CALCULATING THE ODDS

ROB HYNDMAN

INTRODUCTION

In the 17th century, a French nobleman was playing games at Monte-Carlo and tried unsuccessfully to predict the relative frequency that certain bets would be won. He mentioned the problem to Pascal and Fermat, and so initiated a famous exchange of letters between the two mathematicians, which is now recognized as the foundations of probability theory. It also demonstrates the prime motivation over the years, to mathematically analyse games. Players have long wished to know the best strategy of play to maximize their return.

However, today much of the motivation for analysis comes from the manufacturers and administrators of games. They wish to establish an upper bound on the expected return to the player and thus ensure a minimum profit margin. This is where much of my own research has had its origins. The Statistical Consulting Centre is part of the Department of Statistics at The University of Melbourne, drawing clients from both within and without the university. One of our clients is Olympic Amusements Pty. Ltd. - a company manufacturing computerized versions of traditional games, as well as some new games. We design games for Olympic so that an optimal strategy will return approximately a given amount to the player. This can involve the design of a prize structure, setting probabilities for the various prize categories and developing the rules of the game, so that over a long period of time, one can predict the maximum return a player will get, and hence the minimum profit the management can expect.

THE GAME STRUCTURE

Many of these computer games have a very simple structure, due to the fact that there is only one player, playing against the machine. In this paper I wish to examine some of the methods that can be used in the analysis of games of this type. The game structure can be summarized as follows:

Prize Category	Prize Value	Probability
0	$x_0 = 0$	p_0
1	x_1	p_1
2	x_2	p_2
.	.	.
.	.	.
.	.	.
n	x_n	p_n

Table 1

where the prize categories are all mutually exclusive (that is, it is only possible to win one prize at a time). The prize values $(x_0,x_1,\ldots,x_n)$ are constant, and the probability of obtaining each is dependent on the strategy employed. Category 0 is the result when no prize is won and hence the prize value is 0. Since the prizes are mutually exclusive,

$$\sum_{k=0}^{n} P_k = 1.$$

The actual payout to the player is the prize value multiplied by his/her bet.

Examples of games having this structure are Poker, Keno, Blackjack etc. Where there is only one player playing the machine.

WHAT WE WANT TO KNOW

Essentially, we wish to calculate three things about these games:

1. Pr (winning anything) = $1 - p_0$

$$= \sum_{i=1}^{n} p_i$$

If this figure is too low, no one will play the game. If it is too high, the prizes will have to be very small or the management will lose money. When designing games, a rough rule of thumb is to keep it between 20% and 30%.

2. Expected return = $\left(\sum_{i=0}^{n} x_i p_i \right)$ x amount bet

Since the probabilities p_i of obtaining each prize, vary with the strategy, and the amount bet may vary within the strategy (for example, in Stud Poker), the expected return may be considered a "function" having the strategy as the independent "variable". (I use the terms very loosely.)

By the law of large numbers, the actual return approaches the expected return to a player over many games, for any fixed strategy. Hence, we need to find a strategy that will maximize the player's expected return, which we call the ....

3. Optimal strategy

If we are designing the game, we require that when the optimal strategy is employed, the expected return is less than 100% to ensure the game makes a profit. But it must not be too low or no one will play. The best level (giving the greatest profit to the management) is dependent largely on the psychology of the consumer. We have found that a maximum expected return of about 90% is quite good. The New South Wales Liquor Board sets a lower limit of 85% for the games in hotels. In reality, the average return is significantly <u>less</u> than this as very few players play close to the optimal strategy. It may drop down to as low as 50%. What this says about the average IQ of the users of these games is open to interpretation!

AN EXACT ANALYSIS

If the game is simple enough to analyse comprehensively using mathematical tools, the usual method is: [1]

1. given prizes;
2. determines possible strategies;
3. find the optimal strategy;
4. calculate probabilities $p_0 \ldots p_n$ based on the optimal strategy;
5. calculate the maximum expected return of the game.

Example: Draw Poker

As a vehicle of illustration of this type of analysis, we will look at a version of Draw Poker having a second draw and including a wild card. Cards are discarded after the first draw and replaced in the second draw from the depleted pack. For purposes of simplicity, the game we will consider will not include prizes for a straight or straight flush.

The prize categories and values are as follows:

Prize Category	Prize Value	Probability
Royal Straight Flush	500	?
Five of a kind	250	?
Four of a kind	15	?
Full House	10	?
Flush	7	?
Three of kind	2	?
Two pairs	1	?

Table 2

The bets will be made before the game begins so the amount is independent of the strategy, and we will assume it is 1 with no loss of generality.

THE OPTIMAL STRATEGY

The strategies involve retaining cards after the first draw - which cards do we keep in our hand, which do we discard? We want to keep the cards that we expect will return us the most money.

Let us list the possible distributions of cards we may hold after the first draw:

CARD DISTRIBUTION

1. Royal Straight Flush
2. Four to RSF + Joker
3. Three to RSF + Joker
4. Two to RSF + Joker
5. Four to RSF
6. Three to RSF
7. Two to RSF
8. Four of a kind + Joker
9. Four of a kind
10. Three of a kind +Joker
11. Three of a kind
12. Pair + Joker
13. Full House
14. Two Pairs + Joker
15. Two Pairs
16. Pair
17. Flush
18. Four to Flush + Joker
19. Four to Flush
20. Three to Flush +Joker
21. Three to Flush
22. Two to Flush + Joker
23. Two to Flush
24. One Joker
25. Anything else

TABLE 3

The question is: in what order of precedence should we place these? As there are 25 categories, there are 25! ways of arranging them (approximately 10^{25}). The decision is made on which hands have the greater expected return after the second draw. Fortunately, most of it is common sense. For instance, if we have a Royal Straight Flush, we retain it; we always keep the Joker, etc. Also the problem is simplified when one considers that the hands containing a joker mutually exclude those without a Joker; hands with a flush come above four to flush, three to flush, etc. Furthermore, many of the hands exclude most of the others, which makes the choice of strategy a little easier.

Of course, the expected return of each first draw category hand depends on the order of precedence of these categories as the probability of each varies when the order is changed. This creates an unsolvable cycle - we can't work out the order of precedence of the categories until we know the expected return from each, and we can't calculate the expected return from each unless we know their order of precedence. The solution, as usual, is to guess a strategy and to calculate the return. The strategy is then modified and the calculations rehashed. This process is repeated until it is apparent that we have the best strategy. As the calculations take several hours work and are very tedious, I do not intend to go through this procedure here. However, I have done it and come up with what I think is an optimal strategy for the game.

It is summarized in Table 4 with the categories given in order of precedence.

CARD DISTRIBUTION	ACTION
1. Royal Straight Flush	Retain all 5 cards
2. Four to RSF + Joker	" " " "
3. Four of a kind + Joker	" " " "
4. Three to RSF + Joker	Retain the 4, discard 1
5. Four to RSF	" " " " "
6. Four of a kind	" " " " "
7. Three of a kind + Joker	" " " " "
8. Full House	Retain all 5 cards
9. Two pairs + Joker	" " " "
10. Flush	" " " "
11. Four to Flush + Joker	" " " "
12. Three of a kind	Retain the 3, discard 2
13. Any pair + Joker	" " " "
14. Two pairs	Retain the 4, discard 1
15. Three to Flush + Joker	" " " "
16. Three to RSF	Retain the 3, discard 2
17. Four to Flush	Retain the 4, discard 1
18. One Joker	Retain Joker Only
19. Any pair	Retain pair, discard 3
20. Three to Flush	Retain 3, discard 2
21. Any other hand	Discard all 5.

TABLE 4

You may notice we now have only 21 categories. It was found that with four of the original categories it was better to either start again than to persevere with the first draw hand or just to keep the joker. The action to be taken is obvious. For instance, if you have one pair and a joker and not much else of interest, you keep the pair and joker and discard the rest as indicated. There are a couple of interesting quirks in the strategy. For instance, if you have

7 Hearts 10 Hearts 3 Spades J Hearts A Hearts

it is better to keep the 10, J and A and discard the 7 of hearts since three to Royal Straight Flush came above four to flush in order of increasing expected return.

Of course different prize values would yield different optimal strategies. If the prize for the Royal Straight Flush was smaller and the prize for a flush larger, this anomaly may not occur.

PROBABILITY CALCULATIONS

The next stage in the analysis is to calculate the probability of obtaining a hand in each of the first draw categories. This is a simple combinatory exercise although care must be taken to exclude the preceding categories in the strategy.

We calculate the total number of hands in each first draw category:

CATEGORY	CALCULATION	NUMBER
Royal Straight Flush	4	4
Four to RSF + Joker	$4\binom{5}{4}$	20
Four of a kind + Joker	13	13
Three to RSF + Joker	$4\binom{5}{3}\binom{47}{1}$	1880
Four to RSF	$4\binom{5}{4}\binom{47}{1}$	940
Four of a kind	13.48	624
Three of a kind + Joker	$\binom{13}{1}\binom{4}{3}\binom{12}{1}\binom{4}{1}$	2496
Full House	$\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}$	3744
Two Pairs + Joker	$\binom{13}{2}\binom{4}{2}\binom{4}{2}$	2808
Flush	$4\binom{13}{5}-4-4\binom{5}{4}\binom{13-5}{1}$	4984
Four to Flush + Joker	$4\binom{13}{4}-20-4\binom{5}{3}\binom{13-5}{1}$	2520
Three of a kind	$\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}\binom{4}{1}$	54912
Pair + Joker	$\binom{13}{1}\binom{4}{2}\binom{12}{2}\binom{4}{1}^2-4\binom{5}{3}\binom{3}{1}^2$	82008
Two Pairs	$\binom{13}{2}\binom{4}{2}\binom{4}{2}\binom{11}{1}\binom{4}{1}$	123552
Three to Flush + Joker	$4[\binom{13}{3}-\binom{5}{3}]\,3\binom{10}{1}$	33120
Three to RSF	$4\binom{5}{3}[\binom{8}{1}\binom{13}{1}\binom{3}{1}+\binom{3}{1}\binom{10}{1}\binom{3}{1}^2$ $+\binom{10}{2}\binom{3}{1}^2+\binom{10}{1}\binom{3}{2}]$	40680
Four to Flush	$4\binom{13}{4}.3.13-940$ $-4\binom{5}{3}\binom{8}{1}\binom{13}{1}\binom{3}{1}$	98120
Joker	$\binom{52}{4}-33120-82008-2520$ $-2808-1880-2496-13-20$	145860
Pair	$\binom{13}{1}\binom{4}{2}\binom{12}{3}\binom{4}{1}^3-4\binom{13}{4}3.4$ $-4\binom{5}{3}\binom{3}{1}.3.\binom{10}{1}.4$	1049520
Three to Flush	$[\binom{4}{1}\binom{13}{3}-\binom{4}{1}\binom{5}{3}][\binom{3}{1}^2\binom{10}{2}]$	447120
Anything else	$\binom{52}{4}-447120-1049520-98120$ $-40680-123552-54912-4984$ $-3744-940-624-4$	774760

Table 5

The probability of each type of draw one hand occurring can now be easily calculated by dividing the number of hands by $\binom{53}{5} = 2,869,685$. The results are given in Table 6.

HAND	PROBABILITY
Royal Straight Flush	1.39388×10^{-6}
Four To RSF + Joker	6.96941×10^{-6}
Five of a kind	4.53011×10^{-6}
Three to RSF + Joker	6.55124×10^{-4}
Four to RSF	3.27562×10^{-4}
Four of a kind	2.17445×10^{-4}
Three of a kind + Joker	8.69782×10^{-4}
Full House	1.30467×10^{-3}
Two pairs + Joker	9.78505×10^{-4}
Flush	1.73678×10^{-3}
Four to Flush + Joker	8.78145×10^{-4}
Three of a kind	0.0191352
Pair + Joker	0.0285774
Two pairs	0.0430542
Three to Flush + Joker	0.0115413
Three to RSF	0.0141758
Four to Flush	0.0341919
One Joker	0.0508279
Pair	0.365727
Three to Flush	0.155808
Anything else	0.269981

Table 6

Now we need to calculate the conditional probability of getting each prize winning card distribution from each of these first draw categories. Obviously if the action is to retain five cards after the first draw (as with Royal Straight Flush), the conditional probability of obtaining that prize is simply equal to 1.

When cards are discarded, a number of prizes may be possible. We haven't time to look at the possibilities in all first draw categories, but I'll go through one in detail as an example.

Suppose we have in our hand after the first draw, three cards of a Royal Straight Flush and a Joker. In this case one card is discarded. The possible prize winning hands after the second draw are a Royal Straight Flush, a Flush or Three of a kind (Pair Joker).

The following conditional probabilities are calculated:

Royal Straight Flush

We need one of the other two cards that make up the Royal Straight Flush. So the conditional probability is 2/48.

Flush

There are 8 cards that could be used to make up the Flush. But we may have discarded one of them. The probability that one was discarded is 8/49 and so the conditional probability of getting a Flush after the second draw is

$$8/49 \times 7/48 + 41/49 \times 8/48 = 8/49.$$

<u>Three of a kind</u>

By similar reasoning, the conditional probability is 69/392.

Hence, the expected return of the first draw hand is the sum of these probabilities multiplied by their prize values which is

$$\begin{aligned} &500 \times 2/48 + 7 \times 8/49 + 2 \times 69/392 \\ &= 22\ {}^{193}/_{588} \\ &= 22.328. \end{aligned}$$

These calculations are done for each type of first draw hand. We calculate the expected return here, because this is the determining figure in the establishment of the optimal strategy.

Finally, we want the probability of getting each prize winning category after the first draw, regardless of our hand. Therefore, we must sum the probability of obtaining each prize from each draw hand over all first draw hands.

That is,

Pr (hand type is ß on second draw)

= ∑Pr (hand type ß on second draw / hand ∂ on first draw)
x Pr(hand ∂ on first draw)

where the sum is over all possible hands ∂ on the first draw.

When this is done we obtain the following results:

PRIZE CATEGORY	PRIZE VALUE	PROBABILITY
Royal Straight Flush	500	0.0001
Five of a Kind	250	0.0001
Four of a Kind	15	0.0086
Full House	10	0.0155
Flush	7	0.0218
Three of a Kind	2	0.1335
Two Pairs	1	0.1088

Table 7

RESULTS OF ANALYSIS

From here on all that is needed is simple arithmetic. The probability of winning anything is the sum of the probabilities in the third column. This gives,

$$\text{Pr (winning a prize)} = 28.9\%.$$

The expected return from the game to the player is then obtained by multiplying the entries of the second and third columns and adding the results. This gives,

$$\text{E (Return)} = 88\%.$$

This is evidently a well designed game as the probability of winning any prize is quite high at about 29%. Since the expected return is 88%, the management can expect to make at least 12% of all the money bet on the game.

ANALYSIS BY MONTE-CARLO METHODS

While it is desirable to be able to carry out an exact analysis of a game of this type, it is not always expedient. Some games are very difficult to analyse using the methods we have just discussed. I used a very simple game to demonstrate the analysis, yet the calculations were very tedious and often quite messy. With a game like Blackjack, the analysis requires much more work. Although a comprehensive analysis has been done for the traditional game (the 'Las Vegas' version), any modified games of Blackjack would require a whole new calculation.

Therefore, we use what are called Monte-Carlo methods to give an approximate analysis of the game, or an initial diagnostic analysis before spending time doing it exactly. Monte-Carlo methods comprise the branch of experimental mathematics which is concerned with experiments on random numbers. For the current problem, this involves the random simulation of the game many times by computer, and examining the results.

Pseudo-Random Number Generators

First of all, we need to generate a random environment. If cards are involved in the game, we want to begin with a randomly dealt hand of cards. This is easily produced with a pseudo-random number generator which is usually a built in function on most computers. However there are some hidden pitfalls associated with random number generators!

For example, there was once an Australian company manufacturing computer games using a rather unusual form of generator involving the random generation of bit patterns. From these, a uniform random number was produced and then a random hand of cards for a Poker game. The results of many games were checked and it was found that in Draw Poker with a five card hand, the results were as predicted but in Seven Card Stud Poker, when seven cards were selected at a time, the results were strange. There were more of some prizes won than was reasonable and less of others. The cause was the pseudo-random numbers were not truly random when taking blocks of seven, and hadn't been tested sufficiently before use. In another famous case, NASA was doing some research using 3-dimensional random numbers. Three random numbers were chosen and used as coordinates of a point in 3-space. The resulting distribution of points was not uniform as desired but actually formed a tiered roof like shape.

The generators generally in use are either "linear congruential" generators of the form,

$$x_{n+1} = a\, x_n + c \pmod{m}, \qquad (x_0, a, m, c > o),$$

or "multiplicative (power residue)" generators of similar form having c = o. There is always a danger that some experiment will generate a new statistical test which will find a defect in the generators being used. The moral of the story is not to simply use the 'nearest' pseudo-random number generator, but to find one that is a proven performer.

Simulating the Game

Having generated a random hand of cards, the computer must play the game according to a given strategy. Obviously, we'd like this to be optimal. If that is unknown, one should try several strategies that may be close to optimal and choose the best.

Example: Draw Poker again

To demonstrate the use of Monte-Carlo methods, I have simulated the game we analysed earlier using the optimal strategy we calculated. In this situation, the simulation merely serves as a check on the calculations and vice-versa. However, when the calculations are complex, it is handy to have an idea about the magnitude of the probabilities.

In this simulation, I have used a pseudo-random number generator taken from Applied Statistics, Vol. 31, p. 188. It actually sums three simple multiplicative congruential generators to 'iron out' any imperfections in them.

The computer played 800,000 games and the results given in Figure 1 were obtained.

HAND	TOTAL	PRIZE	EMPIRICAL RETURN	EMPIRICAL PROBABILITY
Anything else	569048	0	.00%	.7113100
Two Pairs	87110	1	10.89%	.1088875
Three of a kind	106948	2	26.74%	.1336850
Flush	17470	7	15.29%	.0218375
Full House	12414	10	15.52%	.0155175
Four of a kind	6874	15	12.89%	.0085925
Five of a kind	62	250	1.94%	.0000775
Royal Straight Flush	74	500	4.63%	.0000925

DRAW ONE RESULT	TOTAL	THEORETICAL PROBABILITY	EMPIRICAL PROBABILITY	CHI SQ STAT.
Anything else	215653	.2699809	.2695663	0.51
Three to flush	125132	.1558080	.1564150	1.89
Any pair excl. Joker	292439	.3657266	.3655488	0.07
One joker	40524	.0508279	.0506550	0.47
Four to flush	27518	.0341919	.0343975	0.99
Three to RSF	11269	.0141758	.0140863	0.45
Three to flush + Joker	9150	.0115413	.0114375	0.75
Two pairs	34589	.0430542	.0432363	0.62
Any pair + Joker	22787	.0285774	.0284838	0.25
Three of a kind	15276	.0191352	.0190950	0.07
Four to flush + Joker	720	.0008781	.0009000	0.44
Flush	1425	.0017368	.0017813	0.91
Two pairs + Joker	769	.0009785	.0009613	0.24
Full House	1017	.0013047	.0012713	0.68
Three of a kind + Joker	718	.0008698	.0008975	0.71
Four of a kind	201	.0002175	.0002513	4.20
Four to RSF	277	.0003276	.0003463	0.85
Three to RSF + Joker	523	.0006551	.0006538	0.00
Five of a kind	4	.0000045	.0000050	0.04
Four to RSF + Joker	8	.0000070	.0000100	1.05
Royal Straight Flush	1	.0000014	.0000013	0.01

TOTAL HANDS ANALYSED: 800,000

EMPIRICAL RETURN: 87.88%
Standard Deviation (Return): .64%

CHI STAT. ON DRAW1 TOTALS: 15.20

Figure 1

The first thing we have calculated is the empirical probability of each draw one category. These are compared with the calculated result by use of a X^2 goodness of fit test. That is, we calculate the value of

$$\frac{(\text{expected total} - \text{observed total})^2}{\text{expected total}}$$

for each hand. These are given in the column CHI SQ. STAT. If there is an error in either the calculations or computer program, this figure should be large. The total of these values for each hand is given as the X^2 distribution and so we can test to see if it is too large or too small. With 21 categories, the statistic has 20 degrees of freedom. So it is known that it has 95% probability of being between 9.59 and 34.17. Hence we accept these results as being reasonable. A similar test can be carried out on the prize totals.

Next we look at the average return of the games. Since the bet is always one here, the average return is simply the average prize won, which is given as 87.88%. The standard deviation of the return can also be calculated.

Let $R = (\text{prize} \times 100)\%$ be the percentage return

where prize is the amount won and $\bar{R}$ = mean return.

Then the sample variance of R is

$$S^2 = (1/n-1)\{\Sigma(R-\bar{R})^2\}$$

$$= (1/n-1)\{\Sigma R^2-(\Sigma R)^2/n\}$$

where n is the number of games played.

An estimate of the standard deviation of $\bar{R}$ is $S/\sqrt{n}$.

By the Central Limit Theorem

$$\frac{\bar{R}-E(R)}{S/\sqrt{n}} \overset{d}{\sim} N(0,1)$$

Hence 95% confidence interval for $\mu_R = E(R)$ is $\bar{R} \pm 1.96\, s/\sqrt{n}$.

For our simulation, this is $87.88 \pm 1.96\,(0.64)$
$= [86.6,\ 89.1]$.

By the law of large numbers

$$\bar{R} \xrightarrow{P} \mu_R \text{ as } n \to \infty$$

In our simulation, there was little difference between the two values.

Confidence Interval on the Expected Return

The above results are only valid when the bet is made before the game. If it is made during the game, the amount bet becomes part of the strategy and must be treated as a random variable.

We define x_i as the payout on the i^{th} game and y_i as the bet on the i^{th} game.

The expected return is then $\mu_R = E(R) = E(X_1)/E(Y_1)$

This is a more appropriate expected return than $E(X_1/Y_1)$ as the strong law of large numbers guarantees that the actual return of a player over a long period of time approaches EX_1/EY_1 whereas $E(X_1/Y_1)$ will be the average over games of his return on <u>each</u> game.

So the best estimate for the expected return is $\bar{X} / \bar{Y}$

where $\bar{X}$ is the mean of the payouts and $\bar{Y}$ the mean of the bets.

Let $\quad Z_i = X_i - \mu_R Y_i$

Then $\quad \bar{X} / \bar{Y} - \mu_R = \bar{Z} / \bar{Y}$

Thus $\quad \text{var}\ (\bar{X}/\bar{Y}) = \text{var}\ (\bar{Z}/\bar{Y}) = \text{var}\ (Z)/\{n\ E(Y)^2\} = \acute{O}_z{}^2/n\mu_Y{}^2 \quad \dots 2.$

and by the Central Limit Theorem and Continuity Theorem

$$(\bar{X}/\bar{Y} - \mu_R) \overset{d}{\sim} N\ (0, \text{Var}\ (\bar{X}/\bar{Y})) \quad \dots 3.$$

we need to estimate $\text{Var}\ (\bar{X}/\bar{Y}) = \acute{O}_Z{}^2/n\mu_Y{}^2$

$$\acute{O}_Z{}^2 = \text{Var}\ (X_1 - \mu_R Y_1) = \text{Var}\ (X_1) + \mu_R{}^2\ \text{Var}\ (Y_1) - 2\mu_R\ \text{Cov}\ (X_1, Y_1)$$

which can be estimated by replacing

$$\text{Var}\ X_1,\ \text{Var}\ Y_1,\ \text{Cov}\ (X_1, Y_1) \text{ and } \mu_R$$

by the usual sample estimates.

μ_Y can be estimated by $\bar{Y}$.

Thus a 95% confidence interval for μ_R is $\bar{X} / \bar{Y} \pm 1.96\acute{O}_Z/\sqrt{n}\,\bar{Y}$.

CONCLUSION

We have now examined and demonstrated some of the tools used by the statistician in the analysis of games. The exact analysis of a game has been carried out and verified by Monte-Carlo methods.

To a mathematician, the exact approach is far more satisfying and should be done if at all possible. However, from a practical point of view, the analysis by simulation is usually sufficient and almost always easier for non-trivial games. Both approaches requires a substantial amount of probability theory and therefore may help in developing this area of mathematics further.

When Pascal and Fermat conducted their origianl research into games, they could not have realized that it would develop into an enormous branch of mathematics in its own right. I hope that current research in this field will not only further our knowledge of games, but add to the accumulated body of knowledge of probability theory.

FOOTNOTES

1. If designing a game, go back to step 1 after step 5 if the return is not suitable and adjust the prize values $x_1, ..., x_n$.

2. For proof of this result see Cochran, W.G. Sampling Techniques. Wiley, 1977, p. 153.

3. See Billingsley P. Convergence of Probability Measures. New York : Wiley, 1968, p. 31.