score)

Energy score (multivariate forecasts)

- $\text{ES}(F_t, \boldsymbol{y}_t) = \text{E}_F||\boldsymbol{Y} - \boldsymbol{y}_t|| - \frac{1}{2}\text{E}_F||\boldsymbol{Y} - \boldsymbol{Y}^*||$

Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecsts)

Forecast distribution F_t and observation y_t.

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t)dp = \text{E}_F|Y - y_t| - \frac{1}{2}\text{E}_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t.
- Optimal when F_t is true distribution (i.e., it is a proper score)

Energy score (multivariate forecasts)

- $\text{ES}(F_t, \boldsymbol{y}_t) = \text{E}_F||\boldsymbol{Y} - \boldsymbol{y}_t|| - \frac{1}{2}\text{E}_F||\boldsymbol{Y} - \boldsymbol{Y}^*||$

Log score (multivariate forecasts)

- $\text{LS}(F_t, \boldsymbol{y}_t) = -\log f(\boldsymbol{y}_t)$

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule	Coherent v Incoherent	Coherent v Coherent
Log Score	Not proper	• Ordering preserved if compared using bottom-level only
Energy Score	Proper	• Full hierarchy should be used. • Rankings may change otherwise.

Score optimal reconciliation

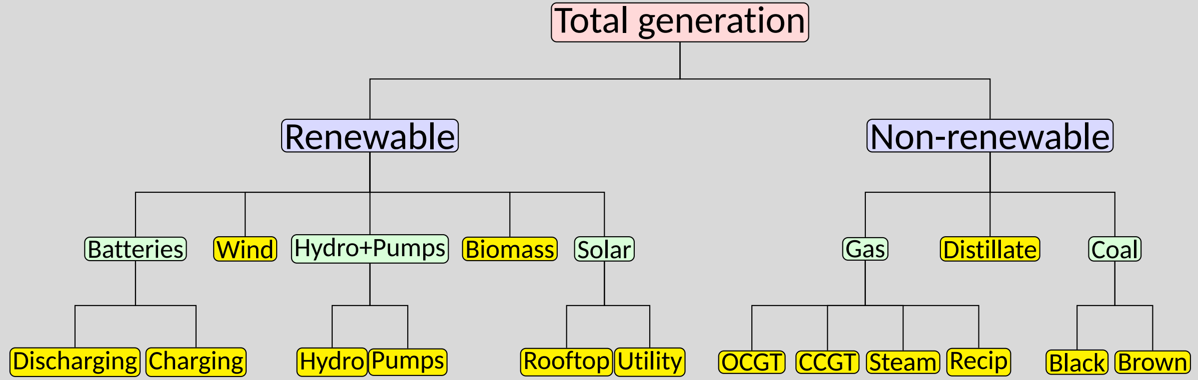
Algorithm proposed by Panagiotelis et al (2020) for optimizing $\boldsymbol{G}$ using stochastic gradient descent to optimize Energy Score.

1. Compute base forecasts over a test set.
2. Compute OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
3. Iteratively update $\boldsymbol{G}$ using SGD with Adam method and ES objective over a test set

Outline

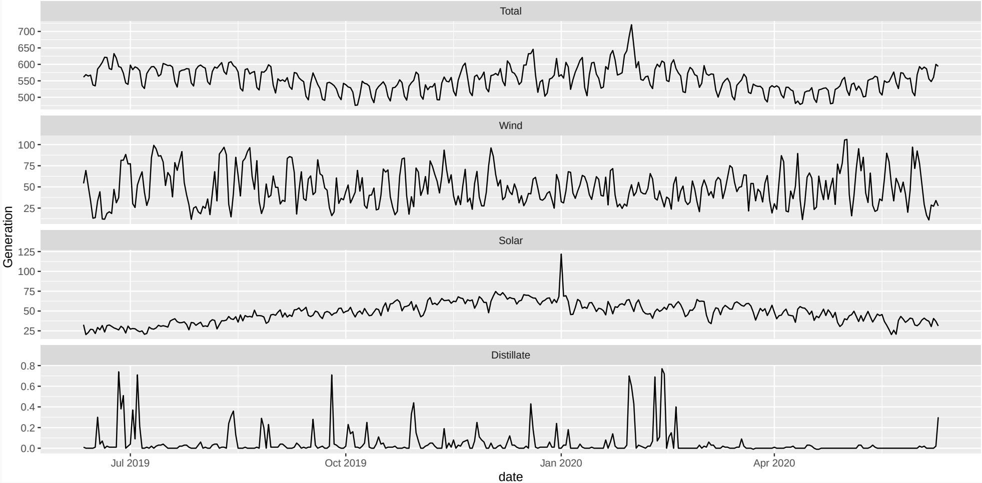
Example: Australian electricity generation

Daily time series from `opennem.org.au`



n = 23 series m = 15 bottom-level series

Example: Australian electricity generation

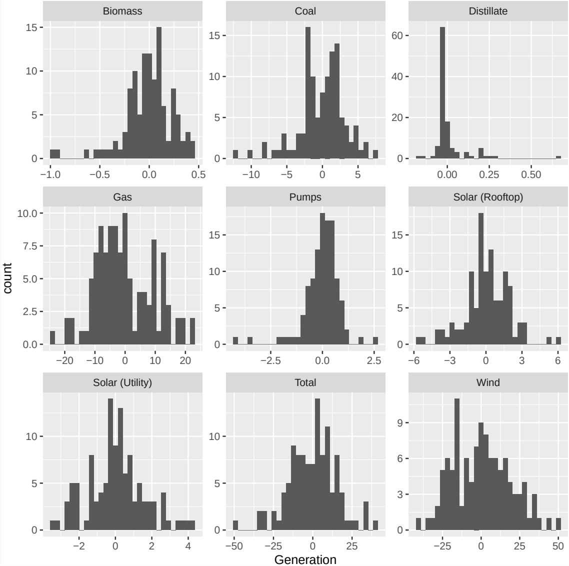


Example: Australian electricity generation

Forecast evaluation

- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using `NNETAR()` function in `fable` package.
- Model could be improved with temperature predictor.

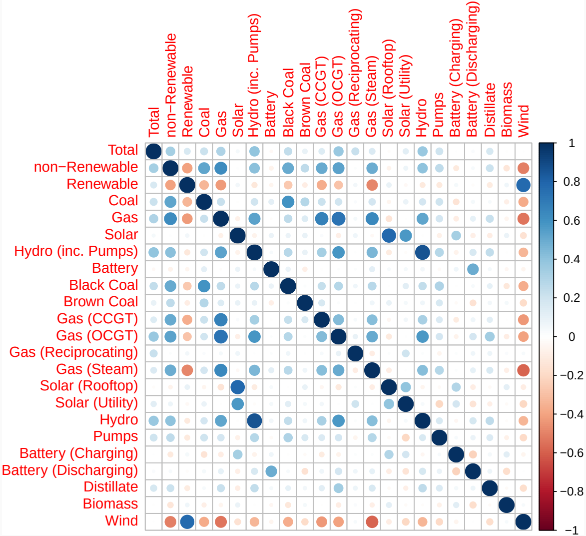
Example: Australian electricity generation



Histogram of residuals: 2 Oct 2019 – 21 Jan 2020

Clearly non-Gaussian

Example: Australian electricity generation



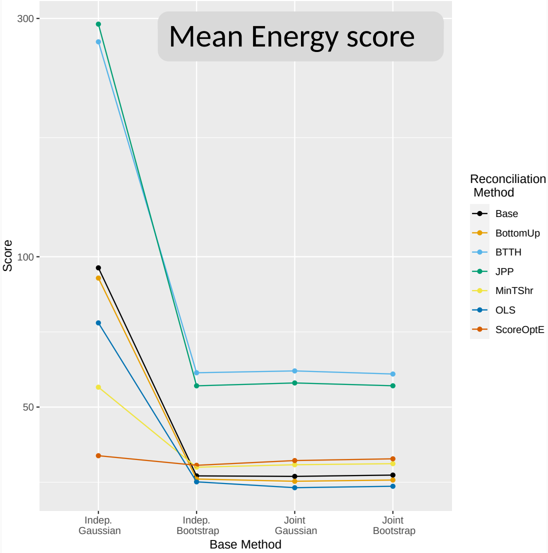
Correlations of residuals:
2 Oct 2019 – 21 Jan 2020

Blue = positive correlation.

Red = negative correlation.

Large = stronger correlations.

Example: Australian electricity generation



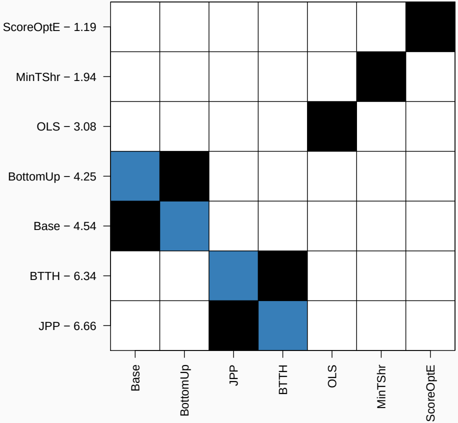
Base residual assumptions

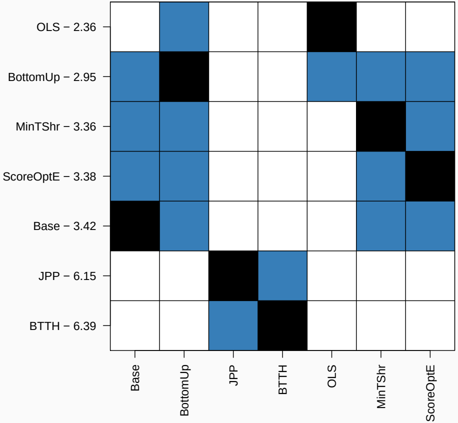
- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

Reconciliation methods

- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS
- MinT(Shrink)
- Score Optimal Reconciliation

Example: Australian electricity generation





Nemenyi test for different scores

Base forecasts are independent and Gaussian.

Nemenyi test for different scores

Base forecasts are obtained by jointly bootstrapping residuals.

Thanks!

More information

- Slides and papers: **robjhyndman.com**
- Packages: **tidyverts.org**
- Forecasting textbook using fable package: **OTexts.com/fpp3**

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