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APPROXIMATIONS AND BOUNDARY CONDITIONS FOR CONTINUOUS-TIME THRESHOLD AUTOREGRESSIVE PROCESSES

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Abstract

Continuous-time threshold autoregressive (CTAR) processes have been developed in the past few years for modelling non-linear time series observed at irregular intervals. Several approximating processes are given here which are useful for simulation and inference. Each of the approximating processes implicitly defines conditions on the thresholds, thus providing greater understanding of the way in which boundary conditions arise.

CONTINUOUS-TIME AUTOREGRESSION; THRESHOLD AUTOREGRESSION; NON-LINEAR STOCHASTIC DIFFERENTIAL EQUATIONS; UNEQUALLY SPACED TIME SERIES

AMS 1991 SUBJECT CLASSIFICATION: PRIMARY 60H10
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1. Introduction

The first-order continuous-time autoregressive (CTAR) process, denoted by CTAR(1), is defined to be a stationary solution of the piecewise linear stochastic differential equation

$$dX(t) + \alpha(X(t))X(t)dt + \beta(X(t))dt = \sigma(X(t))dW(t) \tag{1.1}$$

where

$$\alpha(x) = \alpha_i, \quad \beta(x) = \beta_i \quad \text{and} \quad \sigma(x) = \sigma_i > 0 \qquad \text{for } r_{i-1} < x < r_i, \tag{1.2}$$

and the threshold values $-\infty = r_0 < r_1 < \cdots < r_l = \infty$ partition the real line. The process defined by (1.1) and (1.2) is a diffusion process with drift coefficient $\mu(x) = -\alpha(x)x - \beta(x)$ and diffusion coefficient $\Sigma(x) = \sigma^2(x)/2$.

The CTAR(1) process defined by (1.1) has a stationary distribution if and only if

$$\lim_{x\to-\infty} \mu(x) > 0 \quad \text{and} \quad \lim_{x\to\infty} \mu(x) < 0$$

(see Brockwell et al. (1991)).

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Let $\mathscr{D}(\mathscr{G})$ denote the domain of the generator $\mathscr{G}$ of $\{X(t)\}$. Then, for $f \in \mathscr{D}(\mathscr{G})$, the generator is defined by

$$\mathscr{G}f(x) := \mu(x)f'(x) + \Sigma(x)f''(x), \qquad x \neq r_i, \tag{1.3}$$

with $\mathscr{G}f(r_i)$ determined by continuity of $\mathscr{G}f(x)$.

Note that the process is not uniquely defined until the behaviour of the process at the thresholds is specified. There is no unique way of doing this. Here we consider three possible boundary conditions, each of which has some useful properties. These are specified by placing conditions on functions in $\mathscr{D}(\mathscr{G})$. For $f \in \mathscr{D}(\mathscr{G})$, the conditions are:

Condition A. $f'(x)$ is continuous at $x = r_i$.

Condition B. $\sigma_i f'(x)$ is continuous at $x = r_i$.

Condition C. $\sigma_i^2 f'(x)$ is continuous at $x = r_i$.

Any one of the boundary conditions A, B or C (and these are not the only possibilities) could be used for modelling purposes. Brockwell et al. (1991) considered a CTAR process with boundary condition B and Brockwell and Hyndman (1992) considered a CTAR process with boundary condition A. It is important in specifying the model as a diffusion process to state which of the boundary conditions is being used. Clearly, if $\sigma(x)$ is continuous there is no distinction between the three boundary conditions. Note that specification of the boundary conditions is critical only when the process is formulated directly as a diffusion process.

The distinction between the CTAR(1) processes with different boundary conditions is apparent (unless $\sigma(x)$ is continuous) in the stationary distributions for the different processes. Let $\pi(x)$ be the stationary density for $\{X(t)\}$. Then, following the argument in Brockwell et al. (1991), we obtain

$$\sigma^2(x)\pi(x)\frac{\partial f}{\partial x} \text{ is continuous for all } x.$$

So, under the three boundary conditions given earlier, we obtain corresponding conditions on the stationary density $\pi(x)$:

Condition A. $\sigma_i^2\pi(x)$ is continuous at $x = r_i$.

Condition B. $\sigma_i\pi(x)$ is continuous at $x = r_i$.

Condition C. $\pi(x)$ is continuous at $x = r_i$.

Hence CTAR(1) processes with identical parameters but different boundary conditions have quite different stationary distributions. Hyndman (1992a) gives several examples of the stationary densities of CTAR(1) processes under these three boundary conditions.

Khazen (1961) considered piecewise linear diffusion processes and used the continuity of the transition probability density as a boundary condition. The above analysis shows that, for CTAR(1) processes, this is equivalent to boundary condition C. Atkinson and

Caughey (1968) in their determination of the spectral density for the CTAR(1) process also require continuity of the transition probability density. But since they specify the process with constant variance ($\sigma_i = \sigma$ for all i), there is no distinction between the three boundary conditions considered here.

Processes which approximate a CTAR process are useful for inference (Brockwell and Hyndman (1992)) as well as for simulating the process. Simulations are particularly useful in determining forecast distributions (Hyndman (1992b)).

Brockwell and Hyndman (1992) give an approximating stochastic differential equation whose limit is consistent with its convergence to a process having boundary condition A, but it is not easily modified to allow other boundary conditions. Brockwell et al. (1991) consider (1.1) with $W(t)$ replaced by an approximation and show that it converges to a CTAR(1) process with boundary condition B. But, again, this is not easily modified to allow other boundary conditions.

In this paper, we consider two discrete-time processes which converge to CTAR(1) processes and have the property that small modifications to the approximating process lead to different boundary conditions. Such approximations provide greater understanding of the way in which different boundary conditions arise. The first process, considered in Section 2, is a Markov chain defined on a countable state space. Section 3 considers a random broken line defined by the Euler approximation with piecewise linear interpolation. A brief comment on higher-order processes is given in Section 4.

2. A Markov chain approximation

In this section, we consider a Markov chain which approximates the CTAR(1) process with one threshold. More general results for any number of thresholds are given in Hyndman (1992a).

Let X_t be a discrete-time process with steps of size Δx in time intervals of Δt defined on the state space $\{r, r \pm \Delta x, r \pm 2\Delta x, \cdots\}$. Let $p(x)$ be the probability of step $+\Delta x$ from state x and $q(x) = 1 - p(x)$ be the probability of step $-\Delta x$ from state x. Then choose step size $\Delta x = \sigma(x)\sqrt{\Delta t}$ and let

$$p(x) = \frac{1}{2}\left[1 - \frac{(\alpha(x)x + \beta(x))\sqrt{\Delta t}}{\sigma(x)}\right] \quad \text{and} \quad q(x) = \frac{1}{2}\left[1 + \frac{(\alpha(x)x + \beta(x))\sqrt{\Delta t}}{\sigma(x)}\right].$$

When $\alpha(x)$, $\beta(x)$ and $\sigma(x)$ are constant, the continuous process defined by interpolating the Markov chain process converges weakly as $\Delta t \to 0$ to the linear CAR(1) process defined by (1.1) when $l = 1$ (see, for example, Cox and Miller (1965), pp. 213–214).

Now consider the above Markov chain where $\alpha(x)$, $\beta(x)$ and $\sigma(x)$ are defined as in (1.2) where there is just one threshold, r. Since the parameters are not defined at the threshold, the description of the Markov chain is incomplete and we need to define the transition probabilities at r. Let p^* be the probability of a step in the positive direction from the threshold r. Figure 2.1 depicts a sample path of the Markov chain.

We obtain the boundary conditions of the Markov chain by considering the limiting properties of the generator of the approximating process. The results of Kurtz (1975)

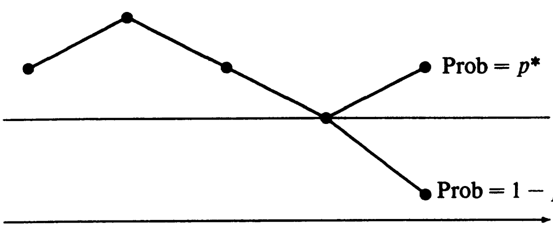

Figure 2.1. Markov chain approximation

guarantee the weak convergence of the approximating process to the CTAR(1) process with the appropriate boundary condition.

Now

$$E[f(X_{\Delta t}) \mid X_0 = r] = p^* f(r + \sigma_2 \sqrt{\Delta t}) + (1 - p^*) f(r - \sigma_1 \sqrt{\Delta t})$$

where f is a function in the domain of $\mathscr{G}$, the generator of $\{X(t)\}$. Thus

$$\begin{aligned}\frac{E[f(X_{\Delta t}) \mid X_0 = r] - f(r)}{\Delta t} &= \frac{1}{\Delta t}[p^* f(r + \sigma_2 \sqrt{\Delta t}) + (1 - p^*) f(r - \sigma_1 \sqrt{\Delta t}) - f(r)] \\ &= \frac{\sqrt{\Delta t}}{\Delta t}[p^* \sigma_2 f'(r + \varepsilon_2) - (1 - p^*)\sigma_1 f'(r - \varepsilon_1)]\end{aligned}$$

for $0 < \varepsilon_1 < \sigma_1 \Delta t$ and $0 < \varepsilon_2 < \sigma_2 \Delta t$ by the mean value theorem. Hence,

$$\mathscr{G}f(r) = \lim_{t \downarrow 0} \frac{E[f(X_t) \mid X_0 = r] - f(r)}{t}$$

exists if and only if

$$p^* \sigma_2 f'(r^+) = (1 - p^*) \sigma_1 f'(r^-) \tag{2.1}$$

where

$$f(x^+) := \lim_{h \to 0, h > 0} f(x + h) \quad \text{and} \quad f(x^-) := \lim_{h \to 0, h > 0} f(x - h).$$

Hence different values of p^* lead to different boundary conditions. Plugging $p^* = \frac{1}{2}$ into (2.1) gives the boundary condition B. Other choices of p^* lead to alternative boundary conditions. For example, $p^* = \sigma_1/(\sigma_1 + \sigma_2)$ leads to boundary condition A and $p^* = \sigma_2/(\sigma_1 + \sigma_2)$ leads to boundary condition C.

3. A random broken line approximation

The CTAR(1) process can also be approximated by a random broken line with vertices defined by a discrete-time Markov chain on the times $0 = t_0 < t_1 < t_2 < \cdots$. Such approximations are frequently used in the numerical study of diffusion processes (see, for example, Kloeden and Platen (1992)).

One approach is to approximate a CTAR(1) process using the Euler method of time discretisation. This yields the Markov chain

$$X_n(t+n^{-1}) = X_n(t) + n^{-1}\mu(X_n(t)) + n^{-1/2}\sigma(X_n(t))Z_n(t), \tag{3.1}$$

where $t = 0, 1/n, 2/n, \cdots$, $\{Z_n(t)\}$ is a sequence of i.i.d. binary increments taking the values ± 1 each with probability $\frac{1}{2}$, and $X_n(0)$ is independent of $\{Z_n(t)\}$ and has the same distribution as $X(0)$. Sample paths of $\{X_n(t)\}$ between $t = j/n$ and $t = (j+1)/n$, $(j = 0, 1, 2, \cdots)$ are defined by linear interpolation. To complete the definition, arbitrarily define the values of $\mu(r_i)$ and $\sigma(r_i)$ as $\mu(r_i^+)$ and $\sigma(r_i^+)$ respectively.

Higher-order approximations are available (Kloeden and Platen (1992)) but will not be considered here.

Now let f be a function in the domain of the generator of $\{X_n(t)\}$. Then

$$\begin{aligned} &E[f(X_n(s+\delta)) \mid X_n(s) = x] \\ &\quad = \tfrac{1}{2}f(x + \mu(x)\delta + \sigma(x)\delta\sqrt{n}) + \tfrac{1}{2}f(x + \mu(x)\delta - \sigma(x)\delta\sqrt{n}) \end{aligned}$$

where $0 < \delta \leqq n^{-1}$. Using the mean value theorem, we obtain

$$\begin{aligned} &\frac{E[f(X_n(s+\delta)) \mid X_n(s) = x] - f(x)}{\delta} \\ &\quad = \tfrac{1}{2}(\mu(x) + \sigma(x)\sqrt{n})f'(x+\varepsilon_1) + \tfrac{1}{2}(\mu(x) - \sigma(x)\sqrt{n})f'(x-\varepsilon_2), \end{aligned}$$

where, if n is sufficiently large, $0 < \varepsilon_1 < \mu(x)\delta + \sigma(x)\delta\sqrt{n}$ and $0 < \varepsilon_2 < \mu(x)\delta - \sigma(x)\delta\sqrt{n}$. Hence, the limit of the generator of $\{X_n(t)\}$,

$$\mathscr{G}f(x) = \lim_{n\to\infty}\lim_{\delta\to 0}\frac{E[f(X_n(s+\delta)) \mid X_n(s) = x] - f(x)}{\delta},$$

exists if and only if $f'(x)$ is continuous for all x. That is, boundary condition A.

Under this condition, the generator of $\{X_n(t)\}$ is

$$\mu(x)f'(x) + \Sigma(x)f''(x) + O(n^{-1/2})$$

which converges uniformly to the right-hand side of equation (1.3). Thus, by Theorem 4.29 of Kurtz (1975), $\{X_n(t)\}$ converges weakly to the CTAR(1) process with boundary condition A.

This approximation consists of random broken lines with vertices determined by the Markov chain $\{X_n(j/n); j = 0, 1, 2, \cdots\}$. Define for $j/n < t < (j+1)/n$

$$m(t) = \begin{cases} \mu(X_n(t)) + \sigma(X_n(t))\sqrt{n} & \text{if } Z_n(j/n) = 1. \\ \mu(X_n(t)) - \sigma(X_n(t))\sqrt{n} & \text{if } Z_n(j/n) = -1. \end{cases}$$

Then, the slope of the line segment at time $j/n < t < (j+1)/n$ is $m(j/n)$. Thus,

$$X_n(j/n + \delta) = X_n(j/n) + m(j/n)\delta, \qquad 0 < \delta \leqq 1/n.$$

Hence, when the approximating process crosses into a new regime, the parameters of the process do not change until after the first vertex is observed in the new regime. A sample path of the process is depicted on the left of Figure 3.1.

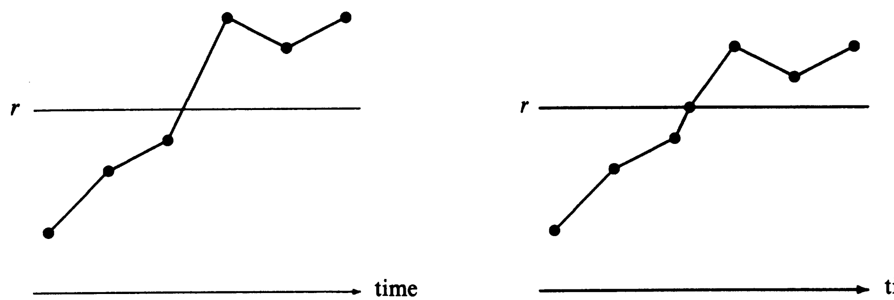

Figure 3.1. Sample paths of the random broken line approximations with a binary increment. Left, broken line approximation gives boundary condition A; right, modified broken line approximation gives boundary condition B

A natural modification of this approximating process is to introduce an extra vertex whenever the process crosses the threshold and change the slope of the new segment, between the threshold and the next vertex, according to the parameters of the regime entered. Thus, whenever $X_n(t+n^{-1})$ is on the opposite side of the threshold, r_i, from $X_n(t)$, redefine $X_n(t+\delta)$ as

$$X_n^*(t+\delta)=\begin{cases} r_i+m(t+\tau)(\delta-\tau), & 0<\tau\leqq\delta\leqq n^{-1} \\ X_n(t)+m(t)\delta, & 0<\delta<\tau \text{ or } 0=\tau<\delta \end{cases}$$

where $\tau=[r_i-X_n(t)]/m(t)$ is the time from t before the threshold is crossed. A sample path of this modified process is shown on the right of Figure 3.1.

Using this modification, and following the same argument as above, we find the limit of the generator of the modified process exists if and only if

$$[\sigma(r_i^+)f'(r_i+\varepsilon_1)-\sigma(r_i^-)f'(r_i-\varepsilon_2)]\to 0 \qquad \text{as } n\to\infty.$$

That is, boundary condition B.

Other increment processes, $\{Z_n(t)\}$, in (3.1) lead to similar results (see Hyndman (1992a)).

4. Higher-order processes

Unfortunately, neither the boundary conditions of Section 1 nor the approximations of Sections 2 and 3 are easily generalised to CTAR processes of higher order. The only published work in this direction is that of Brockwell and Hyndman (1992) who consider an approximation of the CTAR(1) process defined by (1.1) by approximating $\alpha(x)$, $\beta(x)$ and $\sigma(x)$ with continuous functions. This approximation is easily generalised to higher-order processes. For the first-order process, they show that the approximation is consistent with a process with boundary condition A.

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