

# Ten years of forecast reconciliation

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[robjhyndman.com/isf2020](http://robjhyndman.com/isf2020)

# Outline

- 1 Hierarchical forecasting 20 years ago
- 2 Point forecast reconciliation
- 3 Example: Australian tourism
- 4 Probabilistic forecast reconciliation
- 5 Example: Australian electricity generation
- 6 Extensions

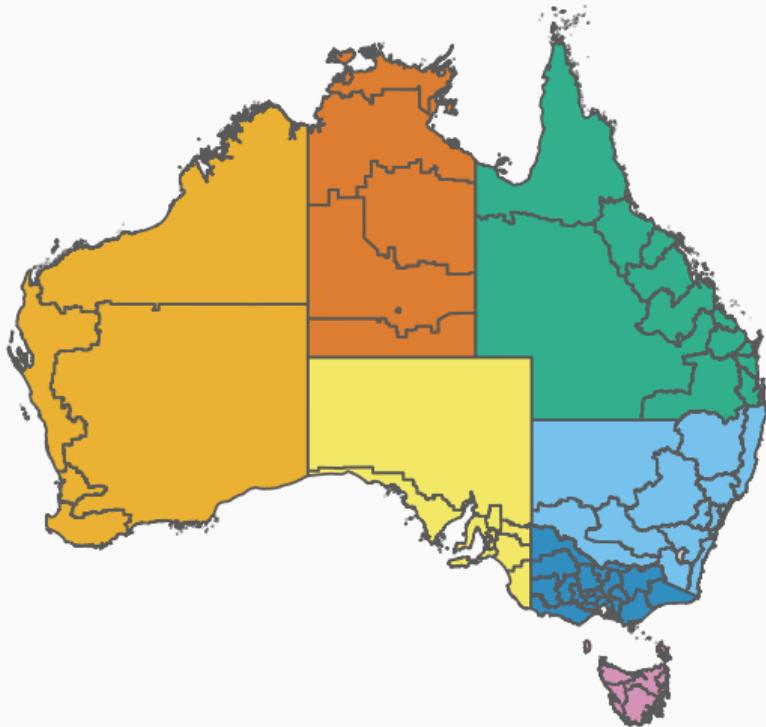
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# Australian tourism regions



# Australian tourism regions



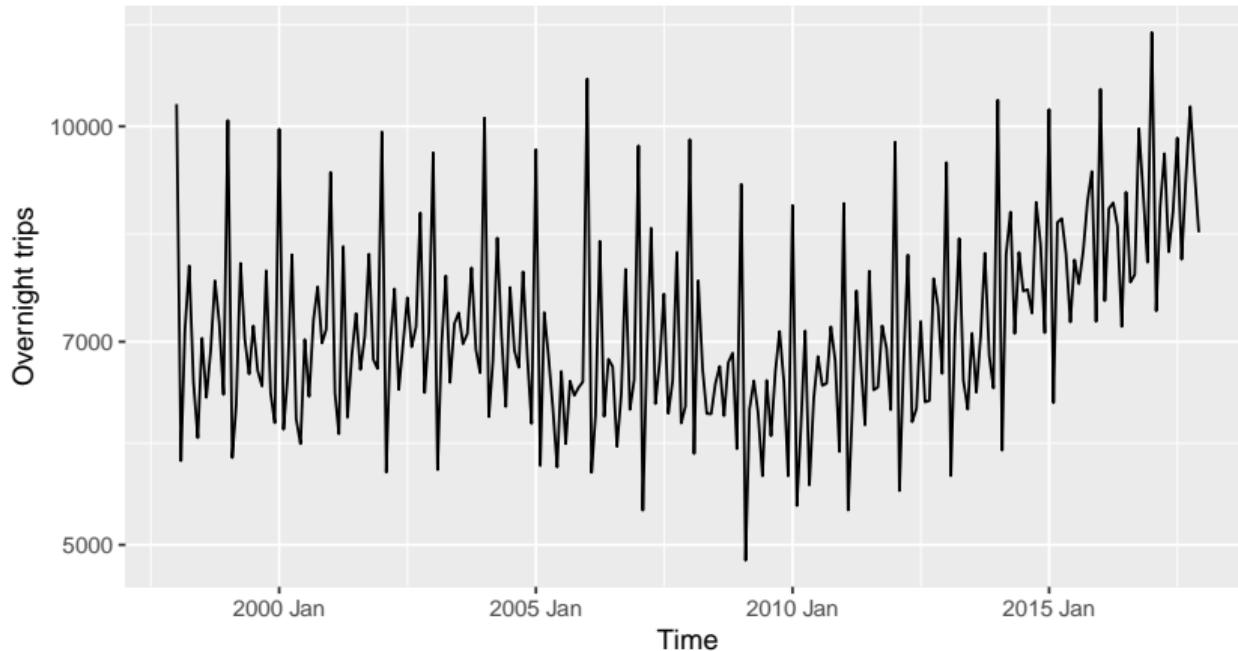
- Monthly data on visitor night from 1998 – 2017
- From *National Visitor Survey*, annual interviews of 120,000 Australians aged 15+.
- Geographical hierarchy split by
  - ▶ 7 states
  - ▶ 27 zones
  - ▶ 75 regions

# Australian tourism data

```
## # A tsibble: 18,000 x 5 [1M]
## # Key:      state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>      <dbl>
## 1 1998 Jan NSW Metro NSW Sydney      926.
## 2 1998 Feb NSW Metro NSW Sydney      647.
## 3 1998 Mar NSW Metro NSW Sydney      716.
## 4 1998 Apr NSW Metro NSW Sydney      621.
## 5 1998 May NSW Metro NSW Sydney      598.
## 6 1998 Jun NSW Metro NSW Sydney      601.
## 7 1998 Jul NSW Metro NSW Sydney      720.
## 8 1998 Aug NSW Metro NSW Sydney      645.
## 9 1998 Sep NSW Metro NSW Sydney      633.
## 10 1998 Oct NSW Metro NSW Sydney      771.
## # ... with 17,990 more rows
```

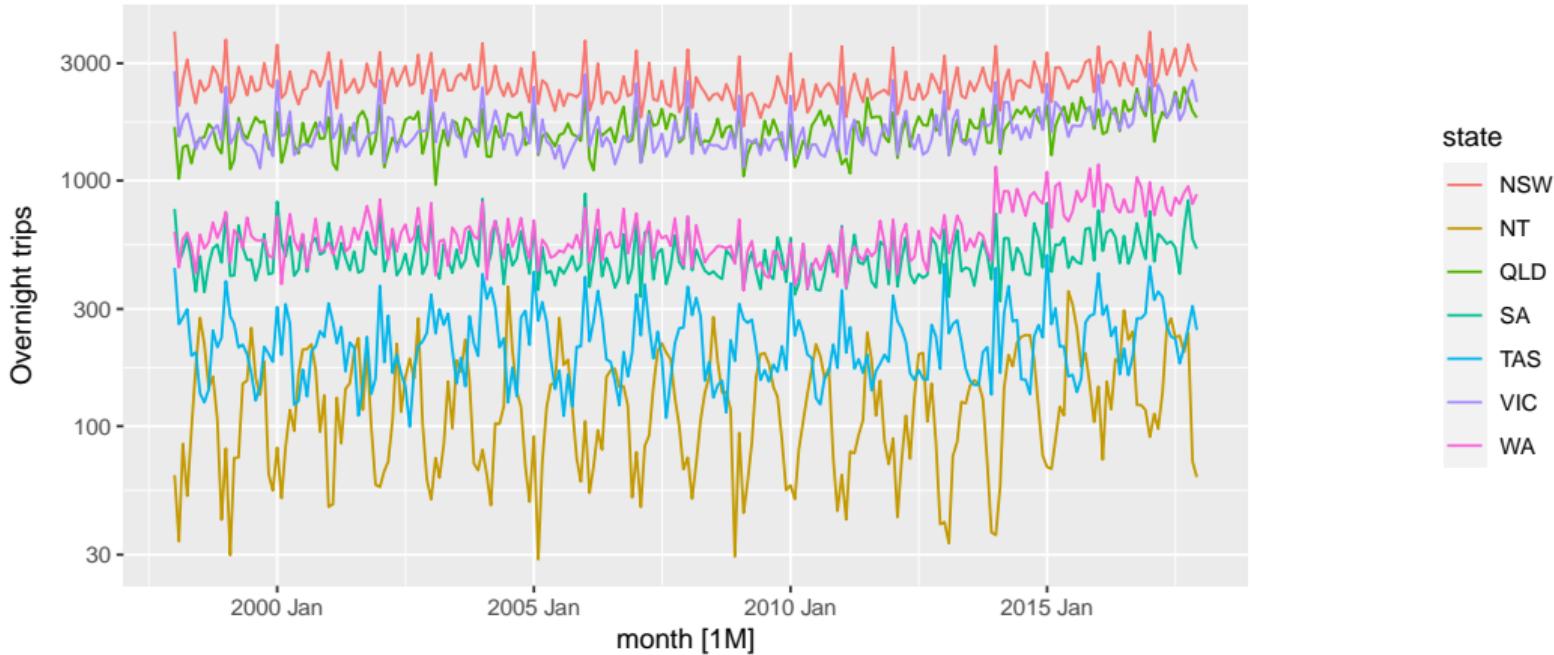
# Australian tourism data

Total domestic travel: Australia



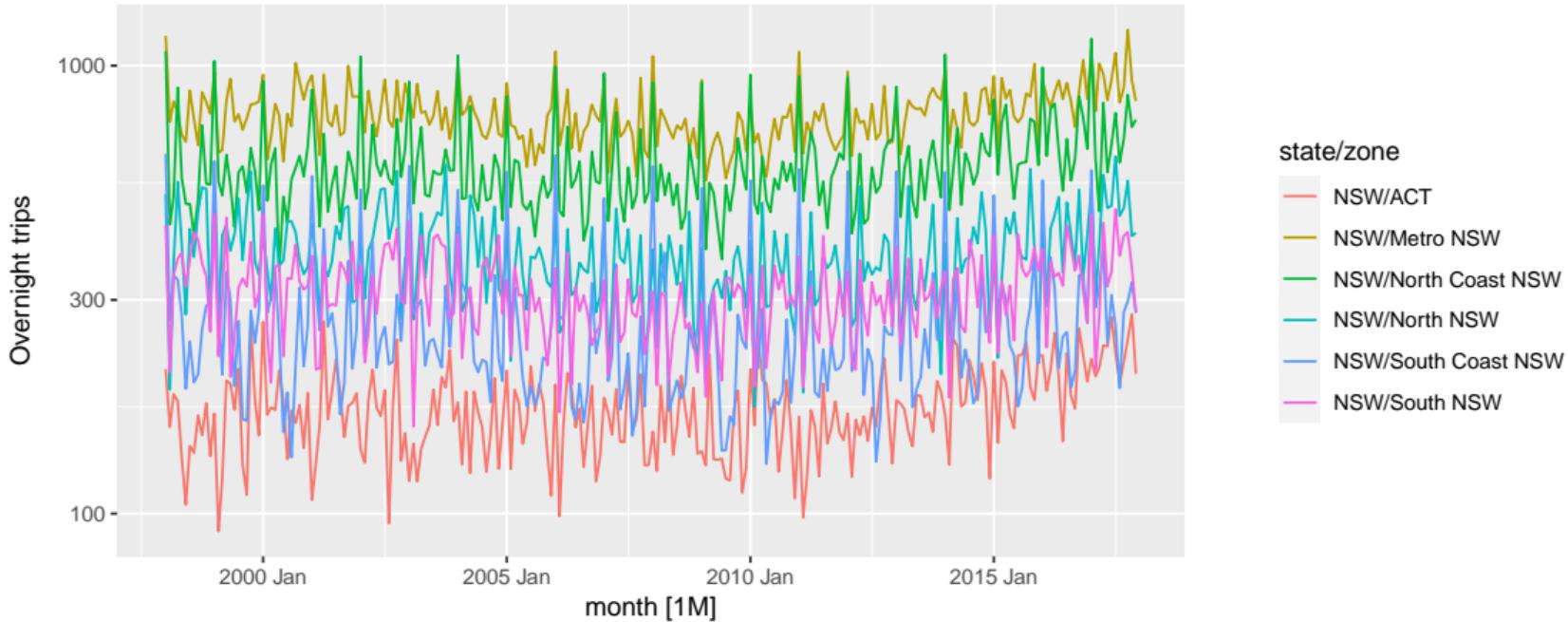
# Australian tourism data

Total domestic travel: by state



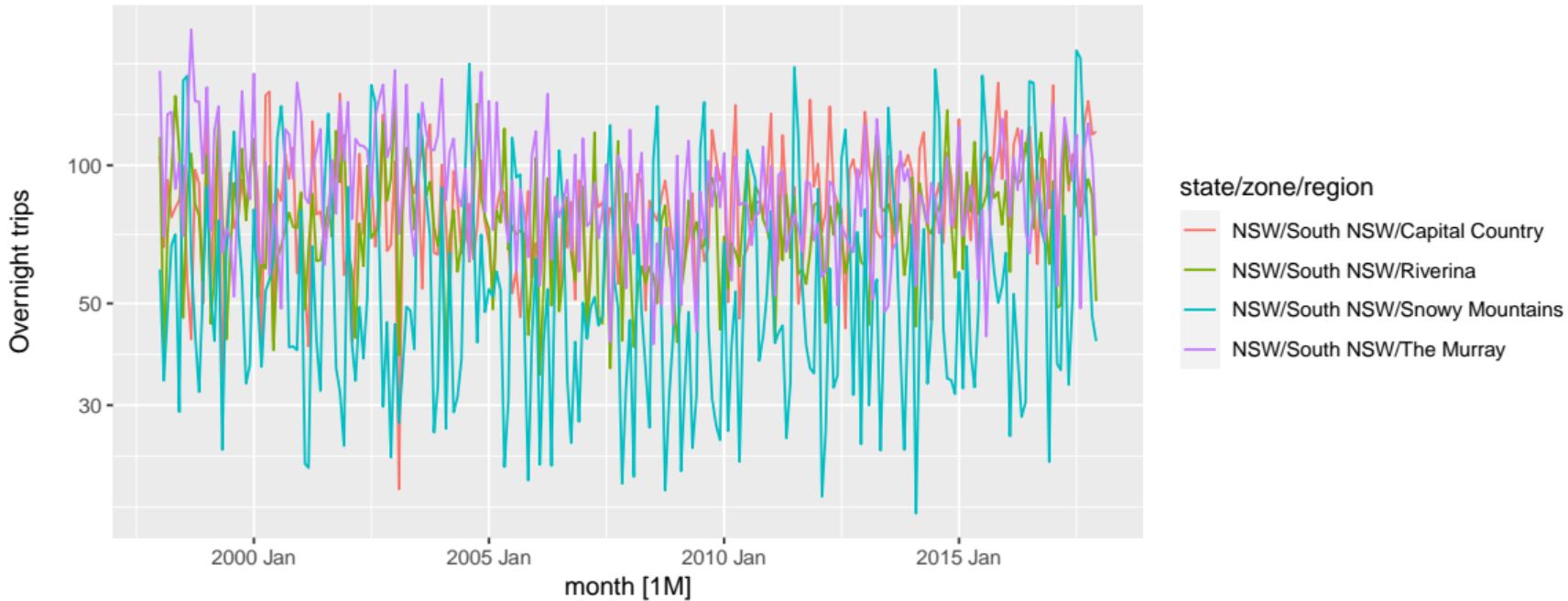
# Australian tourism data

Total domestic travel: NSW by zone

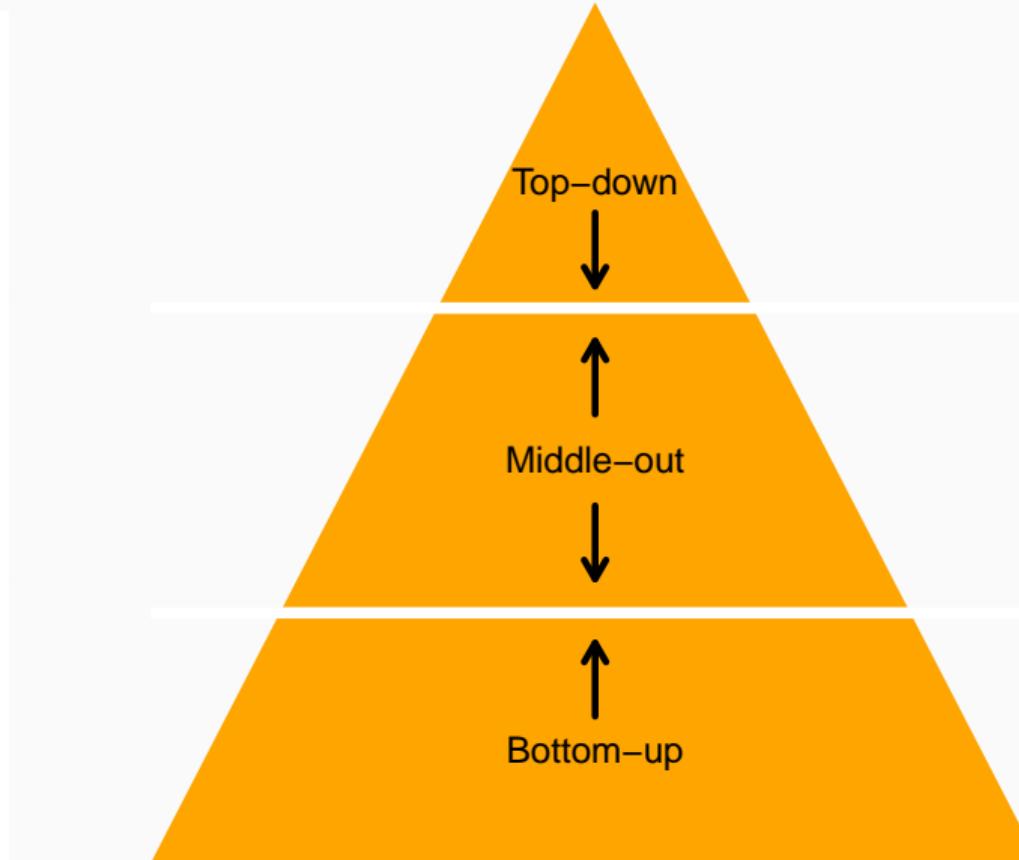
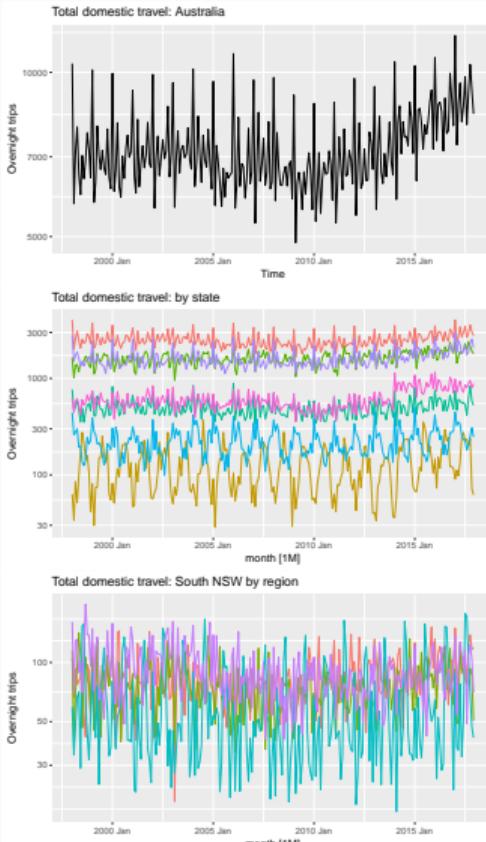


# Australian tourism data

Total domestic travel: South NSW by region



# Hierarchical forecasting 20 years ago



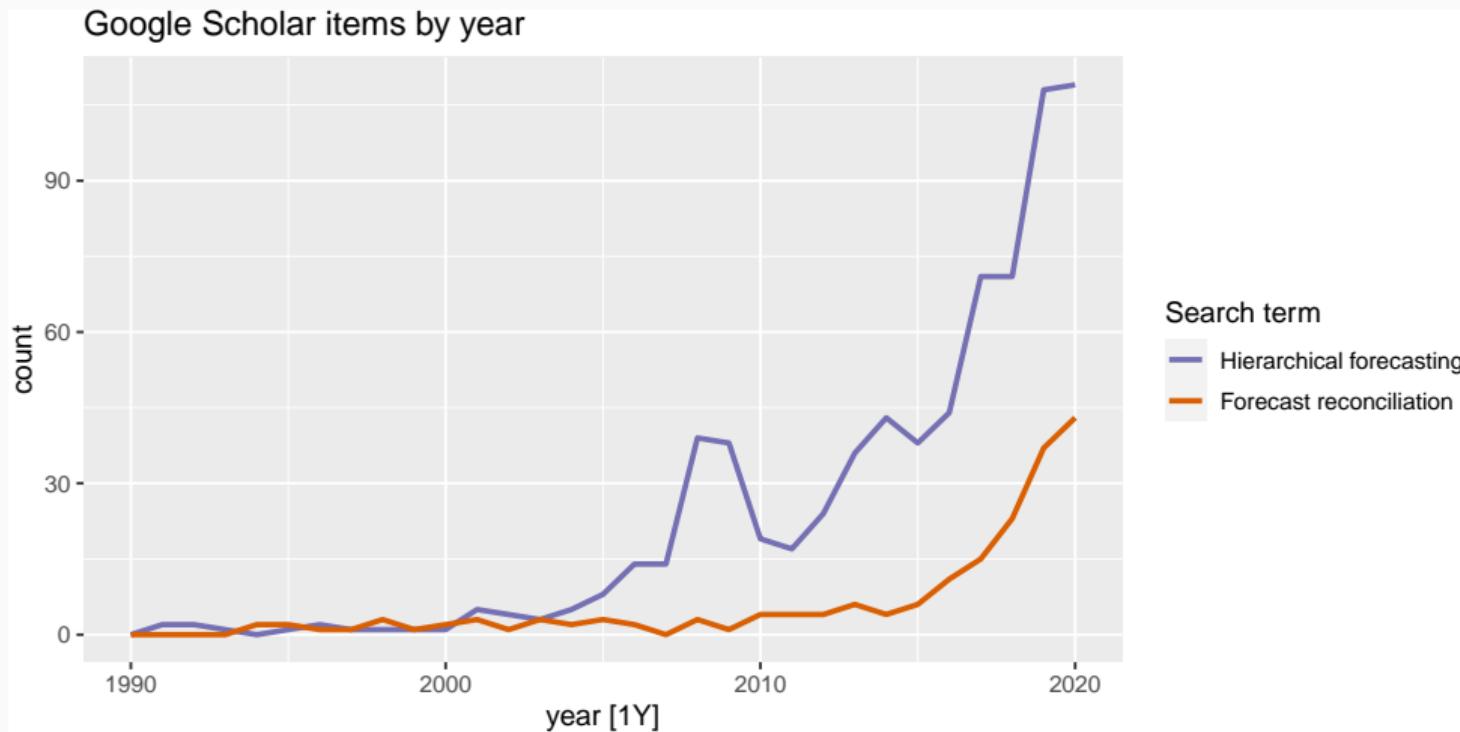
# Forecast reconciliation

- Forecast all series at all levels of aggregation.
- Reconcile forecasts using least squares optimization.

## History

- 2001:** Idea to use all available series to forecast Australia's labour market by occupation.
- 2004:** PhD student Roman Ahmed begins, co-supervised with George Athanasopoulos.
- 2006:** Presentation at ISF, Santander.
- 2007:** Pre-print of "Optimal combination forecasts for hierarchical time series".
- 2009:** Application to Australian tourism published in IJF.
- 2010:** First version of hts package on CRAN.
- 2011:** "Optimal combination forecasts for hierarchical time series" appears in CSDA.

# Forecast reconciliation research



# Forecast reconciliation research



# Outline

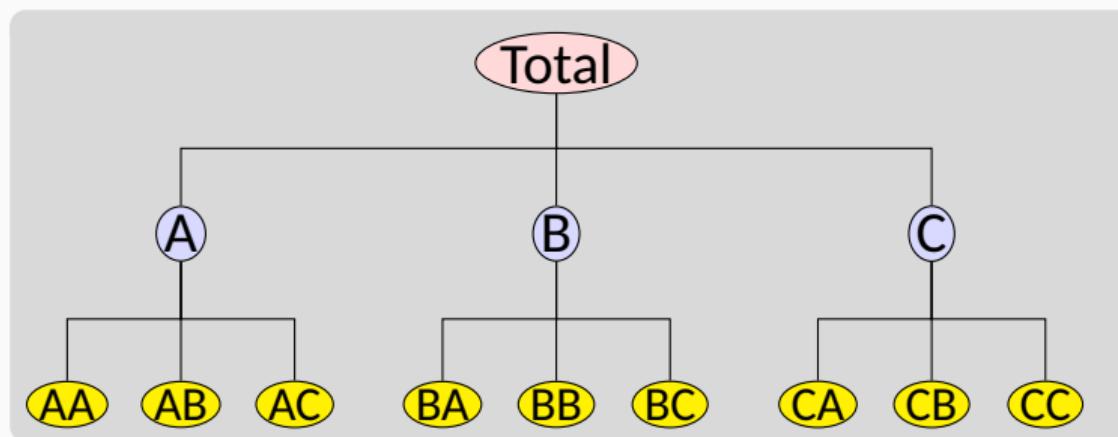
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# Point forecast reconciliation papers

- Hyndman, Ahmed, Athanasopoulos, Shang (2011 *CSDA*) Optimal combination forecasts for hierarchical time series.
- Hyndman, Lee, Wang (2016 *CSDA*) Fast computation of reconciled forecasts for hierarchical and grouped time series.
- Wickramasuriya, Athanasopoulos, Hyndman (2019 *JASA*) Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization.
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020 *IJF*) Forecast reconciliation: A geometric view with new insights on bias correction.

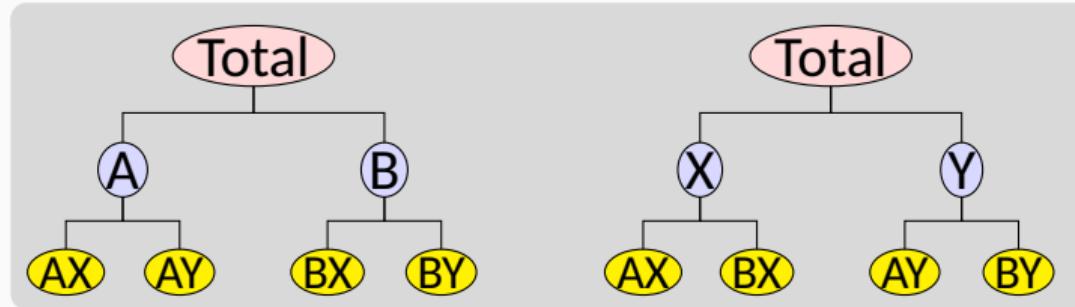
# Hierarchical time series

A **hierarchical time series** is a collection of several time series that are linked together in a hierarchical structure.



# Grouped time series

A **grouped time series** is a collection of time series that can be grouped together in a number of non-hierarchical ways.



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## Examples

- Tourism by state and purpose of travel
- Retail sales by product groups/sub groups, and by countries/regions

# Hierarchical and grouped time series

Every collection of time series with linear constraints can be written as

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

- $\mathbf{y}_t$  = vector of all series at time  $t$
- $y_t$  = aggregate of all series at time  $t$ .
- $y_{X,t}$  = value of series  $X$  at time  $t$ .
- $\mathbf{b}_t$  = vector of most disaggregated series at time  $t$
- $\mathbf{S}$  = “summing matrix” containing the linear constraints.

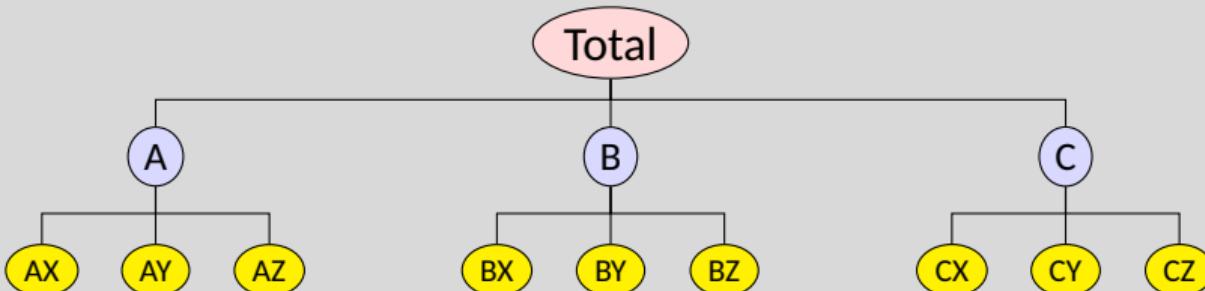


$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_S \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \\ y_{C,t} \end{pmatrix}}_{\mathbf{b}_t} \end{aligned}$$

# Hierarchical time series

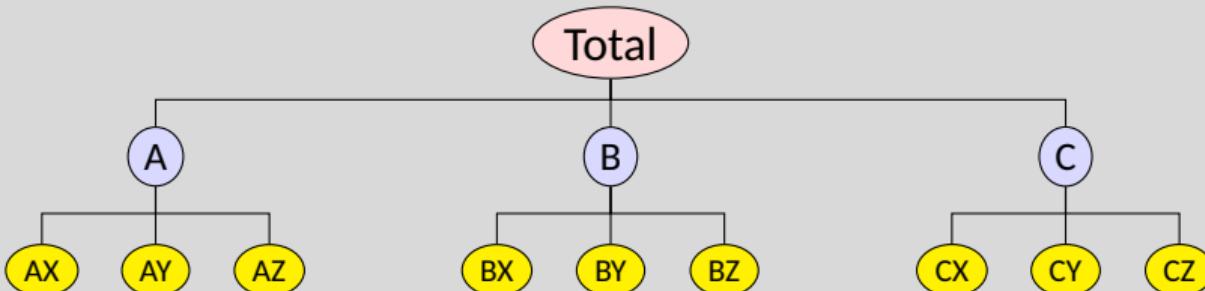


# Hierarchical time series



$$\mathbf{y}_t = \begin{pmatrix} y_t \\ y_{A,t} \\ y_{B,t} \\ y_{C,t} \\ y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y_{AX,t} \\ y_{AY,t} \\ y_{AZ,t} \\ y_{BX,t} \\ y_{BY,t} \\ y_{BZ,t} \\ y_{CX,t} \\ y_{CY,t} \\ y_{CZ,t} \end{pmatrix}$$

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$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

# Definitions

## Coherent subspace

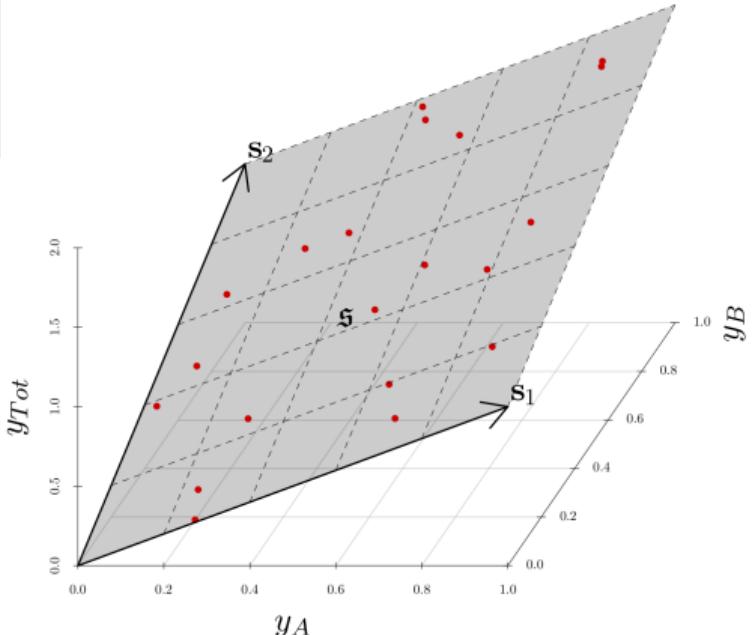
$m$ -dimensional linear subspace  $\mathfrak{s} \subset \mathbb{R}^n$  for which linear constraints hold for all  $\mathbf{y} \in \mathfrak{s}$ .

## Hierarchical time series

An  $n$ -dimensional multivariate time series such that  $\mathbf{y}_t \in \mathfrak{s} \quad \forall t$ .

## Coherent point forecasts

$\tilde{\mathbf{y}}_{t+h|t}$  is coherent if  $\tilde{\mathbf{y}}_{t+h|t} \in \mathfrak{s}$ .



$$Y_{Tot} = Y_A + Y_B$$

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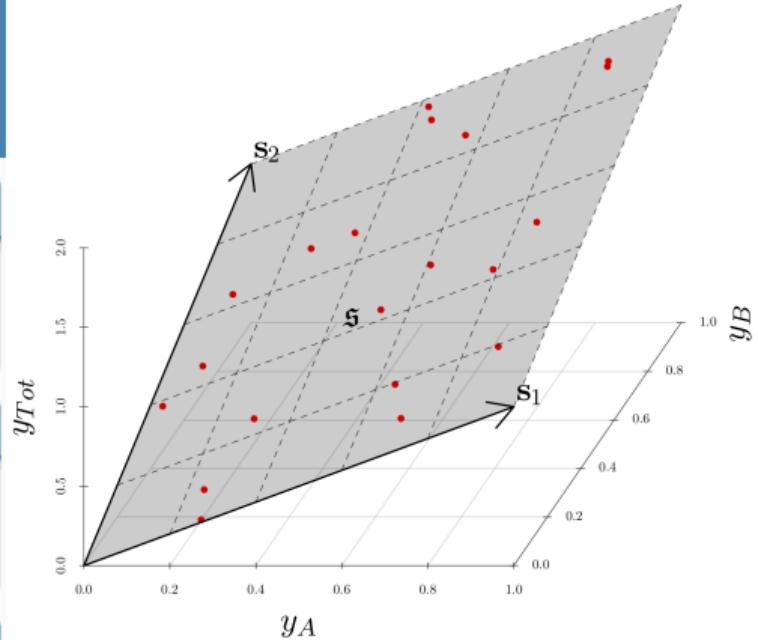
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$$Y_{Tot} = Y_A + Y_B$$

## Reconciled forecasts

Let  $\psi$  be a mapping,  $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$ .  
 $\tilde{\mathbf{y}}_{t+h|t} = \psi(\hat{\mathbf{y}}_{t+h|t})$  “reconciles”  $\hat{\mathbf{y}}_{t+h|t}$ .

# Linear reconciliation

If  $\psi$  is a linear function, then  $\tilde{\mathbf{y}}_{t+h|t} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{t+h|t}$

- $\mathbf{G}$  combines base forecasts  $\hat{\mathbf{y}}_{T+h|T}$  to get bottom-level forecasts.
- $\mathbf{S}$  creates linear combinations.

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## Mean

$$E[\tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] = E[\mathbf{y}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$$

provided  $\mathbf{S}\mathbf{G}\mathbf{S}' = \mathbf{S}$  and

$$E[\hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] = E[\mathbf{y}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$$

i.e., reconciled forecasts are unbiased if base forecasts are unbiased and  $\mathbf{S}\mathbf{G}$  is a projection.

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## Variance

$$\begin{aligned} \mathbf{V}_h &= \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n] \\ &= \mathbf{S}\mathbf{G}\mathbf{W}_h\mathbf{G}'\mathbf{S}' \end{aligned}$$

where

$$\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$$

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where

$$\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{T+h|T} \mid \mathbf{y}_1, \dots, \mathbf{y}_n]$$

## Minimum trace (MinT) reconciliation

If  $\mathbf{S}\mathbf{G}$  is a projection, then the trace of  $\mathbf{V}_h$  is minimized when

$$\mathbf{G} = (\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}$$

# Linear projections

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}\mathbf{G}\hat{\mathbf{y}}_{T+h|T}$$

## Reconciliation method    G

OLS	$(\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
WLS(var)	$(\mathbf{S}'\Lambda_v\mathbf{S})^{-1}\mathbf{S}'\Lambda_v$
WLS(struct)	$(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s$
MinT(sample)	$(\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{sam}}^{-1}$
MinT(shrink)	$(\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}\mathbf{S})^{-1}\mathbf{S}'\hat{\mathbf{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming  $\mathbf{W}_h = k_h \mathbf{W}_1$ .

- $\Lambda_v = \text{diag}(\mathbf{W}_1)^{-1}$
- $\Lambda_s = \text{diag}(\mathbf{S}\mathbf{1})^{-1}$
- $\hat{\mathbf{W}}_{\text{sam}}$  is sample estimate of the residual covariance matrix
- $\hat{\mathbf{W}}_{\text{shr}}$  is shrinkage estimator  $\tau \text{diag}(\hat{\mathbf{W}}_{\text{sam}}) + (1 - \tau)\hat{\mathbf{W}}_{\text{sam}}$   
where  $\tau$  selected optimally.

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# Example: Australian tourism

tourism

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## # A tsibble: 18,000 x 5 [1M]
## # Key:      state, zone, region [75]
##       month state zone      region visitors
##       <mth> <chr> <chr>     <chr>      <dbl>
## 1 1998 Jan NSW Metro NSW Sydney      926.
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## 10 1998 Oct NSW Metro NSW Sydney      771.
```

# Example: Australian tourism

```
tourism_agg <- tourism %>%
  aggregate_key(state(zone/region, visitors = sum(visitors)))
```

```
## # A tsibble: 26,400 x 5 [1M]
## # Key:      state, zone, region [110]
##       month state      zone      region     visitors
##       <mth> <chr>    <chr>    <chr>      <dbl>
## 1 1998 Jan <aggregated> <aggregated> <aggregated> 10376.
## 2 1998 Feb <aggregated> <aggregated> <aggregated> 5746.
## 3 1998 Mar <aggregated> <aggregated> <aggregated> 7129.
## 4 1998 Apr <aggregated> <aggregated> <aggregated> 7939.
## 5 1998 May <aggregated> <aggregated> <aggregated> 6552.
## 6 1998 Jun <aggregated> <aggregated> <aggregated> 5969.
## 7 1998 Jul <aggregated> <aggregated> <aggregated> 7041.
## 8 1998 Aug <aggregated> <aggregated> <aggregated> 6382.
## 9 1998 Sep <aggregated> <aggregated> <aggregated> 6907.
```

# Example: Australian tourism

```
fit <- tourism_agg %>%
  filter(year(month) <= 2015) %>%
  model(ets = ETS(visitors))

## # A mable: 110 x 4
## # Key:      state, zone, region [110]
##   state zone           region          ets
##   <chr> <chr>           <chr>          <model>
## 1 NSW   <aggregated> <aggregated> <ETS(M,N,A)>
## 2 NSW   Metro NSW     <aggregated> <ETS(M,N,A)>
## 3 NSW   North Coast NSW <aggregated> <ETS(M,N,M)>
## 4 NSW   South Coast NSW <aggregated> <ETS(A,N,A)>
## 5 NSW   South NSW     <aggregated> <ETS(M,N,M)>
## 6 NSW   North NSW     <aggregated> <ETS(M,N,A)>
## 7 NSW   ACT            <aggregated> <ETS(M,N,A)>
## 8 NSW   Metro NSW     Sydney         <ETS(M,N,A)>
```

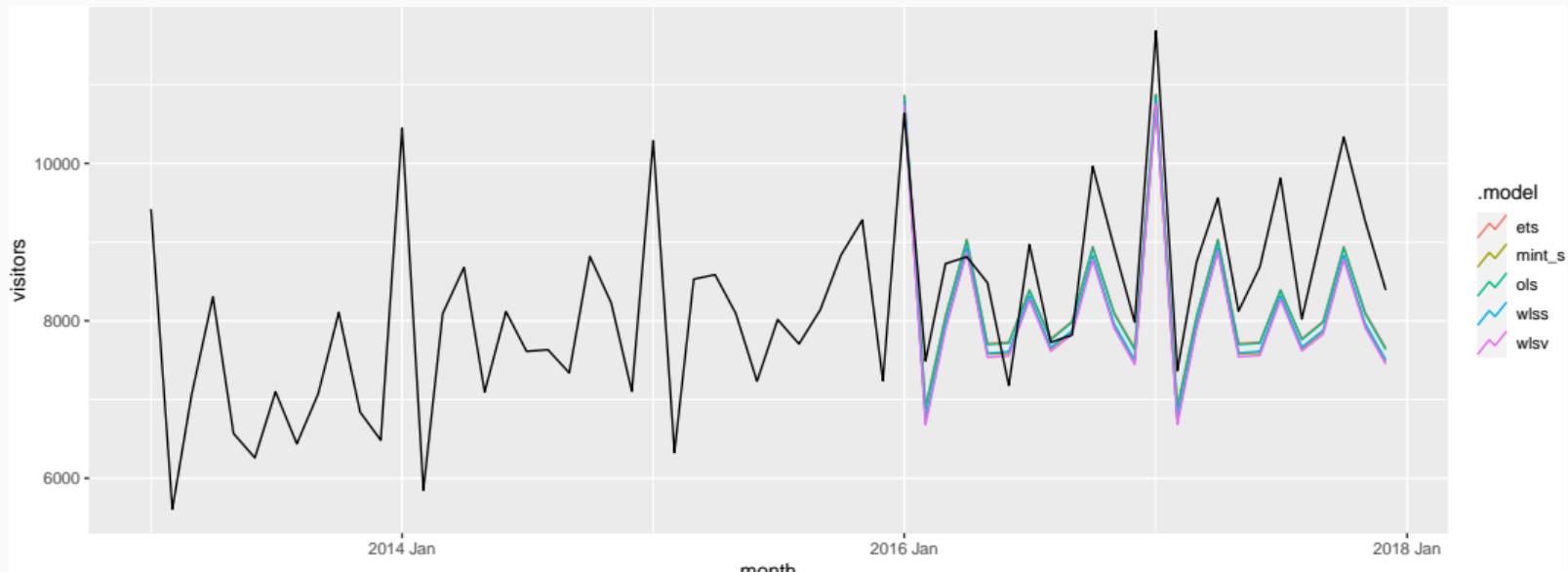
# Example: Australian tourism

```
fc <- fit %>%
  reconcile(
    ols = min_trace(ets, method="ols"),
    wlsv = min_trace(ets, method="wls_var"),
    wlss = min_trace(ets, method="wls_struct"),
    #mint_c = min_trace(ets, method="mint_cov"),
    mint_s = min_trace(ets, method="mint_shrink"),
  ) %>%
  forecast(h = "2 years")
```

```
## # A fable: 5,280 x 7 [1M]
## # Key:      state, zone, region, .model [220]
##   state zone       region     .model     month   visitors .mean
##   <chr> <chr>       <chr>     <chr>     <mth>       <dist> <dbl>
## 1 NSW  <aggregated> <aggregated> ets     2016 Jan N(3679, 71136) 3679.
## 2 NSW  <aggregated> <aggregated> ets     2016 Feb N(2241, 27912) 2241.
```

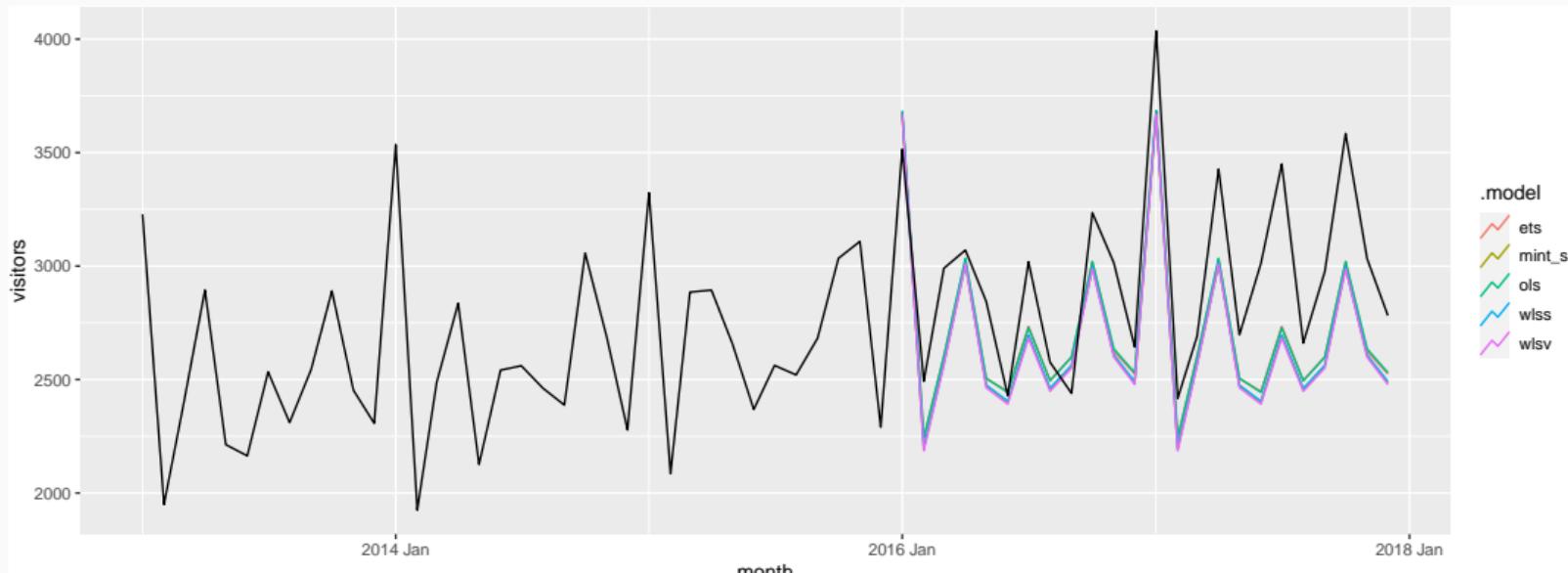
# Example: Australian tourism

```
fc %>%
  filter(is_aggregated(state)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



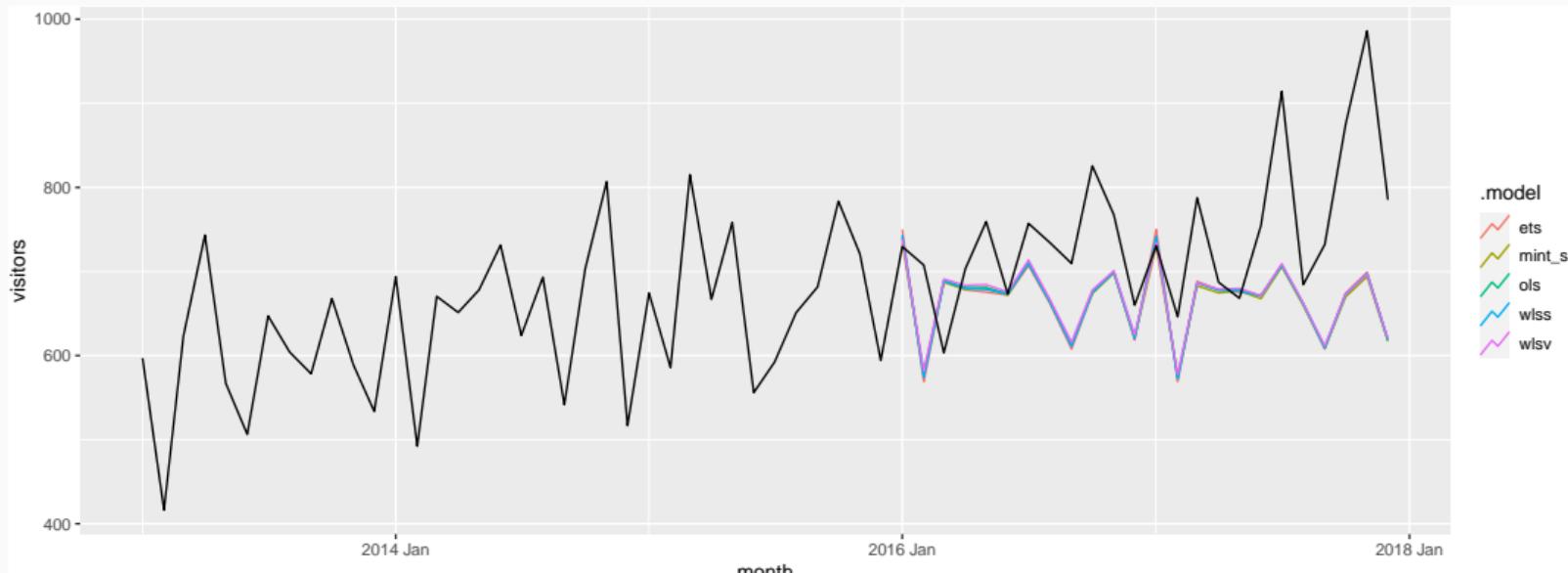
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```
fc %>%
  filter(state == "NSW" & is_aggregated(zone)) %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



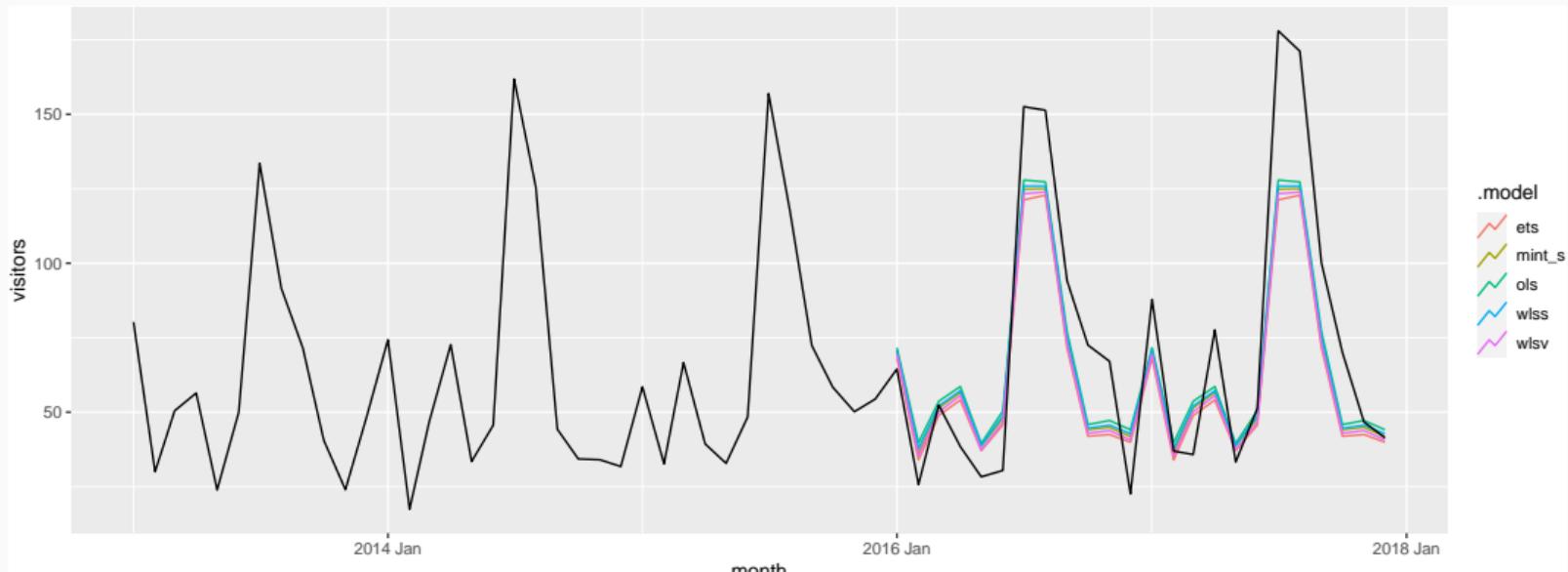
# Example: Australian tourism

```
fc %>%
  filter(region == "Melbourne") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



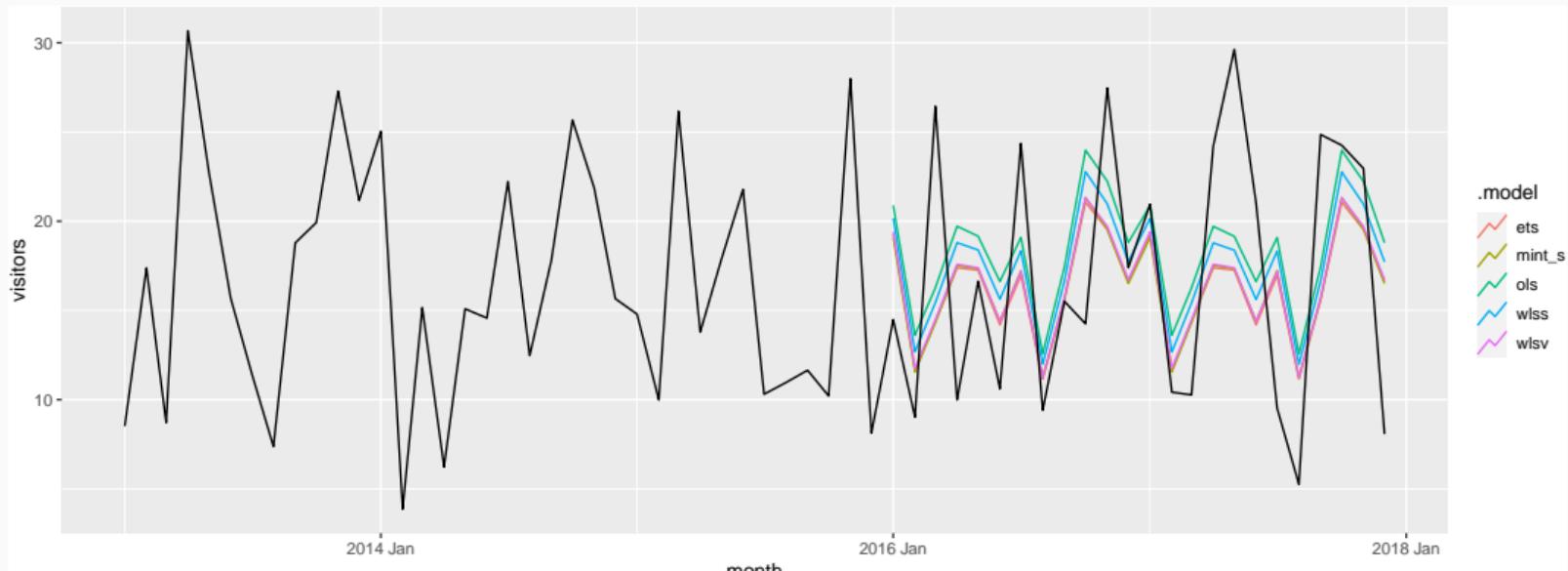
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```
fc %>%
  filter(region == "Snowy Mountains") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



# Example: Australian tourism

```
fc %>%
  filter(region == "Barossa") %>%
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



# Example: Australian tourism

```
fc %>%  
  accuracy(data = tourism_agg,  
            measures = list(rmsse = RMSSE))
```

```
## # A tibble: 550 x 6  
##   .model state zone          region     .type rmsse  
##   <chr>   <chr> <chr>        <chr>     <chr> <dbl>  
## 1 ets     NSW   ACT         Canberra    Test    0.835  
## 2 ets     NSW   ACT         <aggregated> Test    0.835  
## 3 ets     NSW   Metro NSW Central Coast Test    0.747  
## 4 ets     NSW   Metro NSW Sydney      Test    1.16  
## 5 ets     NSW   Metro NSW <aggregated> Test    1.18  
## 6 ets     NSW   North Coast NSW Hunter    Test    1.21  
## 7 ets     NSW   North Coast NSW North Coast NSW Test    0.884  
## 8 ets     NSW   North Coast NSW <aggregated> Test    1.02  
## 9 ets     NSW   North NSW    Blue Mountains Test    1.02
```

# Example: Australian tourism

```
fc %>%
  accuracy(tourism_agg,
            measures = list(mase = MASE, rmsse = RMSSE)) %>%
  group_by(.model) %>%
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) %>%
  arrange(rmsse)
```

```
## # A tibble: 5 x 3
##   .model    mase   rmsse
##   <chr>    <dbl>  <dbl>
## 1 ols      0.930  0.926
## 2 wlss     0.949  0.948
## 3 mint_s   0.953  0.954
## 4 wlsv     0.964  0.965
## 5 ets      0.968  0.968
```

# Example: Australian tourism

```
## # A tibble: 20 x 4
## # Groups:   .model [5]
##       .model level      mase rmsse
##       <chr>   <fct>     <dbl> <dbl>
## 1 ets    National  1.44  1.27
## 2 ols    National  1.46  1.29
## 3 wlss   National  1.61  1.43
## 4 mint_s National  1.64  1.45
## 5 wlsv   National  1.69  1.49
## 6 ols    State     1.07  1.08
## 7 ets    State     1.10  1.11
## 8 wlss   State     1.13  1.14
## 9 mint_s State     1.15  1.15
## 10 wlsv   State     1.18  1.17
## 11 ols    Zone      0.954 0.948
## 12 wlss   Zone      0.987 0.980
## 13 mint_s Zone      0.995 0.988
## 14 ets    Zone      1.01  0.999
## 15 wlsv   Zone      1.01  1.00
## 16 ols    Region    0.901 0.895
## 17 wlss   Region    0.910 0.907
## 18 mint_s Region    0.911 0.911
## 19 wlsv   Region    0.917 0.919
## 20 ets    Region    0.935 0.938
```

- Overall, every reconciliation method is better than the base ETS forecasts.
- OLS is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

# Outline

- 1 Hierarchical forecasting 20 years ago
- 2 Point forecast reconciliation
- 3 Example: Australian tourism
- 4 Probabilistic forecast reconciliation
- 5 Example: Australian electricity generation
- 6 Extensions

# Probabilistic forecast reconciliation

## Key papers

- Ben Taieb, Taylor, Hyndman (*ICML*, 2017)
- Jeon, Panagiotelis, Petropoulos (*EJOR*, 2019)
- Ben Taieb, Taylor, Hyndman (*JASA*, 2020)
- Panagiotelis, Gamakumara, Athanasopoulos, Hyndman (2020).  
[robjhyndman.com/publications/coherentprob/](http://robjhyndman.com/publications/coherentprob/)

# Probabilistic forecast reconciliation

## Key papers

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[robjhyndman.com/publications/coherentprob/](http://robjhyndman.com/publications/coherentprob/)
- The reconciled multivariate density must lie on the coherent subspace.
- The univariate density at each node is a convolution of the densities of its children.

# Construction of reconciled distributions

## Reconciled density of bottom-level

Density of bottom-level series under reconciled distribution is

$$\tilde{f}_b(\mathbf{b}) = |\mathbf{G}^*| \int \hat{f}(\mathbf{G}^- \mathbf{b} + \mathbf{G}_\perp \mathbf{a}) d\mathbf{a}$$

- $\hat{f}$  is density of incoherent base probabilistic forecast
- $\mathbf{G}^-$  is  $n \times m$  generalised inverse of  $\mathbf{G}$  st  $\mathbf{G}\mathbf{G}^- = \mathbf{I}$
- $\mathbf{G}_\perp$  is  $n \times (n - m)$  orthogonal complement to  $\mathbf{G}$  st  $\mathbf{G}\mathbf{G}_\perp = \mathbf{0}$
- $\mathbf{G}^* = (\mathbf{G}^- : \mathbf{G}_\perp)$ , and  $\mathbf{b}$  and  $\mathbf{a}$  are obtained via

the change of variables  $\mathbf{y} = \mathbf{G}^* \begin{pmatrix} \mathbf{b} \\ \mathbf{a} \end{pmatrix}$

# Construction of reconciled distributions

## Reconciled density of full hierarchy

Density of full hierarchy under reconciled distribution is

$$\tilde{f}_y(y) = |S^*| \tilde{f}_b(S^- y) \mathbb{1}\{y \in s\}$$

- $S^* = (S^{-'} \ S_{\perp})'$
- $S^-$  is  $m \times n$  generalised inverse of  $S$  such that  $S^- S = I$ ,
- $S_{\perp}$  is  $n \times (n - m)$  orthogonal complement to  $S$  such that  $S_{\perp}' S = 0$ .

### Gaussian reconciliation

If the incoherent base forecasts are  $N(\hat{\mu}, \hat{\Sigma})$ , then the reconciled density is  $N(SG\hat{\mu}, SG\hat{\Sigma}G'S')$ .

### Bootstrap reconciliation

Reconciling sample paths from incoherent distributions works.

# Evaluating probabilistic forecasts

## Proper scoring rule

optimized when true forecast distribution is used.

# Evaluating probabilistic forecasts

## Proper scoring rule

optimized when true forecast distribution is used.

### Scoring Rule    Coherent v Incoherent    Coherent v Coherent

---

Log Score      Not proper

- Ordering preserved if compared using bottom-level only

Energy Score   Proper

- Full hierarchy should be used.
- Rankings may change otherwise.

## Score optimal reconciliation

Algorithm proposed by Panagiotelis et al (2020) for optimizing  $\mathbf{G}$  using stochastic gradient descent to optimize Energy Score.

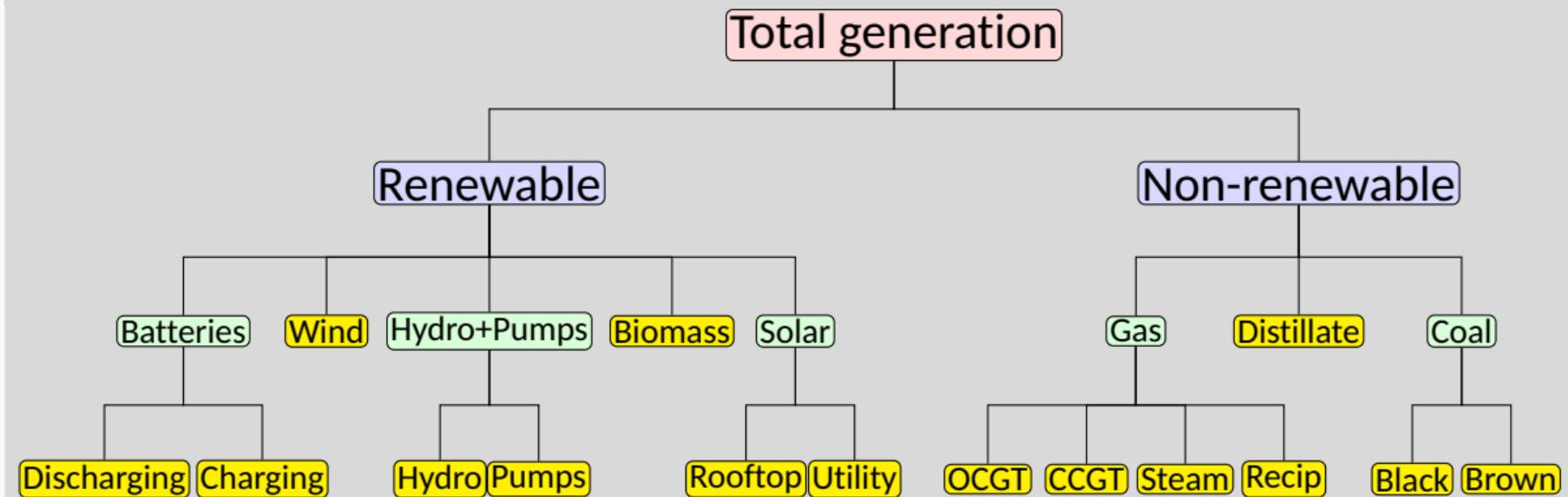
- 1 Compute base forecasts over a test set.
- 2 Compute OLS reconciliation:  $\mathbf{G} = (\mathbf{S}'\mathbf{S})^{-1}\mathbf{S}'$
- 3 Iteratively update  $\mathbf{G}$  using SGD with Adam method and ES objective over a test set

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# Example: Australian electricity generation

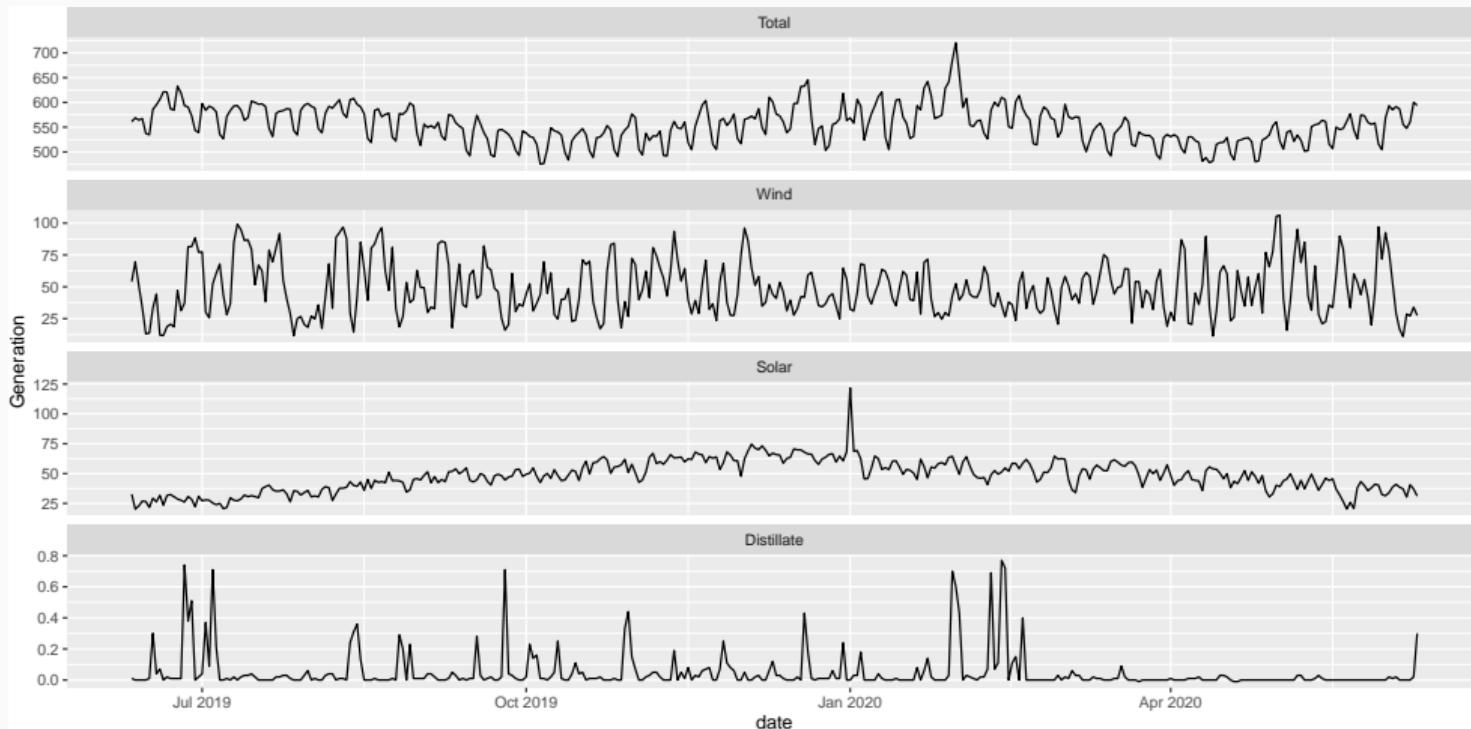
Daily time series from [opennem.org.au](http://opennem.org.au)



$n = 23$  series

$m = 15$  bottom-level series

# Example: Australian electricity generation

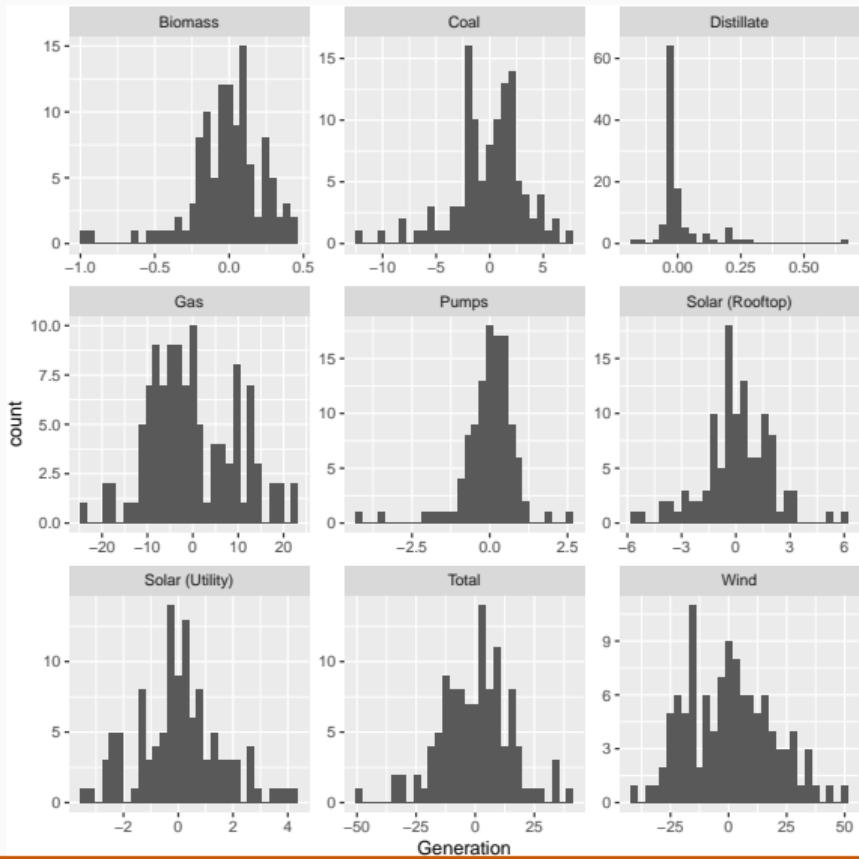


# Example: Australian electricity generation

## Forecast evaluation

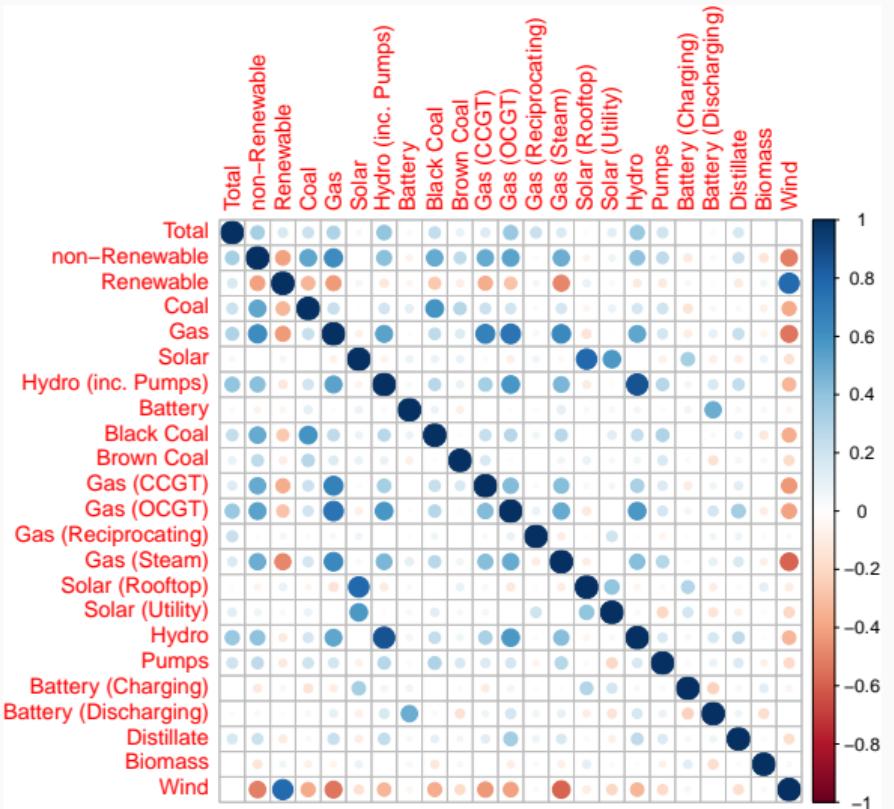
- Rolling window of 140 days training data, and one-step-forecasts for 170 days test data.
- One-layer feed-forward neural network with up to 28 lags of target variable as inputs.
- Implemented using NNETAR() function in fable package.
- Model could be improved with temperature predictor.

# Example: Australian electricity generation



Histogram of residuals:  
2 Oct 2019 - 21 Jan 2020  
Clearly non-Gaussian

# Example: Australian electricity generation



Correlations of residuals:

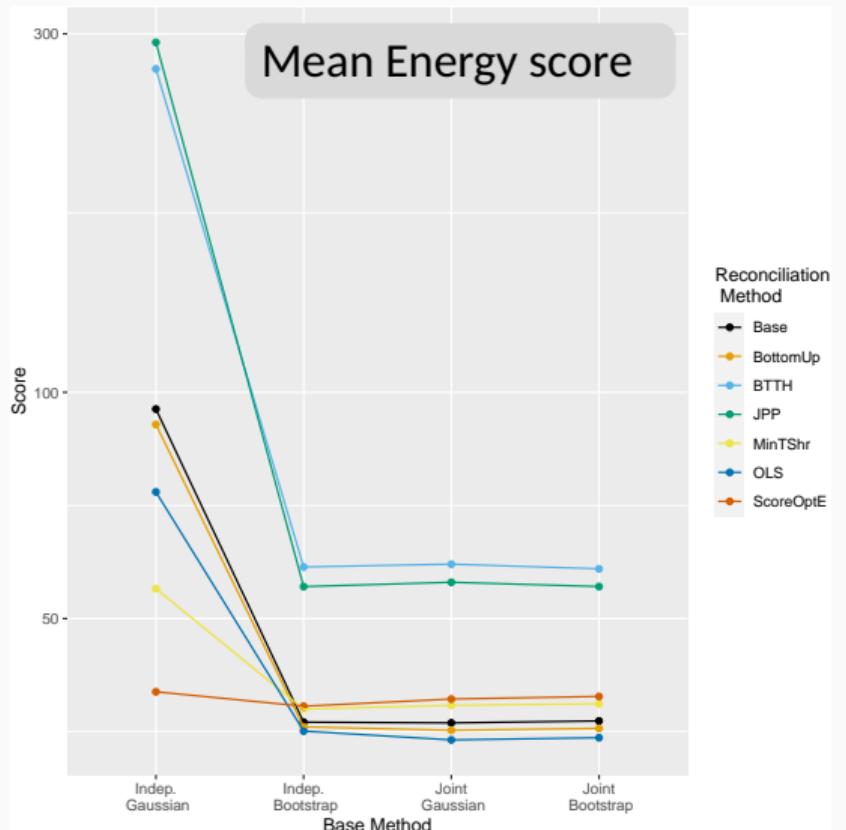
2 Oct 2019 - 21 Jan 2020

Blue = positive correlation.

Red = negative correlation.

Large = stronger correlations.

# Example: Australian electricity generation



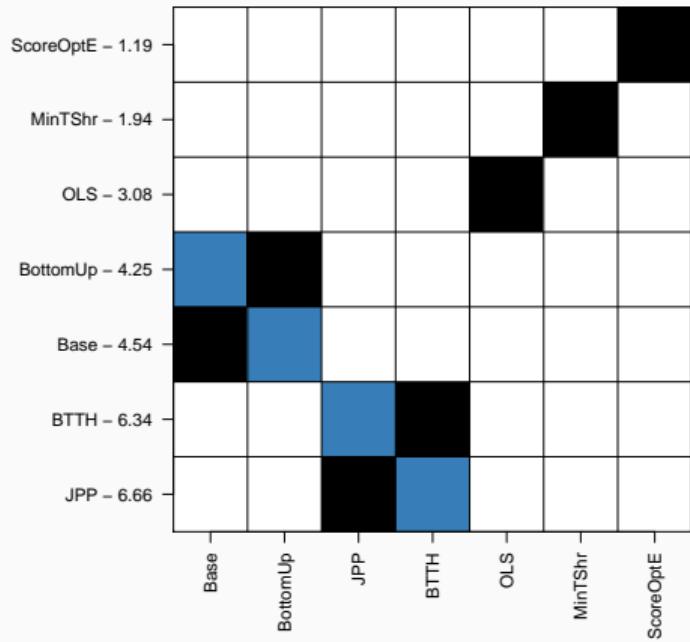
## Base residual assumptions

- Gaussian independent
- Gaussian dependent
- Non-Gaussian independent
- Non-Gaussian dependent

## Reconciliation methods

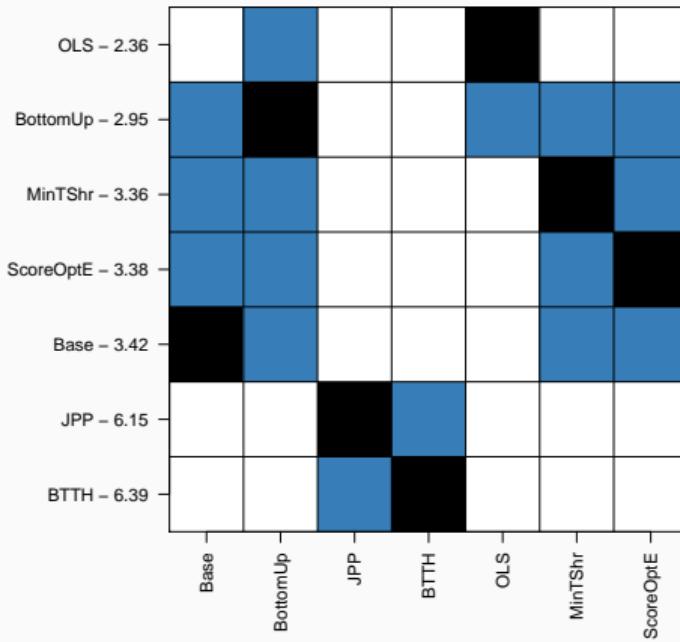
- Base
- BottomUp
- BTTH: Ben Taieb, Taylor, Hyndman
- JPP: Jeon, Panagiotelis, Petropoulos
- OLS
- MinT(Shrink)
- Score Optimal Reconciliation

# Example: Australian electricity generation



## Nemenyi test for different scores

Base forecasts are independent and Gaussian.



## Nemenyi test for different scores

Base forecasts are obtained by jointly bootstrapping residuals.

# Outline

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# Linear combinations

## Optimal linear reconciliation

$$\mathbf{y}_t = \mathbf{S}\mathbf{b}_t$$

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{T+h|T}$$

# Linear combinations

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$\mathbf{S}$  can contain any values, not just 0s and 1s.

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$\mathbf{S}$  can contain any values, not just 0s and 1s.

- Shang, Hyndman (*JCGS*, 2017)
- Athanasopoulos, Gamakumara, Panagiotelis, Hyndman, Affan (*Springer*, 2020)

# Reconciled linear regression forecasts

If the base forecasts are from a linear regression model, then we can produce coherent forecasts in one step:

$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\boldsymbol{\Lambda}_s\mathbf{S})^{-1}\mathbf{S}'\boldsymbol{\Lambda}_s\hat{\mathbf{y}}_{T+h|T} \quad \hat{\mathbf{y}}_{T+h|T} = \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{T}$$

- $\mathbf{X}$  is matrix of predictors for training set
- $\mathbf{X}_{T+h}^*$  is vector of predictors for time  $T + h$

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- $\mathbf{X}$  is matrix of predictors for training set
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$$\mathbf{V}_h = \sigma^2 \mathbf{S}(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'\Lambda_s [1 + \mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}(\mathbf{X}_{T+h}^*)'] \Lambda_s \mathbf{S}'(\mathbf{S}'\Lambda_s\mathbf{S})^{-1}\mathbf{S}'$$

- $\sigma^2$  is variance of base model residuals.

# Reconciled linear regression forecasts

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$$\tilde{\mathbf{y}}_{T+h|T} = \mathbf{S}(\mathbf{S}'\boldsymbol{\Lambda}_s\mathbf{S})^{-1}\mathbf{S}'\boldsymbol{\Lambda}_s\mathbf{X}_{T+h}^*(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}$$

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- $\sigma^2$  is variance of base model residuals.

Reference: Ashouri, Hyndman, and Shmueli (2019).

[robjhyndman.com/publications/lhf/](http://robjhyndman.com/publications/lhf/)

# Non-negative forecast reconciliation

## Minimum trace (MinT) reconciliation

The trace of  $V_h$  is minimized when

$$\tilde{\mathbf{b}}_{T+h|T} = (\mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S})^{-1} \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T}$$

subject to unbiasedness preservation ( $\mathbf{S}\mathbf{G}\mathbf{S}' = \mathbf{S}$ ).

# Non-negative forecast reconciliation

## Minimum trace (MinT) reconciliation

The trace of  $\mathbf{V}_h$  is minimized when

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subject to unbiasedness preservation ( $\mathbf{S}\mathbf{G}\mathbf{S}' = \mathbf{S}$ ).

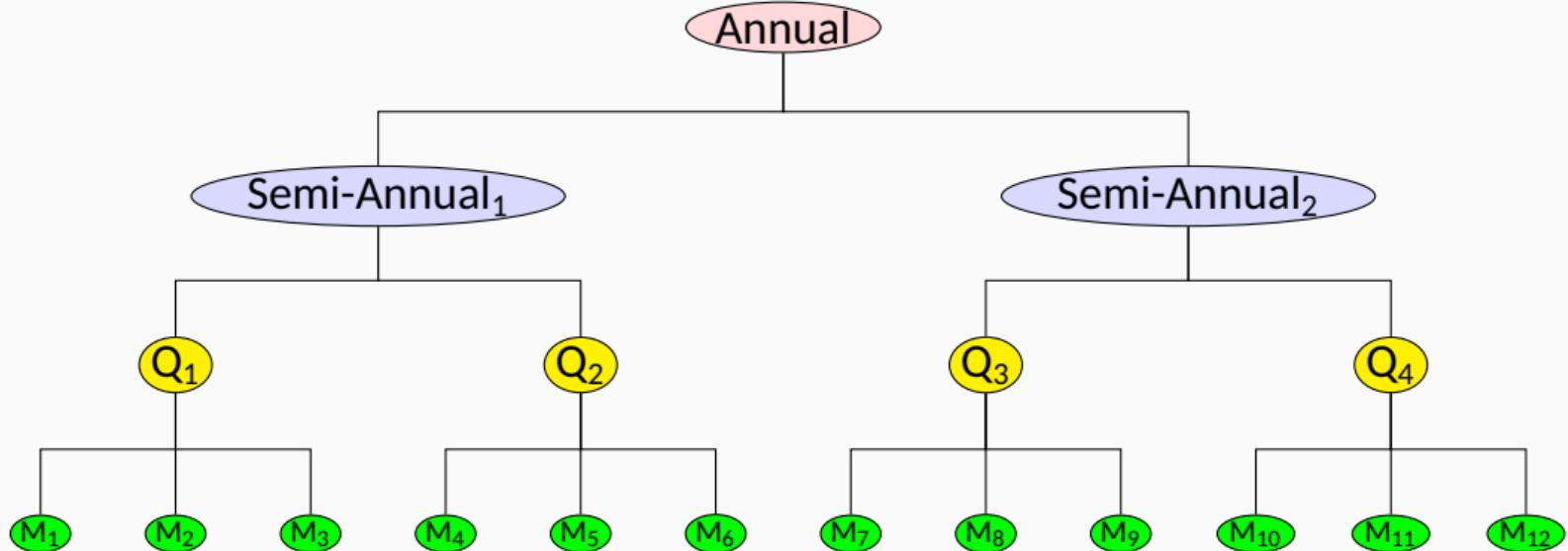
Wickramasuriya, Turlach and Hyndman (S&C, 2020) replace the unbiased constraint by a non-negative constraint:

$$\tilde{\mathbf{b}}_{T+h|T} \geq 0$$

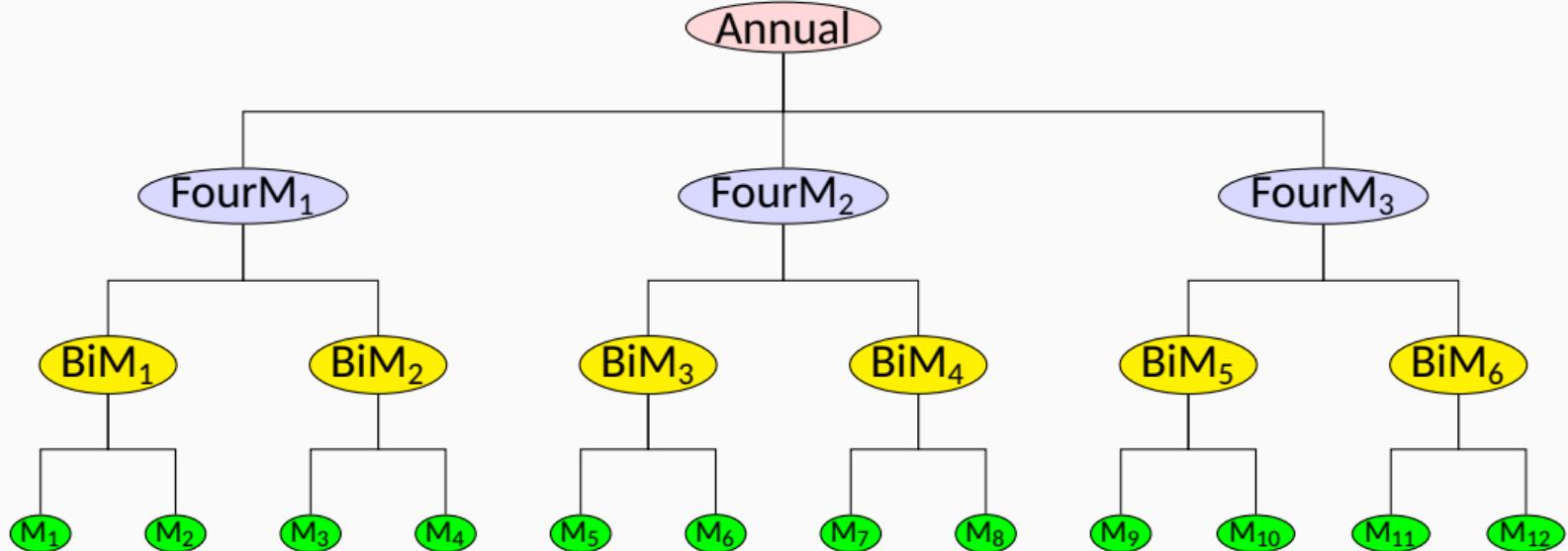
and show that it can be solved via quadratic programming:

$$\min_{\mathbf{b}} \frac{1}{2} \mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \mathbf{S} \mathbf{b} - \mathbf{b}' \mathbf{S}' \mathbf{W}_h^{-1} \hat{\mathbf{y}}_{T+h|T} \quad \text{s.t. } \mathbf{b} \geq 0$$

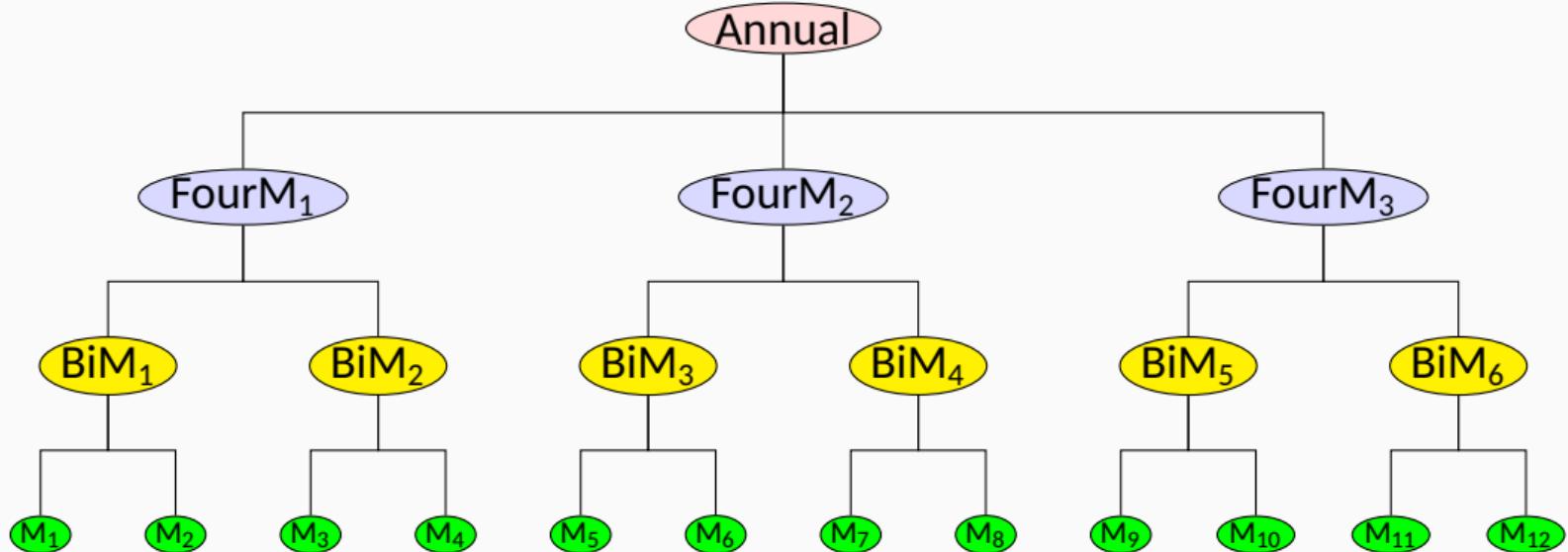
# Temporal reconciliation



# Temporal reconciliation



# Temporal reconciliation



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

# Temporal reconciliation

For a time series  $y_1, \dots, y_T$ , observed at frequency  $m$ , we generate aggregate series

$$y_j^{[k]} = \sum_{t=1+(j-1)k}^{jk} y_t, \quad \text{for } j = 1, \dots, [T/k]$$

- $k \in F(m) = \{\text{factors of } m\}$ .
- A single unique hierarchy is only possible when there are no coprime pairs in  $F(m)$ .
- $M_k = m/k$  is seasonal period of aggregated series.
- Proposed by Athanasopoulos, Hyndman, Kourentzes, Petropoulos (*EJOR*, 2017)

## Cross-temporal reconciliation

- Kourentzes, Athanasopoulos (*ATR*, 2019)
- Punia, Singh, Madaan (*C&IE*, 2020)
- Di Fonzo, Girolimetto (2020)

# Cross-temporal reconciliation

- Kourentzes, Athanasopoulos (ATR, 2019)
- Punia, Singh, Madaan (C&IE, 2020)
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## ISF talks coming up!

- ➡ Mitch O'Hara-Wild: *Probabilistic cross-temporal hierarchies in fable* (in about 3 hours)
- ➡ Tommi Di Fonzo: *Non-negative cross-temporal forecast reconciliation. An application to the Australian domestic tourism flows* (in about 11 hours).

# Bayesian forecast reconciliation

- Park, Nassar (*ICML*, 2014)
- Novak, McGarvie, Garcia (2017)
- Corani, Azzimonti, Zaffalon (2019)
- Corani, Azzimonti, Augusto, Zaffalon (2020)
- Eckert, Hyndman, Panagiotelis (*EJOR*, 2020)

# ML and regularization

- Qiao, Huang (*ICIS*, 2018)
- Ben Taieb, Koo (*ICKDDM*, 2019)
- Yang, Hu, Wang (*ICANN*, 2019)
- Abolghasemi, Hyndman, Tarr, Bergmeir (2019)
- Spiliotis, Abolghasemi, Hyndman, Petropoulos, Assimakopoulos (2020)

# Special IJF issue on hierarchical forecasting



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## Guest Editors

- George Athanasopoulos
- Rob J Hyndman
- Anastasios Panagiotelis
- Nikolaos Kourentzes

## Call for papers

[bit.ly/ijfhierarchical](https://bit.ly/ijfhierarchical)

## Submission deadline

31 August 2021



International Institute of Forecasters

# Thanks!



## More information

- Slides and papers: [robjhyndman.com](http://robjhyndman.com)
- Packages: [tidyverts.org](http://tidyverts.org)
- Forecasting textbook using fable package:  
[OTexts.com/fpp3](http://OTexts.com/fpp3)

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