Ten years of forecast reconciliation

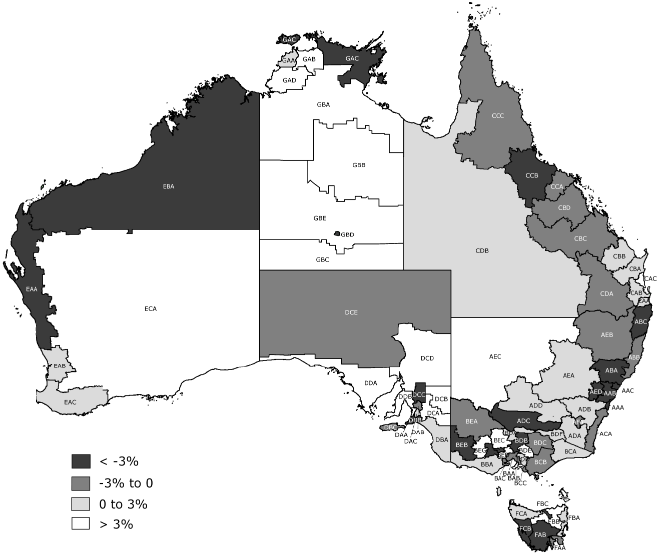
Rob J Hyndman

ISF 2020

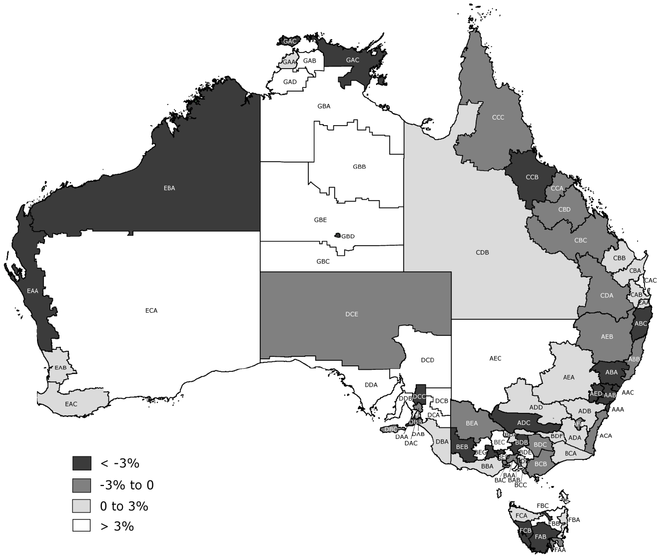
Outline

Outline

Australian tourism



Australian tourism

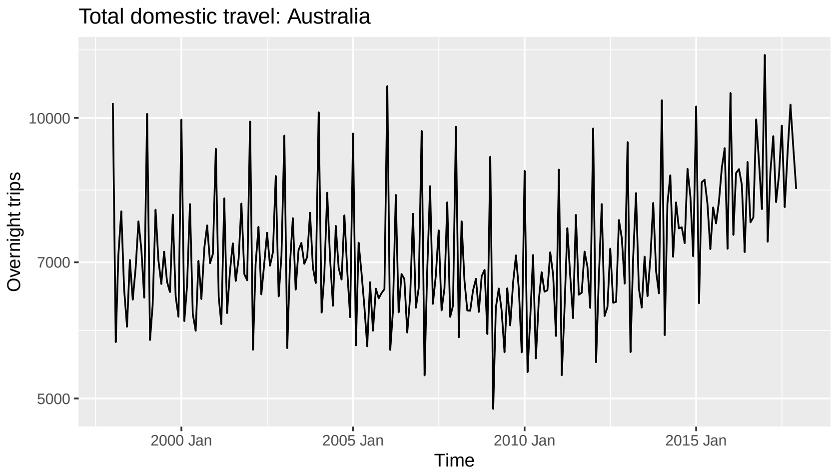


- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - 7 states
  - 27 zones
  - 75 regions

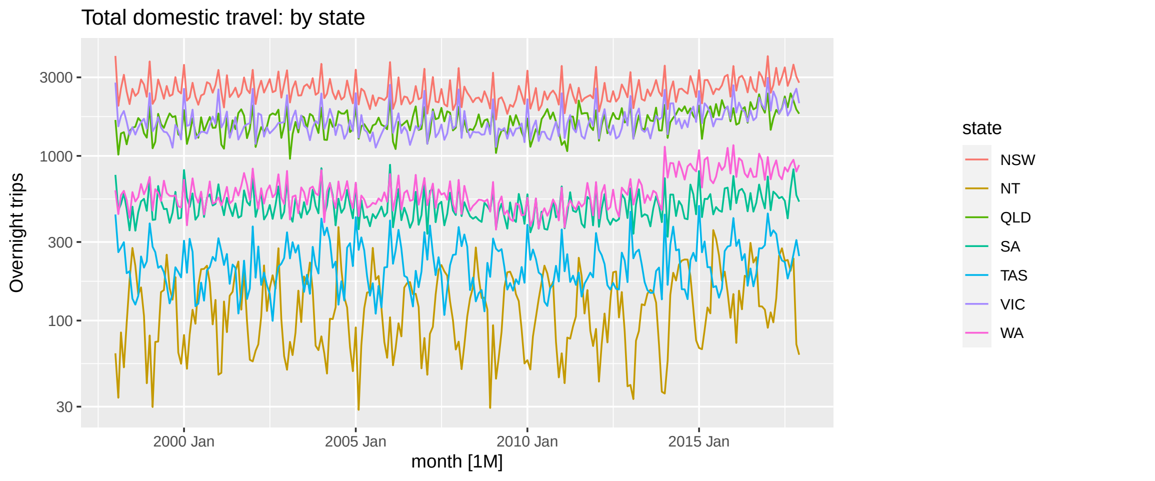
Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney     926.
##  2 1998 Feb NSW   Metro NSW Sydney     647.
##  3 1998 Mar NSW   Metro NSW Sydney     716.
##  4 1998 Apr NSW   Metro NSW Sydney     621.
##  5 1998 May NSW   Metro NSW Sydney     598.
##  6 1998 Jun NSW   Metro NSW Sydney     601.
##  7 1998 Jul NSW   Metro NSW Sydney     720.
##  8 1998 Aug NSW   Metro NSW Sydney     645.
##  9 1998 Sep NSW   Metro NSW Sydney     633.
## 10 1998 Oct NSW   Metro NSW Sydney     771.
## # ... with 17,990 more rows
```

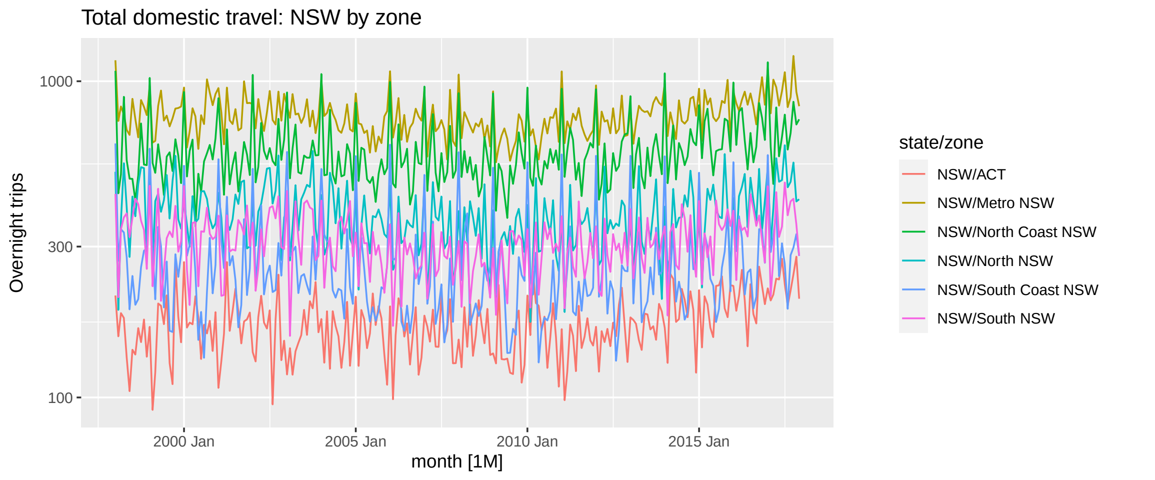
Australian tourism data



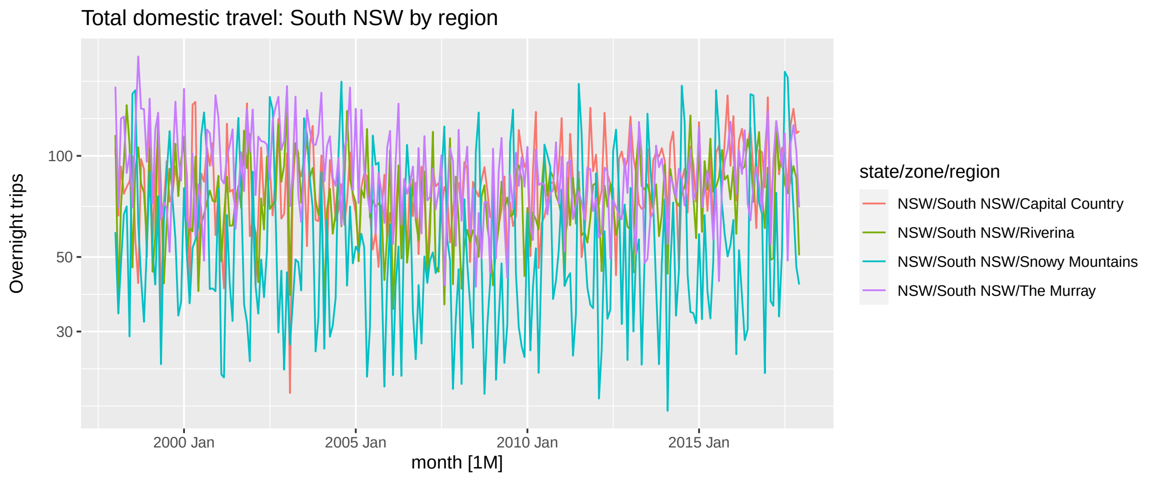
Australian tourism data



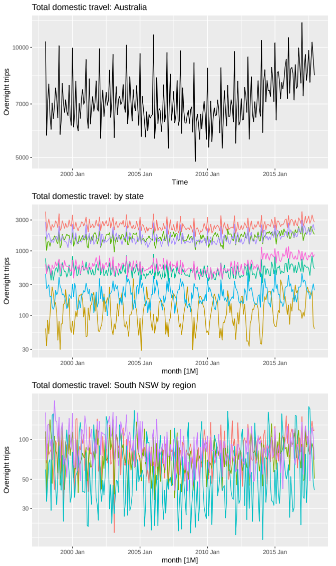
Australian tourism data

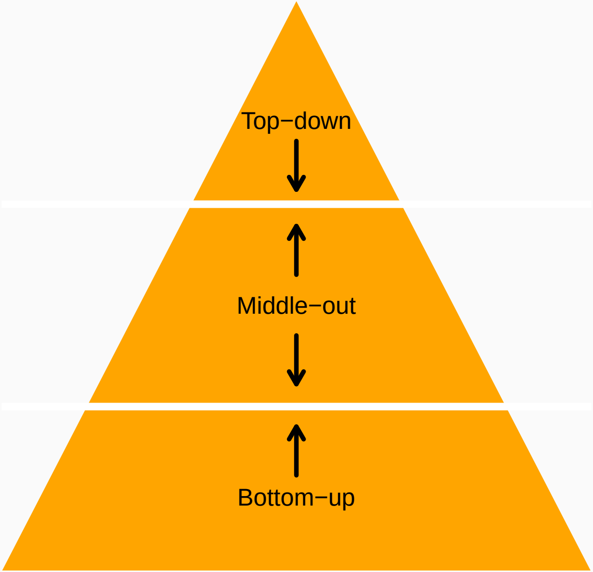


Australian tourism data



Hierarchical forecasting 20 years ago





Forecast reconciliation

- Forecast all series at all levels of aggregation.
- Reconcile forecasts using least squares optimization.

History

2001: Idea to use all available series to forecast Australia’s labour market by occupation.

2004: PhD student Roman Ahmed begins, co-supervised with George Athanasopoulos.

2006: Presentation at ISF, Santander.

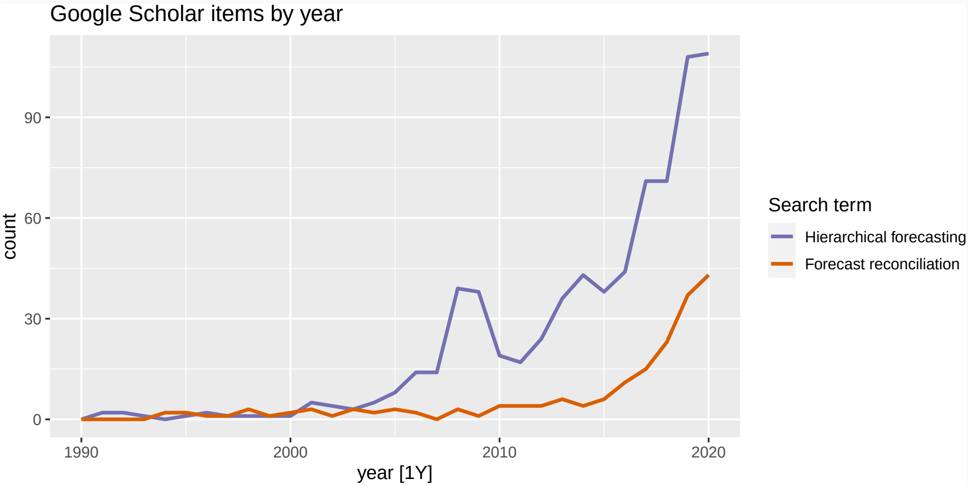
2007: Pre-print of "Optimal combination forecasts for hierarchical time series".

2009: Application to Australian tourism published in IJF.

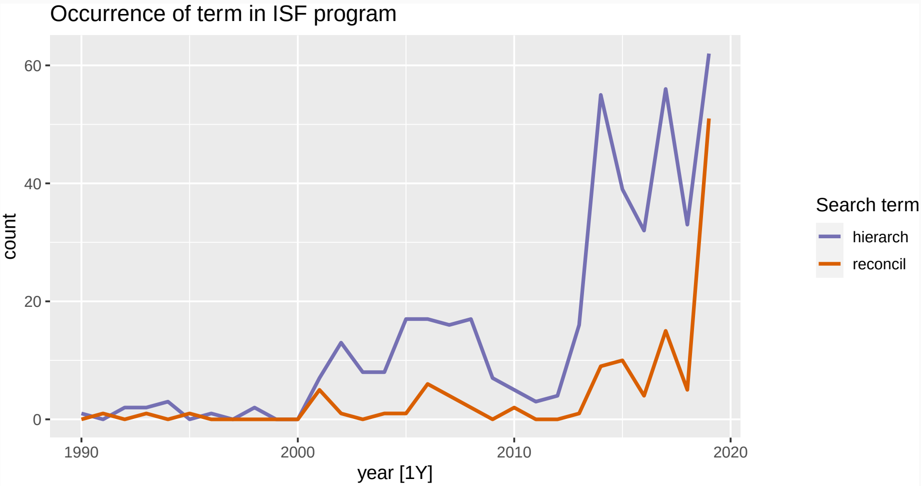
2010: First version of hts package on CRAN.

2011: "Optimal combination forecasts for hierarchical time series" appears in CSDA.

Forecast reconciliation research



Forecast reconciliation research



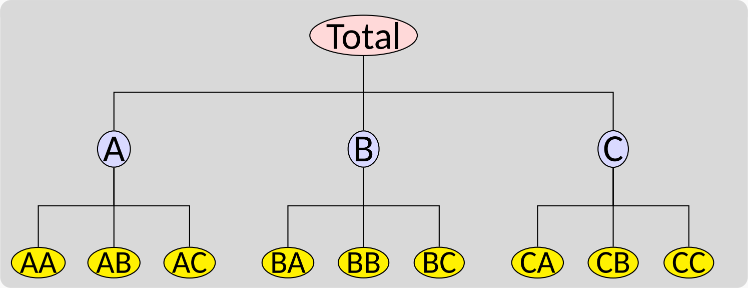
Outline

Point forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 *CSDA*) Optimal combination forecasts for hierarchical time series.
- Hyndman, Lee, Wang (2016 *CSDA*) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 *JASA*) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 *IJF*) Forecast reconciliation: A geometric view with new insights on bias correction.

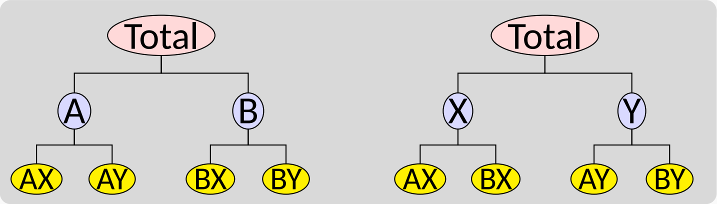
Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



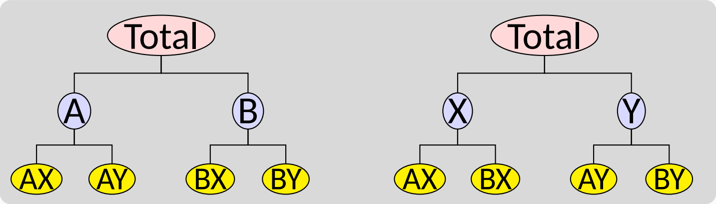
Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Grouped time series

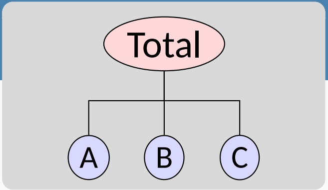
A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

Hierarchical and grouped time series



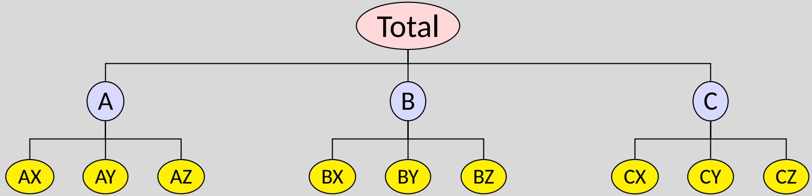
Every collection of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

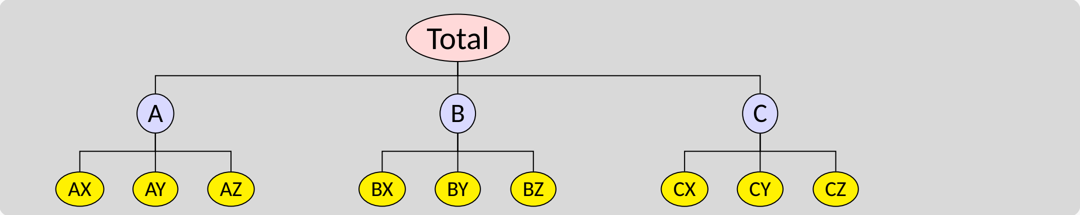
- $\boldsymbol{y}_t$ = vector of all series at time t
- y_t = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Hierarchical time series

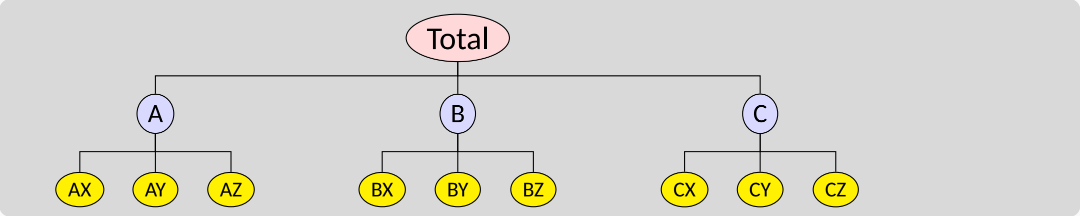


Hierarchical time series



$$
\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}
$$

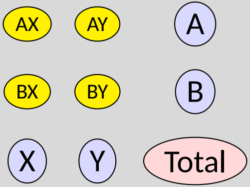
Hierarchical time series



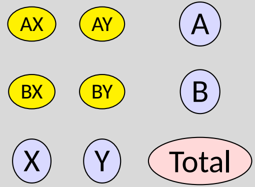
$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Grouped data

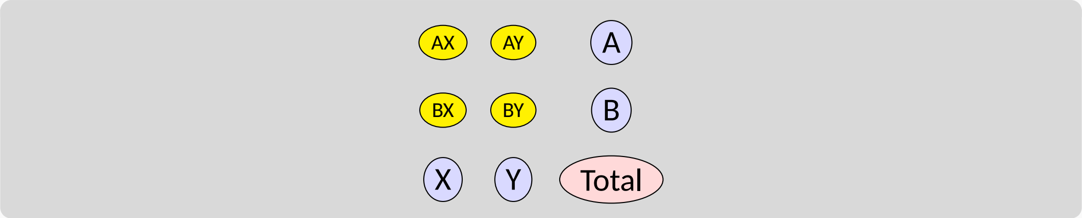


Grouped data



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}$$

Grouped data



$$\boldsymbol{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{X,t} \\ y_{Y,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{BX,t} \\ y_{BY,t} \end{pmatrix}$$

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

Definitions

Coherent subspace

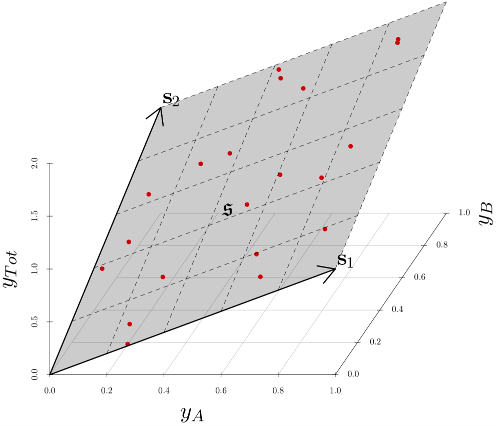
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

Definitions

Coherent subspace

m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

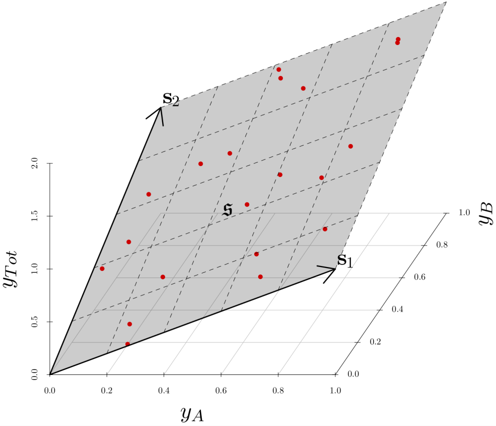
An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.



$y_{Tot} = y_A + y_B$

Definitions

Coherent subspace

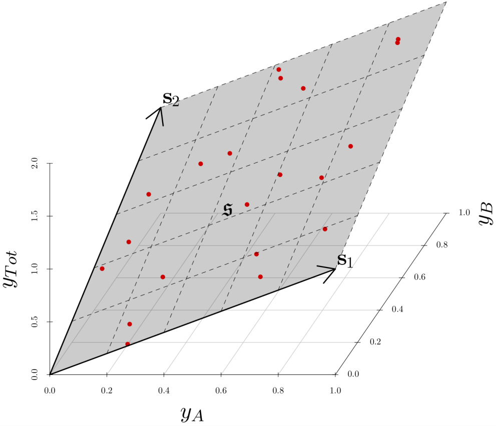
m-dimensional linear subspace $\mathfrak{s} \subset \mathbb{R}^n$ for which linear constraints hold for all $\boldsymbol{y} \in \mathfrak{s}$.

Hierarchical time series

An n-dimensional multivariate time series such that $\boldsymbol{y}_t \in \mathfrak{s} \quad \forall t$.

Coherent point forecasts

$\tilde{\boldsymbol{y}}_{t+h|t}$ is *coherent* if $\tilde{\boldsymbol{y}}_{t+h|t} \in \mathfrak{s}$.



$y_{Tot} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Reconciled forecasts

Let ψ be a mapping, $\psi : \mathbb{R}^n \to \mathfrak{s}$.
$\tilde{\boldsymbol{y}}_{t+h|t} = \psi(\hat{\boldsymbol{y}}_{t+h|t})$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Linear reconciliation

If ψ is a linear function, then $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$

- $\boldsymbol{G}$ combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.

Linear reconciliation

If ψ is a linear function, then $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$

- $\boldsymbol{G}$ combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.

Mean

$$\mathrm{E}[\tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

provided $\boldsymbol{S}\boldsymbol{G}\boldsymbol{S}' = \boldsymbol{S}$ and

$$\mathrm{E}[\hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

i.e., reconciled forecasts are unbiased if base forecasts are unbiased and $\boldsymbol{S}\boldsymbol{G}$ is a projection.

Linear reconciliation

If ψ is a linear function, then $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$

- $\boldsymbol{G}$ combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.

Mean

$$\text{E}[\tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1,\dots,\boldsymbol{y}_n] = \text{E}[\boldsymbol{y}_{T+h|T} \mid \boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$$

provided $\boldsymbol{S}\boldsymbol{G}\boldsymbol{S}' = \boldsymbol{S}$ and

$$\text{E}[\hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1,\dots,\boldsymbol{y}_n] = \text{E}[\boldsymbol{y}_{T+h|T} \mid \boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$$

i.e., reconciled forecasts are unbiased if base forecasts are unbiased and $\boldsymbol{S}\boldsymbol{G}$ is a projection.

Variance

$$\begin{aligned}\boldsymbol{V}_h &= \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1,\dots,\boldsymbol{y}_n] \\ &= \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'\end{aligned}$$

where

$$\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1,\dots,\boldsymbol{y}_n]$$

Linear reconciliation

If ψ is a linear function, then $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{t+h|t}$

- $\boldsymbol{G}$ combines base forecasts $\hat{\boldsymbol{y}}_{T+h|T}$ to get bottom-level forecasts.
- $\boldsymbol{S}$ creates linear combinations.

Mean

$$\mathrm{E}[\tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

provided $\boldsymbol{S}\boldsymbol{G}\boldsymbol{S}' = \boldsymbol{S}$ and

$$\mathrm{E}[\hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] = \mathrm{E}[\boldsymbol{y}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

i.e., reconciled forecasts are unbiased if base forecasts are unbiased and $\boldsymbol{S}\boldsymbol{G}$ is a projection.

Variance

$$\begin{aligned}\boldsymbol{V}_h &= \mathrm{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n] \\ &= \boldsymbol{S}\boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}'\boldsymbol{S}'\end{aligned}$$

where

$$\boldsymbol{W}_h = \mathrm{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{T+h|T} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_n]$$

Minimum trace (MinT) reconciliation

If $\boldsymbol{S}\boldsymbol{G}$ is a projection, then the trace of $\boldsymbol{V}_h$ is minimized when

$$\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}h$$

Linear projections

$$\tilde{\boldsymbol{y}}_{T+h|T} = \boldsymbol{S}\boldsymbol{G}\hat{\boldsymbol{y}}_{T+h|T}$$

Reconciliation method	$\boldsymbol{G}$
OLS	$(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS(var)	$(\boldsymbol{S}'\boldsymbol{\Lambda}_v\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{\Lambda}_v$
WLS(struct)	$(\boldsymbol{S}'\boldsymbol{\Lambda}_s\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{\Lambda}_s$
MinT(sample)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}$
MinT(shrink)	$(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\boldsymbol{W}_h = k_h\boldsymbol{W}_1$.

- $\boldsymbol{\Lambda}_v = \text{diag}(\boldsymbol{W}_1)^{-1}$
- $\boldsymbol{\Lambda}_s = \text{diag}(\boldsymbol{S}\boldsymbol{1})^{-1}$
- $\hat{\boldsymbol{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_{\text{sam}}) + (1-\tau)\hat{\boldsymbol{W}}_{\text{sam}}$ where τ selected optimally.

Reconciled linear regression forecasts

If the base forecasts are from a linear regression model, then we can produce coherent forecasts in one step:

$$\tilde{\boldsymbol{y}}_{T+h} = \boldsymbol{S}(\boldsymbol{S}'\Lambda_s\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_s\boldsymbol{X}^*_{T+h}(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{Y}$$

- $\boldsymbol{X}$ is matrix of predictors for training set.
- $\boldsymbol{X}^*_{T+h}$ is vector of predictors for time $T+h$.

$$\boldsymbol{V}_h = \sigma^2\boldsymbol{S}(\boldsymbol{S}'\Lambda_s\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_s\left[1+\boldsymbol{X}^*_{T+h}(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{X}^*_{T+h})'\right]\Lambda_s\boldsymbol{S}'(\boldsymbol{S}'\Lambda_s\boldsymbol{S})^{-1}\boldsymbol{S}'$$

where σ^2 is the variance of the base model residuals.

Reference: Ashouri, Hyndman, and Shmueli (2019).

`robjhyndman.com/publications/lhf/`

Non-negative forecast reconciliation

Non-negative constraints

Temporal and cross-temporal reconciliation

- Kourentzes and Athanasopoulos (2019)
- Di Fonzo and Girolimetto (2020)
- Punia, Singh, and Madaan (2020)

Outline

Coherent probabilistic forecasts

Coherent probabilistic forecasts

Given the triple $(\mathbb{R}^m, \mathscr{F}_{\mathbb{R}^m}, \nu)$, a coherent probability triple $(\mathfrak{s}, \mathscr{F}_{\mathfrak{s}}, \breve{\nu})$ is such that

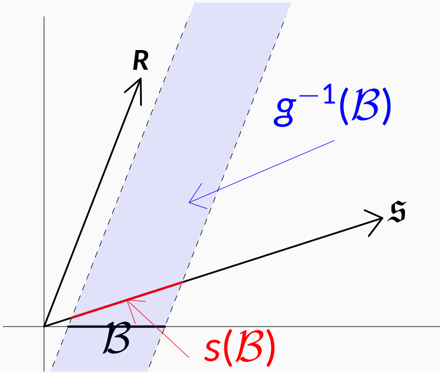
$$\breve{\nu}(s(\mathcal{B})) = \nu(\mathcal{B}) \quad \forall \mathcal{B} \in \mathscr{F}_{\mathbb{R}^m}.$$

Probabilistic forecast reconciliation

The reconciled probability measure of $\hat{\nu}$ wrt $\psi(.)$ is such that

$$\tilde{\nu}(\mathcal{B}) = \hat{\nu}(\psi^{-1}(\mathcal{B})) \qquad \forall \mathcal{B} \in \mathscr{F}_{\mathfrak{s}}\,,$$

where $\psi^{-1}(\mathcal{B}) := \{\boldsymbol{y} \in \mathbb{R}^n : \psi(\boldsymbol{y}) \in \mathcal{B}\}$ is the pre-image of $\mathcal{B}$, that is the set of all points in $\mathbb{R}^n$ that $\psi(.)$ maps to a point in $\mathcal{B}$.



Construction of reconciled distributions

Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_{\boldsymbol{b}}(\boldsymbol{b}) = |\boldsymbol{G}^*| \int \hat{f}(\boldsymbol{G}^- \boldsymbol{b} + \boldsymbol{G}_\perp \boldsymbol{a}) d\boldsymbol{a}$$

- $\hat{f}$ is density of incoherent base probabilistic forecast
- $\boldsymbol{G}^-$ is $n \times m$ generalised inverse of $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}^- = \boldsymbol{I}$
- $\boldsymbol{G}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{G}$ st $\boldsymbol{G}\boldsymbol{G}_\perp = \boldsymbol{0}$
- $\boldsymbol{G}^* = \left(\boldsymbol{G}^- \vdots \boldsymbol{G}_\perp\right)$, and $\boldsymbol{b}$ and $\boldsymbol{a}$ are obtained via the change of variables $\boldsymbol{y} = \boldsymbol{G}^* \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{a} \end{pmatrix}$

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix} \boldsymbol{S}^- \\ \boldsymbol{S}'_\perp \end{pmatrix}$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Construction of reconciled distributions

Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_{\boldsymbol{y}}(\boldsymbol{y}) = |\boldsymbol{S}^*|\tilde{f}_{\boldsymbol{b}}(\boldsymbol{S}^-\boldsymbol{y})\mathbb{1}\{\boldsymbol{y} \in \mathfrak{s}\}$$

- $\boldsymbol{S}^* = \begin{pmatrix} \boldsymbol{S}^- \\ \boldsymbol{S}'_\perp \end{pmatrix}$
- $\boldsymbol{S}^-$ is $m \times n$ generalised inverse of $\boldsymbol{S}$ such that $\boldsymbol{S}^-\boldsymbol{S} = \boldsymbol{I}$,
- $\boldsymbol{S}_\perp$ is $n \times (n-m)$ orthogonal complement to $\boldsymbol{S}$ such that $\boldsymbol{S}'_\perp\boldsymbol{S} = \boldsymbol{0}$.

Gaussian reconciliation

If the incoherent base forecasts are $\text{N}(\hat{\mu}, \hat{\Sigma})$,
then the reconciled density is $\text{N}(\boldsymbol{SG}\hat{\mu}, \boldsymbol{SG}\hat{\Sigma}\boldsymbol{G}'\boldsymbol{S}')$.

Simulation from a reconciled distribution

Suppose that $\left(\hat{\boldsymbol{y}}^{[1]}, \dots, \hat{\boldsymbol{y}}^{[L]}\right)$ is a sample drawn from an incoherent probability measure $\hat{\nu}$. Then $\left(\tilde{\boldsymbol{y}}^{[1]}, \dots, \tilde{\boldsymbol{y}}^{[L]}\right)$ where $\tilde{\boldsymbol{y}}^{[\ell]} := \psi(\hat{\boldsymbol{y}}^{[\ell]})$ for $\ell = 1, \dots, L$, is a sample drawn from the reconciled probability measure $\tilde{\nu}$.

- So reconciling sample paths from incoherent distributions works.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Evaluating probabilistic forecasts

Proper scoring rule

optimized when true forecast distribution is used.

Scoring Rule	Coherent v Incoherent	Coherent v Coherent
Log Score	Not proper	• Ordering preserved if compared using bottom-level only
Energy Score	Proper	• Full hierarchy should be used. • Rankings may change otherwise.

Score optimal reconciliation

Algorithm proposed by Panagiotelis et al (2020) for optimizing $\boldsymbol{G}$ using stochastic gradient descent to optimize Energy Score.

1. Compute base forecasts over a test set.
2. Compute OLS reconciliation: $\boldsymbol{G} = (\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
3. Iteratively update $\boldsymbol{G}$ using SGD with Adam method and ES objective over a test set

Outline

Example: Australian tourism

```
tourism
```

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:       state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>     <dbl>
##  1 1998 Jan NSW   Metro NSW Sydney     926.
##  2 1998 Feb NSW   Metro NSW Sydney     647.
##  3 1998 Mar NSW   Metro NSW Sydney     716.
##  4 1998 Apr NSW   Metro NSW Sydney     621.
##  5 1998 May NSW   Metro NSW Sydney     598.
##  6 1998 Jun NSW   Metro NSW Sydney     601.
##  7 1998 Jul NSW   Metro NSW Sydney     720.
##  8 1998 Aug NSW   Metro NSW Sydney     645.
##  9 1998 Sep NSW   Metro NSW Sydney     633.
## 10 1998 Oct NSW   Metro NSW Sydney     771.
```

Example: Australian tourism

```r
tourism_agg <- tourism %>%
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
## # A tsibble: 26,400 x 5 [1M]
## # Key:       state, zone, region [110]
##       month state        zone         region       visitors
##       <mth> <chr>        <chr>        <chr>           <dbl>
##  1 1998 Jan <aggregated> <aggregated> <aggregated>   10376.
##  2 1998 Feb <aggregated> <aggregated> <aggregated>    5746.
##  3 1998 Mar <aggregated> <aggregated> <aggregated>    7129.
##  4 1998 Apr <aggregated> <aggregated> <aggregated>    7939.
##  5 1998 May <aggregated> <aggregated> <aggregated>    6552.
##  6 1998 Jun <aggregated> <aggregated> <aggregated>    5969.
##  7 1998 Jul <aggregated> <aggregated> <aggregated>    7041.
##  8 1998 Aug <aggregated> <aggregated> <aggregated>    6382.
##  9 1998 Sep <aggregated> <aggregated> <aggregated>    6907.
```

Example: Australian tourism

```r
fit <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(ets = ETS(visitors))
```

```
## # A mable: 110 x 4
## # Key:     state, zone, region [110]
##    state zone            region                ets
##    <chr> <chr>           <chr>             <model>
##  1 NSW   <aggregated>    <aggregated>  <ETS(M,N,A)>
##  2 NSW   Metro NSW       <aggregated>  <ETS(M,N,A)>
##  3 NSW   North Coast NSW <aggregated>  <ETS(M,N,M)>
##  4 NSW   South Coast NSW <aggregated>  <ETS(A,N,A)>
##  5 NSW   South NSW       <aggregated>  <ETS(M,N,M)>
##  6 NSW   North NSW       <aggregated>  <ETS(M,N,A)>
##  7 NSW   ACT             <aggregated>  <ETS(M,N,A)>
##  8 NSW   Metro NSW       Sydney        <ETS(M,N,A)>
```

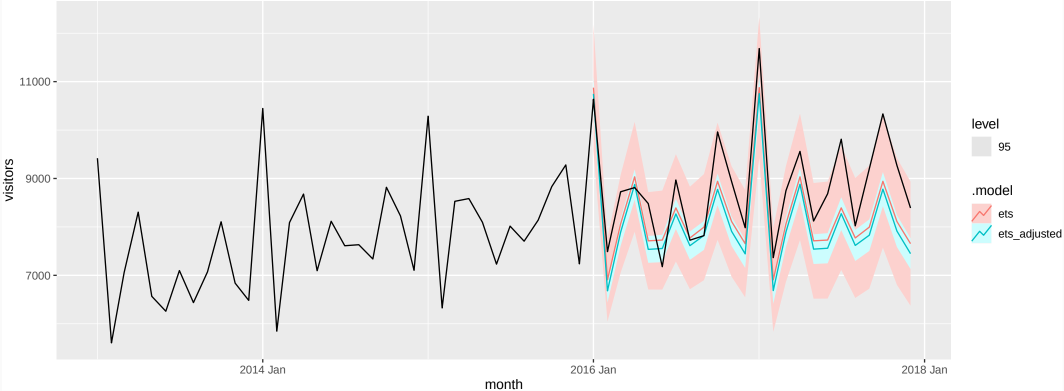
Example: Australian tourism

```r
fc <- fit %>%
  reconcile(ets_adjusted = min_trace(ets)) %>%
  forecast(h = "2 years")
```

```
## # A fable: 5,280 x 7 [1M]
## # Key:     state, zone, region, .model [220]
##    state zone         region       .model    month          visitors .mean
##    <chr> <chr>        <chr>        <chr>     <mth>            <dist> <dbl>
##  1 NSW   <aggregated> <aggregated> ets    2016 Jan N(3679, 71136) 3679.
##  2 NSW   <aggregated> <aggregated> ets    2016 Feb N(2241, 27912) 2241.
##  3 NSW   <aggregated> <aggregated> ets    2016 Mar N(2602, 37643) 2602.
##  4 NSW   <aggregated> <aggregated> ets    2016 Apr N(3027, 50976) 3027.
##  5 NSW   <aggregated> <aggregated> ets    2016 May N(2504, 36795) 2504.
##  6 NSW   <aggregated> <aggregated> ets    2016 Jun N(2447, 36005) 2447.
##  7 NSW   <aggregated> <aggregated> ets    2016 Jul N(2734, 44488) 2734.
##  8 NSW   <aggregated> <aggregated> ets    2016 Aug N(2496, 38775) 2496.
```

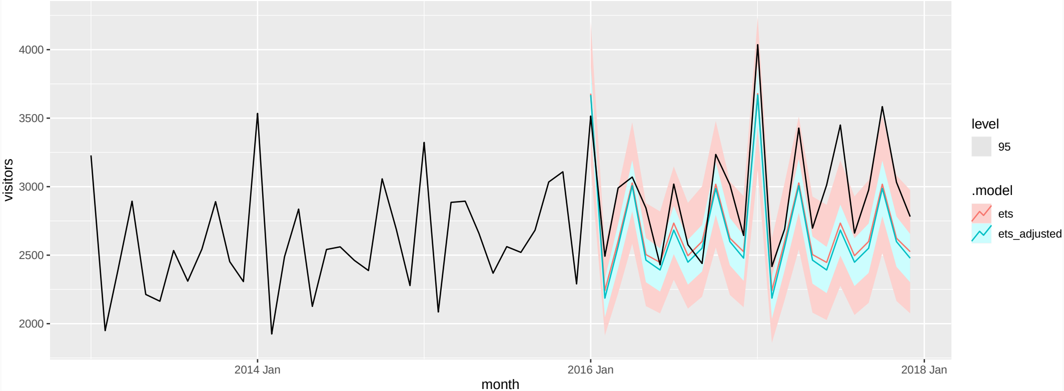
Example: Australian tourism

```
fc %>%
  filter(is_aggregated(state)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



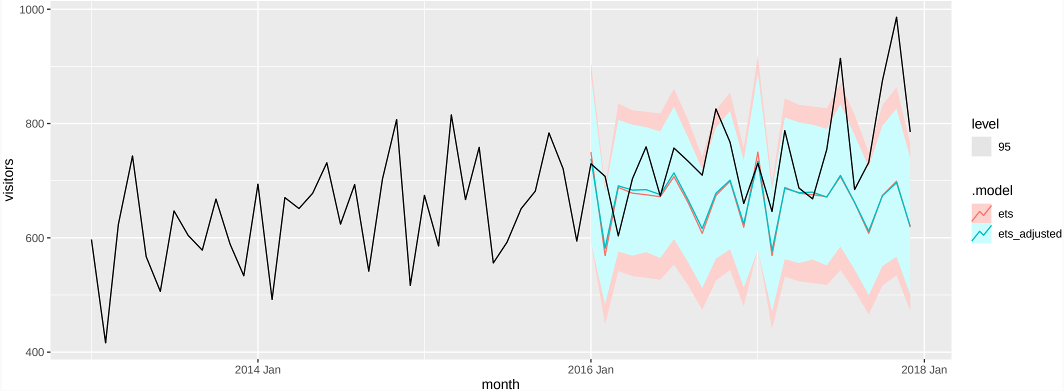
Example: Australian tourism

```
fc %>%
  filter(state == "NSW" & is_aggregated(zone)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



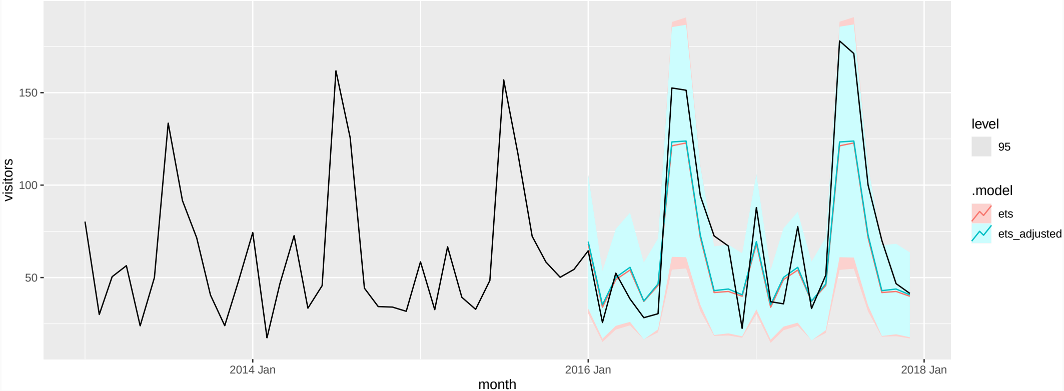
Example: Australian tourism

```
fc %>%
  filter(region == "Melbourne") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



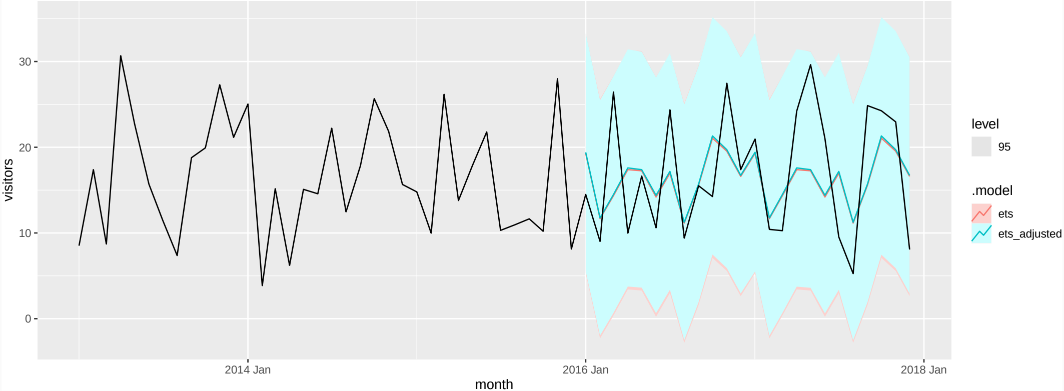
Example: Australian tourism

```r
fc %>%
  filter(region == "Snowy Mountains") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



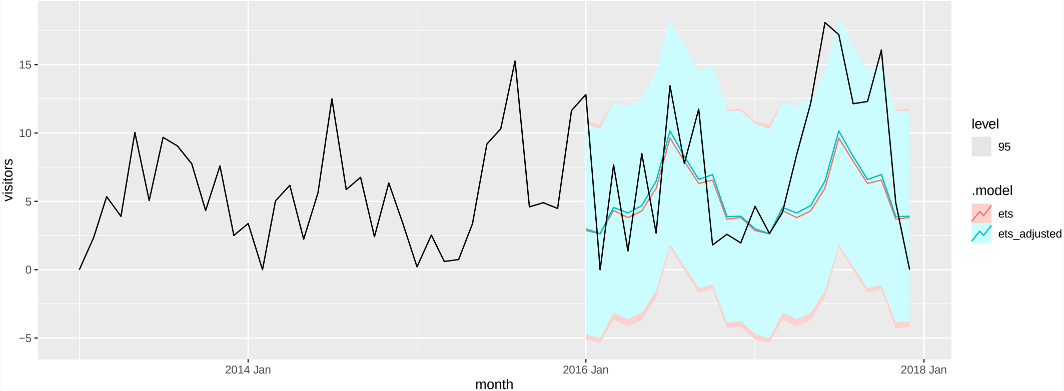
Example: Australian tourism

```
fc %>%
  filter(region == "Barossa") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



Example: Australian tourism

```r
fc %>%
  filter(region == "MacDonnell") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = 95)
```



Example: Australian tourism

```r
fc <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(
    ets = ETS(visitors),
    arima = ARIMA(visitors)
  ) %>%
  mutate(
    comb = (ets + arima) / 2
  ) %>%
  reconcile(
    ets_adj = min_trace(ets),
    arima_adj = min_trace(arima),
    comb_adj = min_trace(comb)
  ) %>%
  forecast(h = "2 years")
```

Example: Australian tourism

```r
fc %>%
  accuracy(data = tourism_agg,
           measures = list(crps = CRPS, ss=skill_score(CRPS)))
```

```
## # A tibble: 660 x 7
##    .model state zone             region         .type  crps      ss
##    <chr>  <chr> <chr>            <chr>          <chr> <dbl>   <dbl>
##  1 arima  NSW   <aggregated>     <aggregated>   Test  158.   0.277
##  2 arima  NSW   Metro NSW        <aggregated>   Test   69.1  0.152
##  3 arima  NSW   North Coast NSW  <aggregated>   Test   58.5  0.0577
##  4 arima  NSW   South Coast NSW  <aggregated>   Test   24.2  0.147
##  5 arima  NSW   South NSW        <aggregated>   Test   25.4  0.277
##  6 arima  NSW   North NSW        <aggregated>   Test   57.0  0.0321
##  7 arima  NSW   ACT              <aggregated>   Test   34.5 -0.221
##  8 arima  NSW   Metro NSW        Sydney         Test   62.4  0.139
##  9 arima  NSW   Metro NSW        Central Coast  Test   13.9  0.196
```

Example: Australian tourism

```
fc %>%
  accuracy(tourism_agg,
           measures = list(crps = CRPS, ss=skill_score(CRPS))) %>%
  group_by(.model) %>%
  summarise(sspc = mean(ss) * 100) %>%
  arrange(sspc)
```

```
## # A tibble: 6 x 2
##   .model     sspc
##   <chr>     <dbl>
## 1 arima_adj  11.9
## 2 arima      12.0
## 3 comb_adj   17.0
## 4 ets_adj    17.7
## 5 comb       18.2
## 6 ets        19.1
```

Outline

1. Hierarchical forecasting 20 years ago
2. Point forecast reconciliation
3. Probabilistic forecast reconciliation
4. Example: Australian tourism
5. Example: Australian electricity generation
6. Bayesian forecast reconciliation
7. ML and regularization

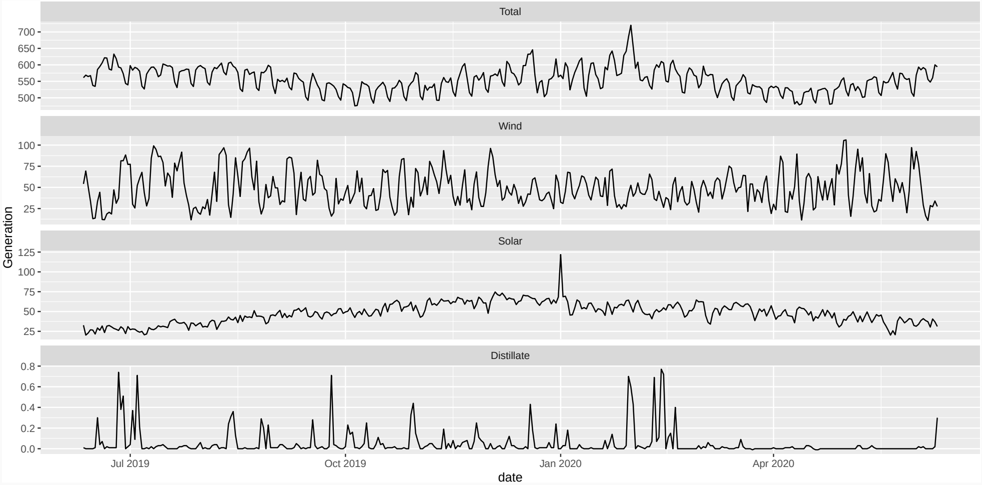
Example: Australian electricity generation

Daily time series from `opennem.org.au`

1. Total = Renewable + Non-renewable
2. Renewable = Batteries + Hydro + Solar + Wind + Biomass
   Non-Renewable = Coal + Gas + Distillate
3. Battery = Battery (Discharging) + Battery (Charging)
   Solar = Solar (Rooftop) + Solar (Utility)
   Coal = Black Coal + Brown Coal
   Gas = Gas (OCGT) + Gas (CCGT) + Gas (Steam) + Gas (Recip)

n = 23 series; m = 15 bottom-level series.

Example: Australian electricity generation

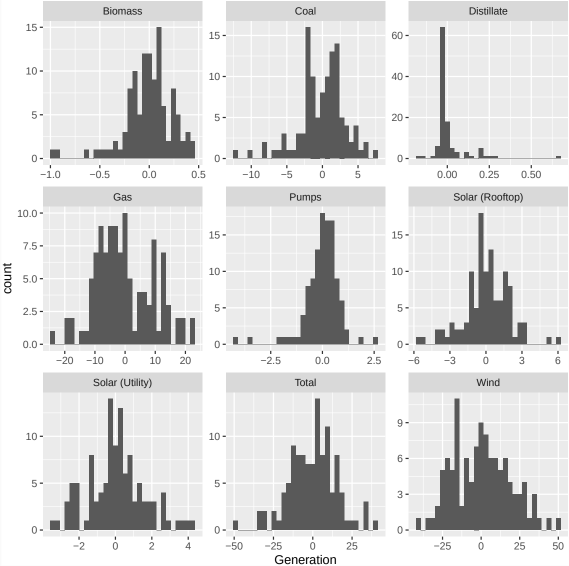


Example: Australian electricity generation

Forecast evaluation

- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using `NNETAR()` function in `fable` package.
- Model could be improved with temperature predictor.

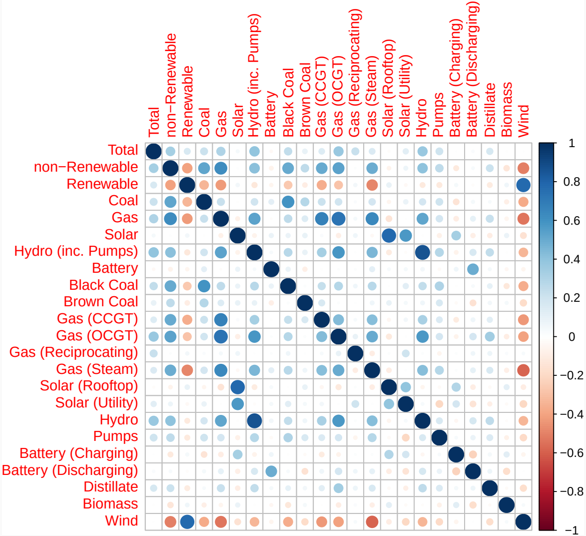
Example: Australian electricity generation



Histogram of residuals:
2 Oct 2019 – 21 Jan 2020

Clearly non-Gaussian

Example: Australian electricity generation



Correlations of residuals:

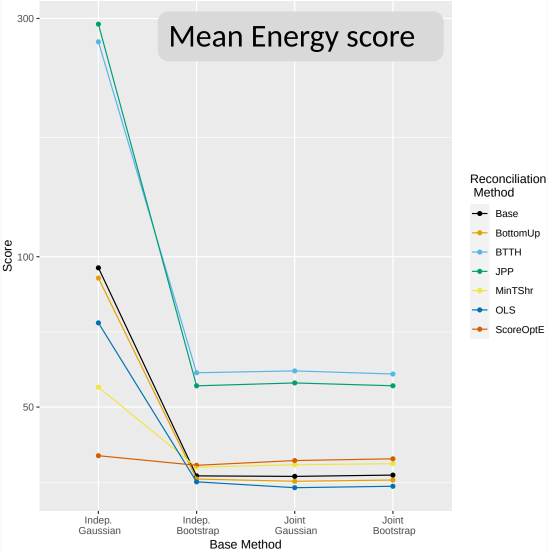
2 Oct 2019 – 21 Jan 2020

Blue = positive correlation.

Red = negative correlation.

Large = stronger correlations.

Example: Australian electricity generation



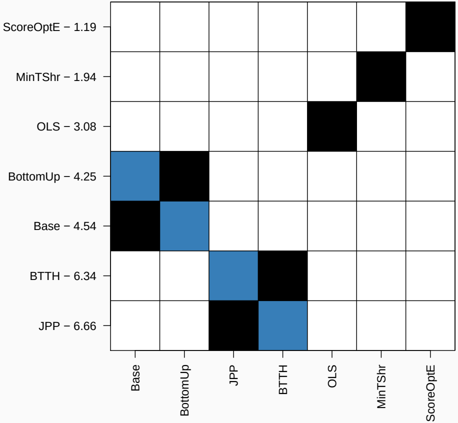
Base residual assumptions

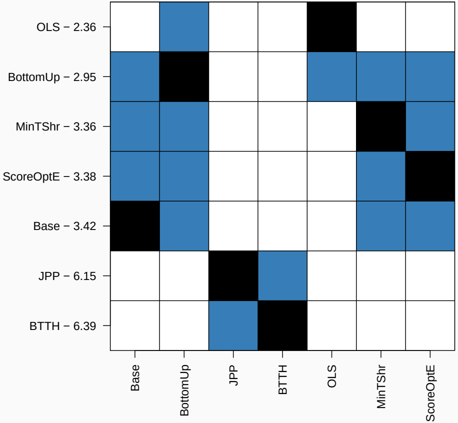
- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

Reconciliation methods

- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS
- MinT(Shrink)
- Score Optimal Reconciliation

Example: Australian electricity generation





Nemenyi test for different scores

Base forecasts are independent and Gaussian.

Nemenyi test for different scores

Base forecasts are obtained by jointly bootstrapping residuals.

Outline

Bayesian forecast reconciliation

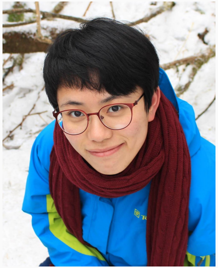
- Park and Nassar (2014)
- Novak, McGarvie, and Garcia (2017)
- Ellison, Dodd, and Forster (2020)
- Eckert, Hyndman, and Panagiotelis (2020)

Outline

ML and regularization

- Qiao and Huang (2018)
- M. Yang, Hu, and Wang (2019)
- Yagli, D. Yang, and Srinivasan (2019)
- Abolghasemi et al. (2019a)
- Spiliotis et al. (2020)
- Punia, Singh, and Madaan (2020)
- Abolghasemi et al. (2019b)

Thanks



More information

- Slides and papers: **robjhyndman.com**
- Packages: **tidyverts.org**
- Forecasting textbook using fable package: **OTexts.com/fpp3**

Find me at ...

@robjhyndman

@robjhyndman

robjhyndman.com

rob.hyndman@monash.edu

Abolghasemi, Mahdi et al. (Dec. 2019a). *Machine learning applications in time series hierarchical forecasting*. arXiv: 1912.00370 [cs.LG]. URL: http://arxiv.org/abs/1912.00370.

– (2019b). *Machine learning applications in time series hierarchical forecasting*. URL: arxiv.org/abs/1912.00370.

Ashouri, Mahsa, Rob J Hyndman, and Galit Shmueli (2019). *Fast forecast reconciliation using linear models*. Working Paper 29/19. Department of Econometrics & Business Statistics, Monash University. URL: robjhyndman.com/publications/lhf.

Di Fonzo, Tommaso and Daniele Girolimetto (2020). *Cross-temporal forecast reconciliation: Optimal combination method and heuristic alternatives*. arXiv: 2006.08570 [stat.ME]. URL: http://arxiv.org/abs/2006.08570.

Eckert, Florian, Rob J Hyndman, and Anastasios Panagiotelis (2020). "Forecasting Swiss exports using Bayesian forecast reconciliation". In: *European J Operational Research*. to appear. URL: robjhyndman.com/publications/swiss-exports/.

Ellison, Joanne, Erengul Dodd, and Jonathan J Forster (June 2020). "Forecasting of cohort fertility under a hierarchical Bayesian approach". In: *Journal of the Royal Statistical Society. Series A*, 183.3, pp. 829–856. URL: https://doi.org/10.1111/rssa.12566.

Kourentzes, Nikolaos and George Athanasopoulos (Mar. 2019). "Cross-temporal coherent forecasts for Australian tourism". In: *Annals Of Tourism Research* 75, pp. 393–409. URL: http://www.sciencedirect.com/science/article/pii/S0160738319300167.

Novak, Julie, Scott McGarvie, and Beatriz Etchegaray Garcia (Nov. 2017). *A Bayesian Model for Forecasting Hierarchically Structured Time Series*. arXiv: 1711.04738 [stat.AP]. URL: http://arxiv.org/abs/1711.04738.

Park, Mijung and Marcel Nassar (2014). "Variational Bayesian inference for forecasting hierarchical time series". In: *International Conference on Machine Learning*. Bejing, China. URL: http://www.gatsby.ucl.ac.uk/~mijung/ICMLworkshop_PARK_NASSAR.pdf.

Punia, Sushil, Surya P Singh, and Jitendra K Madaan (Nov. 2020). "A cross-temporal hierarchical framework and deep learning for supply chain forecasting". In: *Computers & Industrial Engineering* 149, p. 106796. URL: http://www.sciencedirect.com/science/article/pii/S0360835220305040.

Qiao, Mengke and Ke-Wei Huang (2018). "Hierarchical accounting variables forecasting by deep learning methods". In: *Thirty ninth International Conference on Information Systems*. URL: https://aisel.aisnet.org/cgi/viewcontent.cgi?article=1257&context=icis2018.

- Spiliotis, Evangelos et al. (June 2020). *Hierarchical forecast reconciliation with machine learning*. arXiv: 2006.02043 [cs.LG]. URL: http://arxiv.org/abs/2006.02043.
- Yagli, Gokhan Mert, Dazhi Yang, and Dipti Srinivasan (Feb. 2019). "Reconciling solar forecasts: Sequential reconciliation". In: *Solar Energy* 179, pp. 391–397. URL: http://www.sciencedirect.com/science/article/pii/S0038092X18312726.

Yang, Maoxin, Qinghua Hu, and Yun Wang (2019). “Multi-task Learning Method for Hierarchical Time Series Forecasting”. In: *Artificial Neural Networks and Machine Learning – ICANN 2019: Text and Time Series*. Springer International Publishing, pp. 474–485. URL: http://dx.doi.org/10.1007/978-3-030-30490-4_38.