

Ten years of forecast reconciliation

Rob J Hyndman ISF 2020



MONASH University

Photo by Edvard Alexander Rølvaag on Unsplash

robjhyndman.com/isf2020

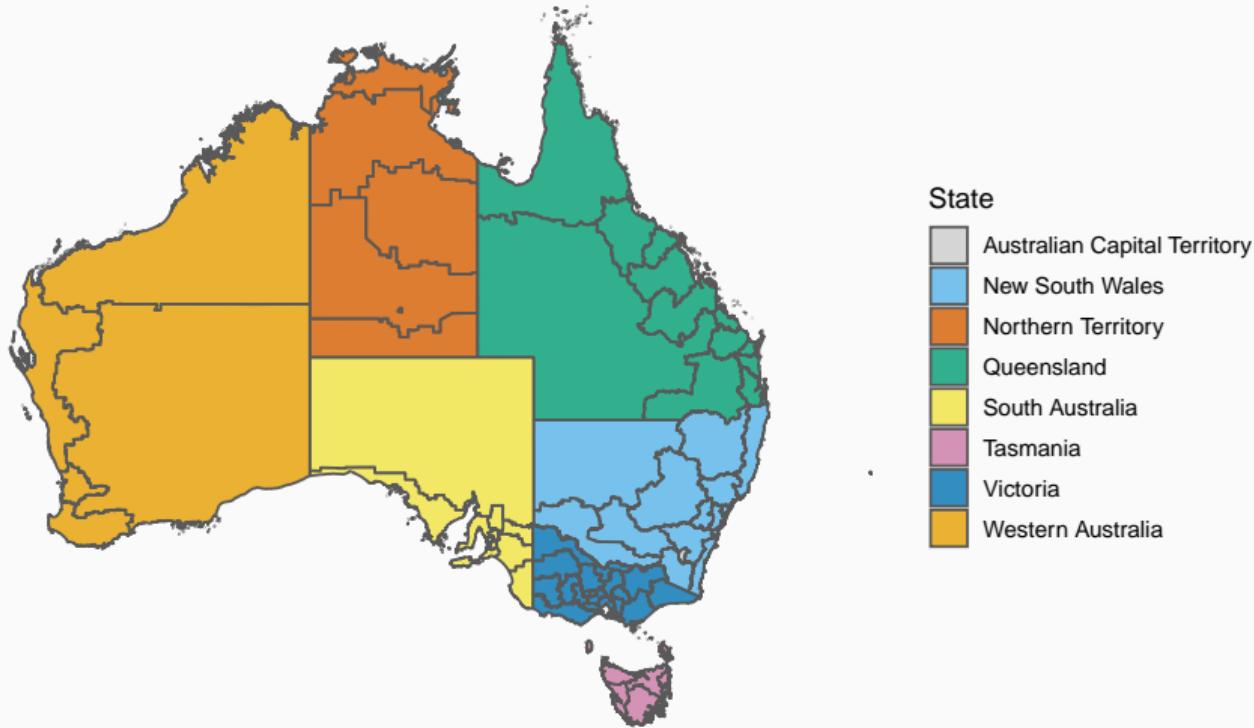
Outline

- 1 Hierarchical forecasting 20 years ago
- 2 Point forecast reconciliation
- 3 Example: Australian tourism
- 4 Probabilistic forecast reconciliation
- 5 Example: Australian electricity generation
- 6 Extensions

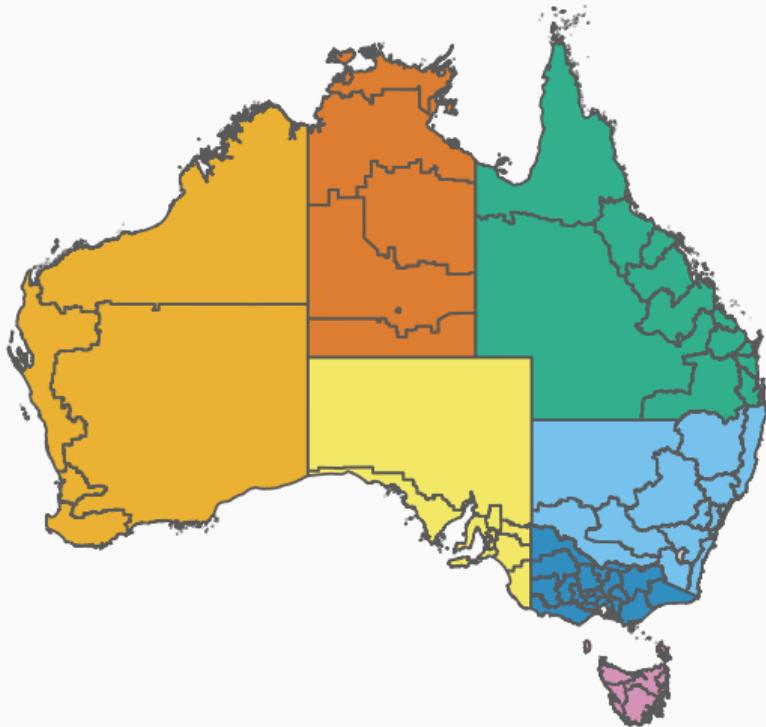
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Australian tourism regions



Australian tourism regions



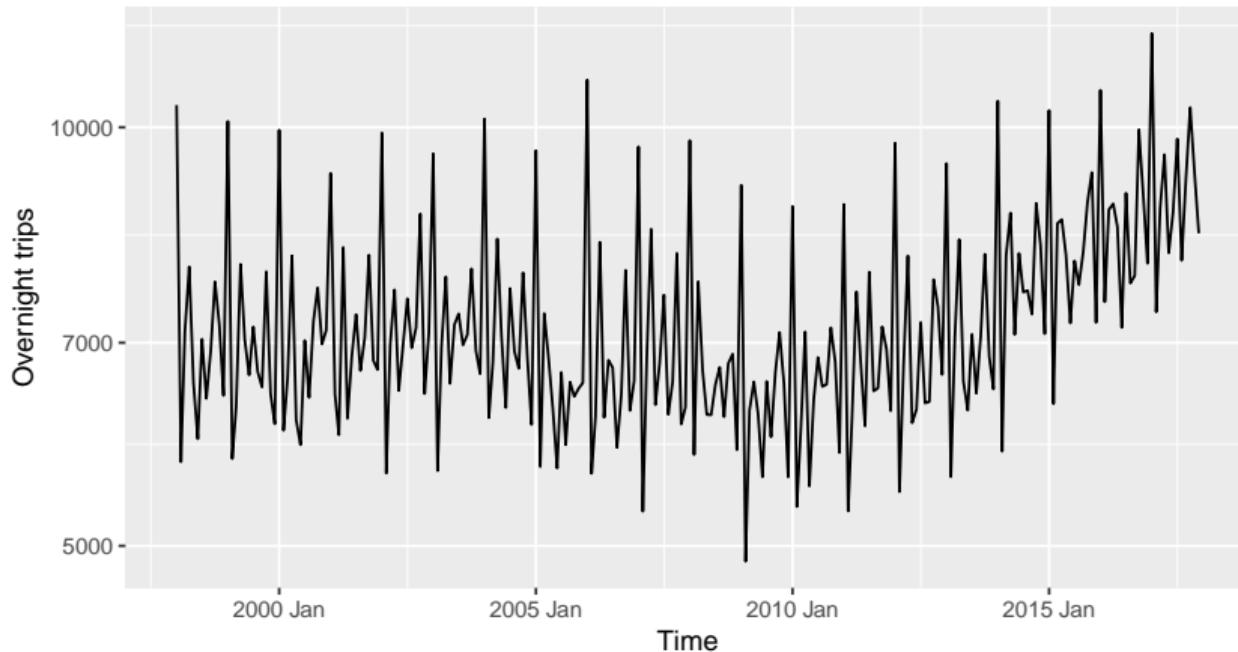
- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
 - ▶ 7 states
 - ▶ 27 zones
 - ▶ 75 regions

Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:      state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>      <dbl>
## 1 1998 Jan NSW Metro NSW Sydney      926.
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## 7 1998 Jul NSW Metro NSW Sydney      720.
## 8 1998 Aug NSW Metro NSW Sydney      645.
## 9 1998 Sep NSW Metro NSW Sydney      633.
## 10 1998 Oct NSW Metro NSW Sydney      771.
## # ... with 17,990 more rows
```

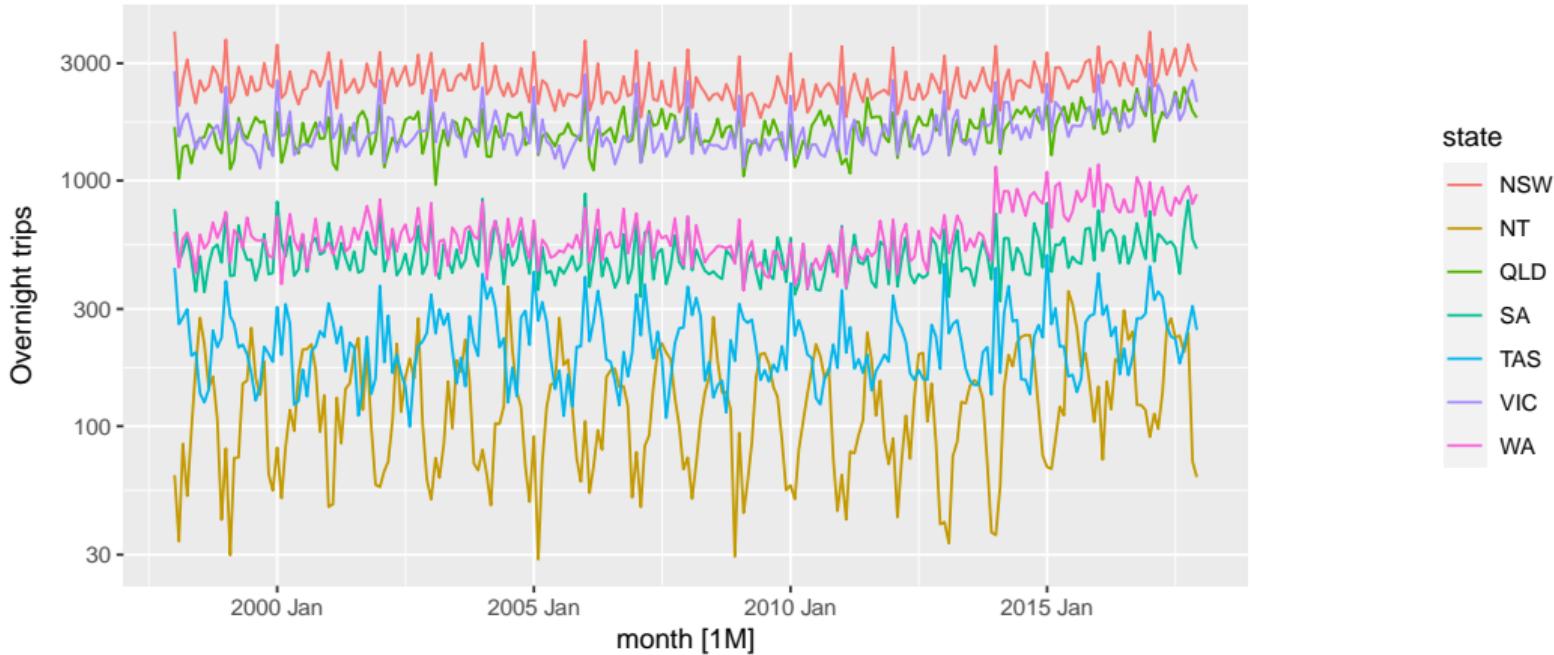
Australian tourism data

Total domestic travel: Australia



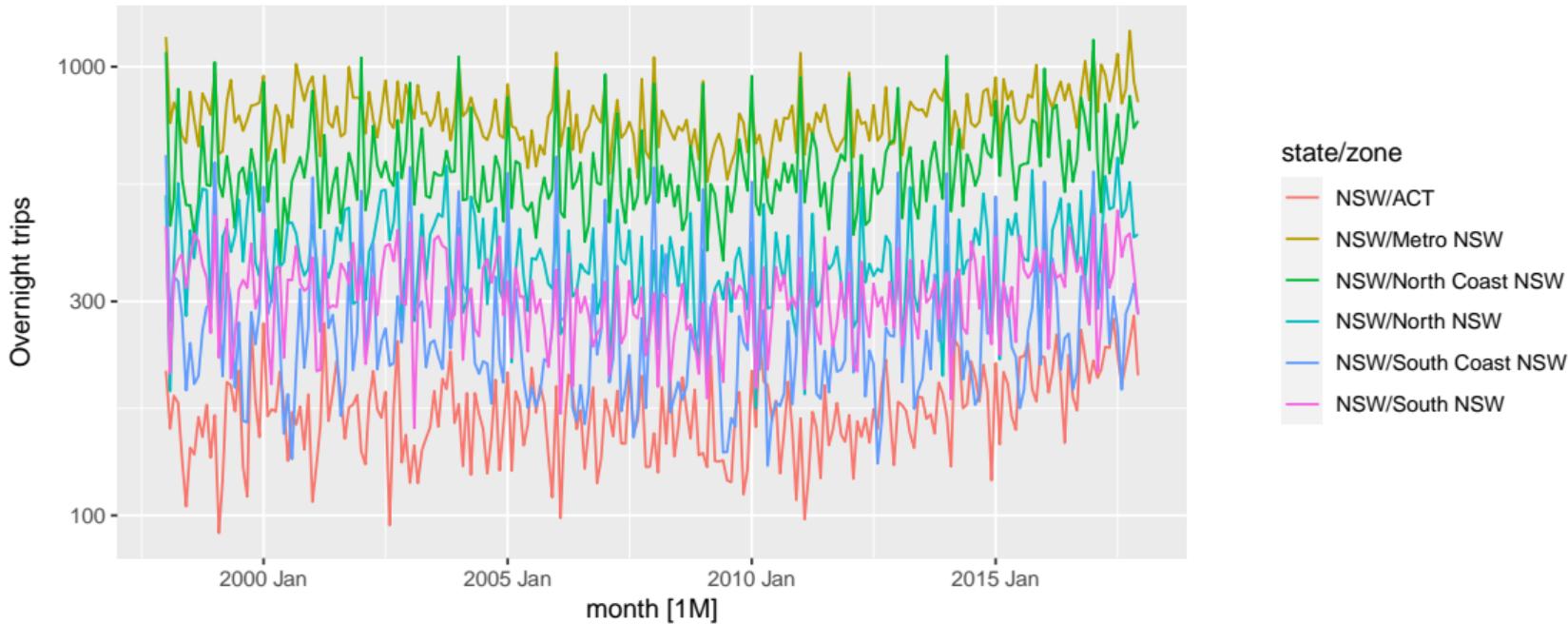
Australian tourism data

Total domestic travel: by state



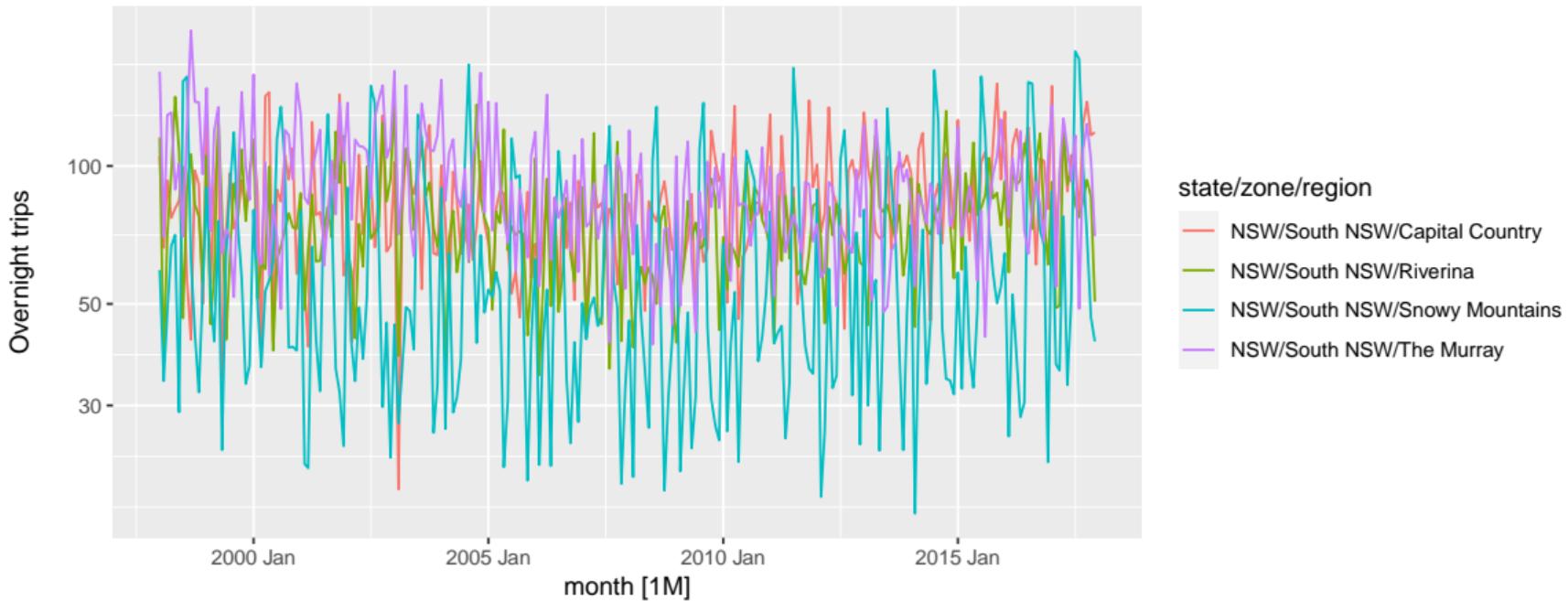
Australian tourism data

Total domestic travel: NSW by zone

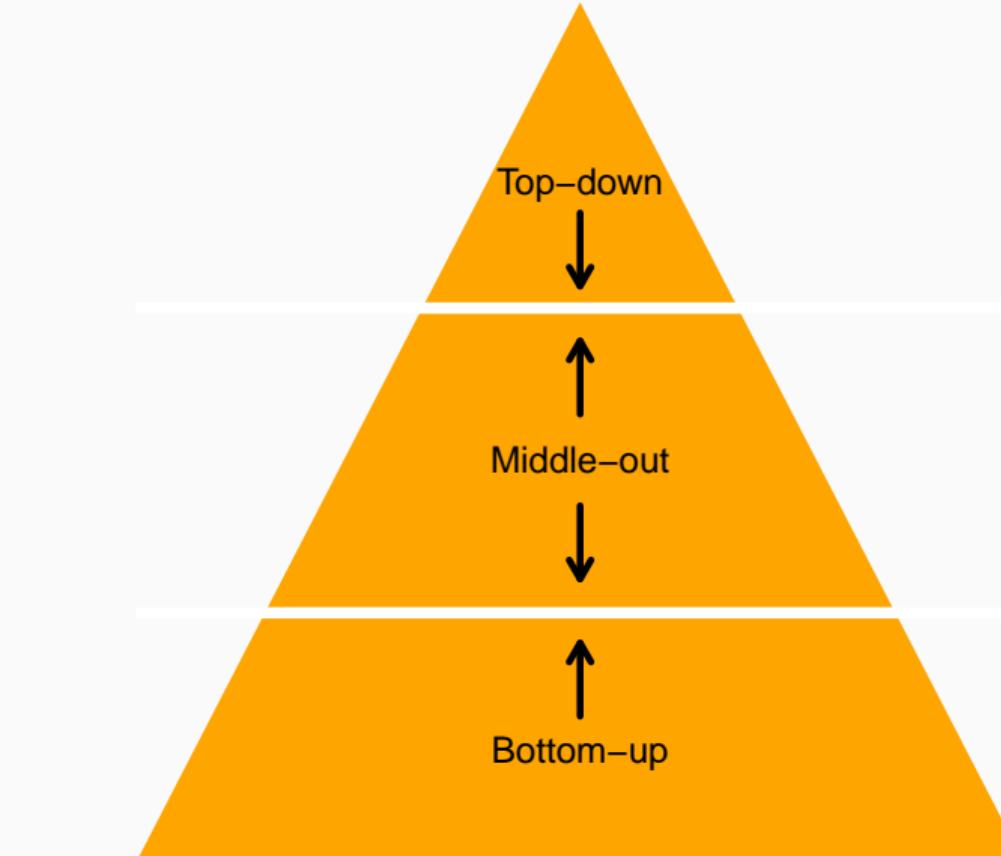
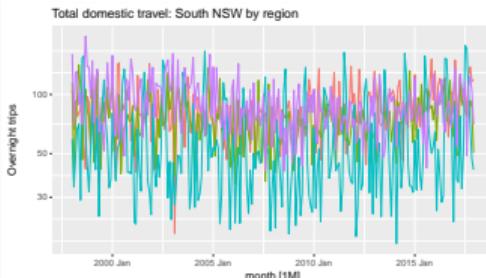
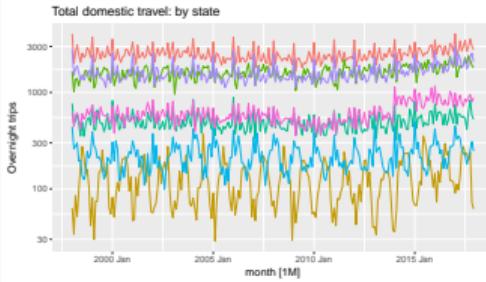
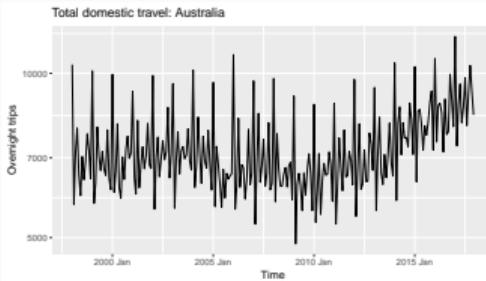


Australian tourism data

Total domestic travel: South NSW by region



Hierarchical forecasting 20 years ago



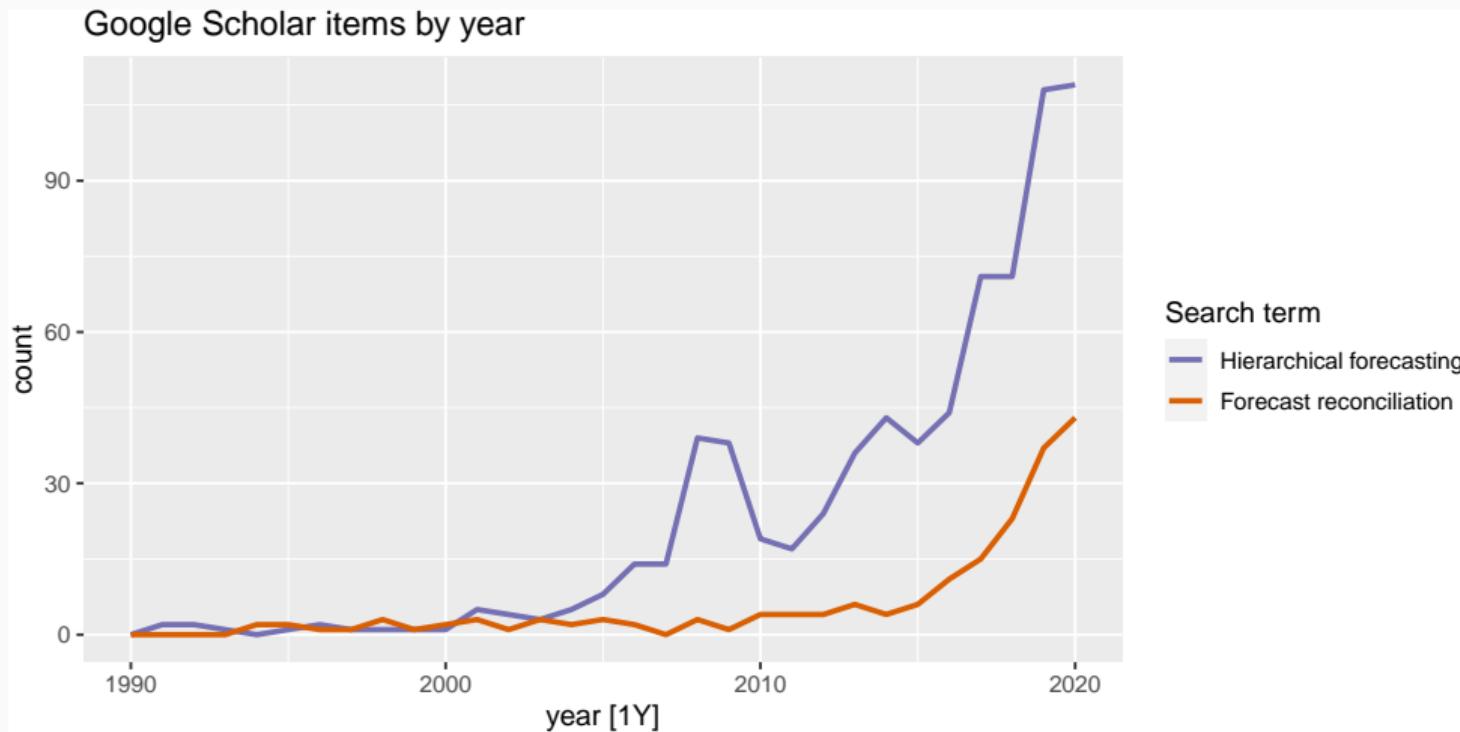
Forecast reconciliation

- Forecast all series at all levels of aggregation.
- Reconcile forecasts using least squares optimization.

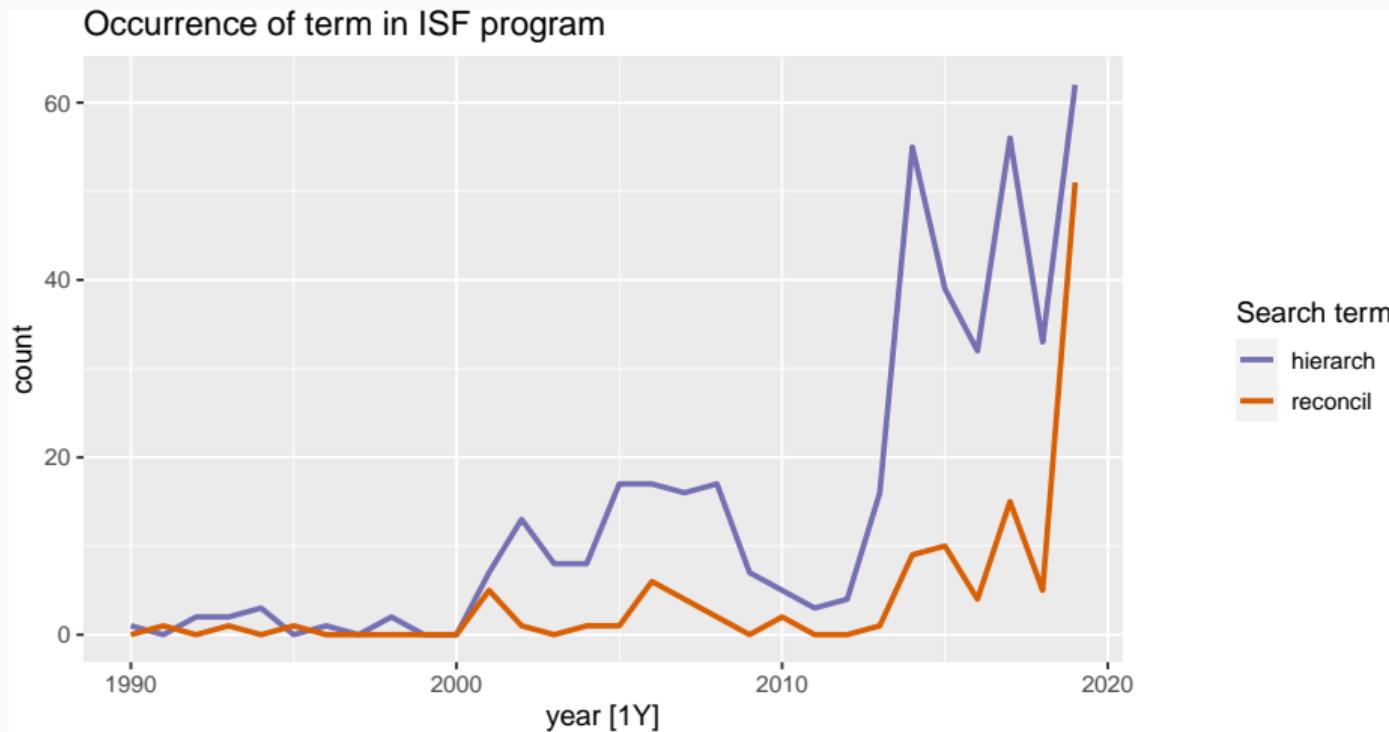
History

- 2001:** Idea to use all available series to forecast Australia's labour market by occupation.
- 2004:** PhD student Roman Ahmed begins, co-supervised with George Athanasopoulos.
- 2006:** Presentation at ISF, Santander.
- 2007:** Pre-print of "Optimal combination forecasts for hierarchical time series".
- 2009:** Application to Australian tourism published in IJF.
- 2010:** First version of hts package on CRAN.
- 2011:** "Optimal combination forecasts for hierarchical time series" appears in CSDA.

Forecast reconciliation research



Forecast reconciliation research



Outline

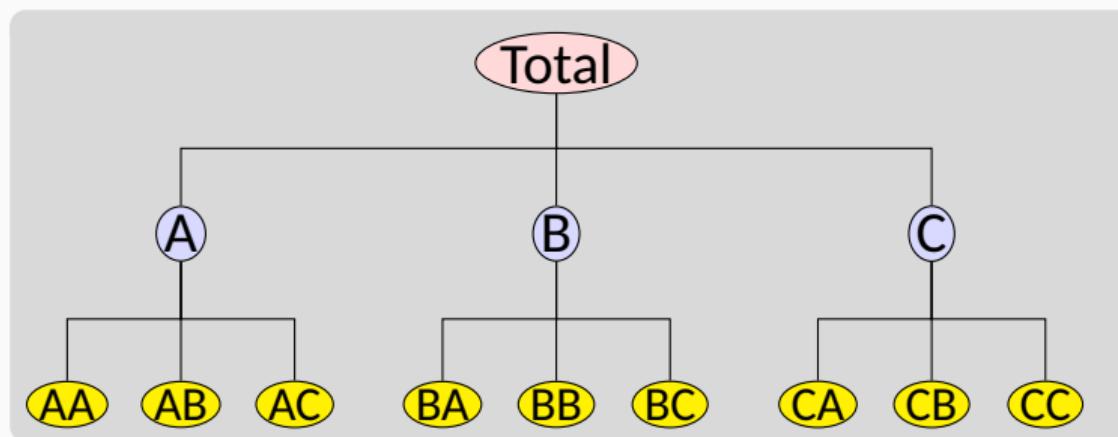
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Point forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 CSDA) Optimal combination forecasts for hierarchical time series.
- Hyndman, Lee, Wang (2016 CSDA) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 JASA) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 IJF) Forecast reconciliation: A geometric view with new insights on bias correction.

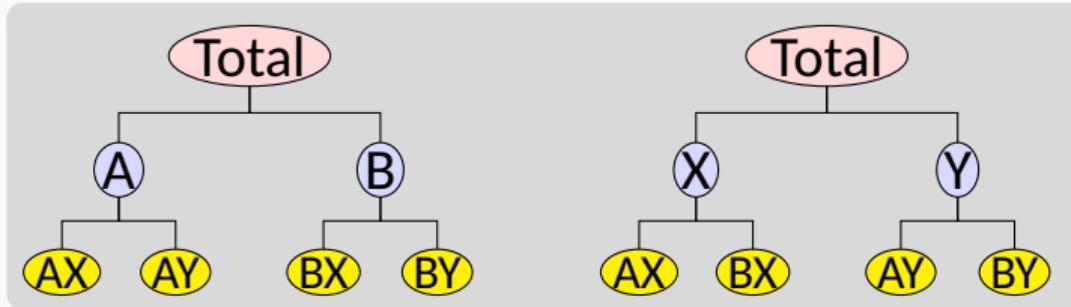
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



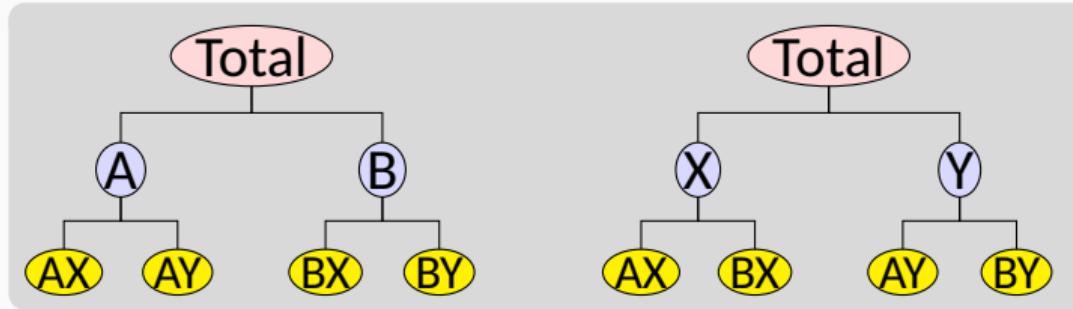
Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



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Examples

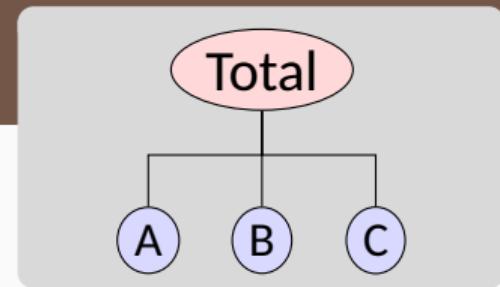
- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Hierarchical and grouped time series

Every collection of time series with linear constraints can be written as

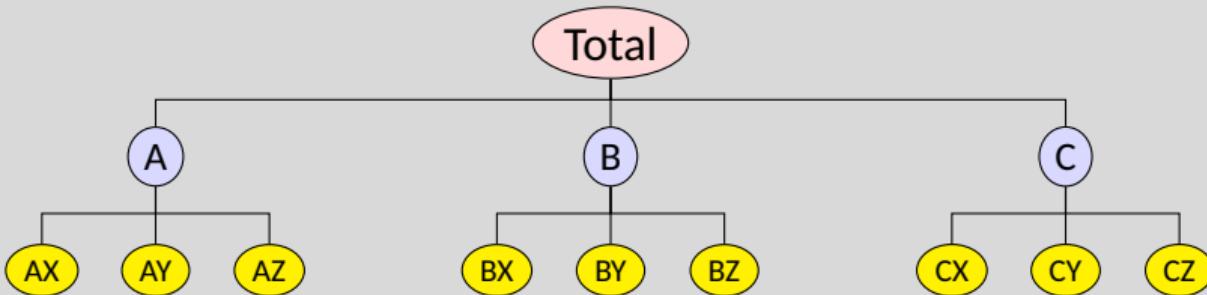
$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

- \mathbf{y}_t = vector of all series at time t
- y_t = aggregate of all series at time t .
- $y_{X,t}$ = value of series X at time t .
- \mathbf{b}_t = vector of most disaggregated series at time t
- \mathbf{S} = “summing matrix” containing the linear constraints.

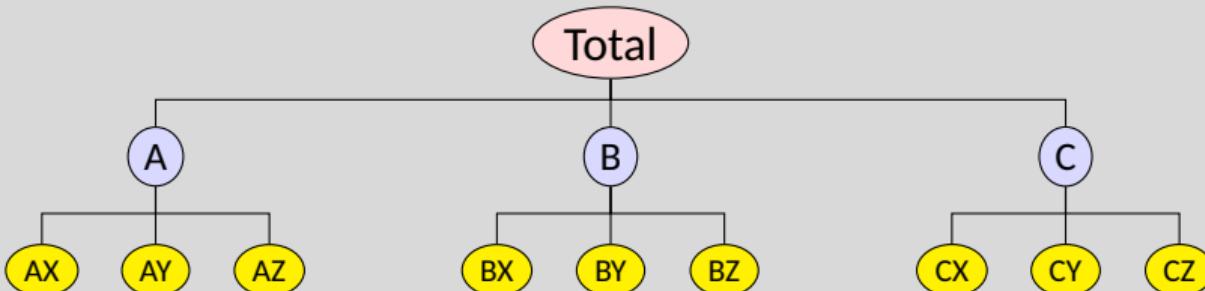


$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t} \end{aligned}$$

Hierarchical time series

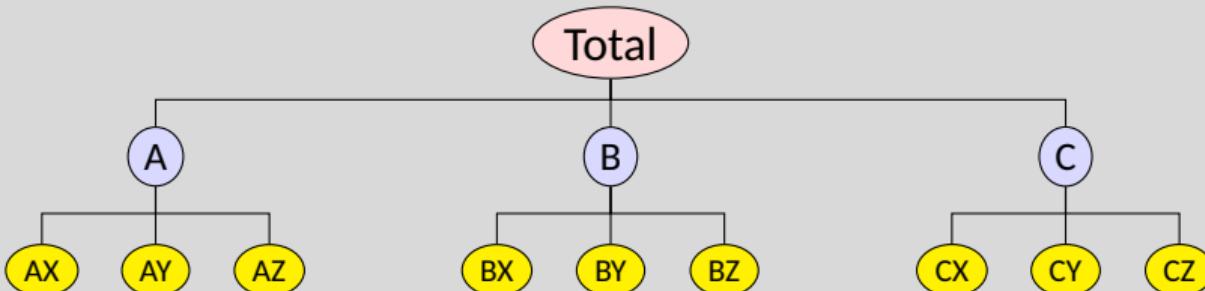


Hierarchical time series



$$\mathbf{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

Hierarchical time series



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Grouped data



Grouped data



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$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

Definitions

Coherent subspace

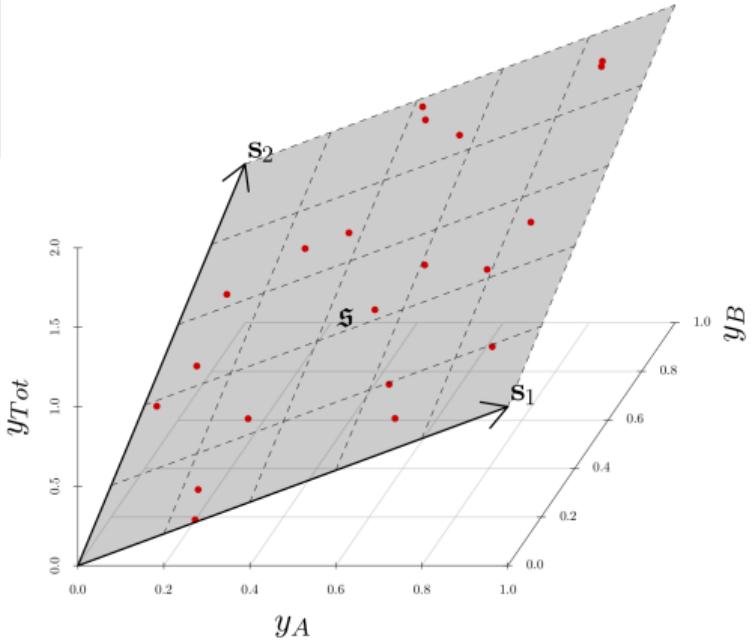
m -dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\mathbf{y} \in \mathfrak{s}$.

Hierarchical time series

An n -dimensional multivariate time series such that $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$ is coherent if $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$.



$$Y_{Tot} = Y_A + Y_B$$

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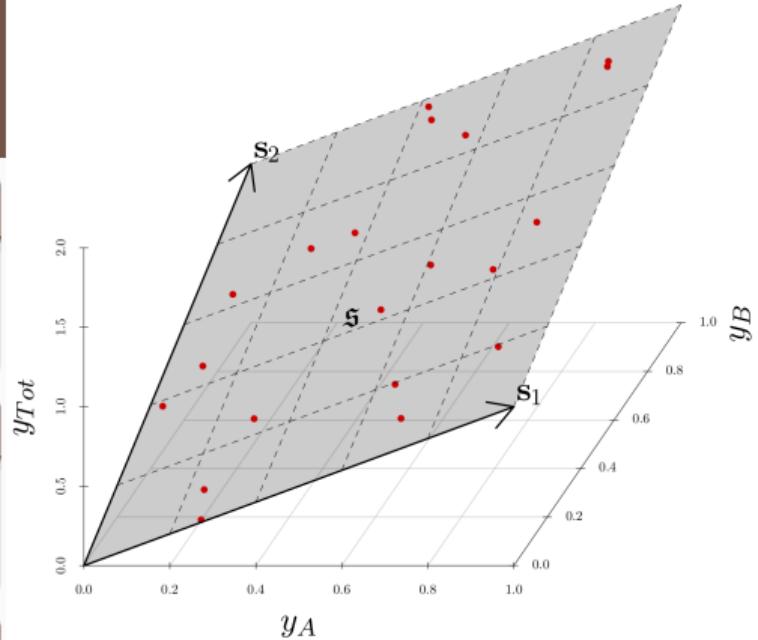
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Base forecasts

Let $\hat{\mathbf{y}}_{t+h|t}$ be vector of *incoherent* initial h -step forecasts.



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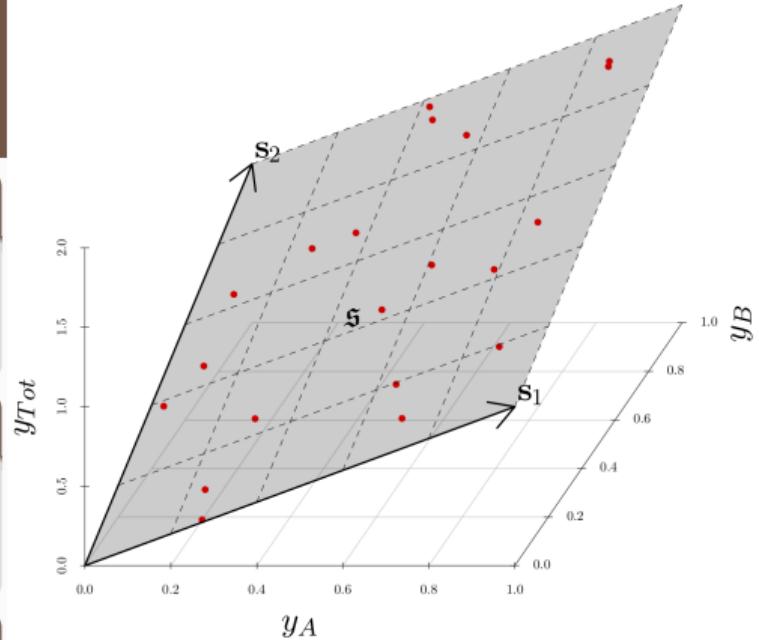
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Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$.
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$ “reconciles” $\hat{\mathbf{y}}_{t+h|t}$.

Linear reconciliation

If ψ is a linear function, then $\tilde{\mathbf{y}}_{t+h|t} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{t+h|t}$

- \mathbf{G} combines base forecasts $\hat{\mathbf{y}}_{T+h|T}$ to get bottom-level forecasts.
- \mathbf{S} creates linear combinations.

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Mean

$$E[\tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] = E[\mathbf{y}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$$

provided $\mathbf{S}\mathbf{G}\mathbf{S}' = \mathbf{S}$ and

$$E[\hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] = E[\mathbf{y}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$$

i.e., reconciled forecasts are unbiased if base forecasts are unbiased and $\mathbf{S}\mathbf{G}$ is a projection.

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Variance

$$\begin{aligned} \mathbf{V}_h &= \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] \\ &= \mathbf{S}\mathbf{G}\mathbf{W}_h\mathbf{G}'\mathbf{S}' \end{aligned}$$

where

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where

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Minimum trace (MinT) reconciliation

If $\mathbf{S}\mathbf{G}$ is a projection, then the trace of \mathbf{V}_h is minimized when

$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}$$

Linear projections

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$

Reconciliation method \mathbf{G}

OLS	$(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
WLS(var)	$(\mathbf{S}'\Lambda_v\mathbf{S})^{-1}\mathbf{S}'\Lambda_v$
WLS(struct)	$(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s$
MinT(sample)	$(\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}$
MinT(shrink)	$(\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\mathbf{W}_h = k_h \mathbf{W}_1$.

- $\Lambda_v = \text{diag}(\mathbf{W}_1)^{-1}$
- $\Lambda_s = \text{diag}(\mathbf{S}\mathbf{1})^{-1}$
- $\hat{\mathbf{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\mathbf{W}}_{\text{shr}}$ is shrinkage estimator $\tau \text{diag}(\hat{\mathbf{W}}_{\text{sam}}) + (1 - \tau)\hat{\mathbf{W}}_{\text{sam}}$
where τ selected optimally.

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Example: Australian tourism

tourism

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## # A tsibble: 18,000 x 5 [1M]
## # Key:      state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>      <dbl>
## 1 1998 Jan NSW Metro NSW Sydney      926.
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## 3 1998 Mar NSW Metro NSW Sydney      716.
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## 6 1998 Jun NSW Metro NSW Sydney      601.
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## 10 1998 Oct NSW Metro NSW Sydney      771.
```

Example: Australian tourism

```
tourism_agg <- tourism %>%
  aggregate_key(state(zone/region, visitors = sum(visitors)))
```

```
## # A tsibble: 26,400 x 5 [1M]
## # Key:      state, zone, region [110]
##       month state      zone      region     visitors
##       <mth> <chr>    <chr>    <chr>      <dbl>
## 1 1998 Jan <aggregated> <aggregated> <aggregated> 10376.
## 2 1998 Feb <aggregated> <aggregated> <aggregated> 5746.
## 3 1998 Mar <aggregated> <aggregated> <aggregated> 7129.
## 4 1998 Apr <aggregated> <aggregated> <aggregated> 7939.
## 5 1998 May <aggregated> <aggregated> <aggregated> 6552.
## 6 1998 Jun <aggregated> <aggregated> <aggregated> 5969.
## 7 1998 Jul <aggregated> <aggregated> <aggregated> 7041.
## 8 1998 Aug <aggregated> <aggregated> <aggregated> 6382.
## 9 1998 Sep <aggregated> <aggregated> <aggregated> 6907.
```

Example: Australian tourism

```
fit <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(ets = ETS(visitors))

## # A mable: 110 x 4
## # Key:      state, zone, region [110]
##   state zone           region          ets
##   <chr> <chr>           <chr>          <model>
## 1 NSW   <aggregated> <aggregated> <ETS(M,N,A)>
## 2 NSW   Metro NSW     <aggregated> <ETS(M,N,A)>
## 3 NSW   North Coast NSW <aggregated> <ETS(M,N,M)>
## 4 NSW   South Coast NSW <aggregated> <ETS(A,N,A)>
## 5 NSW   South NSW     <aggregated> <ETS(M,N,M)>
## 6 NSW   North NSW     <aggregated> <ETS(M,N,A)>
## 7 NSW   ACT            <aggregated> <ETS(M,N,A)>
## 8 NSW   Metro NSW     Sydney          <ETS(M,N,A)>
```

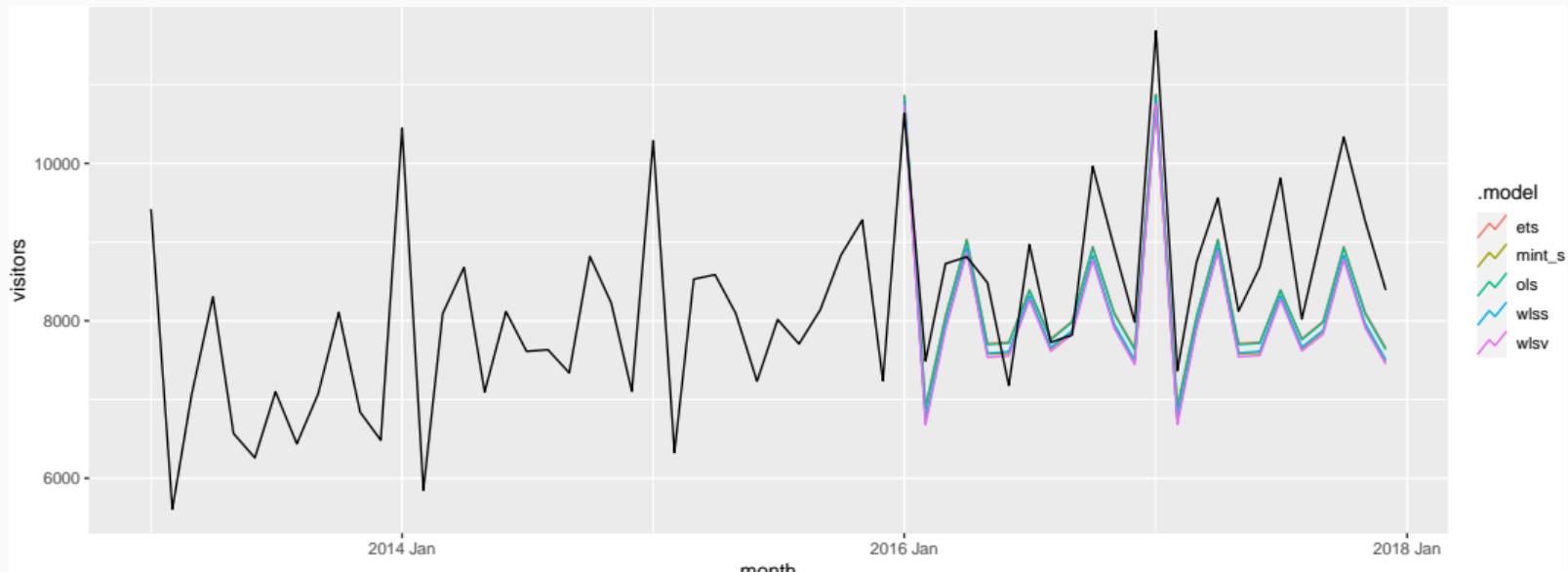
Example: Australian tourism

```
fc <- fit %>%
  reconcile(
    ols = min_trace(ets, method="ols"),
    wlsv = min_trace(ets, method="wls_var"),
    wlss = min_trace(ets, method="wls_struct"),
    #mint_c = min_trace(ets, method="mint_cov"),
    mint_s = min_trace(ets, method="mint_shrink"),
  ) %>%
  forecast(h = "2 years")
```

```
## # A fable: 5,280 x 7 [1M]
## # Key:      state, zone, region, .model [220]
##   state zone       region     .model     month   visitors .mean
##   <chr> <chr>       <chr>     <chr>     <mth>       <dist> <dbl>
## 1 NSW  <aggregated> <aggregated> ets     2016 Jan N(3679, 71136) 3679.
## 2 NSW  <aggregated> <aggregated> ets     2016 Feb N(2241, 27912) 2241.
```

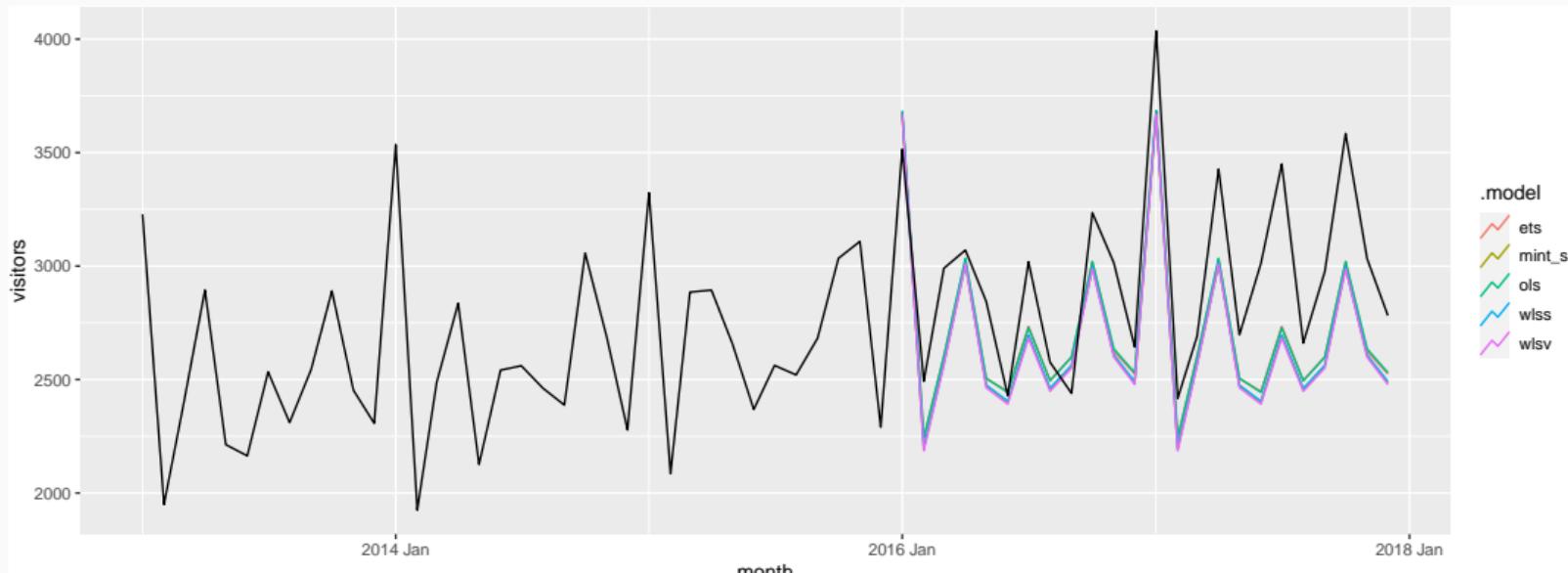
Example: Australian tourism

```
fc %>%
  filter(is_aggregated(state)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



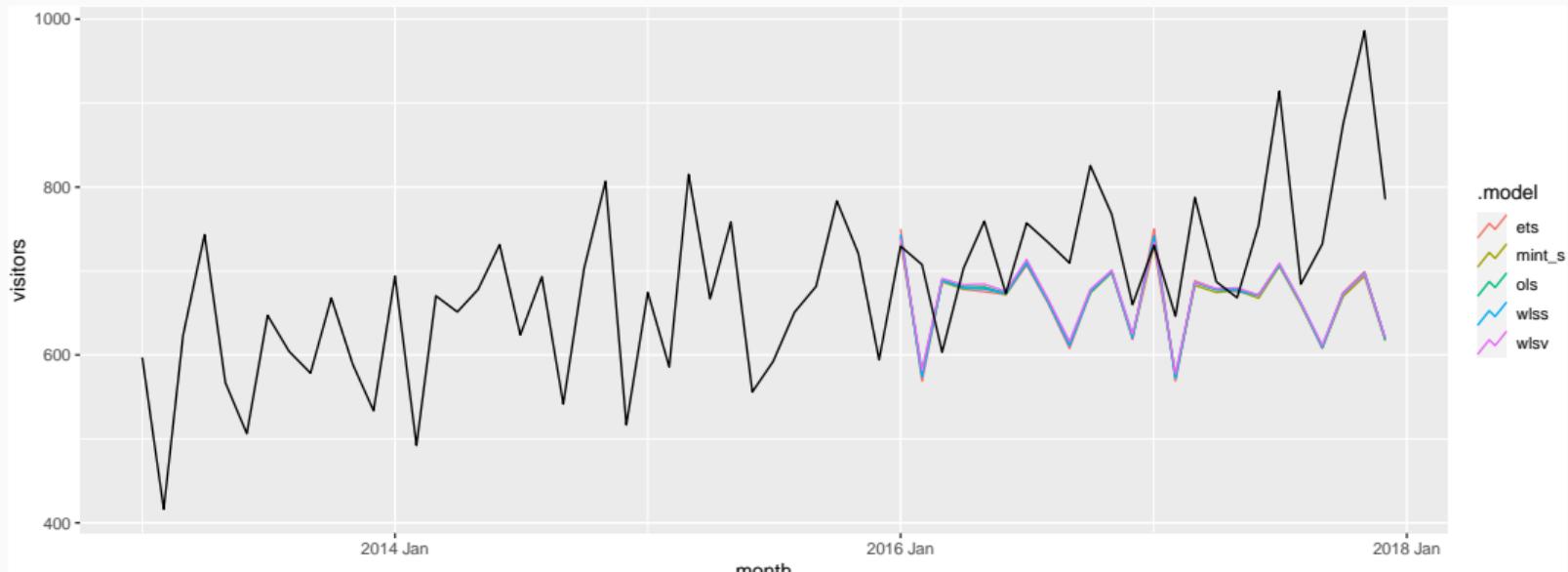
Example: Australian tourism

```
fc %>%
  filter(state == "NSW" & is_aggregated(zone)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



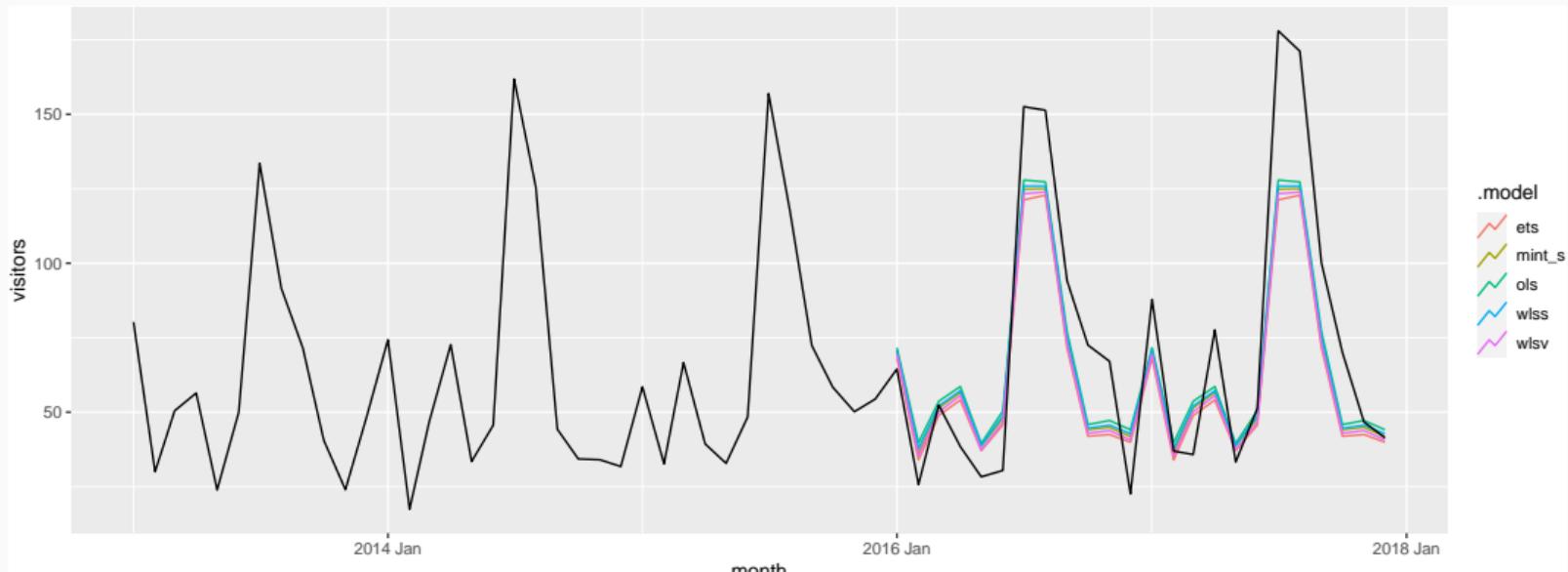
Example: Australian tourism

```
fc %>%
  filter(region == "Melbourne") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



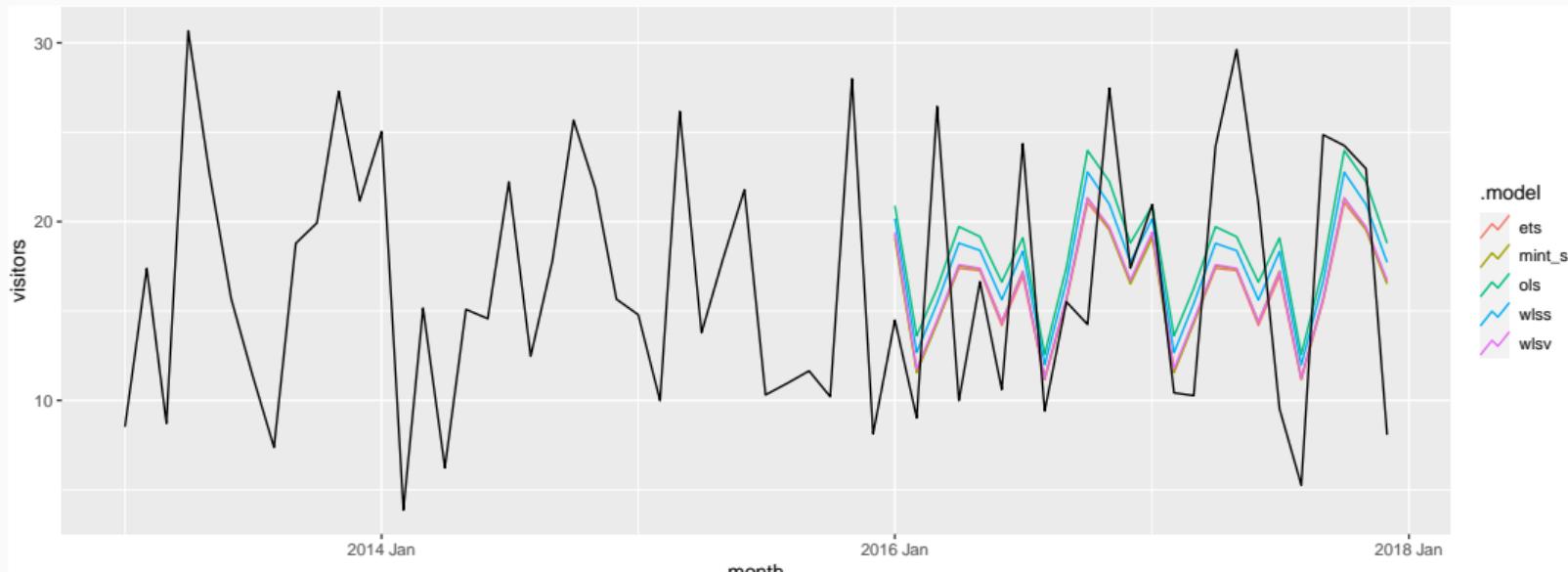
Example: Australian tourism

```
fc %>%
  filter(region == "Snowy Mountains") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Example: Australian tourism

```
fc %>%
  filter(region == "Barossa") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Example: Australian tourism

```
fc %>%  
  accuracy(data = tourism_agg,  
            measures = list(rmsse = RMSSE))
```

```
## # A tibble: 550 x 6  
##   .model state zone          region     .type rmsse  
##   <chr>   <chr> <chr>        <chr>     <chr> <dbl>  
## 1 ets     NSW   ACT         Canberra    Test    0.835  
## 2 ets     NSW   ACT         <aggregated> Test    0.835  
## 3 ets     NSW   Metro NSW Central Coast Test    0.747  
## 4 ets     NSW   Metro NSW Sydney      Test    1.16  
## 5 ets     NSW   Metro NSW <aggregated> Test    1.18  
## 6 ets     NSW   North Coast NSW Hunter    Test    1.21  
## 7 ets     NSW   North Coast NSW North Coast NSW Test    0.884  
## 8 ets     NSW   North Coast NSW <aggregated> Test    1.02  
## 9 ets     NSW   North NSW    Blue Mountains Test    1.02
```

Example: Australian tourism

```
fc %>%
  accuracy(tourism_agg,
            measures = list(mase = MASE, rmsse = RMSSE)) %>%
  group_by(.model) %>%
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) %>%
  arrange(rmsse)
```

```
## # A tibble: 5 x 3
##   .model    mase   rmsse
##   <chr>    <dbl>  <dbl>
## 1 ols      0.930  0.926
## 2 wlss     0.949  0.948
## 3 mint_s   0.953  0.954
## 4 wlsv     0.964  0.965
## 5 ets      0.968  0.968
```

Example: Australian tourism

```
fc %>%  
  accuracy(tourism_agg,  
            measures = list(mase = MASE, rmsse = RMSSE)) %>%  
  group_by(.model) %>%  
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) %>%  
  arrange(rmsse)
```

```
## # A tibble: 5 x 3  
##   .model    mase   rmsse  
##   <chr>    <dbl>  <dbl>  
## 1 ols      0.930  0.926  
## 2 wlss     0.949  0.948  
## 3 mint_s   0.953  0.954  
## 4 wlsv     0.964  0.965  
## 5 ets      0.968  0.968
```

- Overall, every reconciliation method is better than the base ETS forecasts.

Example: Australian tourism

```
## # A tibble: 20 x 4
## # Groups:   .model [5]
##   .model level    mase rmsse
##   <chr>   <fct>   <dbl> <dbl>
## 1 ets     National 1.44  1.27
## 2 ols     National 1.46  1.29
## 3 wlss    National 1.61  1.43
## 4 mint_s  National 1.64  1.45
## 5 wlsv    National 1.69  1.49
## 6 ols     State    1.07  1.08
## 7 ets     State    1.10  1.11
## 8 wlss    State    1.13  1.14
## 9 mint_s  State    1.15  1.15
## 10 wlsv   State    1.18  1.17
## 11 ols    Zone     0.954 0.948
## 12 wlss   Zone     0.987 0.980
## 13 mint_s Zone     0.995 0.988
## 14 ets    Zone     1.01  0.999
## 15 wlsv   Zone     1.01  1.00
## 16 ols    Region   0.901 0.895
## 17 wlss   Region   0.910 0.907
## 18 mint_s Region   0.911 0.911
## 19 wlsv   Region   0.917 0.919
## 20 ets    Region   0.935 0.938
```

Example: Australian tourism

```
## # A tibble: 20 x 4
## # Groups:   .model [5]
##   .model level    mase rmsse
##   <chr>   <fct>   <dbl> <dbl>
## 1 ets     National 1.44  1.27
## 2 ols     National 1.46  1.29
## 3 wlss    National 1.61  1.43
## 4 mint_s  National 1.64  1.45
## 5 wlsv    National 1.69  1.49
## 6 ols     State    1.07  1.08
## 7 ets     State    1.10  1.11
## 8 wlss    State    1.13  1.14
## 9 mint_s  State    1.15  1.15
## 10 wlsv   State    1.18  1.17
## 11 ols     Zone    0.954 0.948
## 12 wlss   Zone    0.987 0.980
## 13 mint_s Zone    0.995 0.988
## 14 ets     Zone    1.01  0.999
## 15 wlsv   Zone    1.01  1.00
## 16 ols     Region   0.901 0.895
## 17 wlss   Region   0.910 0.907
## 18 mint_s Region   0.911 0.911
## 19 wlsv   Region   0.917 0.919
## 20 ets     Region   0.935 0.938
```

- OLS is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

Outline

- 1 Hierarchical forecasting 20 years ago
- 2 Point forecast reconciliation
- 3 Example: Australian tourism
- 4 Probabilistic forecast reconciliation
- 5 Example: Australian electricity generation
- 6 Extensions

Probabilistic forecast reconciliation

Key papers

- Ben Taieb, Taylor, Hyndman (*ICML*, 2017)
- Jeon, Panagiotelis, Petropoulos (*EJOR*, 2019)
- Ben Taieb, Taylor, Hyndman (*JASA*, 2020)
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020).
robjhyndman.com/publications/coherentprob/

Probabilistic forecast reconciliation

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robjhyndman.com/publications/coherentprob/
- The reconciled multivariate density must lie on the coherent subspace.
- The univariate density at each node is a convolution of the densities of its children.

Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_b(\mathbf{b}) = |\mathbf{G}^*| \int \hat{f}(\mathbf{G}^- \mathbf{b} + \mathbf{G}_\perp \mathbf{a}) d\mathbf{a}$$

- \hat{f} is density of incoherent base probabilistic forecast
- \mathbf{G}^- is $n \times m$ generalised inverse of \mathbf{G} st $\mathbf{G}\mathbf{G}^- = \mathbf{I}$
- \mathbf{G}_\perp is $n \times (n - m)$ orthogonal complement to \mathbf{G} st $\mathbf{G}\mathbf{G}_\perp = \mathbf{0}$
- $\mathbf{G}^* = (\mathbf{G}^- : \mathbf{G}_\perp)$, and \mathbf{b} and \mathbf{a} are obtained via

the change of variables $\mathbf{y} = \mathbf{G}^* \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_y(y) = |S^*| \tilde{f}_b(S^- y) \mathbb{1}\{y \in s\}$$

- $S^* = (S^{-'} \ S_{\perp})'$
- S^- is $m \times n$ generalised inverse of S such that $S^- S = I$,
- S_{\perp} is $n \times (n - m)$ orthogonal complement to S such that $S_{\perp}' S = 0$.

Gaussian reconciliation

If the incoherent base forecasts are $N(\hat{\mu}, \hat{\Sigma})$, then the reconciled density is $N(SG\hat{\mu}, SG\hat{\Sigma}G'S')$.

Bootstrap reconciliation

Reconciling sample paths from incoherent distributions works.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule	Coherent v Incoherent	Coherent v Coherent
--------------	-----------------------	---------------------

Log Score	Not proper	<ul style="list-style-type: none">Ordering preserved if compared using bottom-level only
-----------	------------	--

Energy Score	Proper	<ul style="list-style-type: none">Full hierarchy should be used.Rankings may change otherwise.
--------------	--------	---

Score optimal reconciliation

Algorithm proposed by Panagiotelis et al (2020) for optimizing \mathbf{G} using stochastic gradient descent to optimize Energy Score.

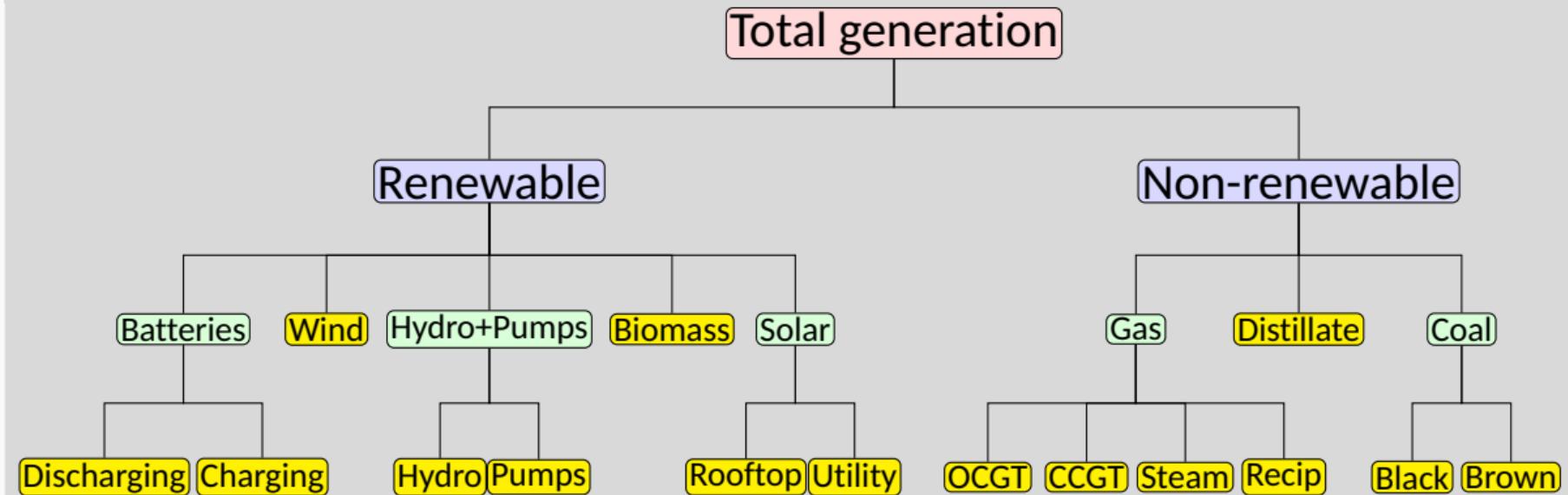
- 1 Compute base forecasts over a test set.
- 2 Compute OLS reconciliation: $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
- 3 Iteratively update \mathbf{G} using SGD with Adam method and ES objective over a test set

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Example: Australian electricity generation

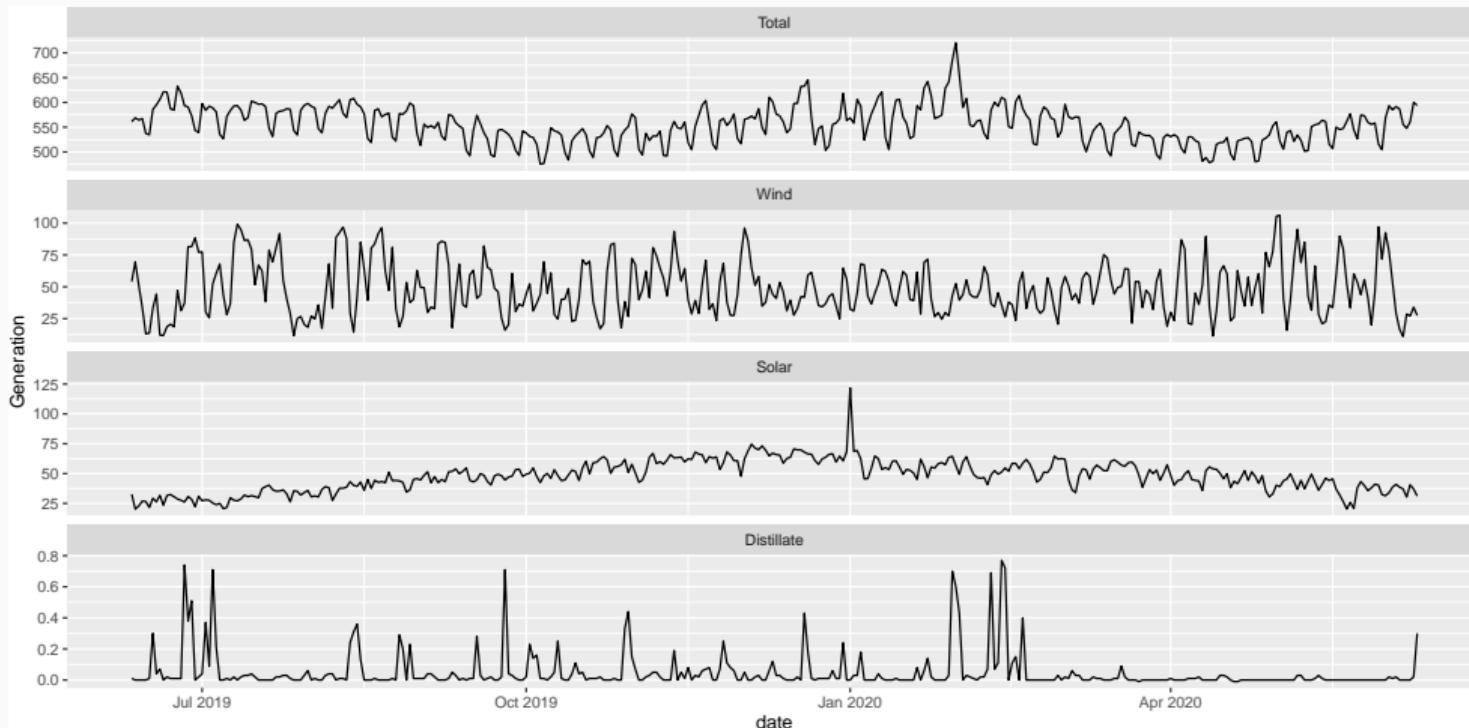
Daily time series from opennem.org.au



$n = 23$ series

$m = 15$ bottom-level series

Example: Australian electricity generation

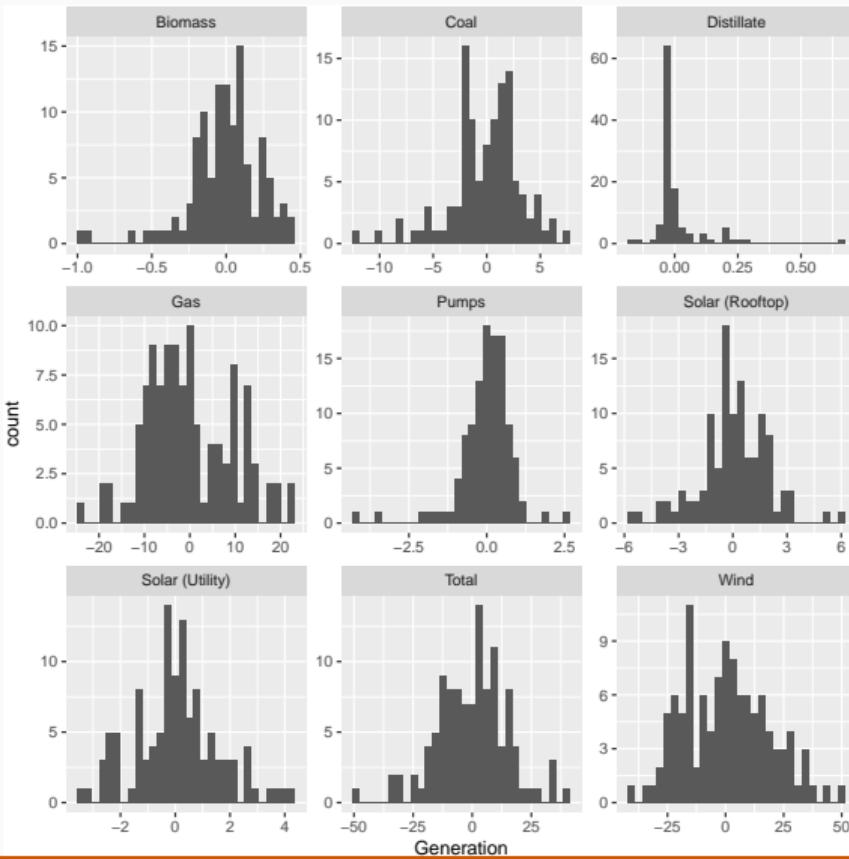


Example: Australian electricity generation

Forecast evaluation

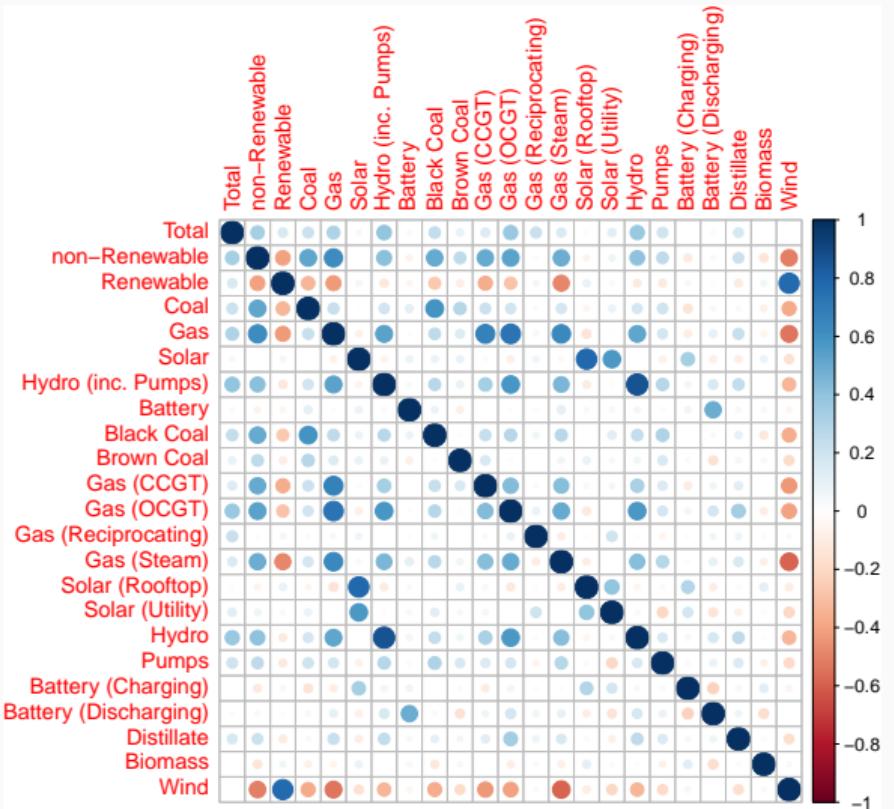
- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using NNETAR() function in fable package.
- Model could be improved with temperature predictor.

Example: Australian electricity generation



Histogram of residuals:
2 Oct 2019 - 21 Jan 2020
Clearly non-Gaussian

Example: Australian electricity generation



Correlations of residuals:

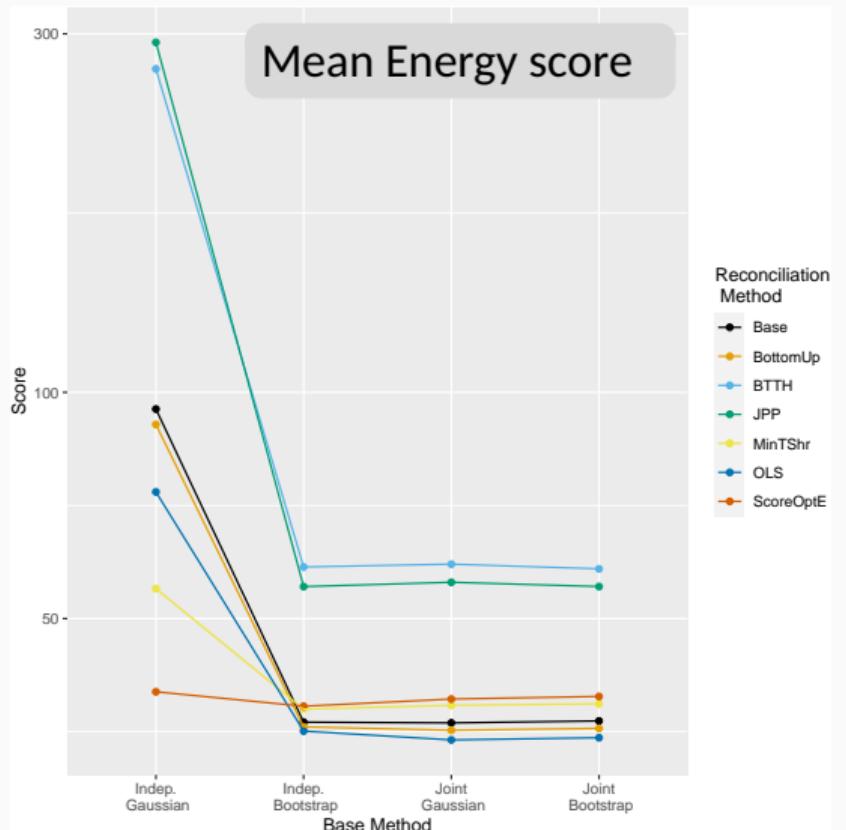
2 Oct 2019 - 21 Jan 2020

Blue = positive correlation.

Red = negative correlation.

Large = stronger correlations.

Example: Australian electricity generation



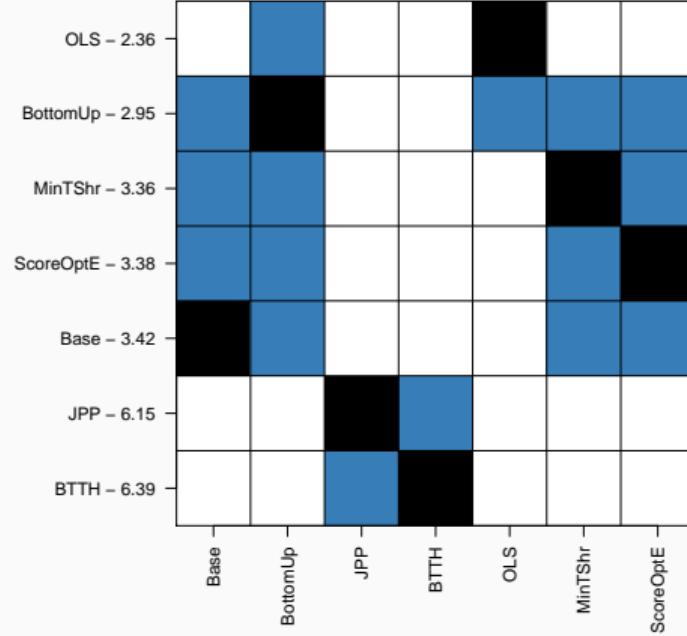
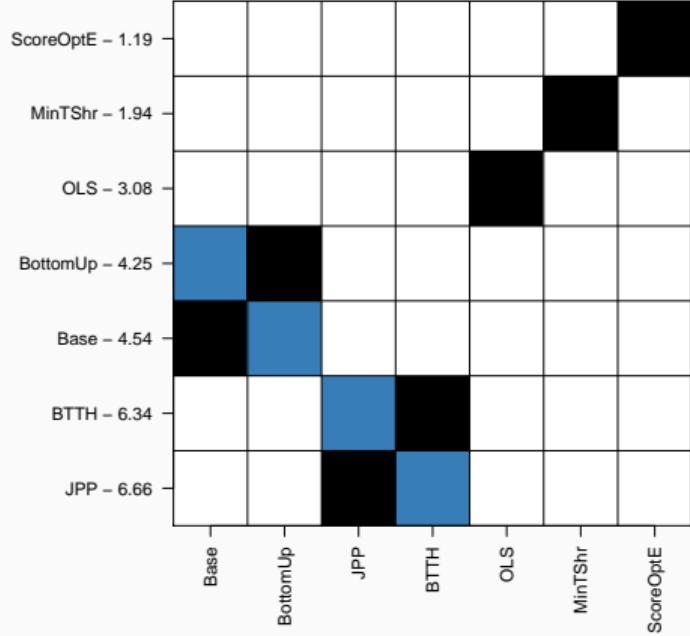
Base residual assumptions

- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

Reconciliation methods

- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS
- MinT(Shrink)
- Score Optimal Reconciliation

Example: Australian electricity generation



Nemenyi test for different scores

Base forecasts are independent and Gaussian.

Nemenyi test for different scores

Base forecasts are obtained by jointly bootstrapping residuals.

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Linear combinations

Optimal linear reconciliation

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{T+h|T}$$

Linear combinations

Optimal linear reconciliation

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{T+h|T}$$

\mathbf{S} can contain any values, not just 0s and 1s.

Linear combinations

Optimal linear reconciliation

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{T+h|T}$$

\mathbf{S} can contain any values, not just 0s and 1s.

- Shang, Hyndman (*JCGS*, 2017)
- Athanasopoulos, Gamakumara, Panagiotelis, Hyndman, Affan (*Springer*, 2020)

Reconciled linear regression forecasts

If the base forecasts are from a linear regression model, then we can produce coherent forecasts in one step:

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\boldsymbol{\Lambda}_s\mathbf{S})^{-1}\mathbf{S}'\boldsymbol{\Lambda}_s\hat{\mathbf{y}}_{T+h|T} \quad \hat{\mathbf{y}}_{T+h|T} = \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{T}$$

- \mathbf{X} is matrix of predictors for training set
- \mathbf{X}_{T+h}^* is vector of predictors for time $T + h$

Reconciled linear regression forecasts

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- \mathbf{X} is matrix of predictors for training set
- \mathbf{X}_{T+h}^* is vector of predictors for time $T + h$

Reconciled linear regression forecasts

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- \mathbf{X} is matrix of predictors for training set
- \mathbf{X}_{T+h}^* is vector of predictors for time $T + h$

$$\mathbf{V}_h = \sigma^2 \mathbf{S}(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s [1 + \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}_{T+h}^*)'] \Lambda_s \mathbf{S}'(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'$$

- σ^2 is variance of base model residuals.

Reconciled linear regression forecasts

If the base forecasts are from a linear regression model, then we can produce coherent forecasts in one step:

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s\mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

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$$\mathbf{V}_h = \sigma^2 \mathbf{S}(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s [1 + \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}_{T+h}^*)'] \Lambda_s \mathbf{S}'(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'$$

- σ^2 is variance of base model residuals.

Reference: Ashouri, Hyndman, and Shmueli (2019).

robjhyndman.com/publications/lhf/

Non-negative forecast reconciliation

Minimum trace (MinT) reconciliation

The trace of V_h is minimized when

$$\tilde{\mathbf{b}}_{T+h|T} = (\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}$$

subject to unbiasedness preservation ($\mathbf{S}\mathbf{G}\mathbf{S}' = \mathbf{S}$).

Non-negative forecast reconciliation

Minimum trace (MinT) reconciliation

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$$\tilde{\mathbf{b}}_{T+h|T} = (\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}$$

subject to unbiasedness preservation ($\mathbf{S}\mathbf{G}\mathbf{S}' = \mathbf{S}$).

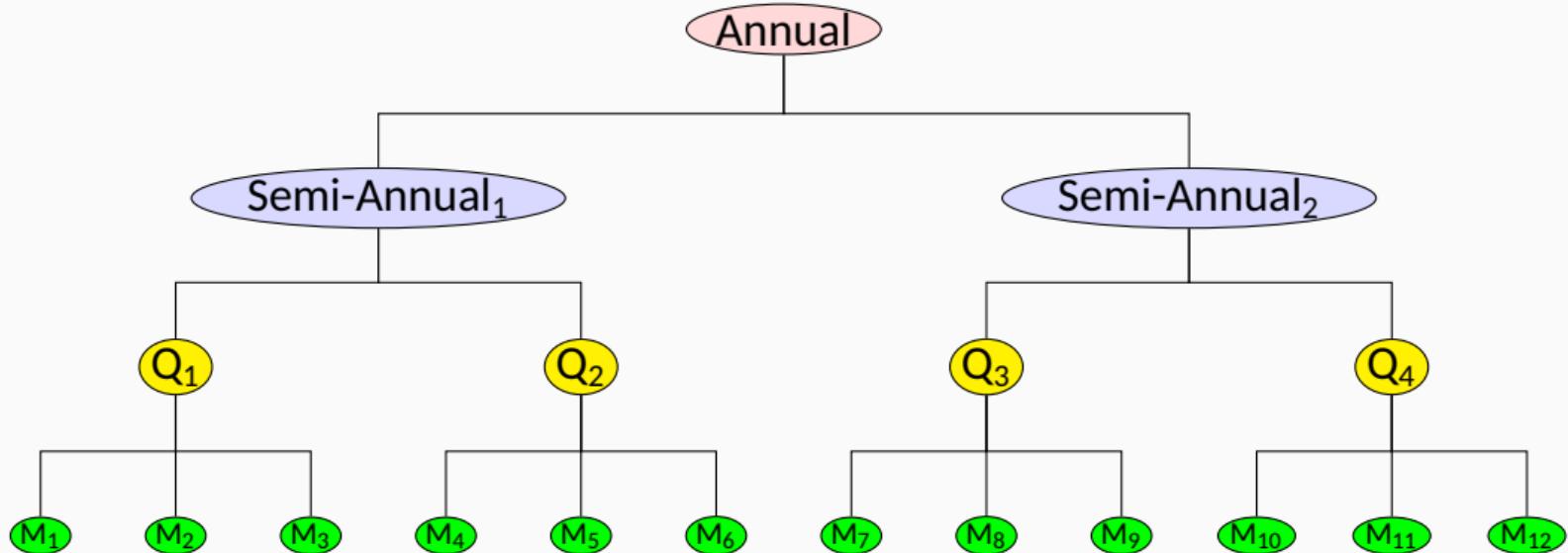
Wickramasuriya, Turlach and Hyndman (S&C, 2020) replace the unbiased constraint by a non-negative constraint:

$$\tilde{\mathbf{b}}_{T+h|T} \geq 0$$

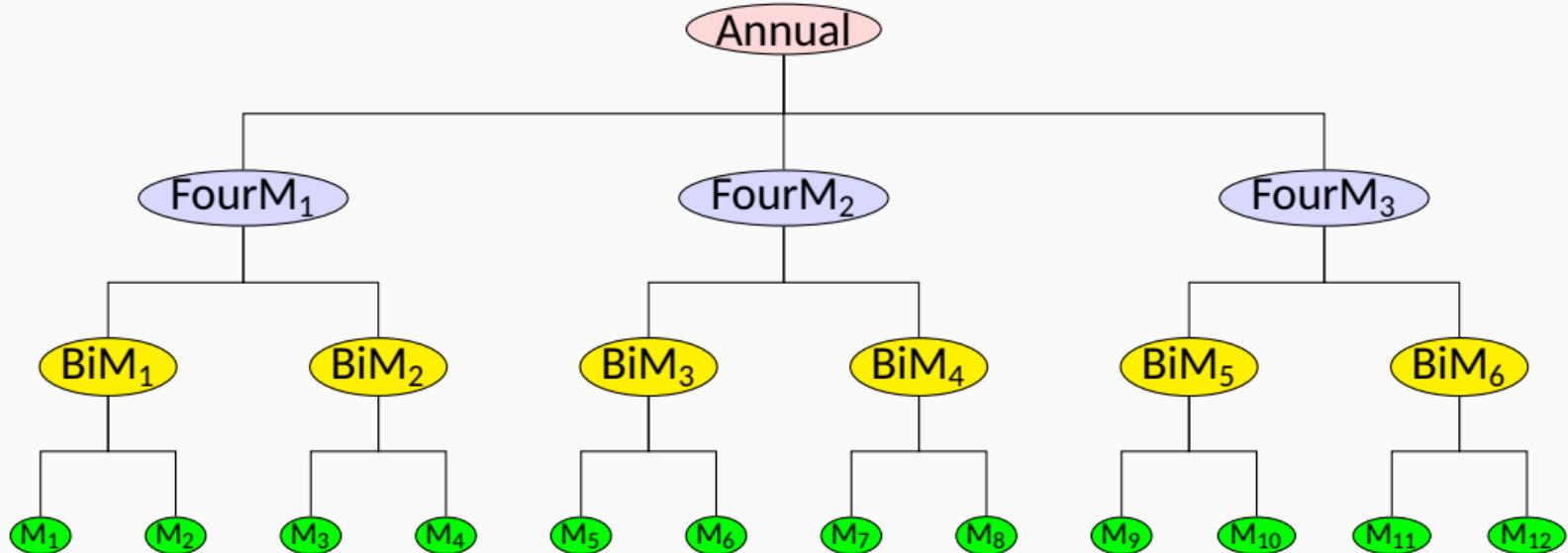
and show that it can be solved via quadratic programming:

$$\min_{\mathbf{b}} \frac{1}{2} \mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - \mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T} \quad \text{s.t. } \mathbf{b} \geq 0$$

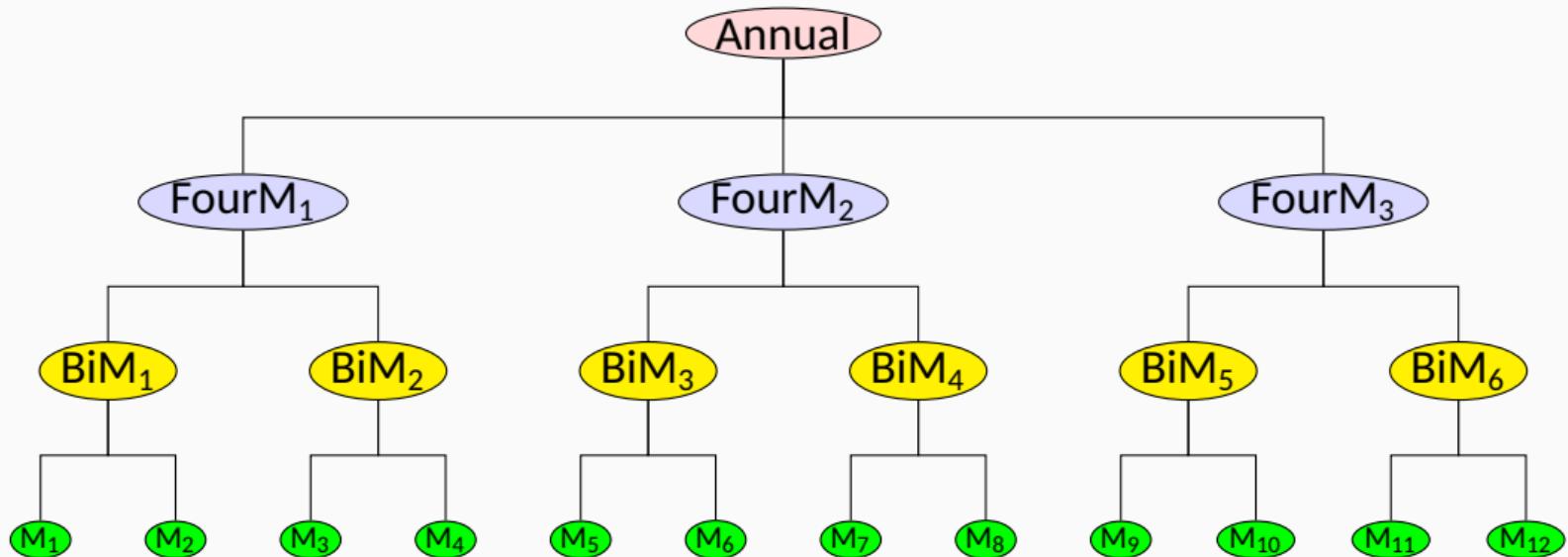
Temporal reconciliation



Temporal reconciliation



Temporal reconciliation



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation

For a time series y_1, \dots, y_T , observed at frequency m , we generate aggregate series

$$y_j^{[k]} = \sum_{t=1+(j-1)k}^{jk} y_t, \quad \text{for } j = 1, \dots, [T/k]$$

- $k \in F(m) = \{\text{factors of } m\}$.
- A single unique hierarchy is only possible when there are no coprime pairs in $F(m)$.
- $M_k = m/k$ is seasonal period of aggregated series.
- Proposed by Athanasopoulos, Hyndman, Kourentzes, Petropoulos (*EJOR*, 2017)

Cross-temporal reconciliation

- Kourentzes, Athanasopoulos (*ATR*, 2019)
- Punia, Singh, Madaan (*C&IE*, 2020)
- Di Fonzo, Girolimetto (2020)

Cross-temporal reconciliation

- Kourentzes, Athanasopoulos (ATR, 2019)
- Punia, Singh, Madaan (C&IE, 2020)
- Di Fonzo, Girolimetto (2020)

ISF talks coming up!

- ➡ Mitch O'Hara-Wild: *Probabilistic cross-temporal hierarchies in fable* (in about 3 hours)
- ➡ Tommi Di Fonzo: *Non-negative cross-temporal forecast reconciliation. An application to the Australian domestic tourism flows* (in about 11 hours).

Bayesian forecast reconciliation

- Park, Nassar (*ICML*, 2014)
- Novak, McGarvie, Garcia (2017)
- Corani, Azzimonti, Zaffalon (2019)
- Corani, Azzimonti, Augusto, Zaffalon (2020)
- Eckert, Hyndman, Panagiotelis (*EJOR*, 2020)

ML and regularization

- Qiao, Huang (*ICIS*, 2018)
- Ben Taieb, Koo (*ICKDDM*, 2019)
- Yang, Hu, Wang (*ICANN*, 2019)
- Abolghasemi, Hyndman, Tarr, Bergmeir (2019)
- Spiliotis, Abolghasemi, Hyndman, Petropoulos, Assimakopoulos (2020)

Special IJF issue on hierarchical forecasting



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international journal of forecasting

Guest Editors

- George Athanasopoulos
- Rob J Hyndman
- Anastasios Panagiotelis
- Nikolaos Kourentzes

Call for papers

bit.ly/ijfhierarchical

Submission deadline

31 August 2021



International Institute of Forecasters

Thanks!



More information

- Slides and papers: robjhyndman.com
- Packages: tidyverts.org
- Forecasting textbook using fable package:
OTexts.com/fpp3

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