The background of the slide features a vibrant, abstract pattern resembling liquid or smoke, with swirling bands of orange, blue, green, and yellow against a dark background.

The geometry of forecast reconciliation

Rob J Hyndman

CUHK: 24 August 2021

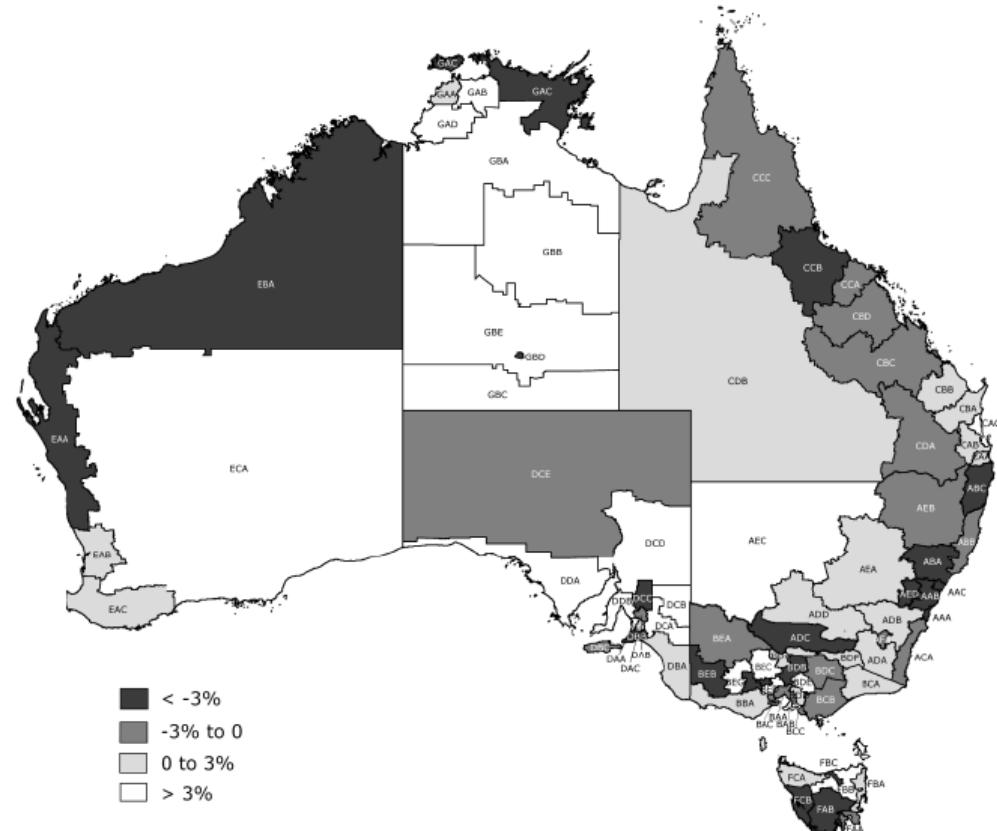
Outline

- 1 Hierarchical and grouped time series
- 2 Forecast reconciliation using projections
- 3 Probabilistic forecast reconciliation
- 4 Evaluating probabilistic forecasts
- 5 Example: Australian tourism
- 6 Example: Australian electricity generation

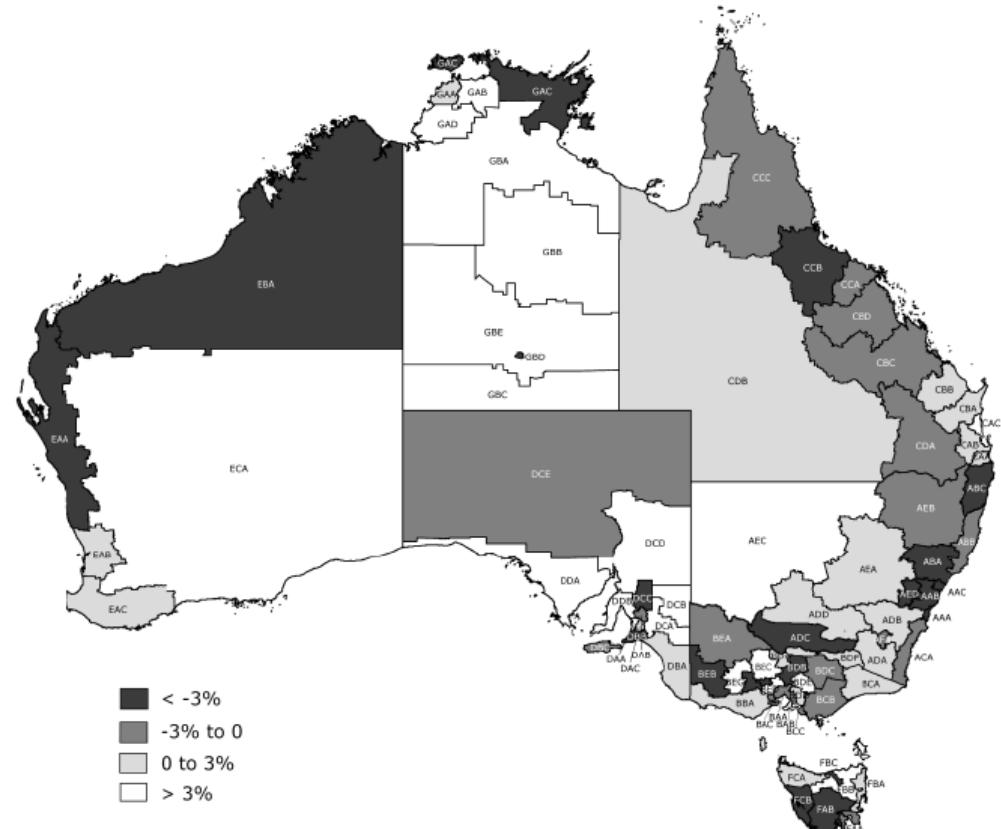
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Australian tourism



Australian tourism



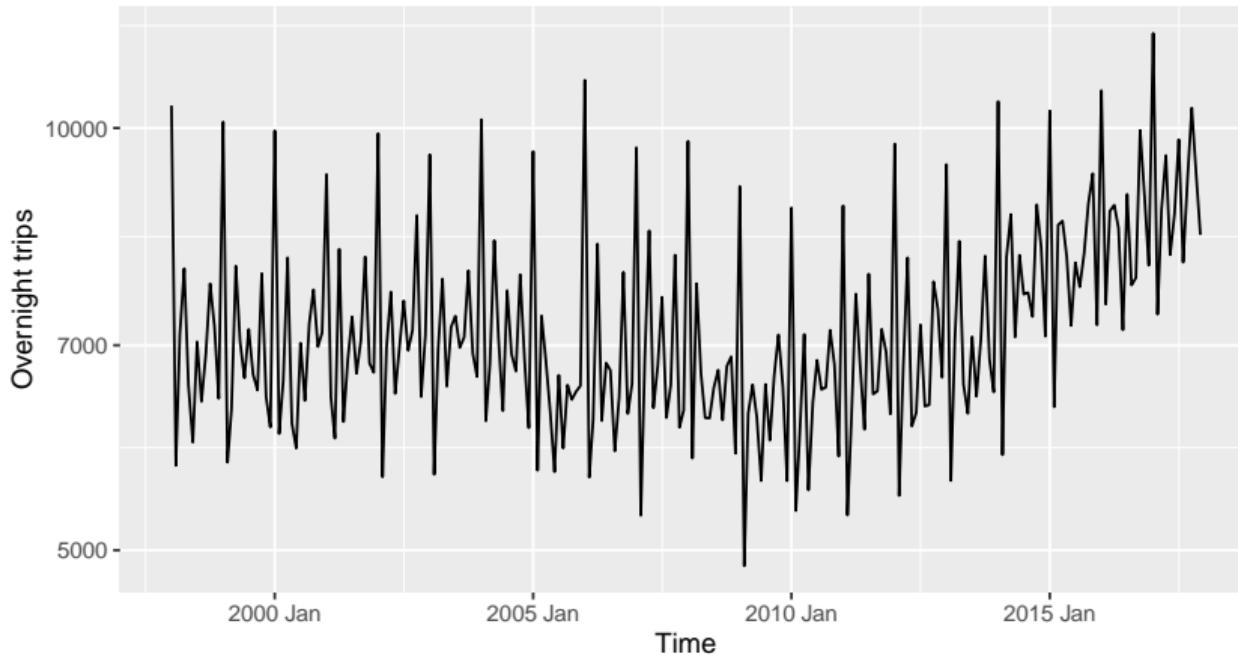
- Monthly data on visitor night from 1998 - 2017
 - From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
 - Geographical hierarchy split by
 - ▶ 7 states
 - ▶ 27 zones
 - ▶ 75 regions

Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:      state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>      <dbl>
## 1 1998 Jan NSW Metro NSW Sydney      926.
## 2 1998 Feb NSW Metro NSW Sydney      647.
## 3 1998 Mar NSW Metro NSW Sydney      716.
## 4 1998 Apr NSW Metro NSW Sydney      621.
## 5 1998 May NSW Metro NSW Sydney      598.
## 6 1998 Jun NSW Metro NSW Sydney      601.
## 7 1998 Jul NSW Metro NSW Sydney      720.
## 8 1998 Aug NSW Metro NSW Sydney      645.
## 9 1998 Sep NSW Metro NSW Sydney      633.
## 10 1998 Oct NSW Metro NSW Sydney      771.
## # ... with 17,990 more rows
```

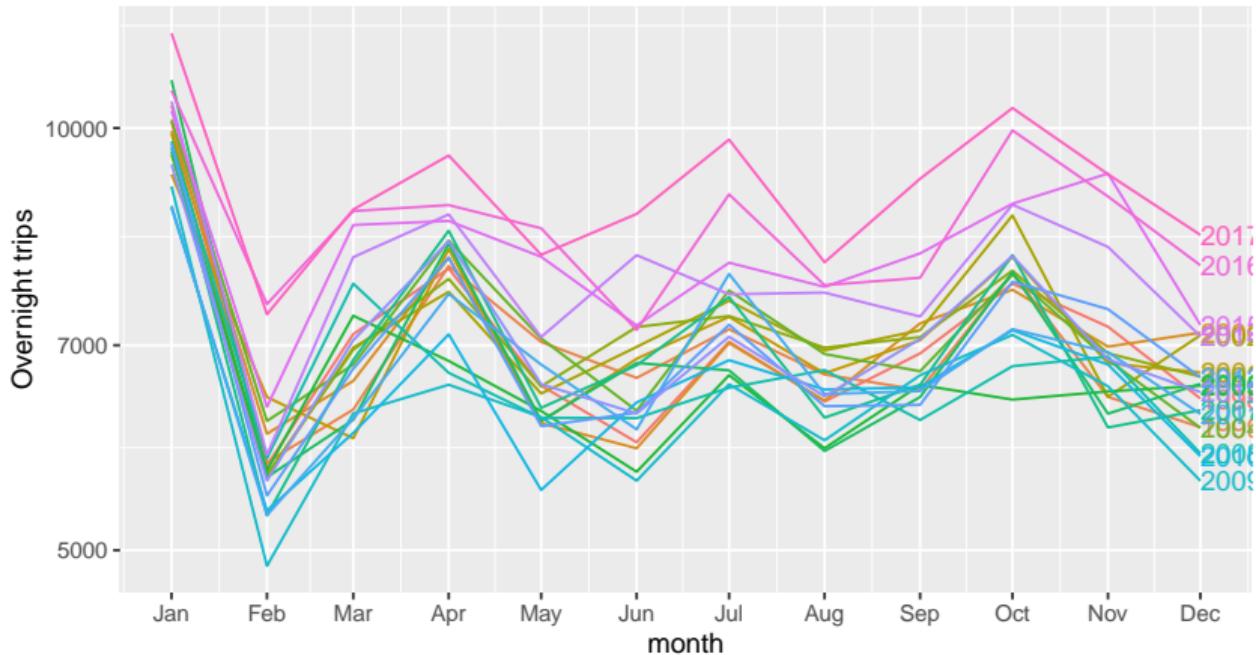
Australian tourism data

Total domestic travel: Australia

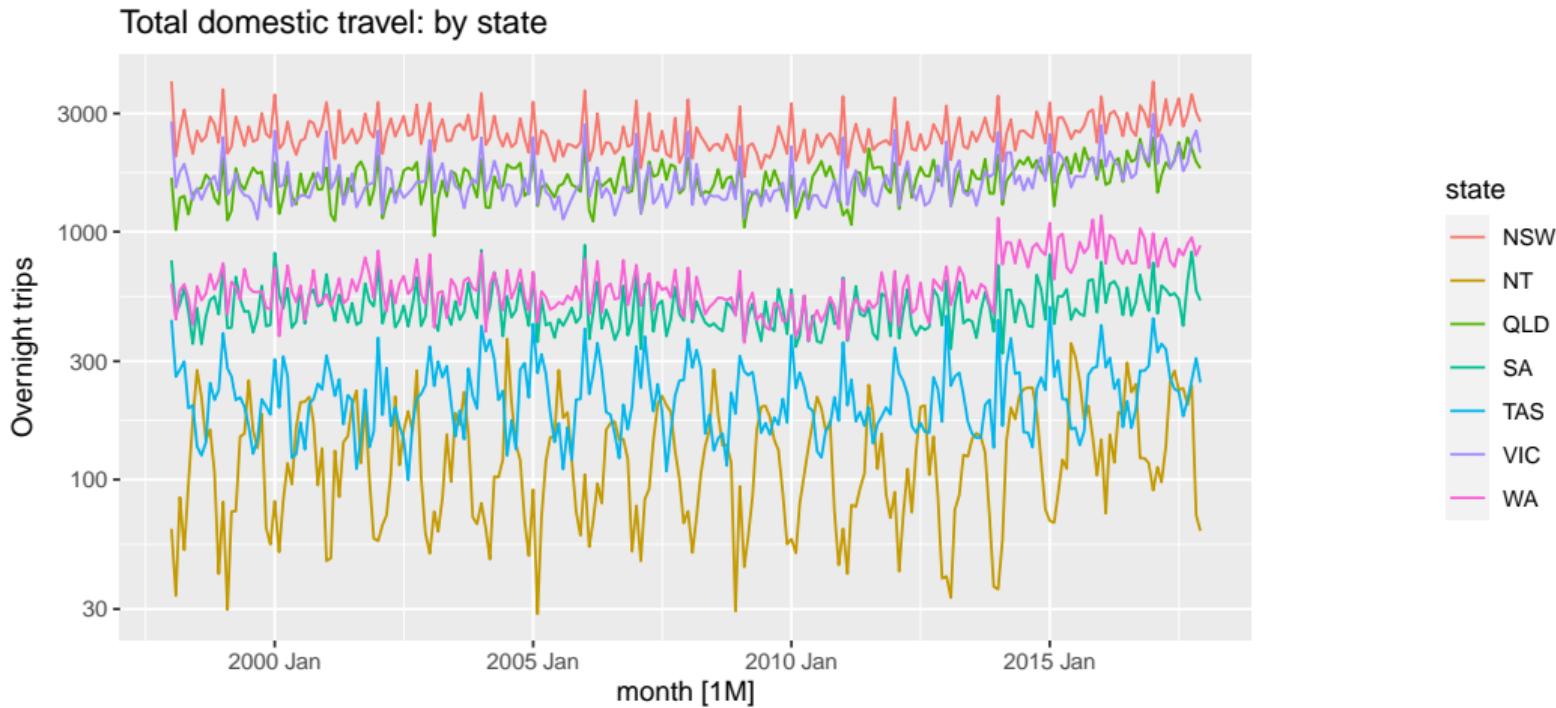


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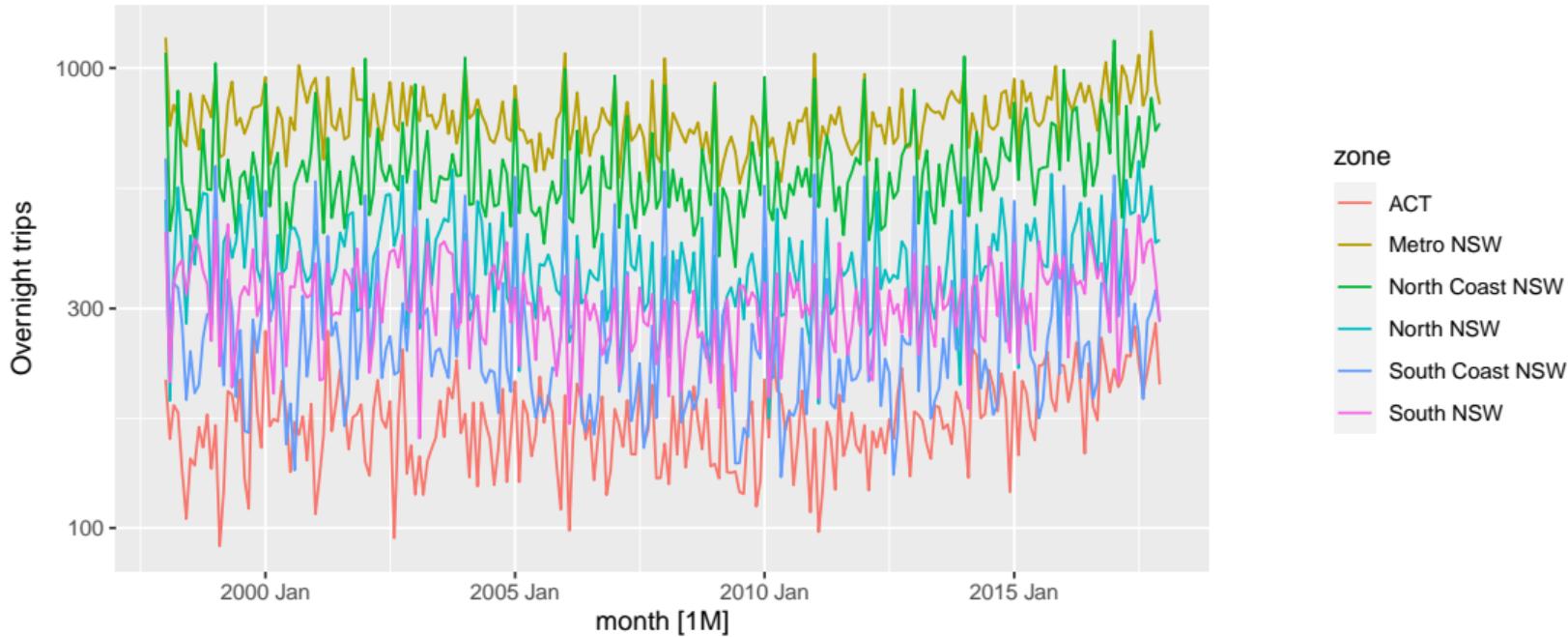


Australian tourism data



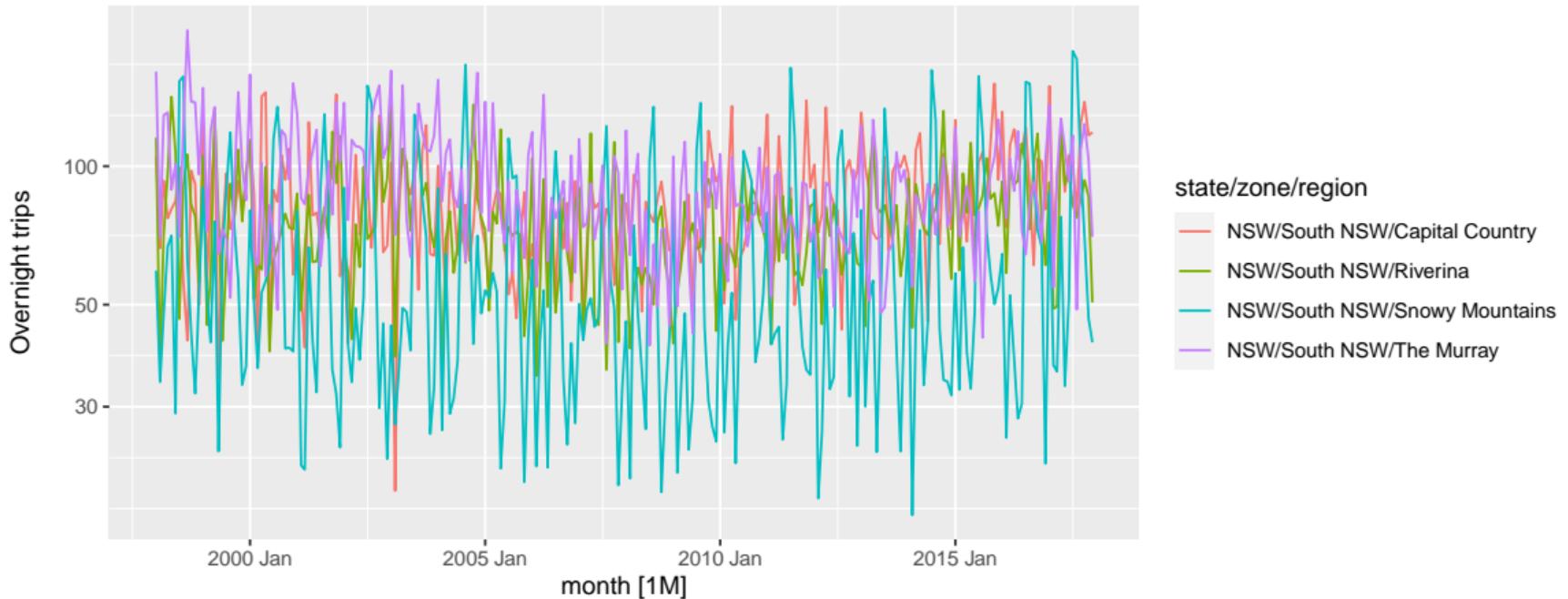
Australian tourism data

Total domestic travel: NSW by zone



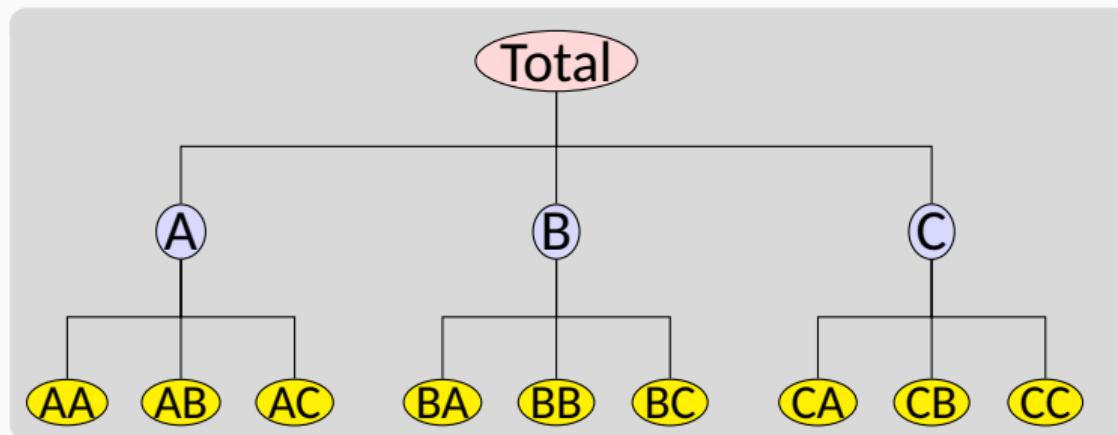
Australian tourism data

Total domestic travel: South NSW by region



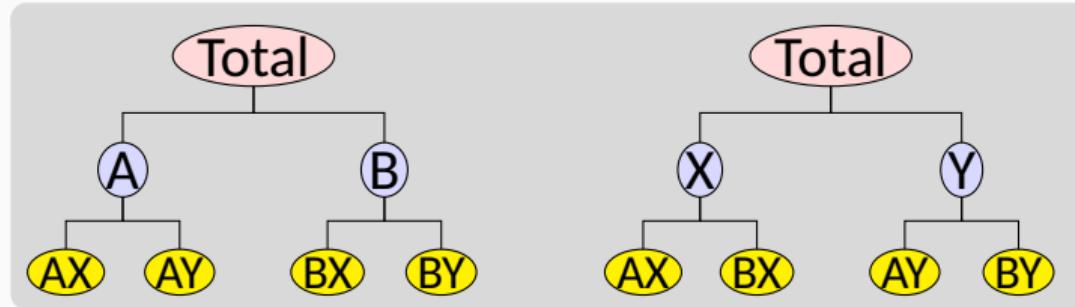
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



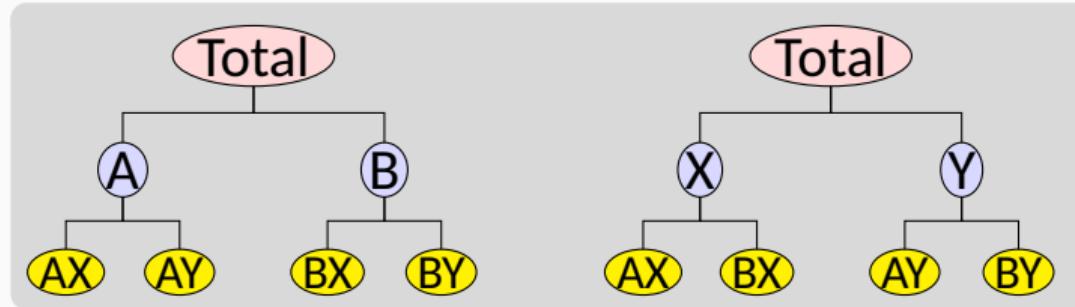
Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



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Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

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The problem

How to produce **coherent** forecasts at all nodes?

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Old approaches (pre 2009)

- Bottom-up forecasting
- Top-down forecasting
- Middle-out forecasting

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Old approaches (pre 2009)

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Forecast reconciliation approach

- 1 Forecast all series at all levels of aggregation using an automatic forecasting algorithm. (e.g., ETS, ARIMA, ...)
- 2 Reconcile the resulting forecasts so they are coherent using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).

Key forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 *CSDA*) Optimal combination forecasts for hierarchical time series.
- Athanasopoulos, Ahmed, Hyndman (2009 *IJF*) Hierarchical forecasts for Australian domestic tourism.
- Hyndman, Lee, Wang (2016 *CSDA*) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 *JASA*) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 *IJF*) Forecast reconciliation: A geometric view with new insights on bias correction.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020) Probabilistic forecast reconciliation: properties, evaluation and score optimisation.

Hierarchical and grouped time series

Every collection of time series with linear constraints can be written as

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

- \mathbf{y}_t = vector of all series at time t
- y_t = aggregate of all series at time t .
- $y_{X,t}$ = value of series X at time t .
- \mathbf{b}_t = vector of most disaggregated series at time t
- \mathbf{S} = “summing matrix” containing the linear constraints.

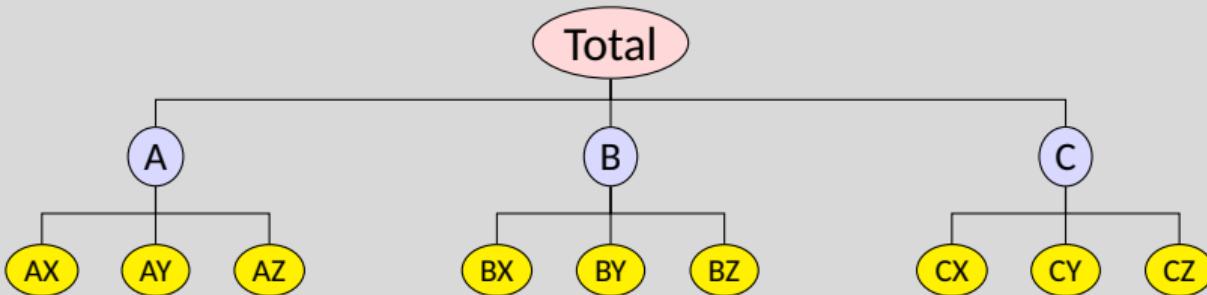


$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t} \end{aligned}$$

Hierarchical time series

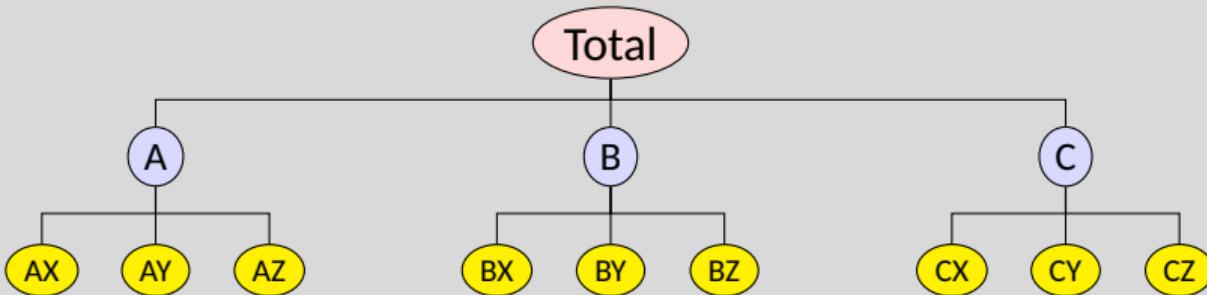


Hierarchical time series



$$\mathbf{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

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Definitions

Coherent subspace

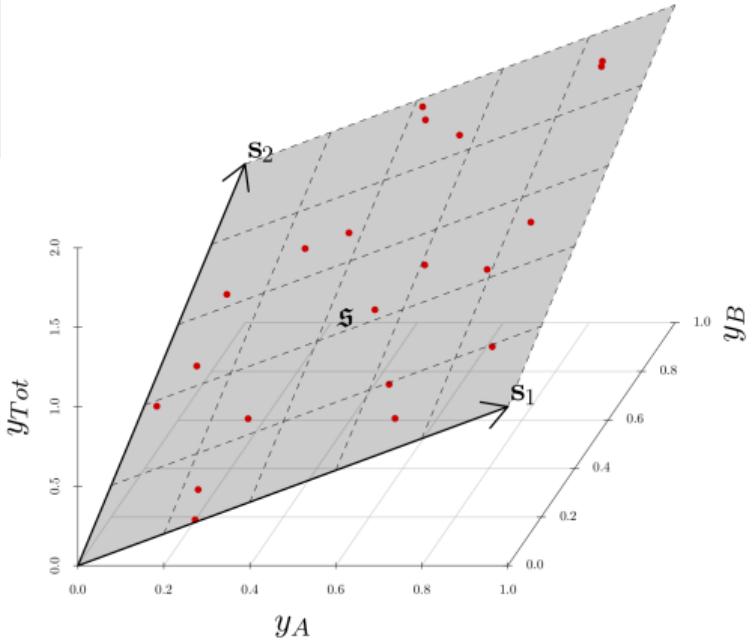
m -dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\mathbf{y} \in \mathfrak{s}$.

Hierarchical time series

An n -dimensional multivariate time series such that $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$ is coherent if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.



$$Y_{Tot} = Y_A + Y_B$$

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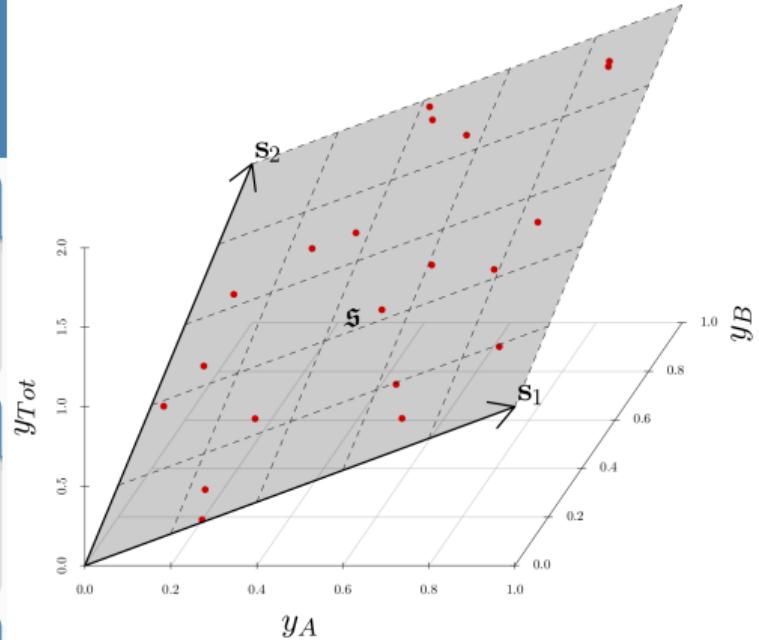
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Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$.
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$ “reconciles” $\hat{\mathbf{y}}_{t+h|t}$.

Linear reconciliation

If ψ is a linear function and \mathbf{G} is some matrix,
then

$$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{t+h|t}$$

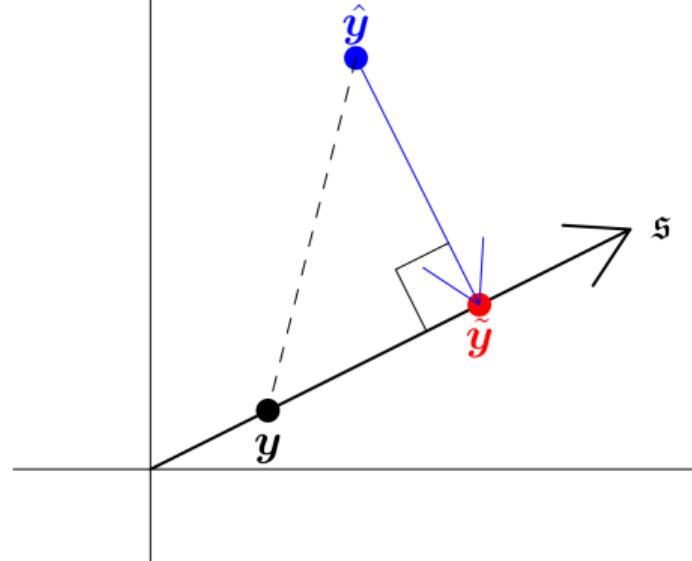
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- \mathbf{S} creates linear combinations.
- e.g., OLS reconciliation: $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$

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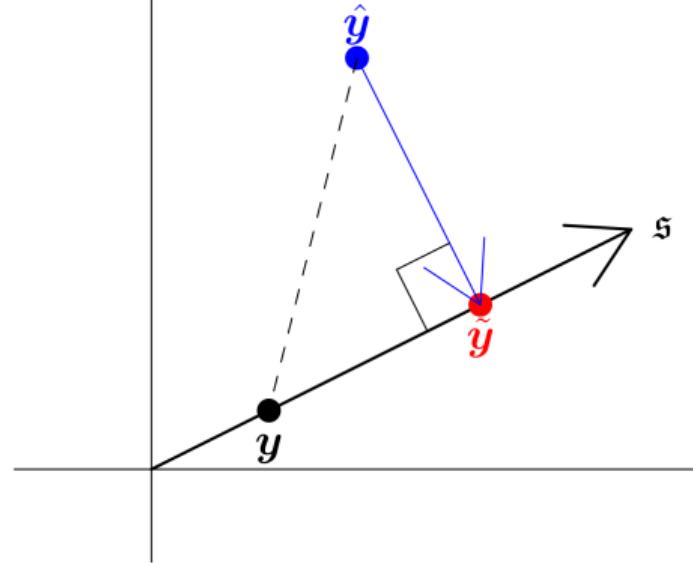
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Projections

Suppose \mathbf{SG} is a projection onto \mathfrak{s} , then

- Coherent base forecasts are unchanged.
- Unbiased base forecasts remain unbiased.



- Orthogonal projections lead to smallest possible adjustments of base forecasts.

Linear reconciliation

If ψ is a linear function and \mathbf{G} is some matrix, then

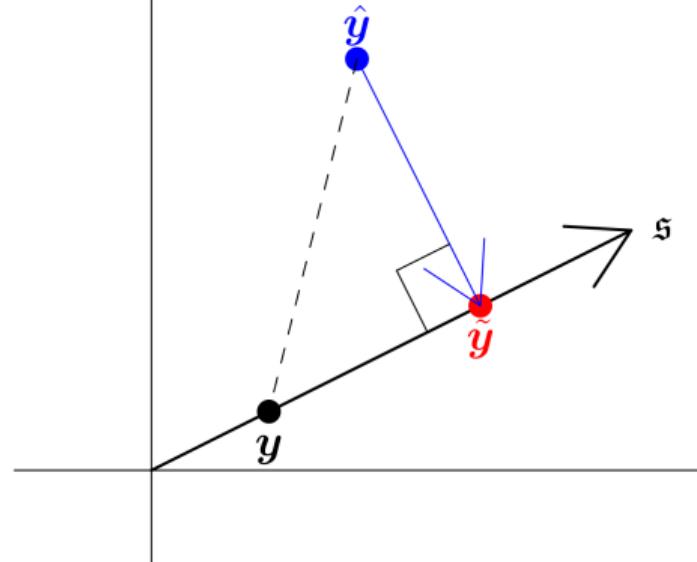
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Distance reducing property

If \mathbf{SG} is an orthogonal projection onto \mathfrak{s} then:

$$\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}\| \leq \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|.$$



- Distance reduction holds for any realisation and any forecast.
- Other measures of forecast accuracy may be worse.
- Not necessarily the optimal reconciliation.

Linear projections

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$

Variance

$$\mathbf{V}_h = \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] = \mathbf{S}\mathbf{G}\mathbf{W}_h\mathbf{G}'\mathbf{S}'$$

where $\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$.

Minimum trace (MinT) reconciliation

If $\mathbf{S}\mathbf{G}$ is a projection, then the trace of \mathbf{V}_h is minimized when

$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{h}$$

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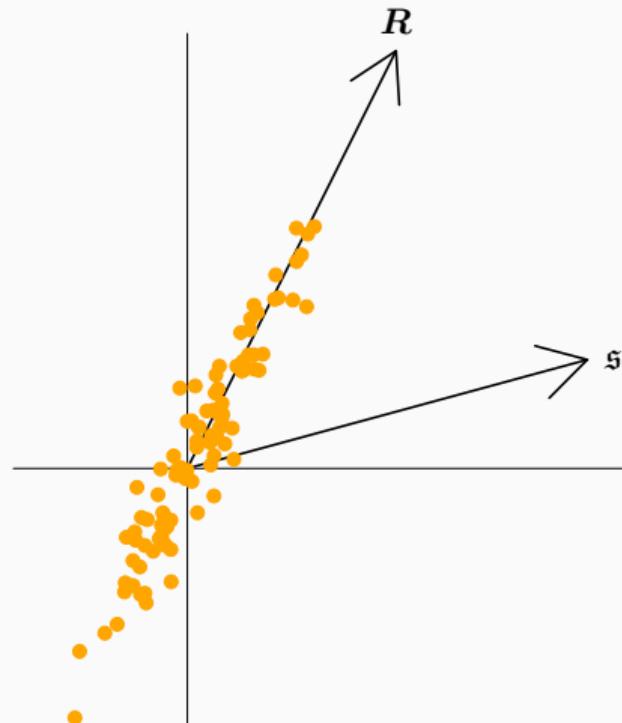
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- R is the most likely direction of deviations from \mathfrak{s} .
- Orange: in-sample errors



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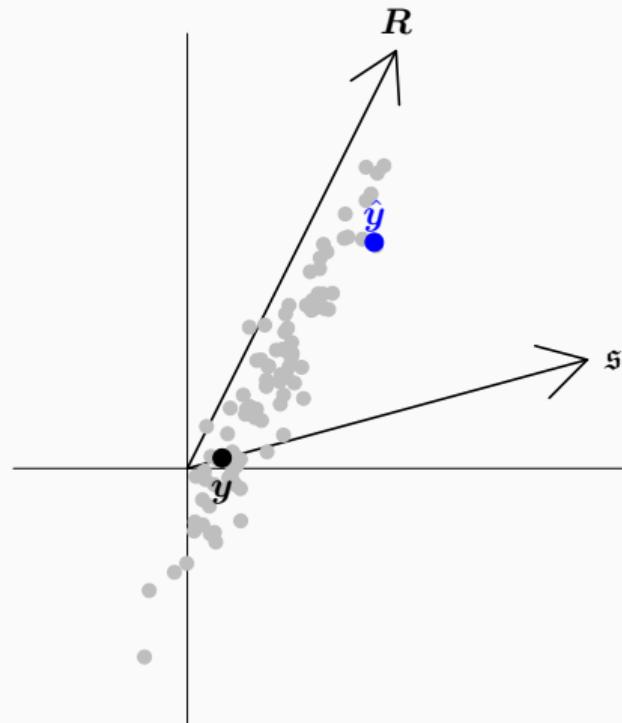
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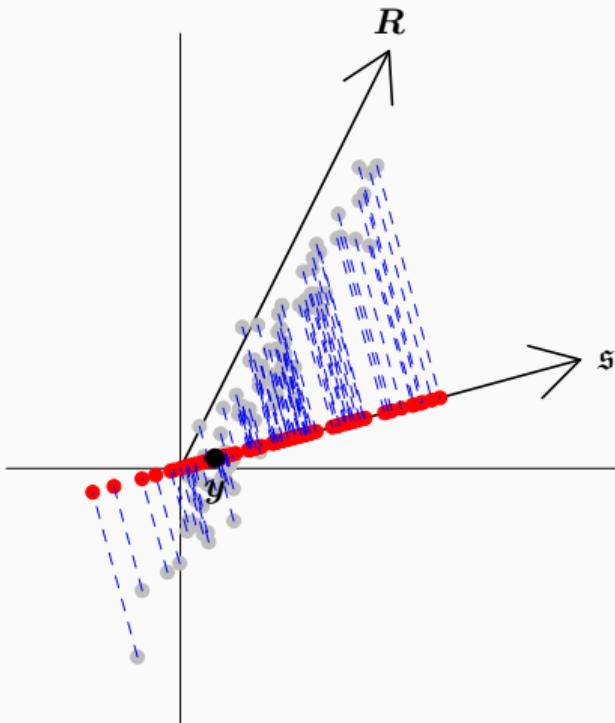
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- Red: reconciled forecasts



Orthogonal projection

Linear projections

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$

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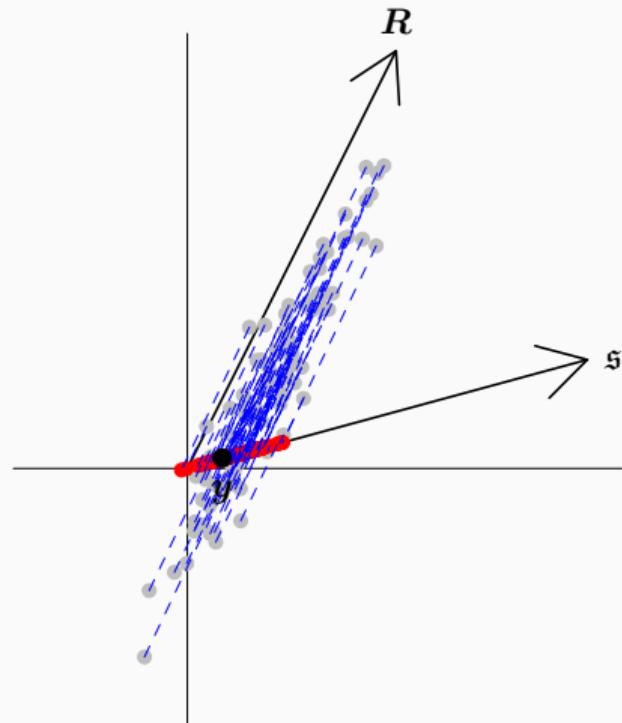
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Oblique projection

Linear projections

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$

Reconciliation method \mathbf{G}

OLS $(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$

WLS $(\mathbf{S}'\Lambda\mathbf{S})^{-1}\mathbf{S}'\Lambda$

MinT(Sample) $(\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}$

MinT(Shrink) $(\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\mathbf{W}_h = k_h \mathbf{W}_1$.

- Λ is diagonal matrix
- $\hat{\mathbf{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\mathbf{W}}_{\text{shr}}$ is shrinkage estimator $\tau \text{diag}(\hat{\mathbf{W}}_{\text{sam}}) + (1 - \tau)\hat{\mathbf{W}}_{\text{sam}}$ where $\tau = \frac{\sum_{i \neq j} \text{Var}(\hat{\sigma}_{ij})}{\sum_{i \neq j} \hat{\sigma}_{ij}^2}$ and σ_{ij} denotes the (i, j) th element of $\hat{\mathbf{W}}_{\text{sam}}$.

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Coherent probabilistic forecasts

Coherent probabilistic forecasts

Given the triple $(\mathbb{R}^m, \mathcal{F}_{\mathbb{R}^m}, \nu)$, a coherent probability triple $(\mathfrak{s}, \mathcal{F}_{\mathfrak{s}}, \check{\nu})$ is such that

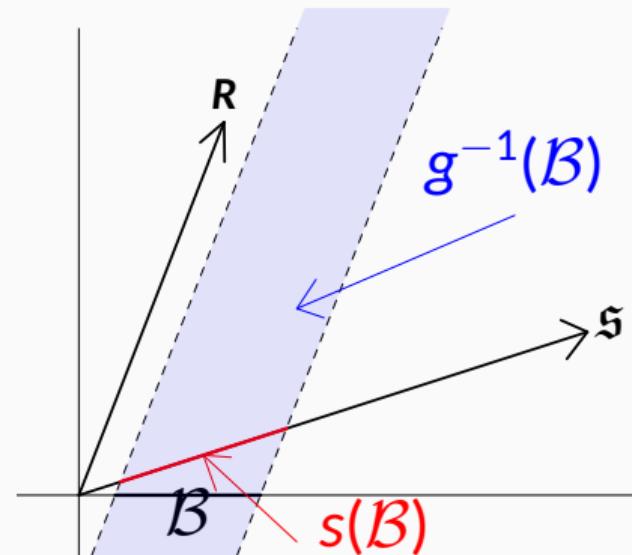
$$\check{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathbb{R}^m}.$$

Probabilistic forecast reconciliation

The reconciled probability measure of $\hat{\nu}$ wrt $\psi(\cdot)$ is such that

$$\hat{\nu}(\mathcal{B}) = \hat{\nu}(\psi^{-1}(\mathcal{B})) \quad \forall \mathcal{B} \in \mathcal{F}_{\mathfrak{s}},$$

where $\psi^{-1}(\mathcal{B}) := \{y \in \mathbb{R}^n : \psi(y) \in \mathcal{B}\}$ is the pre-image of \mathcal{B} , that is the set of all points in \mathbb{R}^n that $\psi(\cdot)$ maps to a point in \mathcal{B} .



Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_b(\mathbf{b}) = |\mathbf{G}^*| \int \hat{f}(\mathbf{G}^- \mathbf{b} + \mathbf{G}_\perp \mathbf{a}) d\mathbf{a}$$

- \hat{f} is density of incoherent base probabilistic forecast
- \mathbf{G}^- is $n \times m$ generalised inverse of \mathbf{G} st $\mathbf{G}\mathbf{G}^- = \mathbf{I}$
- \mathbf{G}_\perp is $n \times (n - m)$ orthogonal complement to \mathbf{G} st $\mathbf{G}\mathbf{G}_\perp = \mathbf{0}$
- $\mathbf{G}^* = (\mathbf{G}^- : \mathbf{G}_\perp)$, and \mathbf{b} and \mathbf{a} are obtained via

the change of variables $\mathbf{y} = \mathbf{G}^* \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_y(y) = |S^*| \tilde{f}_b(S^- y) \mathbb{1}\{y \in \mathfrak{s}\}$$

- $S^* = \begin{pmatrix} S^- \\ S'_\perp \end{pmatrix}$
- S^- is $m \times n$ generalised inverse of S such that $S^- S = I$,
- S_\perp is $n \times (n - m)$ orthogonal complement to S such that $S'_\perp S = 0$.

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_y(y) = |S^*| \tilde{f}_b(S^- y) \mathbb{1}\{y \in s\}$$

- $S^* = \begin{pmatrix} S^- \\ S'_\perp \end{pmatrix}$
- S^- is $m \times n$ generalised inverse of S such that $S^- S = I$,
- S_\perp is $n \times (n - m)$ orthogonal complement to S such that $S'_\perp S = 0$.

Gaussian reconciliation

If the incoherent base forecasts are $N(\hat{\mu}, \hat{\Sigma})$,
then the reconciled density is $N(SG\hat{\mu}, SG\hat{\Sigma}G'S')$.

Simulation from a reconciled distribution

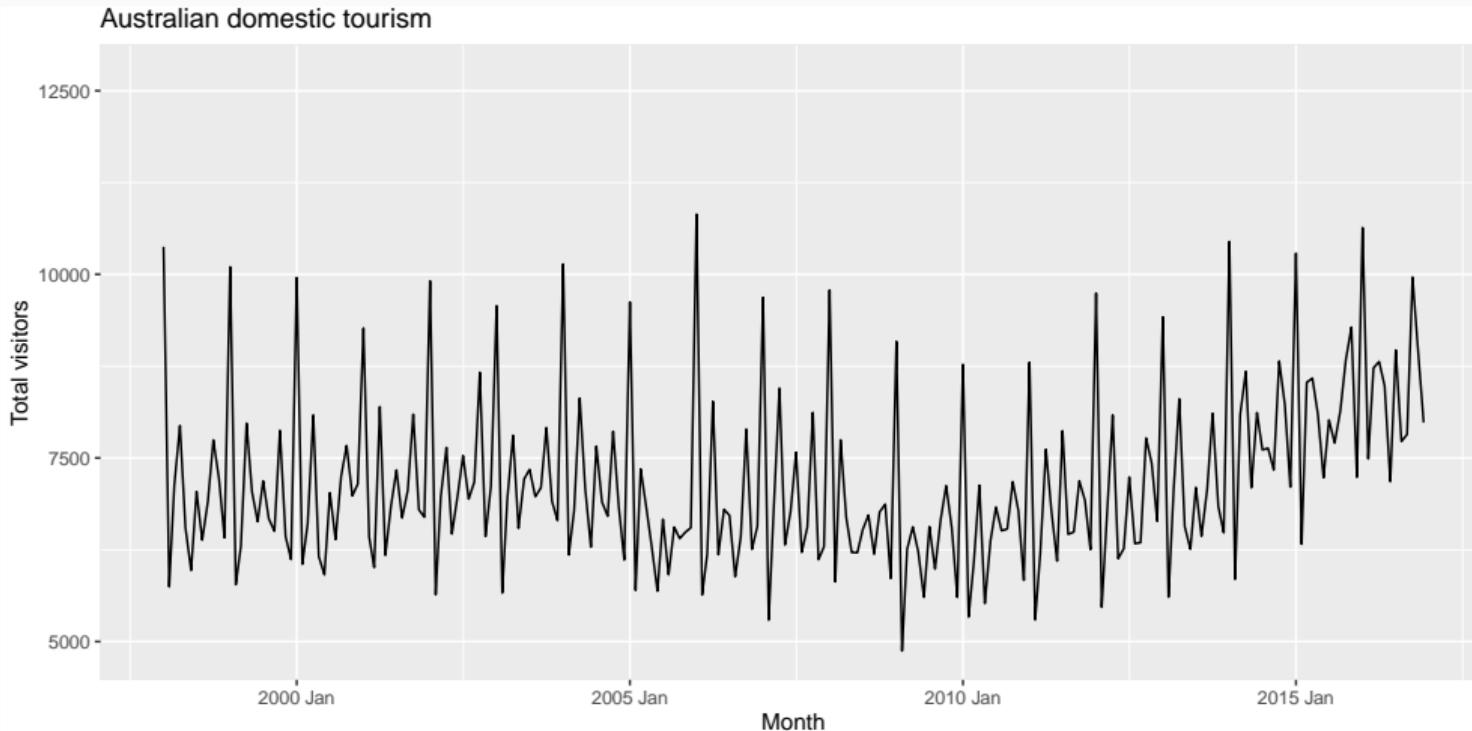
Suppose that $(\hat{\mathbf{y}}^{[1]}, \dots, \hat{\mathbf{y}}^{[L]})$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $(\tilde{\mathbf{y}}^{[1]}, \dots, \tilde{\mathbf{y}}^{[L]})$ where $\tilde{\mathbf{y}}^{[\ell]} := \psi(\hat{\mathbf{y}}^{[\ell]})$ for $\ell = 1, \dots, L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

- So reconciling sample paths from incoherent distributions works.

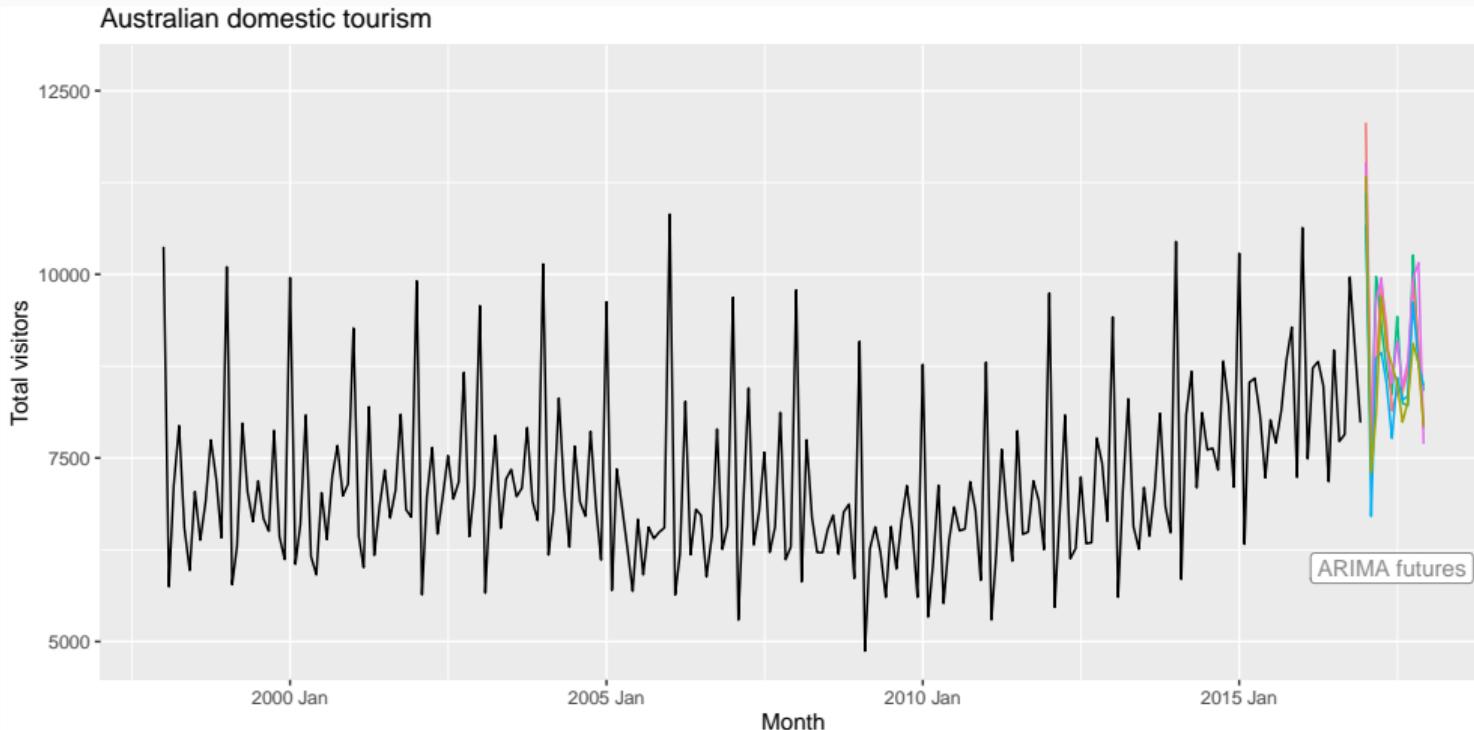
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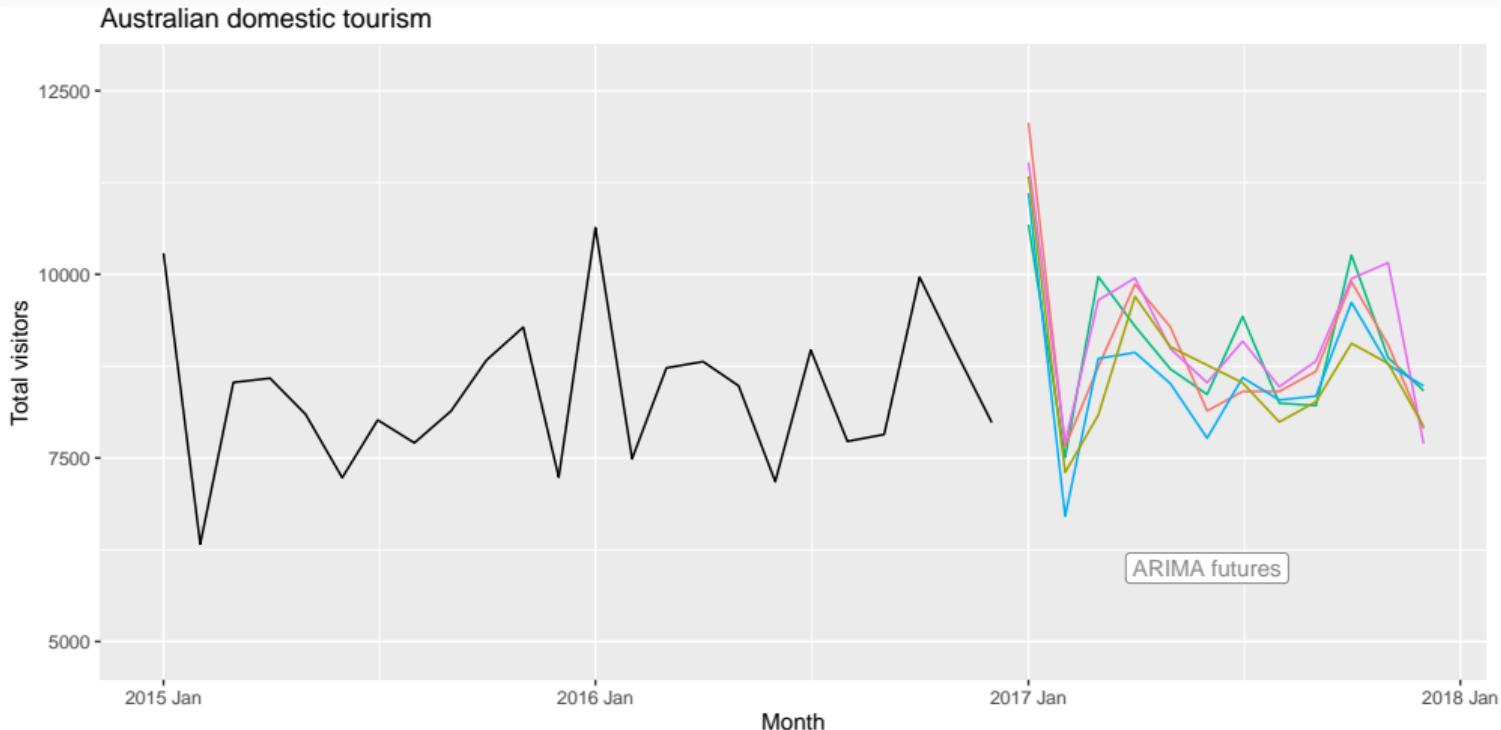
Evaluating probabilistic forecasts



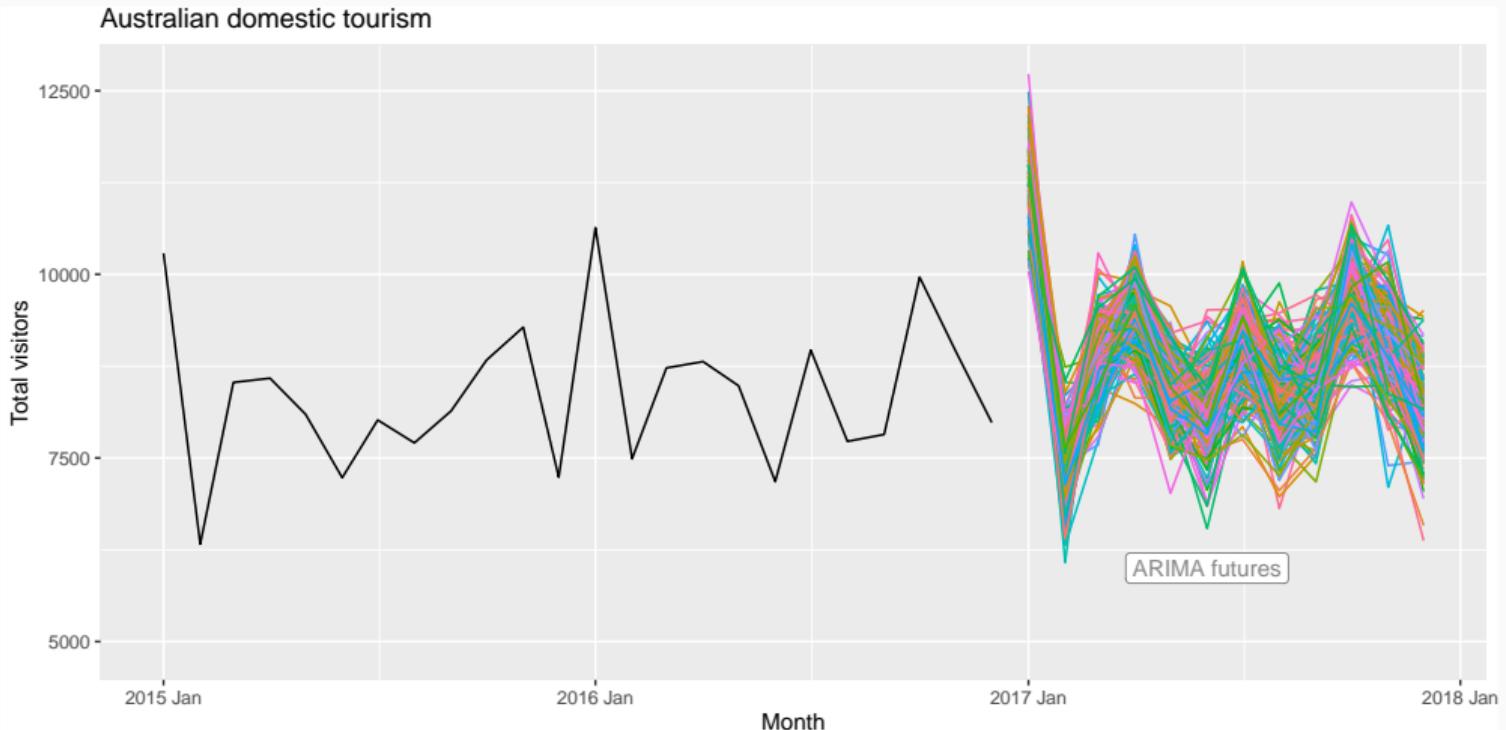
Evaluating probabilistic forecasts



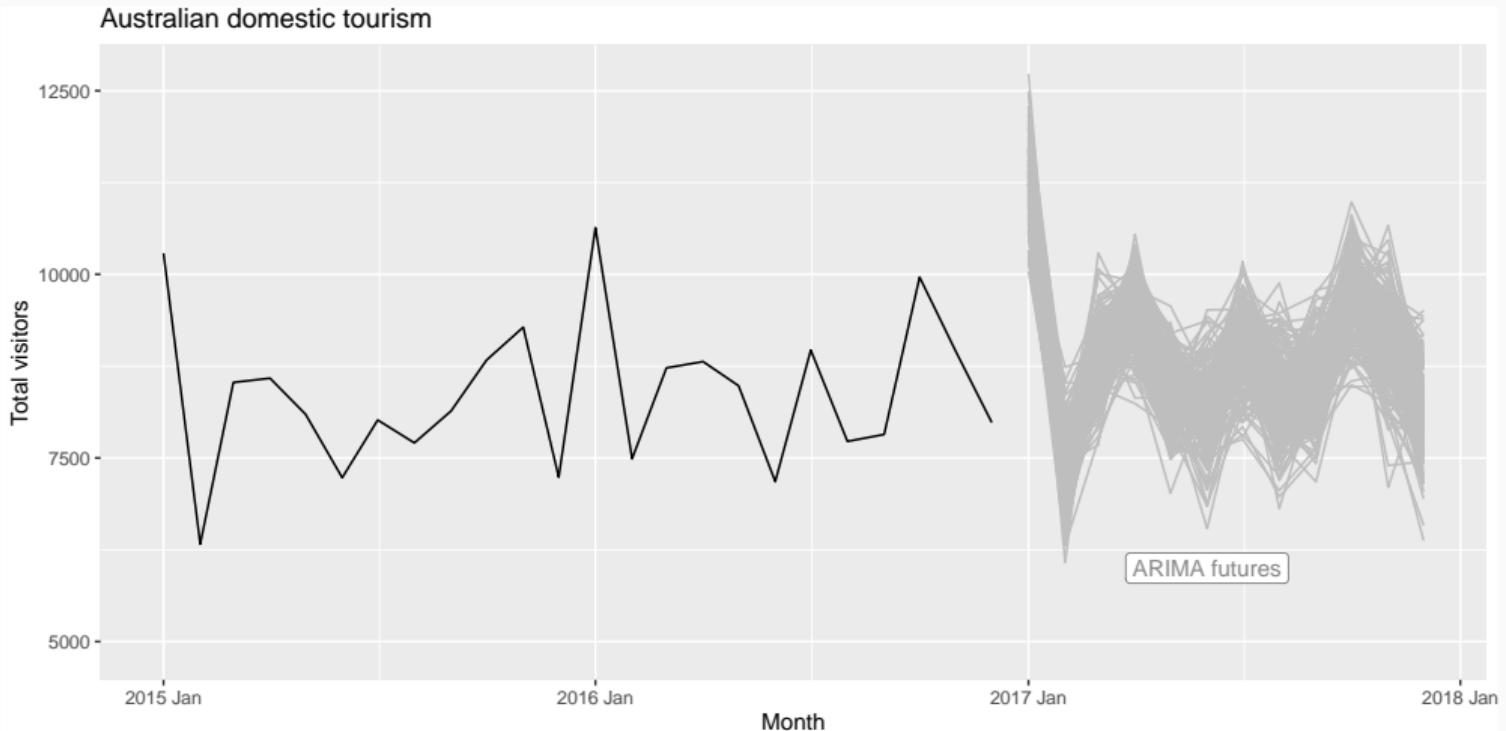
Evaluating probabilistic forecasts



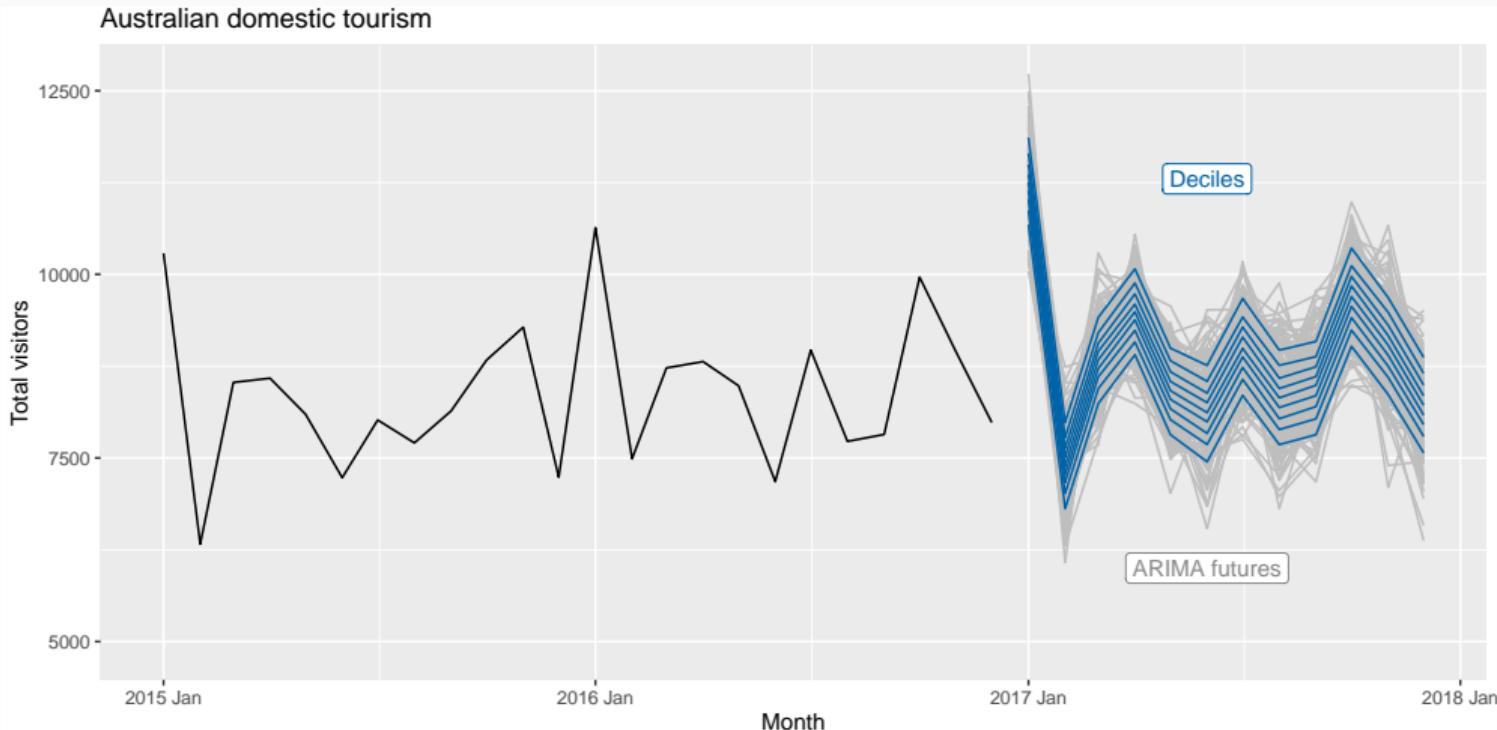
Evaluating probabilistic forecasts



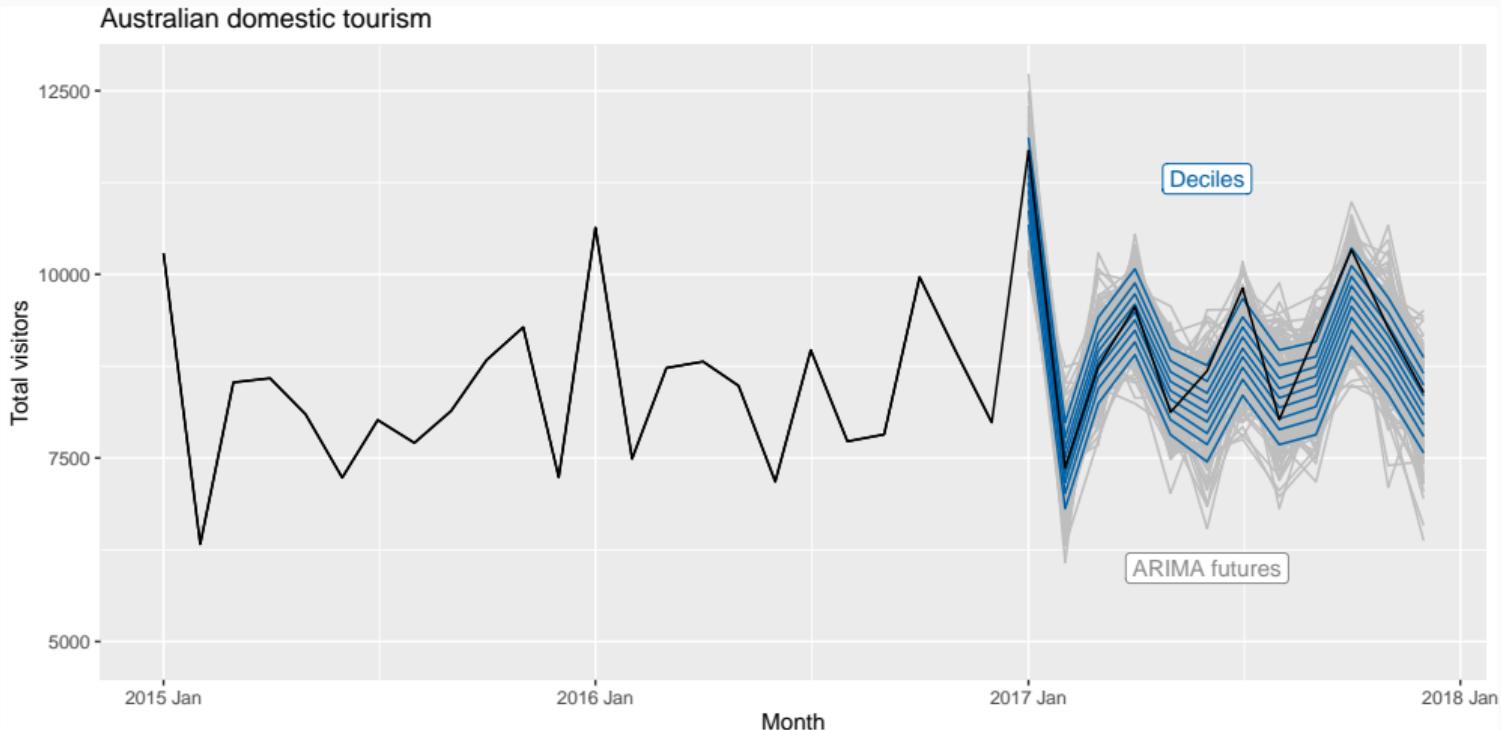
Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



Evaluating probabilistic forecasts

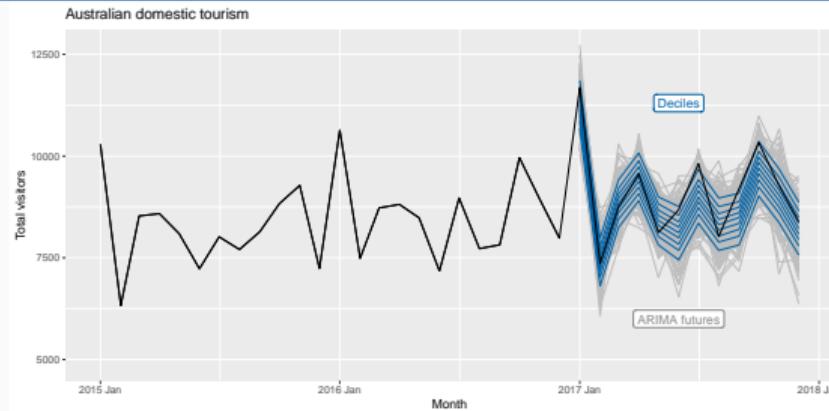
$q_{p,t}$ = quantile forecast with prob. p at time t .

y_t = observation at time t

Quantile score

$$S_t(p, y) = \begin{cases} 2(1 - p)|y_t - q_{p,t}|, & \text{if } y_t < q_{p,t} \\ 2p|y_t - q_{p,t}|, & \text{if } y_t \geq q_{p,t} \end{cases}$$

- Low S_t is good
- Multiplier of 2 often omitted, but useful for interpretation
- S_t like absolute error, weighted to account for likely exceedance
- Average $S_t(p, y)$ over p = CRPS (Continuous Rank Probability Score)



Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecasts)

Forecast distribution F_t and observation y_t .

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t) dp = E_F|Y - y_t| - \frac{1}{2}E_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t .
- Optimal when F_t is true distribution (i.e., it is a proper score)

Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecasts)

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- Y and Y^* are iid draws from F_t .
- Optimal when F_t is true distribution (i.e., it is a proper score)

Energy score (multivariate forecasts)

$$\text{ES}(F_t, \mathbf{y}_t) = E_F||\mathbf{Y} - \mathbf{y}_t|| - \frac{1}{2}E_F||\mathbf{Y} - \mathbf{Y}^*||$$

Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecasts)

Forecast distribution F_t and observation y_t .

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t) dp = E_F|Y - y_t| - \frac{1}{2}E_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t .
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Energy score (multivariate forecasts)

$$\text{ES}(F_t, \mathbf{y}_t) = E_F||\mathbf{Y} - \mathbf{y}_t|| - \frac{1}{2}E_F||\mathbf{Y} - \mathbf{Y}^*||$$

Log score (multivariate forecasts)

$$\text{LS}(F_t, \mathbf{y}_t) = -\log f(\mathbf{y}_t)$$

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule Coherent v Incoherent Coherent v Coherent

Log Score	Not proper	<ul style="list-style-type: none">• Ordering preserved if compared using bottom-level only
Energy Score	Proper	<ul style="list-style-type: none">• Full hierarchy should be used.• Rankings may change otherwise.

Score optimal reconciliation

Algorithm proposed by Panagiotelis et al (2020) for optimizing \mathbf{G} using stochastic gradient descent to optimize Energy Score.

- 1 Compute base forecasts over a test set.
- 2 Compute OLS reconciliation: $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
- 3 Iteratively update \mathbf{G} using SGD with Adam method and ES objective over a test set

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Example: Australian tourism

tourism

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:      state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>      <dbl>
## 1 1998 Jan NSW Metro NSW Sydney      926.
## 2 1998 Feb NSW Metro NSW Sydney      647.
## 3 1998 Mar NSW Metro NSW Sydney      716.
## 4 1998 Apr NSW Metro NSW Sydney      621.
## 5 1998 May NSW Metro NSW Sydney      598.
## 6 1998 Jun NSW Metro NSW Sydney      601.
## 7 1998 Jul NSW Metro NSW Sydney      720.
## 8 1998 Aug NSW Metro NSW Sydney      645.
## 9 1998 Sep NSW Metro NSW Sydney      633.
## 10 1998 Oct NSW Metro NSW Sydney      771.
```

Example: Australian tourism

```
tourism_agg <- tourism %>%
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
## # A tsibble: 26,400 x 5 [1M]
## # Key:      state, zone, region [110]
##       month state      zone      region   visitors
##       <mth> <chr*> <chr*> <chr*>     <dbl>
## 1 1998 Jan <aggregated> <aggregated> <aggregated> 10376.
## 2 1998 Feb <aggregated> <aggregated> <aggregated> 5746.
## 3 1998 Mar <aggregated> <aggregated> <aggregated> 7129.
## 4 1998 Apr <aggregated> <aggregated> <aggregated> 7939.
## 5 1998 May <aggregated> <aggregated> <aggregated> 6552.
## 6 1998 Jun <aggregated> <aggregated> <aggregated> 5969.
## 7 1998 Jul <aggregated> <aggregated> <aggregated> 7041.
## 8 1998 Aug <aggregated> <aggregated> <aggregated> 6382.
## 9 1998 Sep <aggregated> <aggregated> <aggregated> 6907.
```

Example: Australian tourism

```
fit <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(ets = ETS(visitors))
```

```
## # A mable: 110 x 4
## # Key:      state, zone, region [110]
##   state    zone           region          ets
##   <chr*> <chr*>        <chr*>        <model>
## 1 NSW     ACT            Canberra       <ETS(M,N,A)>
## 2 NSW     ACT            <aggregated> <ETS(M,N,A)>
## 3 NSW     Metro NSW     Central Coast <ETS(M,N,M)>
## 4 NSW     Metro NSW     Sydney         <ETS(M,N,A)>
## 5 NSW     Metro NSW     <aggregated> <ETS(M,N,A)>
## 6 NSW     North Coast NSW Hunter       <ETS(M,N,M)>
## 7 NSW     North Coast NSW North Coast NSW <ETS(M,N,M)>
## 8 NSW     North Coast NSW <aggregated> <ETS(M,N,M)>
```

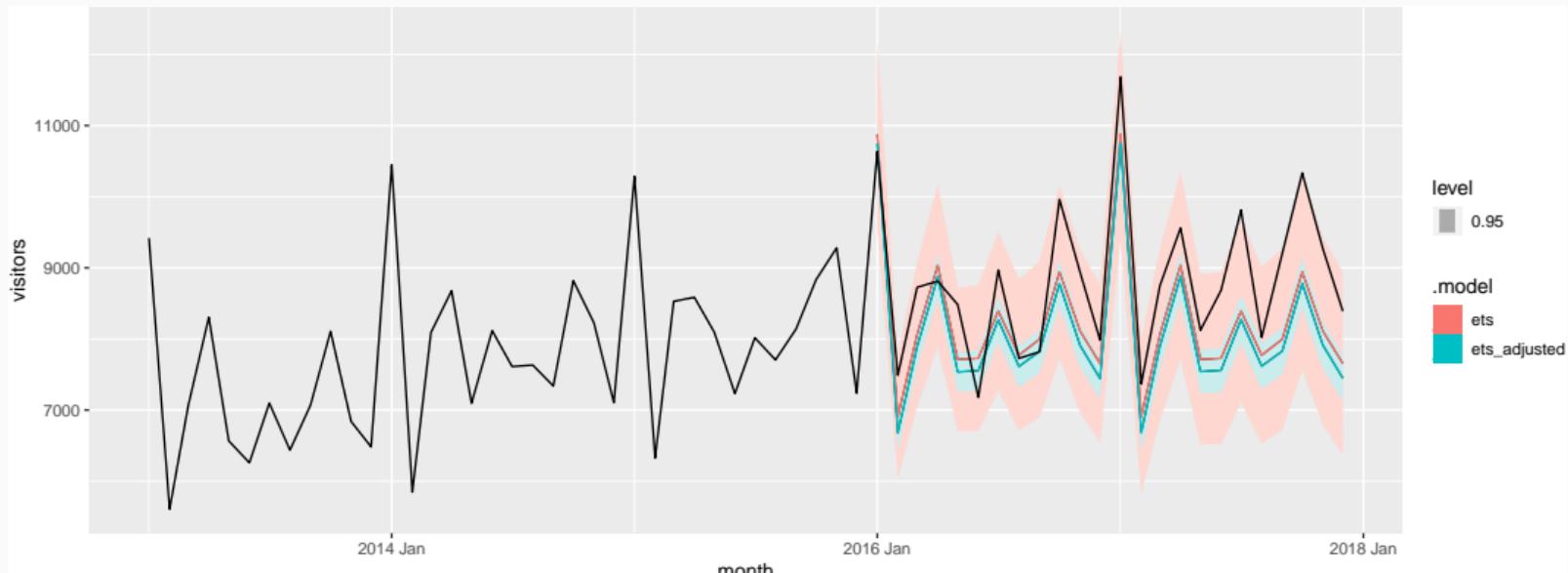
Example: Australian tourism

```
fc <- fit %>%
  reconcile(ets_adjusted = min_trace(ets)) %>%
  forecast(h = "2 years")
```

```
## # A fable: 5,280 x 7 [1M]
## # Key:      state, zone, region, .model [220]
##   state   zone   region   .model   month   visitors   .mean
##   <chr*> <chr*> <chr*>   <chr>    <mth>     <dist>   <dbl>
## 1 NSW     ACT     Canberra  ets     2016 Jan N(202, 1437) 202.
## 2 NSW     ACT     Canberra  ets     2016 Feb N(160, 912) 160.
## 3 NSW     ACT     Canberra  ets     2016 Mar N(204, 1489) 204.
## 4 NSW     ACT     Canberra  ets     2016 Apr N(207, 1527) 207.
## 5 NSW     ACT     Canberra  ets     2016 May N(196, 1380) 196.
## 6 NSW     ACT     Canberra  ets     2016 Jun N(182, 1201) 182.
## 7 NSW     ACT     Canberra  ets     2016 Jul N(207, 1562) 207.
## 8 NSW     ACT     Canberra  ets     2016 Aug N(180, 1191) 180.
```

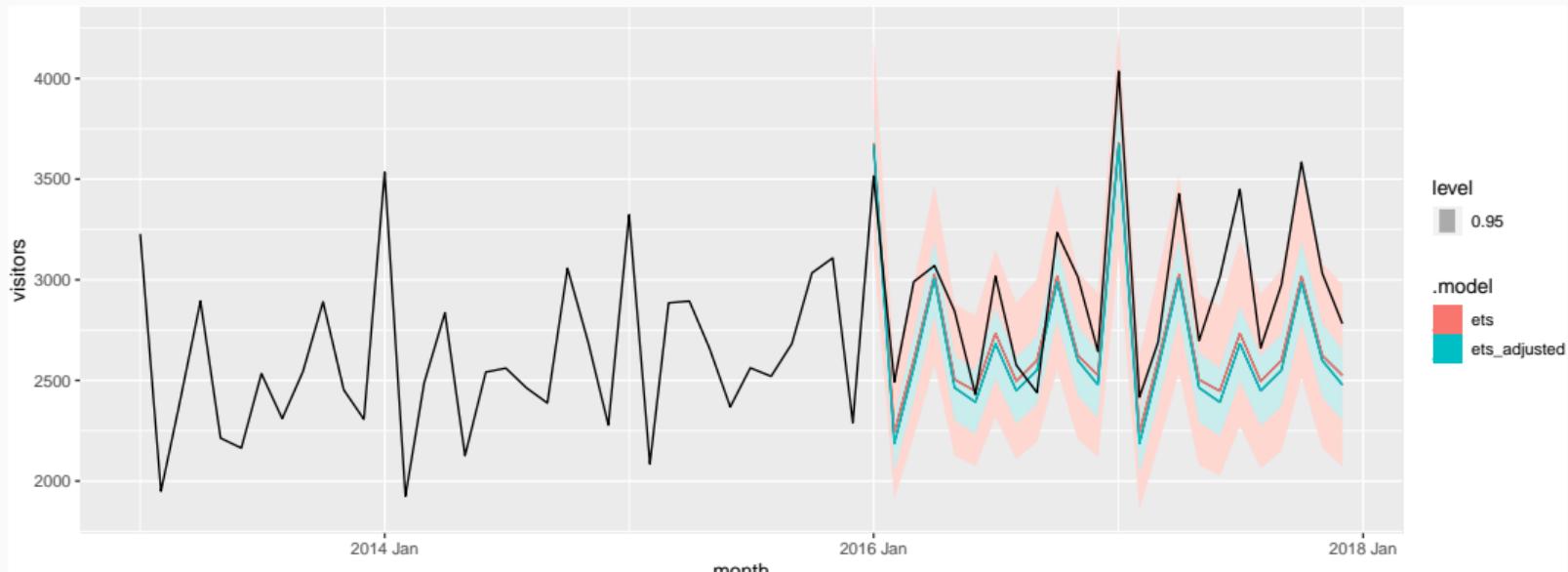
Example: Australian tourism

```
fc %>%
  filter(is_aggregated(state)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



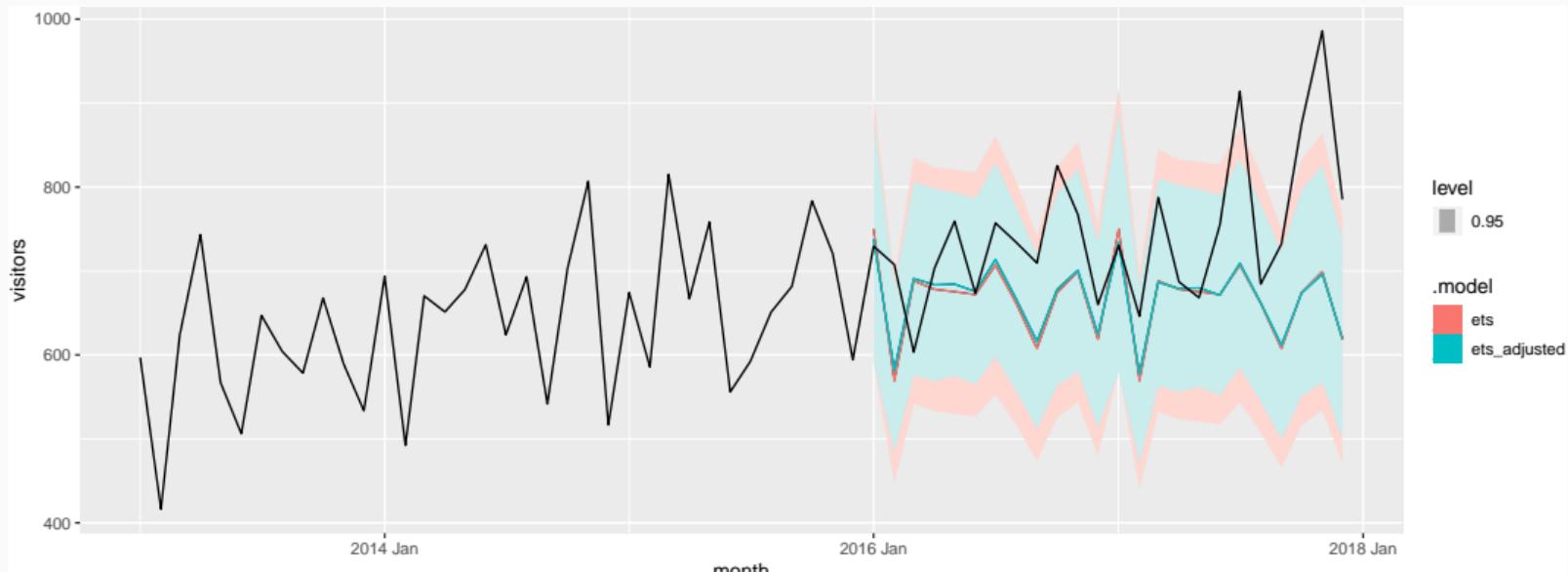
Example: Australian tourism

```
fc %>%
  filter(state == "NSW" & is_aggregated(zone)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



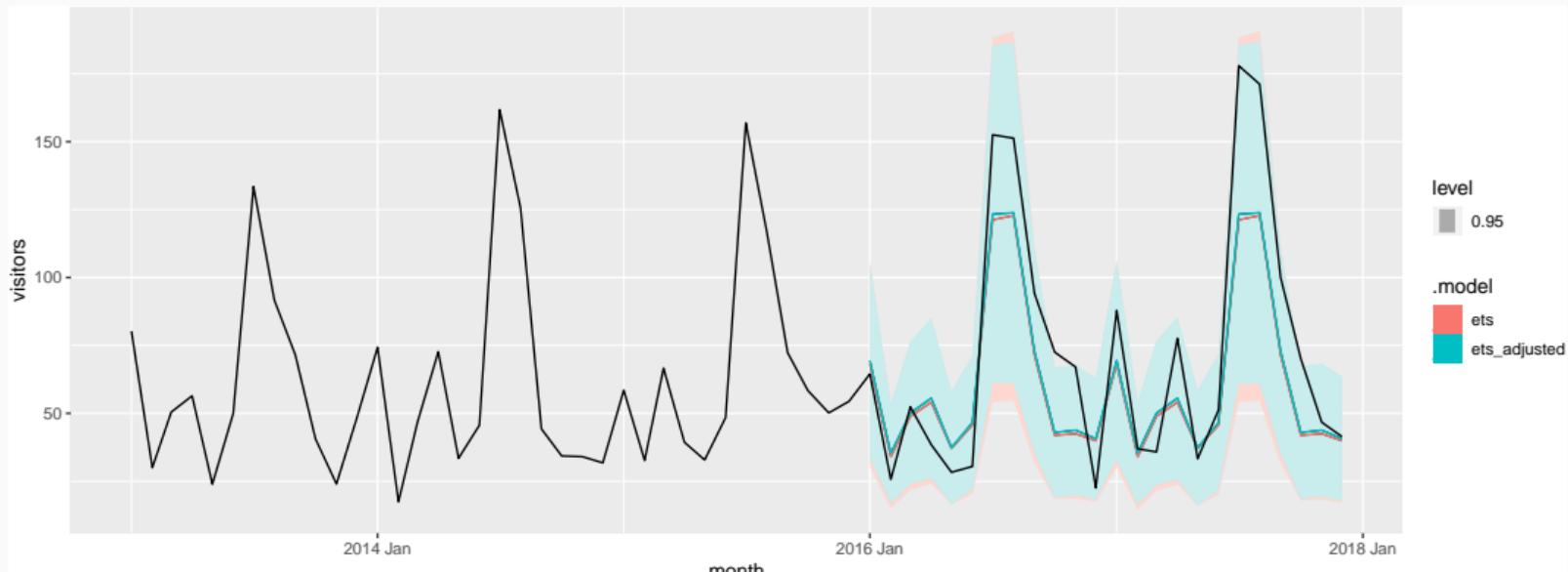
Example: Australian tourism

```
fc %>%
  filter(region == "Melbourne") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



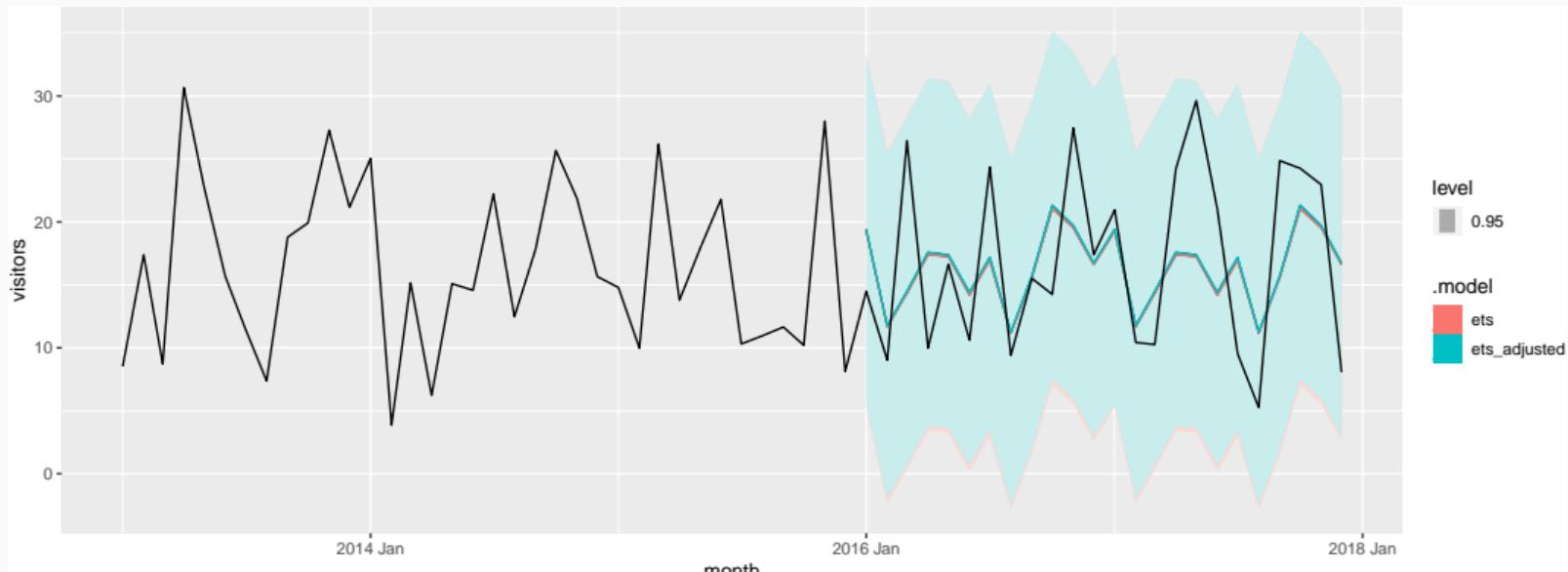
Example: Australian tourism

```
fc %>%
  filter(region == "Snowy Mountains") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



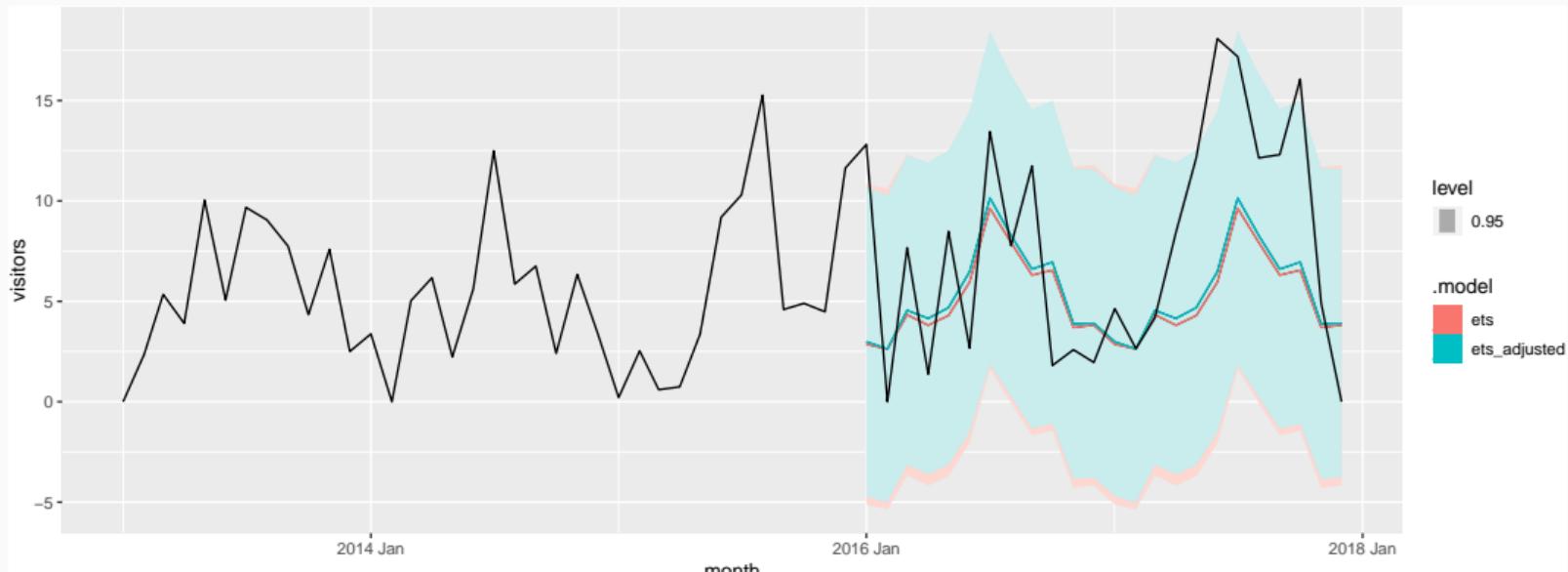
Example: Australian tourism

```
fc %>%
  filter(region == "Barossa") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



Example: Australian tourism

```
fc %>%
  filter(region == "MacDonnell") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



Example: Australian tourism

```
fc <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(
    ets = ETS(visitors),
    arima = ARIMA(visitors)
  ) %>%
  mutate(
    comb = (ets + arima) / 2
  ) %>%
  reconcile(
    ets_adj = min_trace(ets),
    arima_adj = min_trace(arima),
    comb_adj = min_trace(comb)
  ) %>%
  forecast(h = "2 years")
```

Example: Australian tourism

```
fc %>%  
  accuracy(data = tourism_agg,  
            measures = list(crps = CRPS, ss=skill_score(CRPS)))  
  
## # A tibble: 660 x 7  
##   .model state zone          region     .type  crps      ss  
##   <chr>   <chr> <chr*>       <chr*>     <chr> <dbl>    <dbl>  
## 1 arima   NSW   ACT        Canberra     Test  34.5 -0.221  
## 2 arima   NSW   ACT        <aggregated> Test  34.5 -0.221  
## 3 arima   NSW   Metro NSW Central Coast Test  13.9  0.196  
## 4 arima   NSW   Metro NSW Sydney       Test  62.4  0.139  
## 5 arima   NSW   Metro NSW <aggregated> Test  69.1  0.152  
## 6 arima   NSW   North Coast NSW Hunter     Test  40.7 -0.117  
## 7 arima   NSW   North Coast NSW North Coast NSW Test  41.0  0.0349  
## 8 arima   NSW   North Coast NSW <aggregated> Test  58.5  0.0577  
## 9 arima   NSW   North NSW    Blue Mountains Test  13.6  0.0826
```

Example: Australian tourism

```
fc %>%
  accuracy(tourism_agg,
            measures = list(crps = CRPS, ss=skill_score(CRPS))) %>%
  group_by(.model) %>%
  summarise(sspc = mean(ss) * 100) %>%
  arrange(sspc)
```

```
## # A tibble: 6 x 2
##   .model     sspc
##   <chr>     <dbl>
## 1 arima_adj 11.6
## 2 arima      11.8
## 3 comb_adj   16.8
## 4 ets_adj    17.7
## 5 comb       18.1
## 6 ets        19.1
```

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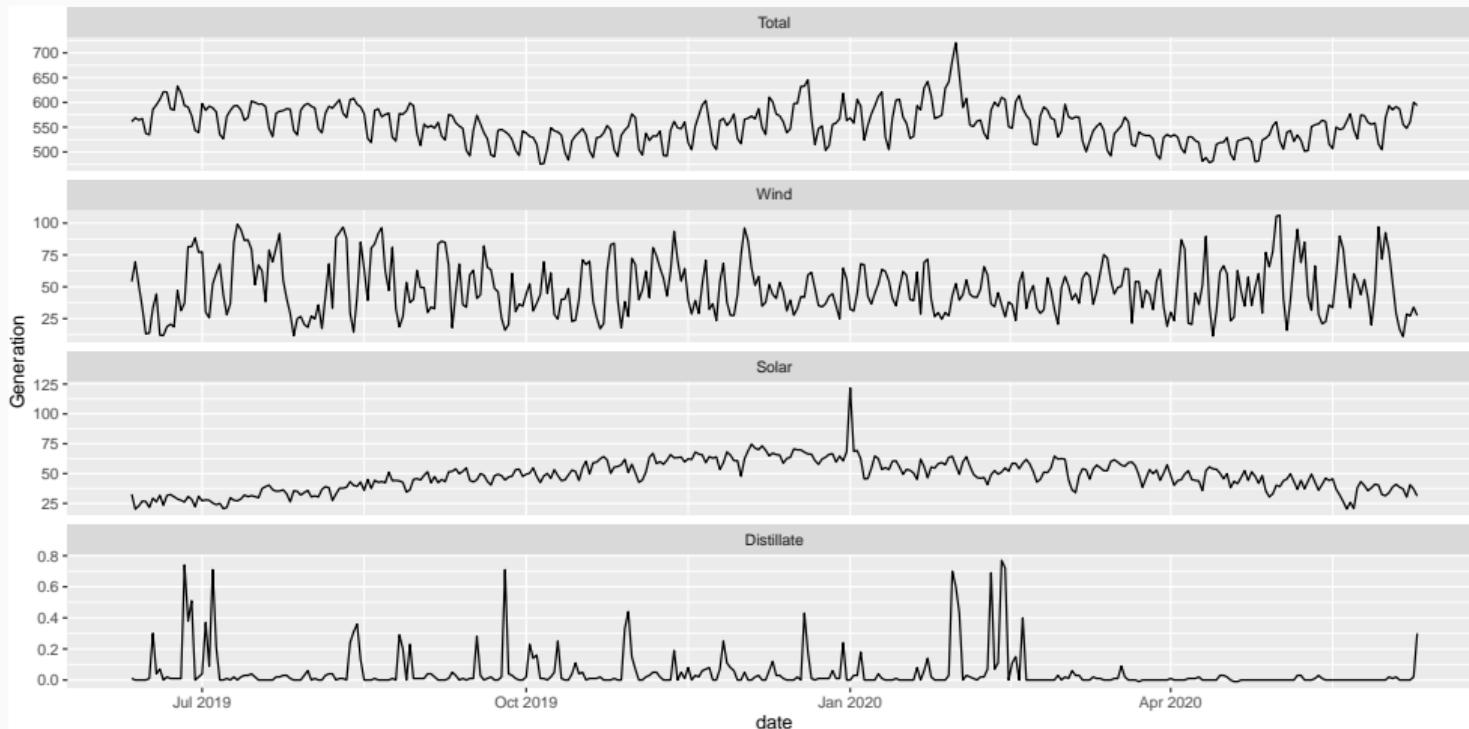
Example: Australian electricity generation

Daily time series from opennem.org.au

- 1 Total = Renewable + Non-renewable
- 2 Renewable = Batteries + Hydro + Solar + Wind + Biomass
Non-Renewable = Coal + Gas + Distillate
- 3 Battery = Battery (Discharging) + Battery (Charging)
Solar = Solar (Rooftop) + Solar (Utility)
Coal = Black Coal + Brown Coal
Gas = Gas (OCGT) + Gas (CCGT) + Gas (Steam) + Gas (Recip)

$n = 23$ series; $m = 15$ bottom-level series.

Example: Australian electricity generation

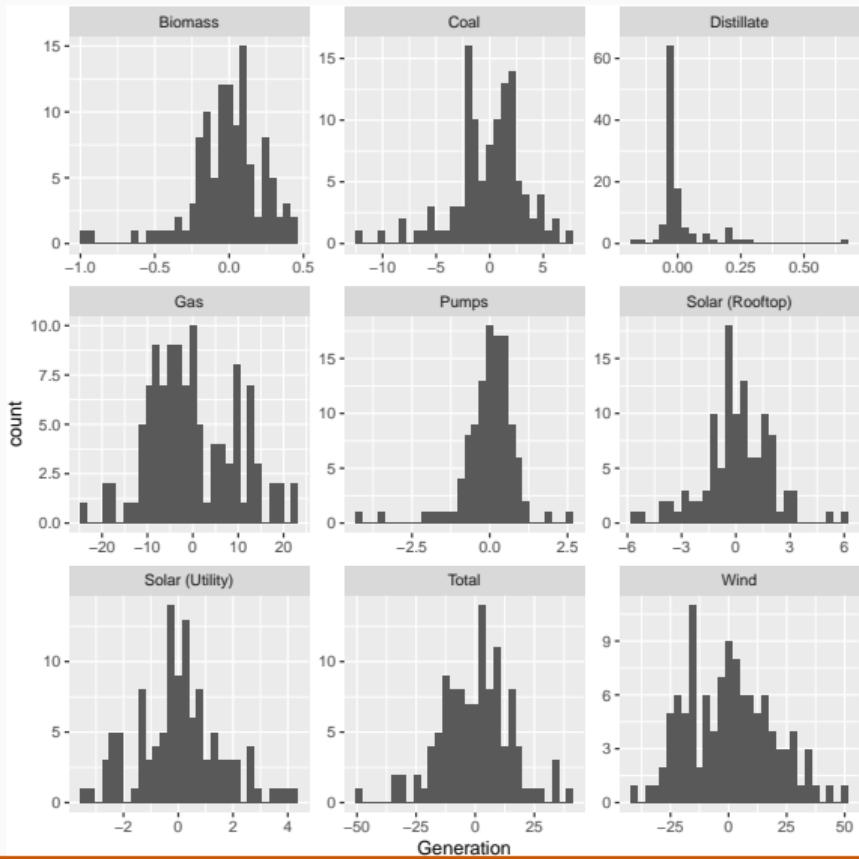


Example: Australian electricity generation

Forecast evaluation

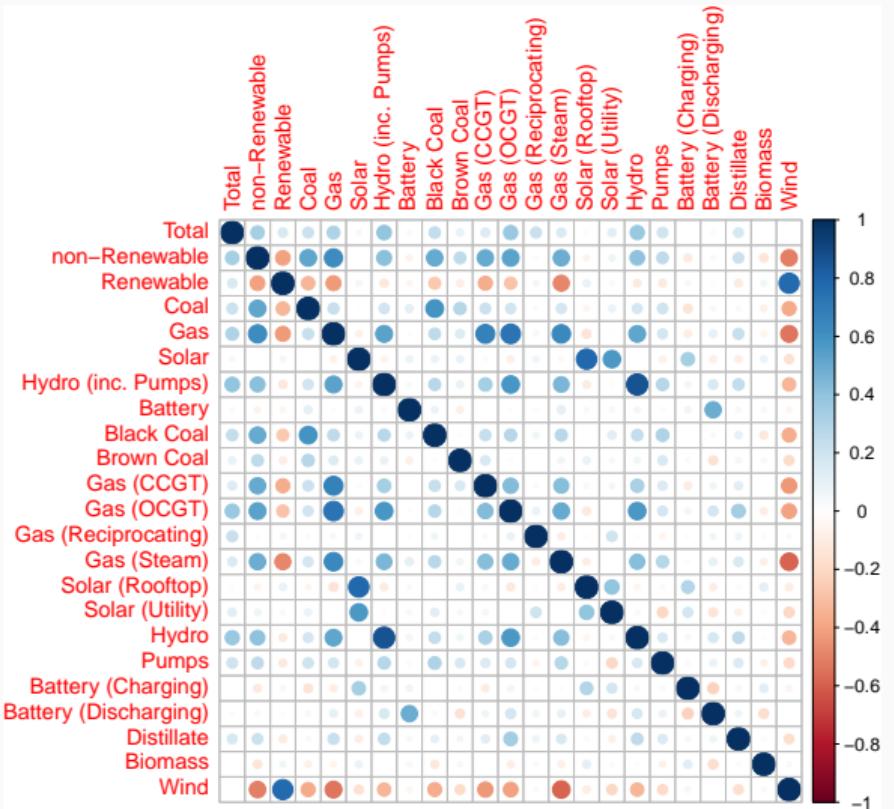
- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using NNETAR() function in fable package.
- Model could be improved with temperature predictor.

Example: Australian electricity generation



Histogram of residuals:
2 Oct 2019 - 21 Jan 2020
Clearly non-Gaussian

Example: Australian electricity generation



Correlations of residuals:

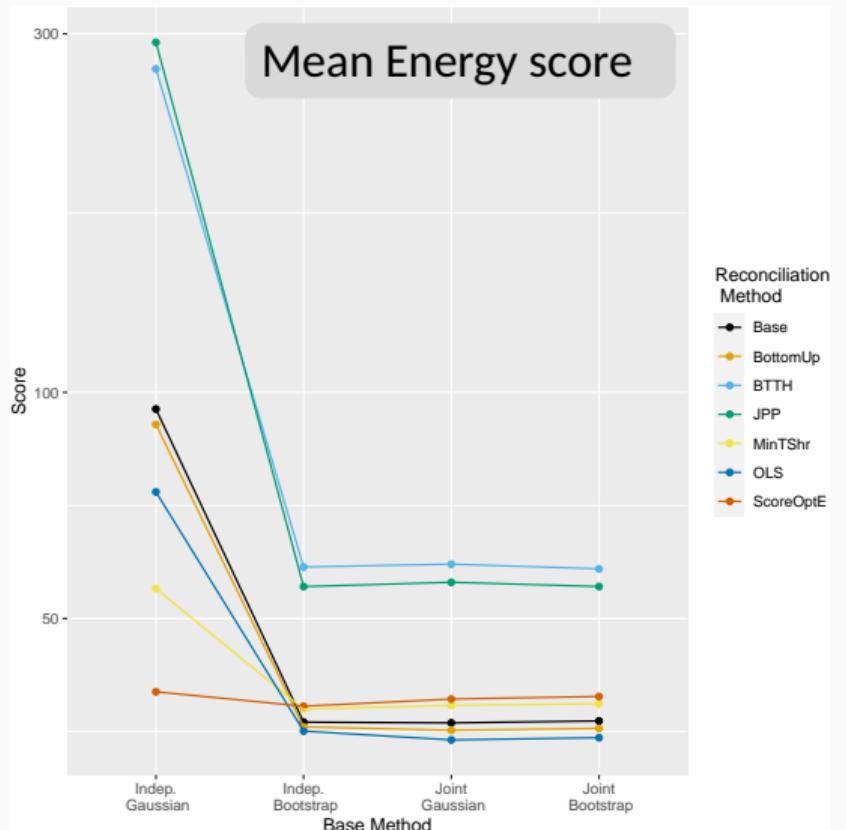
2 Oct 2019 - 21 Jan 2020

Blue = positive correlation.

Red = negative correlation.

Large = stronger correlations.

Example: Australian electricity generation



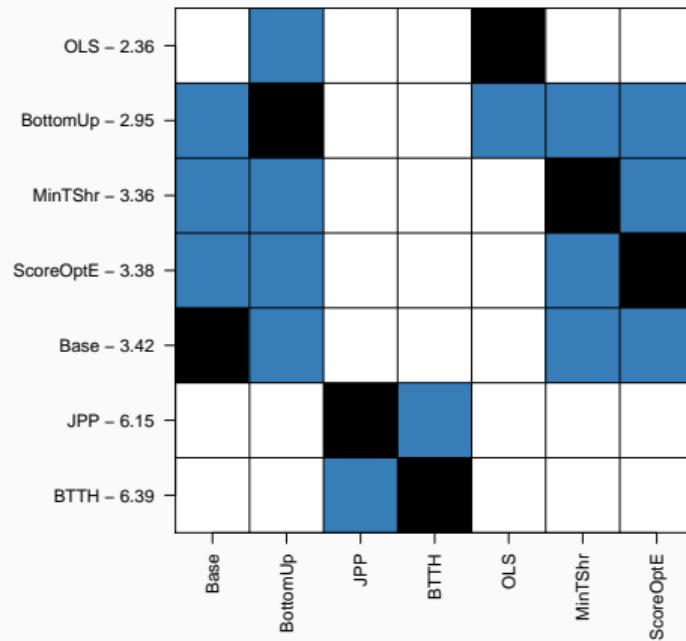
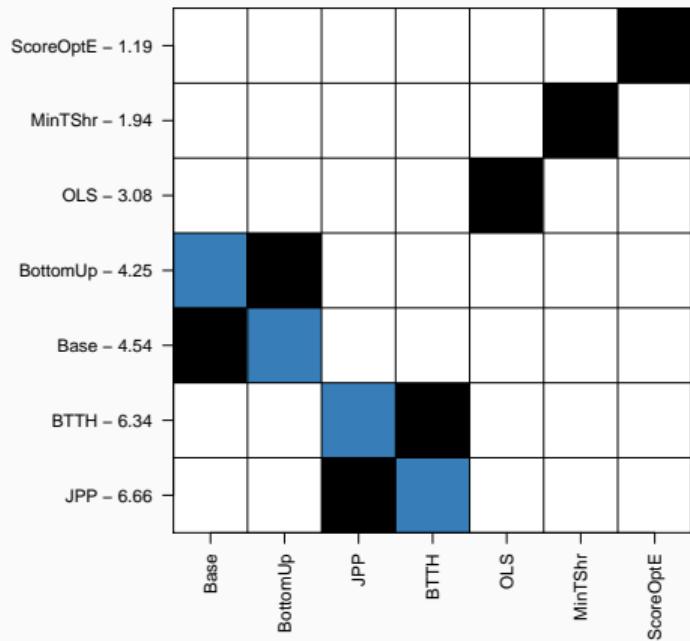
Base residual assumptions

- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

Reconciliation methods

- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS
- MinT(Shrink)
- Score Optimal Reconciliation

Example: Australian electricity generation



Nemenyi test for different scores

Base forecasts are independent and Gaussian.

Nemenyi test for different scores

Base forecasts are obtained by jointly bootstrapping residuals.

Thanks



More information

- Slides and papers: robjhyndman.com
- Packages: tidyverts.org
- Forecasting textbook using fable package:
OTexts.com/fpp3

Find me at ...



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