The geometry of forecast reconciliation

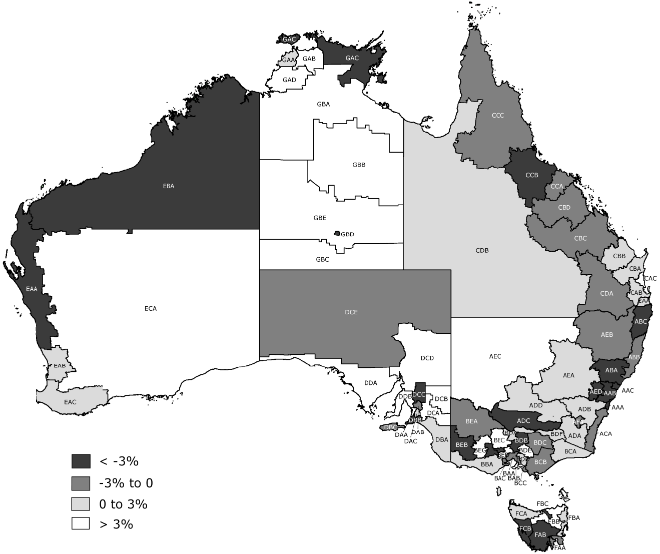
Rob J Hyndman

28 August 2020

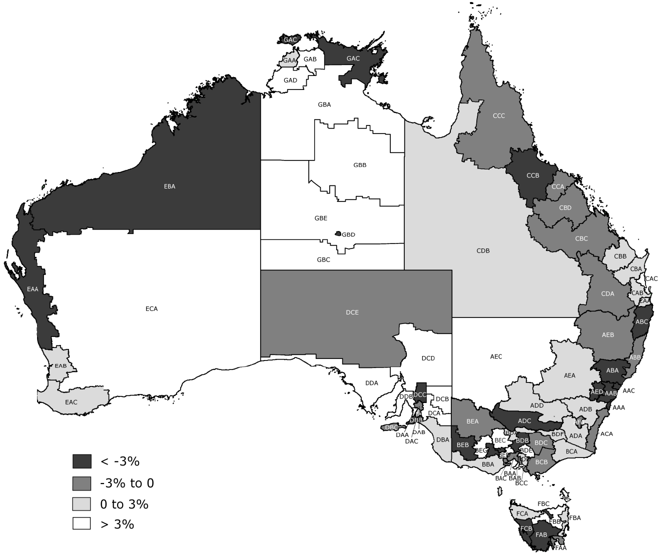
Outline

Outline

Australian tourism



Australian tourism

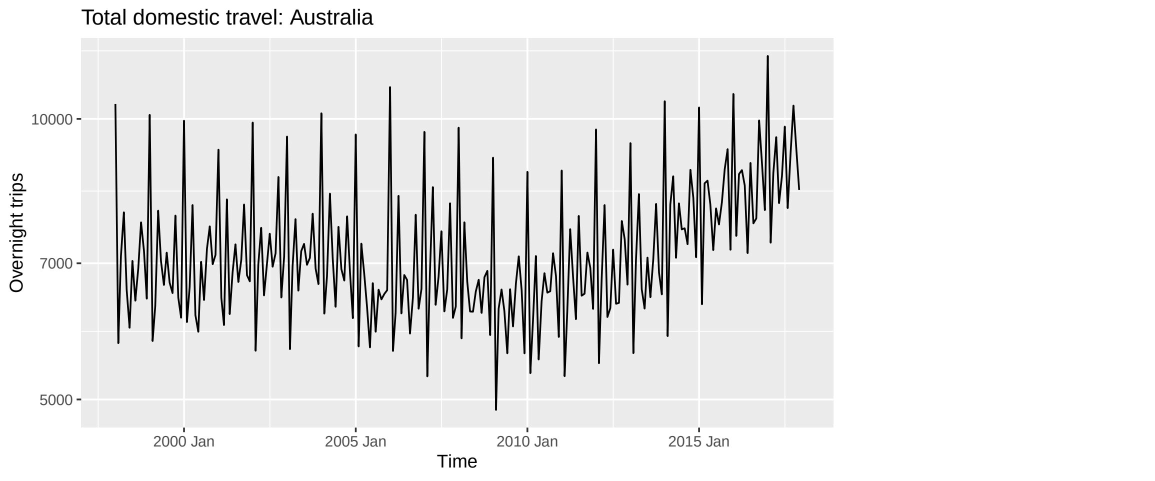


- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - 7 states
  - 27 zones
  - 75 regions

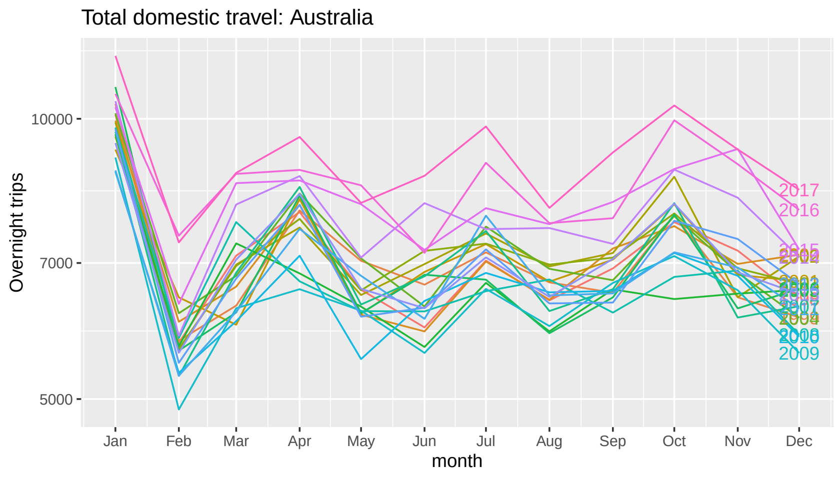
Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney     926.
##  2 1998 Feb NSW   Metro NSW Sydney     647.
##  3 1998 Mar NSW   Metro NSW Sydney     716.
##  4 1998 Apr NSW   Metro NSW Sydney     621.
##  5 1998 May NSW   Metro NSW Sydney     598.
##  6 1998 Jun NSW   Metro NSW Sydney     601.
##  7 1998 Jul NSW   Metro NSW Sydney     720.
##  8 1998 Aug NSW   Metro NSW Sydney     645.
##  9 1998 Sep NSW   Metro NSW Sydney     633.
## 10 1998 Oct NSW   Metro NSW Sydney     771.
## # ... with 17,990 more rows
```

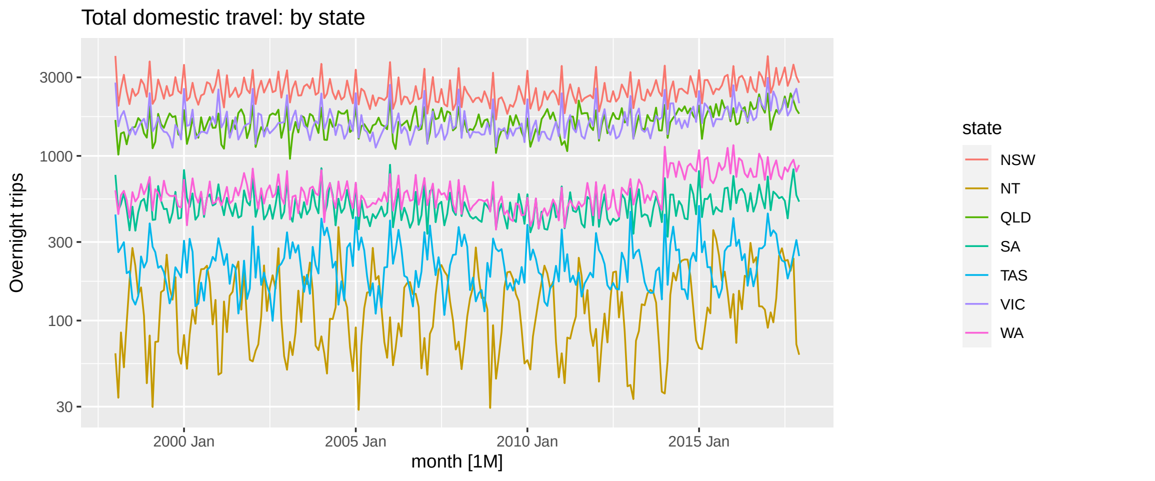
Australian tourism data



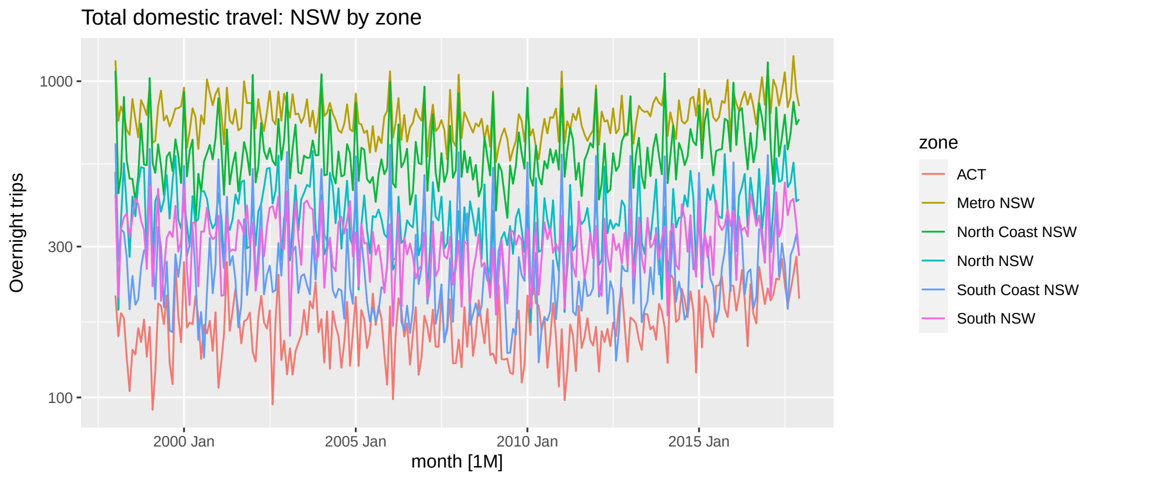
Australian tourism data



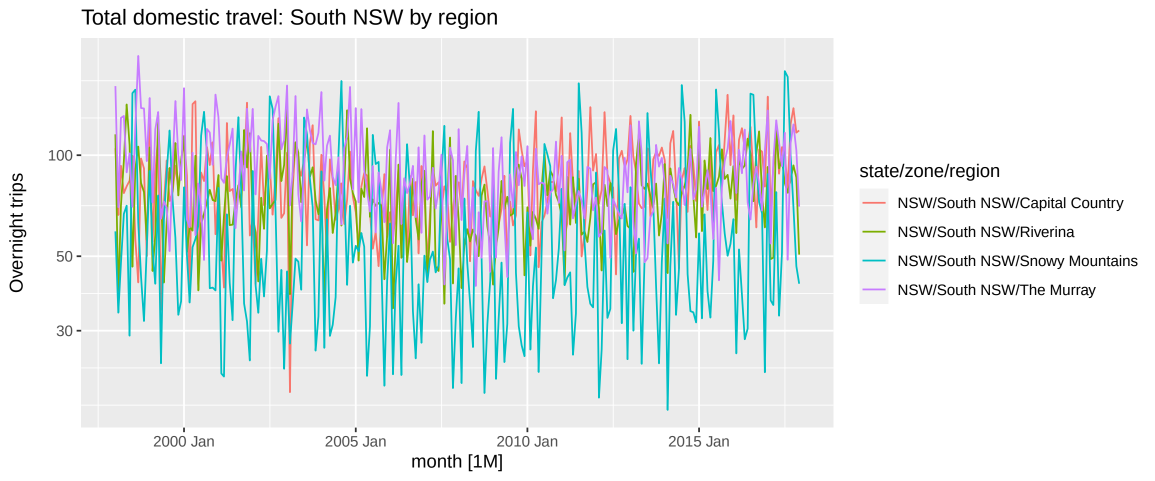
Australian tourism data



Australian tourism data

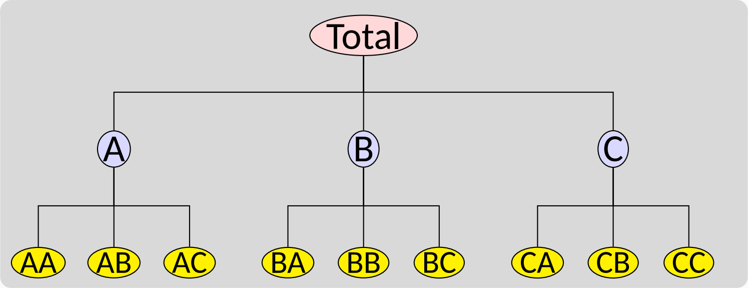


Australian tourism data



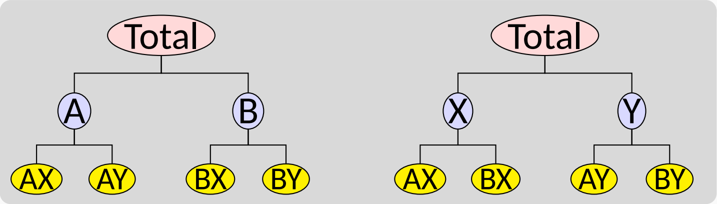
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



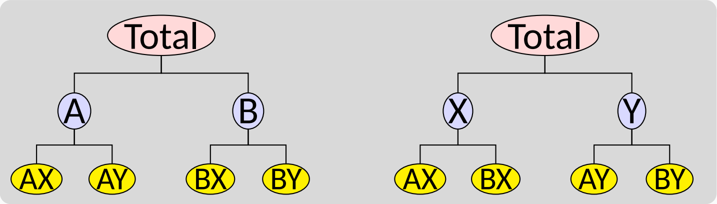
Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Outline

The problem

How to produce **coherent** forecasts at all nodes?

The problem

How to produce **coherent** forecasts at all nodes?

Old approaches (pre 2009)

- Bottom-up forecasting
- Top-down forecasting
- Middle-out forecasting

The problem

How to produce **coherent** forecasts at all nodes?

Old approaches (pre 2009)

- Bottom-up forecasting
- Top-down forecasting
- Middle-out forecasting

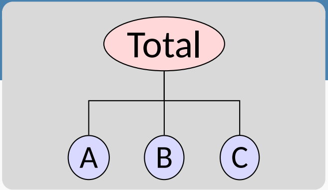
Forecast reconcilation approach

1. Forecast all series at all levels of aggregation using an automatic forecasting algorithm. (e.g., `ETS`, `ARIMA`, ...)
2. Reconcile the resulting forecasts so they are coherent using least squares optimization (i.e., find closest reconciled forecasts to the original forecasts).

Key forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 *CSDA*) Optimal combination forecasts for hierarchical time series.
- Athanasopoulos, Ahmed, Hyndman (2009 *IJF*) Hierarchical forecasts for Australian domestic tourism.
- Hyndman, Lee, Wang (2016 *CSDA*) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 *JASA*) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 *IJF*) Forecast reconciliation: A geometric view with new insights on bias correction.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020) Probabilistic forecast reconciliation: properties, evaluation and score optimisation.

Hierarchical and grouped time series



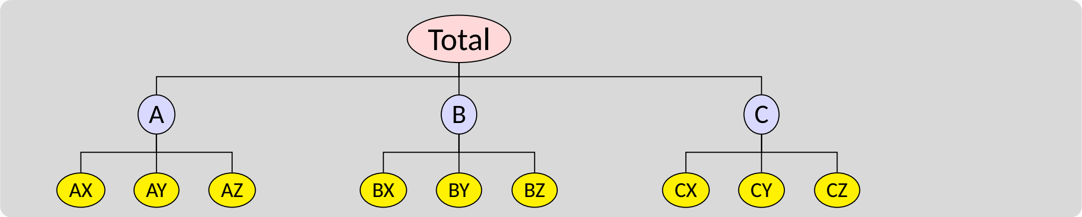
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

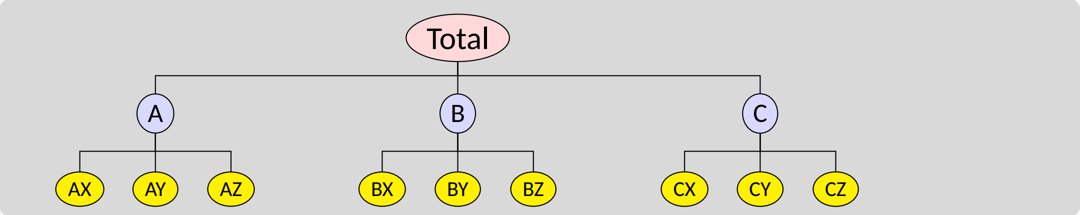
- $\boldsymbol{y}_t$ = vector of all series at time t
- y_t = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Hierarchical time series

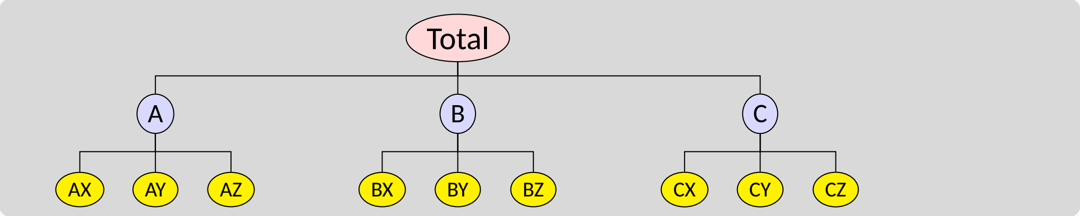


Hierarchical time series



$$
\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}
$$

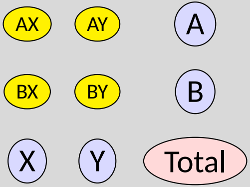
Hierarchical time series



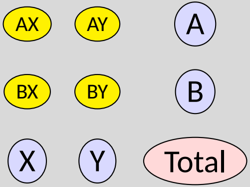
$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Grouped data

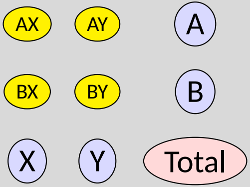


Grouped data



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}$$

Grouped data



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Definitions

Coherent subspace

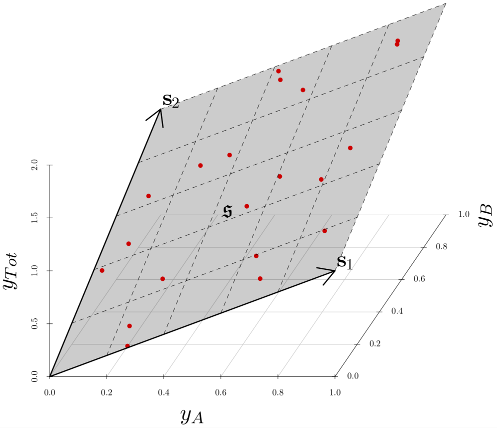
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

Definitions

Coherent subspace

m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

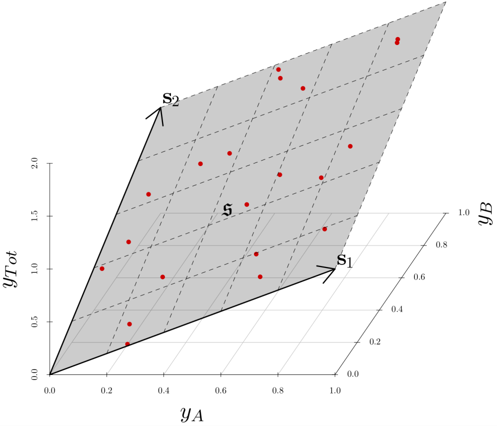
An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



$$y_{Tot} = y_A + y_B$$

Definitions

Coherent subspace

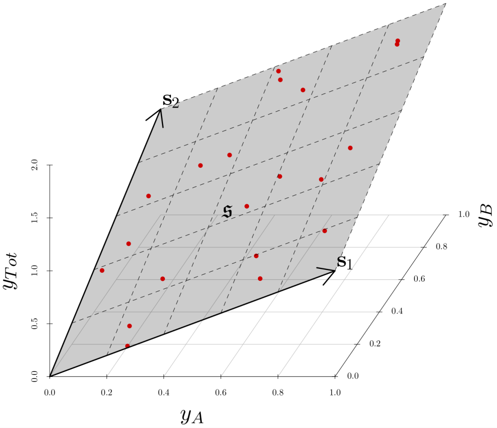
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \to \mathfrak{s}$. $\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear reconciliation

If ψ is a linear function and **G** is some matrix, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$$

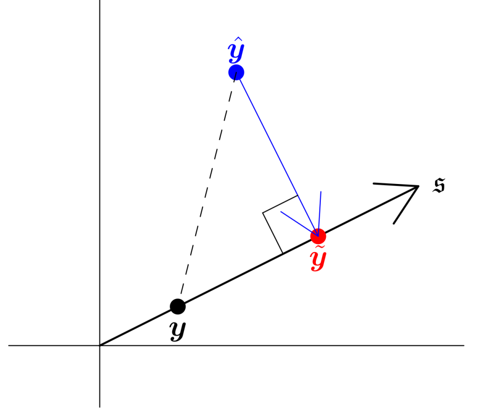
- $\boldsymbol{G}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.
- e.g., OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$

Linear reconciliation

If ψ is a linear function and **G** is some matrix, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$$

- **G** extracts and combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.
- e.g., OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$

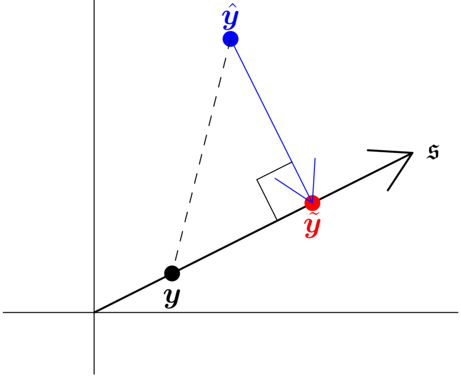


Linear reconciliation

If ψ is a linear function and **G** is some matrix, then

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$$

- **G** extracts and combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.
- e.g., OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$



Projections

Suppose $\boldsymbol{S}\boldsymbol{G}$ is a projection onto $\mathfrak{s}$, then

- Coherent base forecasts are unchanged.
- Unbiased base forecasts remain unbiased.

- Orthogonal projections lead to smallest possible adjustments of base forecasts.

Linear reconciliation

If ψ is a linear function and $\boldsymbol{G}$ is some matrix, then

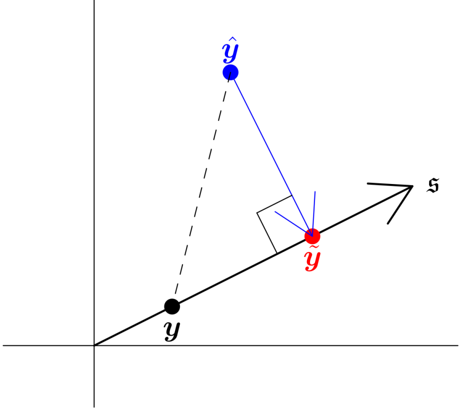
$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$$

- $\boldsymbol{G}$ extracts and combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.
- e.g., OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$

Distance reducing property

If $\boldsymbol{S}\boldsymbol{G}$ is an orthogonal projection onto $\mathfrak{s}$ then:

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\| \le \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|.$$



- Distance reduction holds for any realisation and any forecast.
- Other measures of forecast accuracy may be worse.
- Not necessarily the optimal reconciliation.

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{SG}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

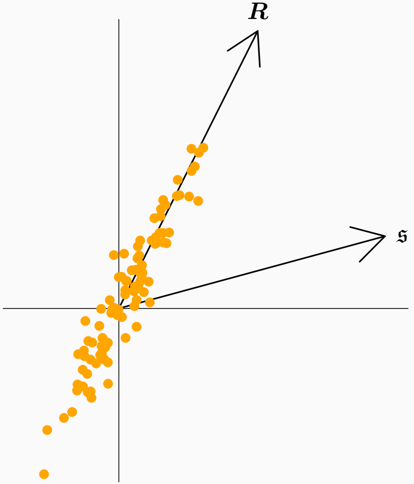
where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Orange: in-sample errors



Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

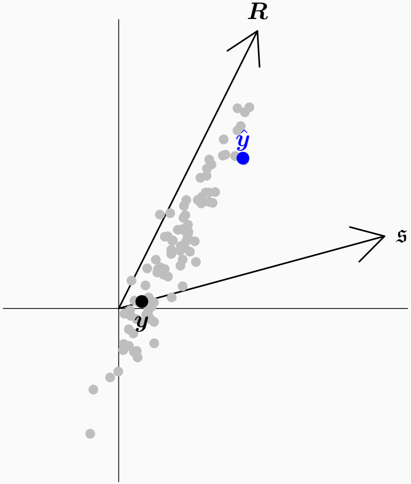
where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{h}$$

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts



Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

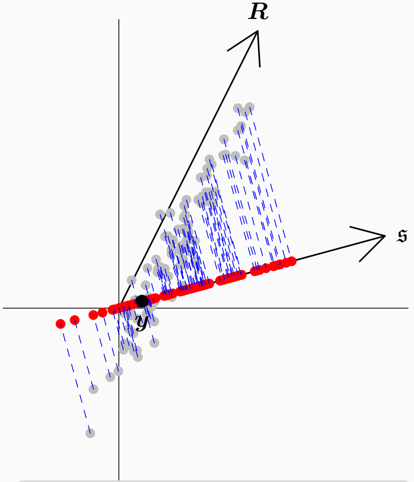
where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecsts



Orthogonal projection

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Variance

$$\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'$$

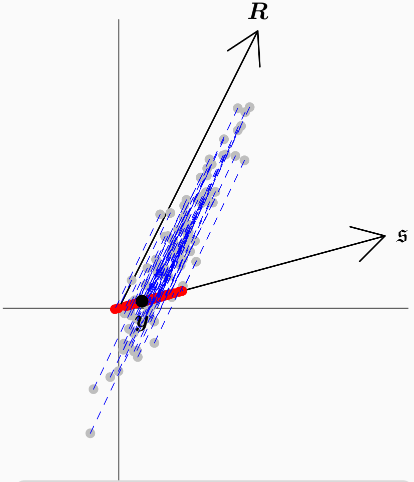
where $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$.

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

- R is the most likely direction of deviations from $\mathfrak{s}$.
- Grey: potential base forecasts
- Red: reconciled forecsts



Oblique projection

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciliation method	$\boldsymbol{G}$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS	$(\boldsymbol{S}'\boldsymbol{\Lambda}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{\Lambda}$
MinT(Sample)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}$
MinT(Shrink)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\boldsymbol{W}_h = k_h\boldsymbol{W}_1$.

- $\boldsymbol{\Lambda}$ is diagonal matrix
- $\hat{\boldsymbol{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_{\text{sam}}) + (1-\tau)\hat{\boldsymbol{W}}_{\text{sam}}$ where $\tau = \dfrac{\sum_{i\neq j}\hat{\text{Var}}(\hat{\sigma}_{ij})}{\sum_{i\neq j}\hat{\sigma}_{ij}^2}$ and σ_{ij} denotes the (i,j)th element of $\hat{\boldsymbol{W}}_{\text{sam}}$.

Outline

1. Hierarchical and grouped time series
2. Forecast reconciliation using projections
3. Probabilistic forecast reconciliation
4. Evaluating probabilistic forecasts
5. Example: Australian tourism

Coherent probabilistic forecasts

Coherent probabilistic forecasts

Given the triple $(\mathbb{R}^m, \mathscr{F}_{\mathbb{R}^m}, \nu)$, a coherent probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is such that

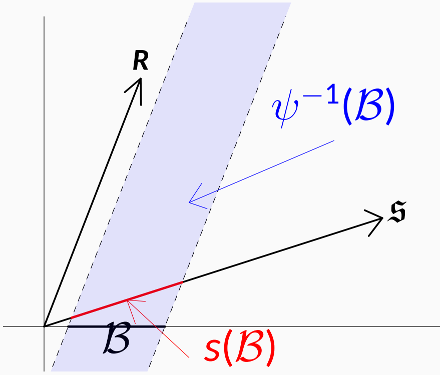
$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\mathbb{R}^m}.$$

Probabilistic forecast reconciliation

The reconciled probability measure of $\hat{\nu}$ wrt $\psi(.)$ is such that

$$\tilde{\nu}(\mathcal{B}) = \hat{\nu}(\psi^{-1}(\mathcal{B})) \qquad \forall \mathcal{B} \in \mathscr{F}_{\mathfrak{s}}\,,$$

where $\psi^{-1}(\mathcal{B}) := \{\boldsymbol{y} \in \mathbb{R}^n : \psi(\boldsymbol{y}) \in \mathcal{B}\}$ is the pre-image of $\mathcal{B}$, that is the set of all points in $\mathbb{R}^n$ that $\psi(.)$ maps to a point in $\mathcal{B}$.



Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_{\boldsymbol{b}}(\boldsymbol{b}) = |\boldsymbol{G}^*| \int \hat{f}(\boldsymbol{G}^- \boldsymbol{b} + \boldsymbol{G}_\perp \boldsymbol{a}) d\boldsymbol{a}$$

- $\hat{f}$ is density of incoherent base probabilistic forecast
- $\boldsymbol{G}^-$ is $n \times m$ generalised inverse of $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}^- = \boldsymbol{I}$
- $\boldsymbol{G}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}_\perp = \boldsymbol{0}$
- $\boldsymbol{G}^* = \left(\boldsymbol{G}^- \vdots \boldsymbol{G}_\perp\right)$, and $\boldsymbol{b}$ and $\boldsymbol{a}$ are obtained via

  the change of variables $\boldsymbol{y} = \boldsymbol{G}^* \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*| \tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y}) \mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix} \boldsymbol{S}^- \\ \boldsymbol{S}'_\perp \end{pmatrix}$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix} \boldsymbol{S}^- \\ \boldsymbol{S}'_\perp \end{pmatrix}$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Gaussian reconciliation

If the incoherent base forecasts are $\text{N}(\hat{\mu}, \hat{\Sigma})$,
then the reconciled density is $\text{N}(\boldsymbol{S}\boldsymbol{G}\hat{\mu}, \boldsymbol{S}\boldsymbol{G}\hat{\Sigma}\boldsymbol{G}'\boldsymbol{S}')$.

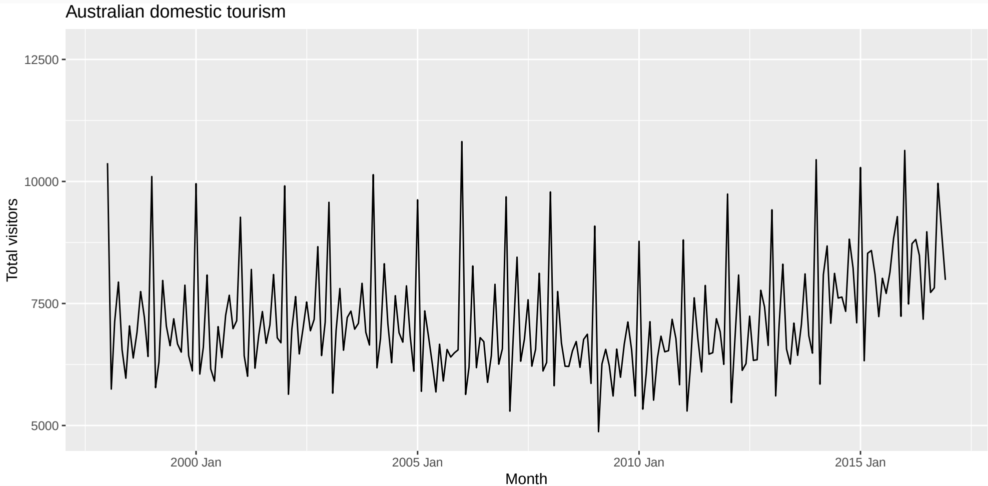
Simulation from a reconciled distribution

Suppose that $\left(\hat{\boldsymbol{y}}^{[1]}, \dots, \hat{\boldsymbol{y}}^{[L]}\right)$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $\left(\tilde{\boldsymbol{y}}^{[1]}, \dots, \tilde{\boldsymbol{y}}^{[L]}\right)$ where $\tilde{\boldsymbol{y}}^{[\ell]} := \psi(\hat{\boldsymbol{y}}^{[\ell]})$ for $\ell = 1, \dots, L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

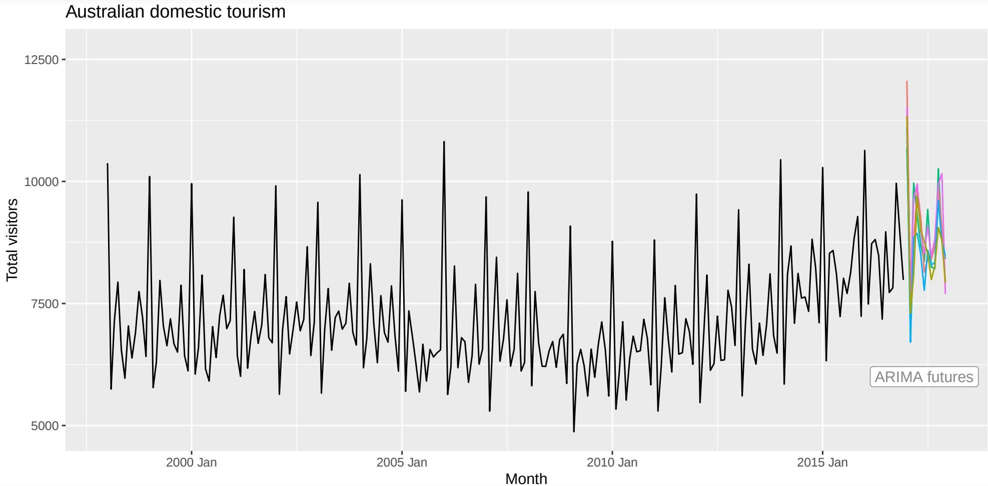
- So reconciling sample paths from incoherent distributions works.

Outline

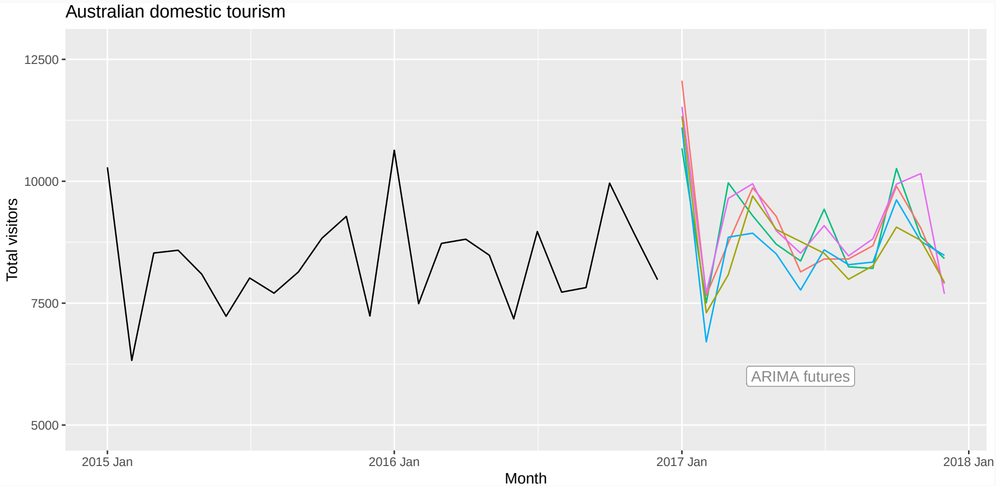
Evaluating probabilistic forecasts



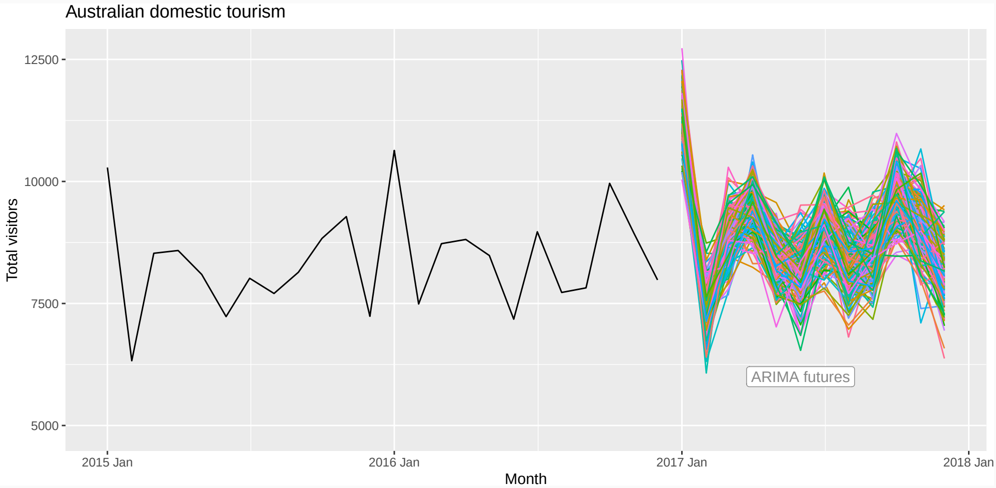
Evaluating probabilistic forecasts



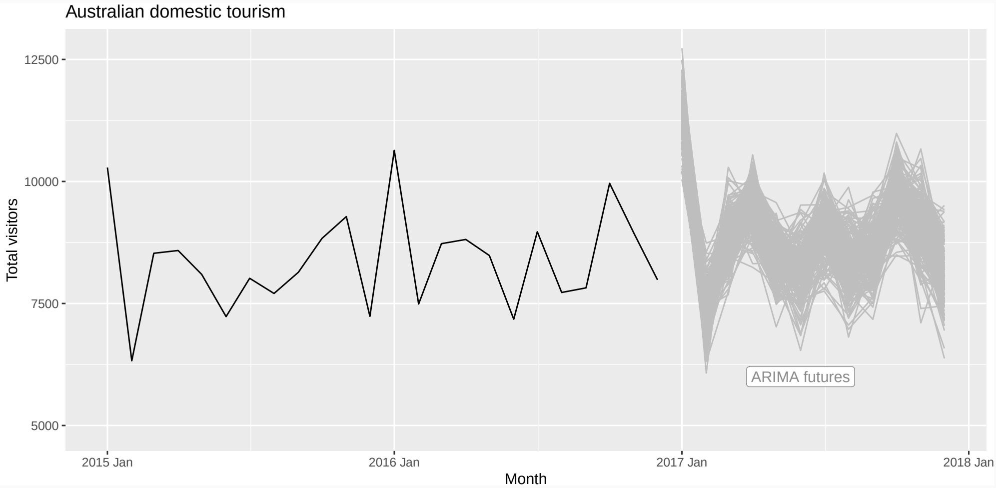
Evaluating probabilistic forecasts



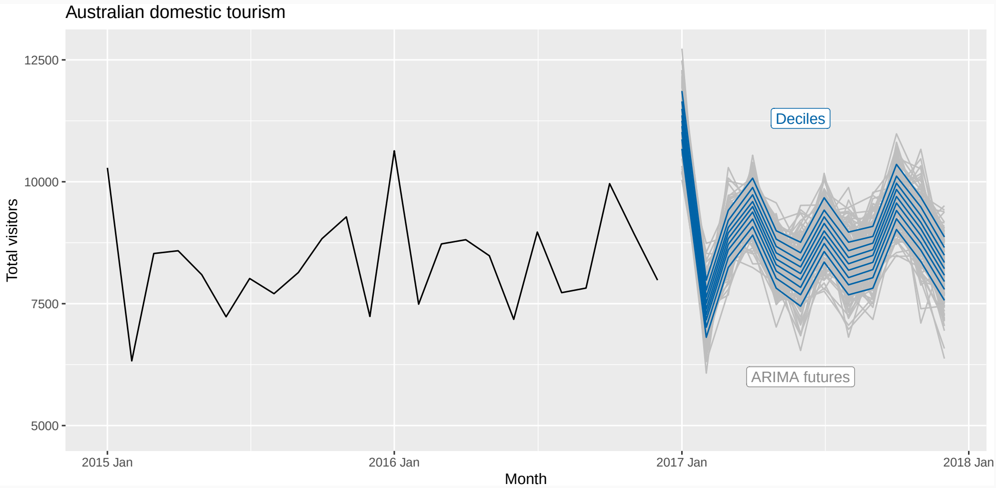
Evaluating probabilistic forecasts



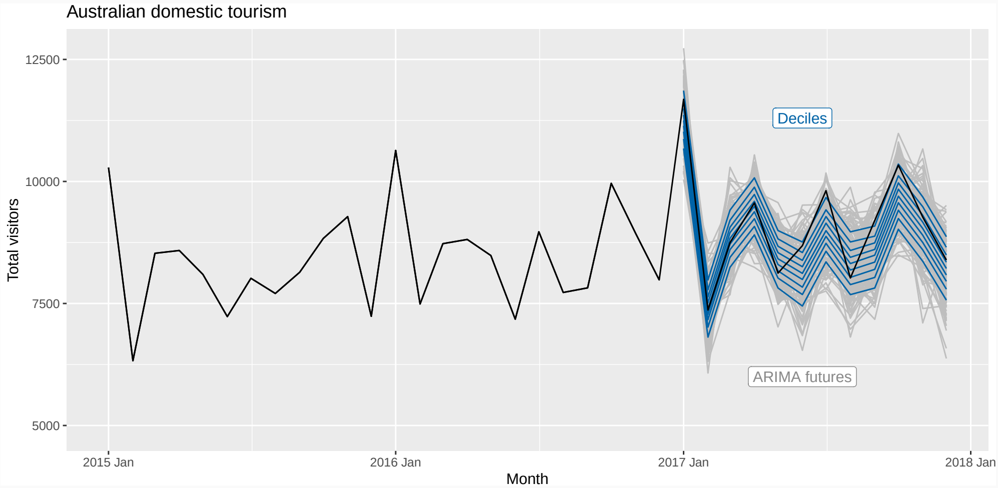
Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



Evaluating probabilistic forecasts



Evaluating probabilistic forecasts

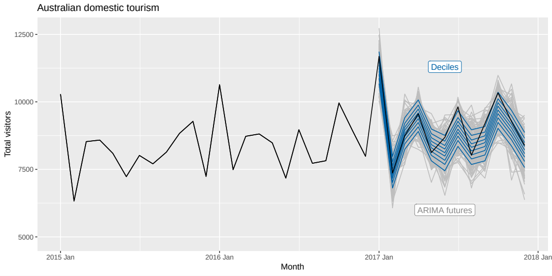
$q_{p,t}$ = quantile forecast with prob. p at time t.

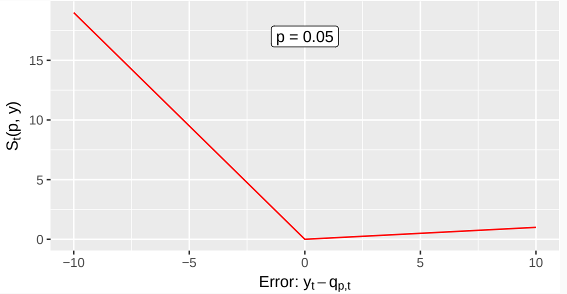
y_t = observation at time t

Quantile score

$$S_t(p, y) = \begin{cases} 2(1-p)|y_t - q_{p,t}|, & \text{if } y_t < q_{p,t} \\ 2p|y_t - q_{p,t}|, & \text{if } y_t \geq q_{p,t} \end{cases}$$

- Low S_t is good
- Multiplier of 2 often omitted, but useful for interpretation
- S_t like absolute error, weighted to account for likely exceedance
- Average $S_t(p, y)$ over p = CRPS (Continuous Rank Probability Score)





Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecsts)

Forecast distribution F_t and observation y_t.

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t)dp = \text{E}_F|Y - y_t| - \frac{1}{2}\text{E}_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t.
- Optimal when F_t is true distribution (i.e., it is a proper score)

Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecests)

Forecast distribution F_t and observation y_t.

$$\text{CRPS}(F_t, y_t) = \int_0^1 S_{p,t}(p, y_t)dp = \text{E}_F|Y - y_t| - \frac{1}{2}\text{E}_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t.
- Optimal when F_t is true distribution (i.e., it is a proper score)

Energy score (multivariate forecasts)

- $\text{ES}(F_t, \boldsymbol{y}_t) = \text{E}_F||\boldsymbol{Y} - \boldsymbol{y}_t|| - \frac{1}{2}\text{E}_F||\boldsymbol{Y} - \boldsymbol{Y}^*||$

Evaluating probabilistic forecasts

Continuous Rank Probability Score (univariate forecsts)

Forecast distribution F_t and observation y_t.

$$\mathrm{CRPS}(F_t, y_t) \;=\; \int_0^1 S_{p,t}(p, y_t)dp \;=\; \mathrm{E}_F|Y - y_t| - \frac{1}{2}\mathrm{E}_F|Y - Y^*|$$

- Y and Y^* are iid draws from F_t.
- Optimal when F_t is true distribution (i.e., it is a proper score)

Energy score (multivariate forecasts)

- $\mathrm{ES}(F_t, \boldsymbol{y}_t) = \mathrm{E}_F||\boldsymbol{Y} - \boldsymbol{y}_t|| - \frac{1}{2}\mathrm{E}_F||\boldsymbol{Y} - \boldsymbol{Y}^*||$

Log score (multivariate forecasts)

- $\mathrm{LS}(F_t, \boldsymbol{y}_t) = -\log f(\boldsymbol{y}_t)$

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule	Coherent v Incoherent	Coherent v Coherent
Log Score	Not proper	• Ordering preserved if compared using bottom-level only
Energy Score	Proper	• Full hierarchy should be used. • Rankings may change otherwise.

To add from probabilistic paper

- Score optimal reconciliation
- Electricity example

Outline

1 Hierarchical and grouped time series

2 Forecast reconciliation using projections

3 Probabilistic forecast reconciliation

4 Evaluating probabilistic forecasts

5 Example: Australian tourism

Example: Australian tourism

```
tourism
```

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##         month state zone      region visitors
##         <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney      926.
##  2 1998 Feb NSW   Metro NSW Sydney      647.
##  3 1998 Mar NSW   Metro NSW Sydney      716.
##  4 1998 Apr NSW   Metro NSW Sydney      621.
##  5 1998 May NSW   Metro NSW Sydney      598.
##  6 1998 Jun NSW   Metro NSW Sydney      601.
##  7 1998 Jul NSW   Metro NSW Sydney      720.
##  8 1998 Aug NSW   Metro NSW Sydney      645.
##  9 1998 Sep NSW   Metro NSW Sydney      633.
## 10 1998 Oct NSW   Metro NSW Sydney      771.
```

Example: Australian tourism

```r
tourism_agg <- tourism %>%
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
## # A tsibble: 26,400 x 5 [1M]
## # Key:       state, zone, region [110]
##       month state        zone         region       visitors
##       <mth> <chr>        <chr>        <chr>           <dbl>
##  1 1998 Jan <aggregated> <aggregated> <aggregated>   10376.
##  2 1998 Feb <aggregated> <aggregated> <aggregated>    5746.
##  3 1998 Mar <aggregated> <aggregated> <aggregated>    7129.
##  4 1998 Apr <aggregated> <aggregated> <aggregated>    7939.
##  5 1998 May <aggregated> <aggregated> <aggregated>    6552.
##  6 1998 Jun <aggregated> <aggregated> <aggregated>    5969.
##  7 1998 Jul <aggregated> <aggregated> <aggregated>    7041.
##  8 1998 Aug <aggregated> <aggregated> <aggregated>    6382.
##  9 1998 Sep <aggregated> <aggregated> <aggregated>    6907.
```

Example: Australian tourism

```r
fit <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(ets = ETS(visitors))
```

```
## # A mable: 110 x 4
## # Key:     state, zone, region [110]
##    state zone            region                ets
##    <chr> <chr>           <chr>             <model>
##  1 NSW   <aggregated>    <aggregated> <ETS(M,N,A)>
##  2 NSW   Metro NSW       <aggregated> <ETS(M,N,A)>
##  3 NSW   North Coast NSW <aggregated> <ETS(M,N,M)>
##  4 NSW   South Coast NSW <aggregated> <ETS(A,N,A)>
##  5 NSW   South NSW       <aggregated> <ETS(M,N,M)>
##  6 NSW   North NSW       <aggregated> <ETS(M,N,A)>
##  7 NSW   ACT             <aggregated> <ETS(M,N,A)>
##  8 NSW   Metro NSW       Sydney       <ETS(M,N,A)>
```

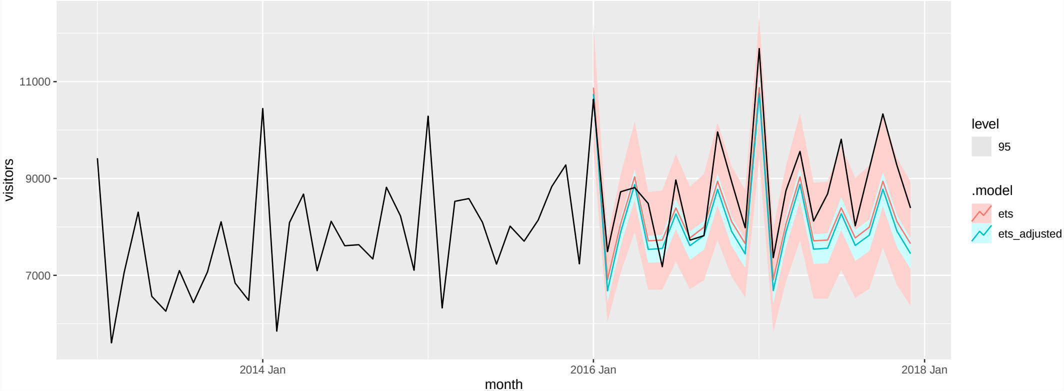
Example: Australian tourism

```
fc <- fit %>%
  reconcile(ets_adjusted = min_trace(ets)) %>%
  forecast(h = "2 years")
```

```
## # A fable: 5,280 x 7 [1M]
## # Key:     state, zone, region, .model [220]
##    state zone         region       .model    month        visitors .mean
##    <chr> <chr>        <chr>        <chr>     <mth>          <dist> <dbl>
##  1 NSW   <aggregated> <aggregated> ets    2016 Jan N(3679, 71136) 3679.
##  2 NSW   <aggregated> <aggregated> ets    2016 Feb N(2241, 27912) 2241.
##  3 NSW   <aggregated> <aggregated> ets    2016 Mar N(2602, 37643) 2602.
##  4 NSW   <aggregated> <aggregated> ets    2016 Apr N(3027, 50976) 3027.
##  5 NSW   <aggregated> <aggregated> ets    2016 May N(2504, 36795) 2504.
##  6 NSW   <aggregated> <aggregated> ets    2016 Jun N(2447, 36005) 2447.
##  7 NSW   <aggregated> <aggregated> ets    2016 Jul N(2734, 44488) 2734.
##  8 NSW   <aggregated> <aggregated> ets    2016 Aug N(2496, 38775) 2496.
```

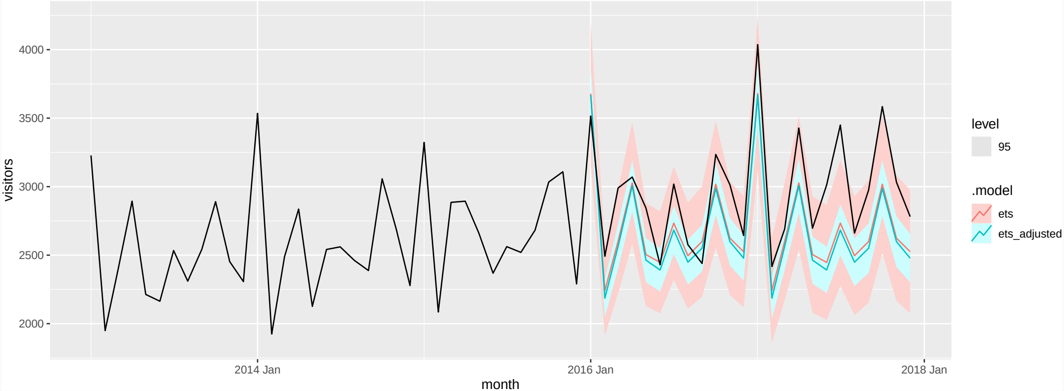
Example: Australian tourism

```r
fc %>%
  filter(is_aggregated(state)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



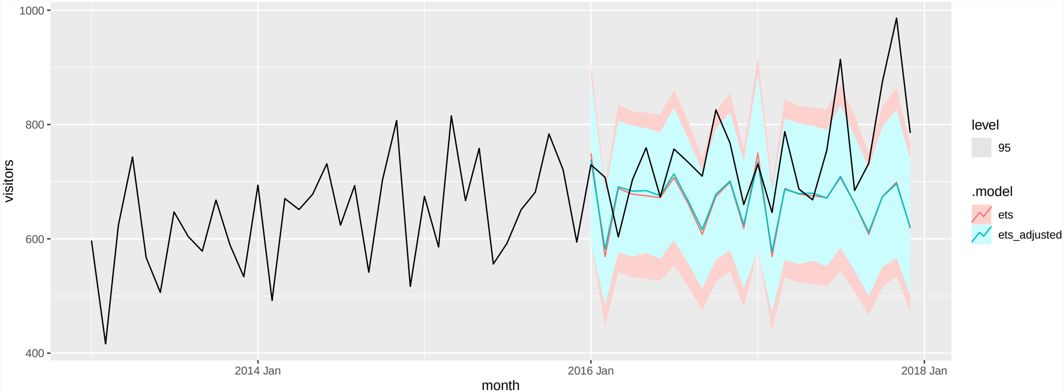
Example: Australian tourism

```
fc %>%
  filter(state == "NSW" & is_aggregated(zone)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



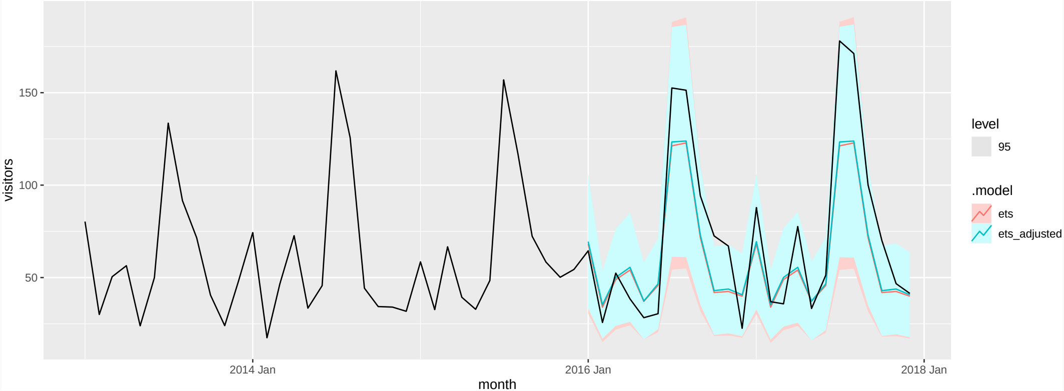
Example: Australian tourism

```r
fc %>%
  filter(region == "Melbourne") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



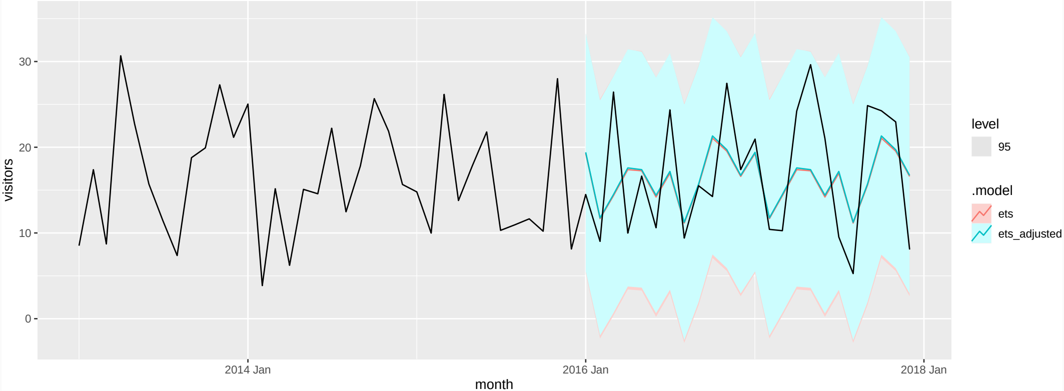
Example: Australian tourism

```
fc %>%
  filter(region == "Snowy Mountains") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



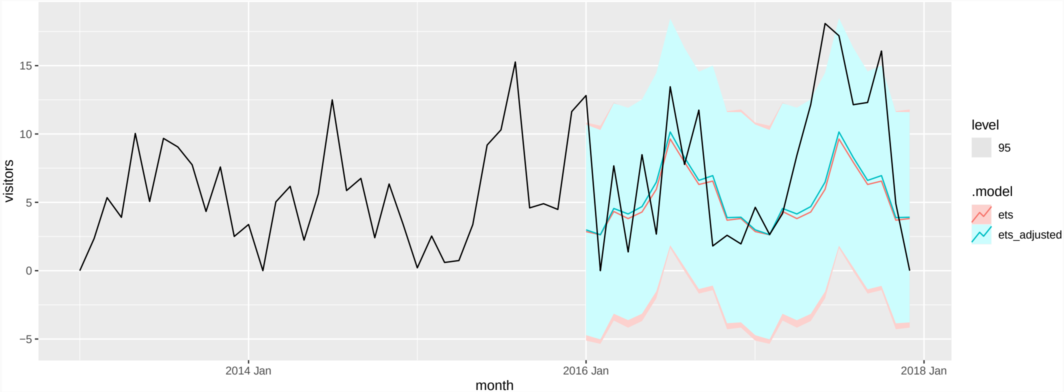
Example: Australian tourism

```
fc %>%
  filter(region == "Barossa") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



Example: Australian tourism

```
fc %>%
  filter(region == "MacDonnell") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



Example: Australian tourism

```
fc <- tourism_agg %>%
  filter_index(. ~ "2015 Dec") %>%
  model(
    ets = ETS(visitors),
    arima = ARIMA(visitors)
  ) %>%
  mutate(
    comb = (ets + arima) / 2
  ) %>%
  reconcile(
    ets_adj = min_trace(ets),
    arima_adj = min_trace(arima),
    comb_adj = min_trace(comb)
  ) %>%
  forecast(h = "2 years")
```

Forecast evaluation

```r
fc %>% accuracy(tourism_agg)
```

```
## # A tibble: 660 x 12
##    .model state        zone         region       .type     ME
##    <chr>  <chr>        <chr>        <chr>        <chr>  <dbl>
##  1 arima  NSW          <aggregat~ <aggregat~ Test   187.
##  2 arima  NSW          Metro NSW~ <aggregat~ Test    81.9
##  3 arima  NSW          North Coa~ <aggregat~ Test    62.2
##  4 arima  NSW          South Coa~ <aggregat~ Test    26.9
##  5 arima  NSW          South NSW~ <aggregat~ Test    18.3
##  6 arima  NSW          North NSW~ <aggregat~ Test    64.5
##  7 arima  NSW          ACT        ~ <aggregat~ Test    46.2
##  8 arima  NSW          Metro NSW~ Sydney     ~ Test    76.8
##  9 arima  NSW          Metro NSW~ Central C~ Test     8.00
## 10 arima  NSW          North Coa~ Hunter     ~ Test    42.5
## # ... with 650 more rows, and 6 more variables: RMSE <dbl>,
```

Forecast evaluation

```r
fc %>%
  accuracy(tourism_agg) %>%
  group_by(.model) %>%
  summarise(MASE = mean(MASE)) %>%
  arrange(MASE)
```

```
## # A tibble: 6 x 2
##   .model      MASE
##   <chr>      <dbl>
## 1 ets_adj    0.964
## 2 ets        0.968
## 3 comb_adj   0.973
## 4 comb       0.984
## 5 arima_adj  1.03
```