Forecast Reconciliation for Quantiles using Bilevel Optimisation

H. Alipour, G. Athanasopoulos, N. Ho-Nguyen and **A. Panagiotelis**

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Joint work with...

. . . and

Probabilistic Forecast Reconciliation

A refresher

- Consider a probabilistic forecast $\hat{F}_{t+h|t}$ for a random n-vector $\hat{\mathbf{y}}_{t+h}$ made using all information up to time t.
- Draw a sample of size J from this probabilistic forecasts

$$\hat{\mathbf{y}}^{(1)}_{t+h|t}, \hat{\mathbf{y}}^{(2)}_{t+h|t}, \ldots, \hat{\mathbf{y}}^{(J)}_{t+h|t} \sim \hat{F}_{t+h|t}$$

- Problem: these draws do not respect (aggregation) constraints.

Reconciliation

- Find any mapping $\psi : \mathbb{R}^n \rightarrow \mathfrak{s}$ where $\mathfrak{s}$ is the linear subspace where constraints hold.
- Usually (including for this talk), this is a linear mapping **SG**, where the columns of **S** span the coherent subspace.
- Letting $\tilde{\mathbf{y}}^j_{t+h|t} = \psi(\hat{\mathbf{y}}^j_{t+h|t})$

$$\tilde{\mathbf{y}}^{(1)}_{t+h|t}, \tilde{\mathbf{y}}^{(2)}_{t+h|t}, \ldots, \tilde{\mathbf{y}}^{(J)}_{t+h|t} \sim \tilde{F}_{t+h|t}$$

- Where $\tilde{F}_{t+h|t}$ is the **reconciled** probabilistic forecast.

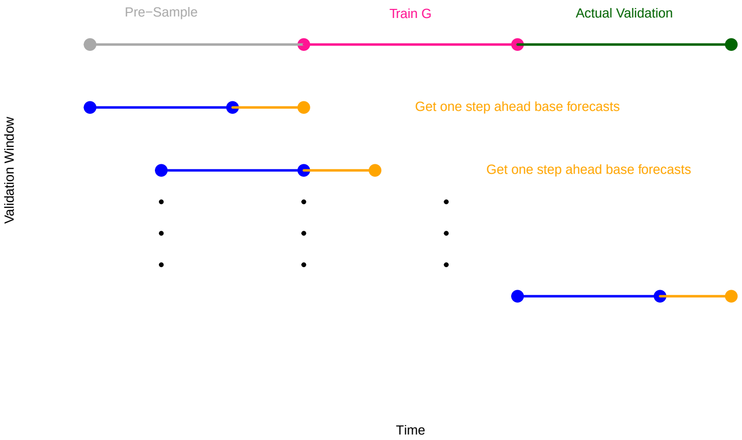
How to choose ψ/G

- There are popular choices from the point forecast reconciliation literature.
- Apart from the Gaussian case, where MinT is optimal w.r.t the logarithmic score (Wickramasuriya, 2023) little is known theoretically.
- Instead we can optimise $\mathbf{G}$ with respect to some loss function.
- This is what we did in Panagiotelis et al (2023), others have done similar.

Data Split

- We require both $\hat{\mathbf{y}}^{(j)}_{t+h|t}$ and $\mathbf{y}$ to train $\mathbf{G}$.
- Data is split into three
  - Pre-sample (used to get $\hat{\mathbf{y}}$ but not train $\mathbf{G}$)
  - Training (used to train $\mathbf{G}$)
  - Validation (once $\mathbf{G}$ is trained, does it give good forecasts?)
- There are variations, what we do so far described on next slide

In a picture


Pre−Sample
Train G
Actual Validation
Get one step ahead base forecasts
Get one step ahead base forecasts
Validation Window
Time

Optimisation

- In general we want to find

$$\underset{\mathbf{G}}{\mathit{argmin}} \sum_{t \in \mathcal{T}_{\mathrm{train}}} L(\hat{\mathbf{y}}_t, \tilde{F}_{t|t-1})$$

- Here, $L(.,.)$ is a loss function such as a scoring rule.
- Note $\tilde{F}_{t|t-1}$ depends on $\mathbf{G}$ through $\tilde{\mathbf{y}}^{(j)}_{t|t-1}$.
- Could generalise to h-step ahead forecasts

Quantiles

Quantile reconciliation

- Quantiles do not need to be coherent!
- If you disagree
  a) we can talk during the coffee break, but
  b) avoid Stephan Kolassa!
- However that does not mean that we cannot borrow information from different series in a hierarchy to improve quantile forecasts.
- That is the motivation for this project.

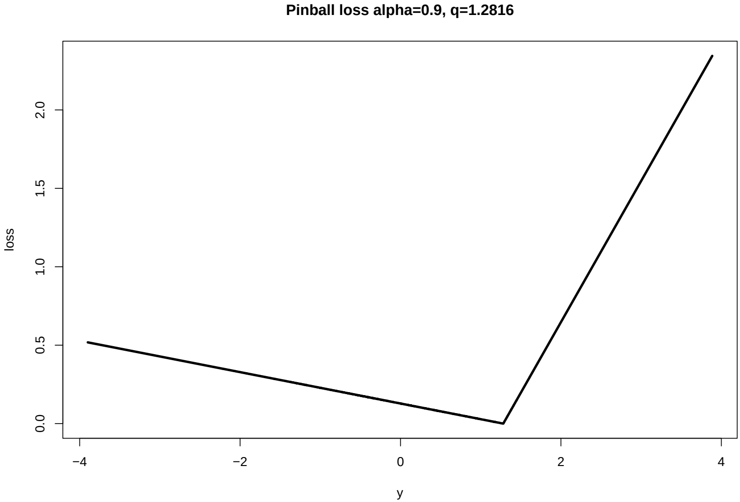
Pinball loss

For the rest of the talk we will consider the **pinball loss** function

$$L_\alpha(y, q) = \alpha(y_i - q)I(y_i \geq q) + (1 - \alpha)(q - y_i)I(y_i < q)$$

- The level of quantile is α and I(.) equals 1 when the statement in parentheses is true, 0 otherwise.
- This is an *consistent* loss function for the α quantile.

In a picture



Simple setting

- Sample quantiles can be recovered using the pinball loss function

$$q^* = \underset{q}{argmin} \sum_i L_\alpha(y_i, q)$$

- Note that these might not exactly match the quantiles that come out of your favorite statistical package, there are alternative definitions of quantiles.

In reconciliation

- Reconciliation is a multivariate problem.
- If we are targeting quantiles we can optimise.

$$\underset{G}{argmin} \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_{\mathrm{train}}} L_\alpha(y_{i,t}, \tilde{q}_{i,t|t-1})$$

- However for each variable we want $\tilde{q}_{i,t|t-1}$ to be the α-quantile of $\tilde{y}^{(1)}_{i,t|t-1}, \ldots \tilde{y}^{(J)}_{i,t|t-1}$ meaning it must satisfy

$$\tilde{q}_{i,t|t-1} = \underset{q}{argmin} \sum_{j} L_\alpha(\tilde{y}^{(j)}_{i,t|t-1}, q)$$

Optimisation

- This is an example of **bi-level optimisation**.
- It is further complicated by the fact that pinball loss is not smooth.
- This is when you phone a friend who knows about optimisation...

Optimisation

Mixed integer programming

- Pinball loss problems can be recast in terms of Mixed Integer Programming (MIP).
- Can be solved with purpose built solvers for these types of problem (e.g. Gurobi)
- The full description of the bilevel problem has 17 equations...
- Some intuition for the simpler problem on the next slide

Pinball loss as MIP

$$
\begin{aligned}
\tilde{y}_j - \tilde{q} &\leq M v_j^-, \quad j \in [J] \\
\tilde{q} - \tilde{y}_j &\leq M v_j^+, \quad j \in [J] \\
0 \leq w_j &\leq 1, \quad j \in [J] \\
w_j &\leq 1 - v_j^-, \quad j \in [J] \\
1 - w_j &\leq 1 - v_j^+, \quad j \in [J] \\
\tfrac{1}{J} \textstyle\sum_{j \in [J]} w_j &= 1 - \alpha \\
v_j^-, v_j^+ &\in \{0, 1\}, \quad j \in [J].
\end{aligned}
$$

Unfortunately...

- Even with modern solvers, this is too slow.
- Even for small problems (3-variable hierarchies, $T = 50$, $J = 100$) it takes a few hours to run.
- It is not feasible for bigger hierarchies.
- So we looked for an alternative.

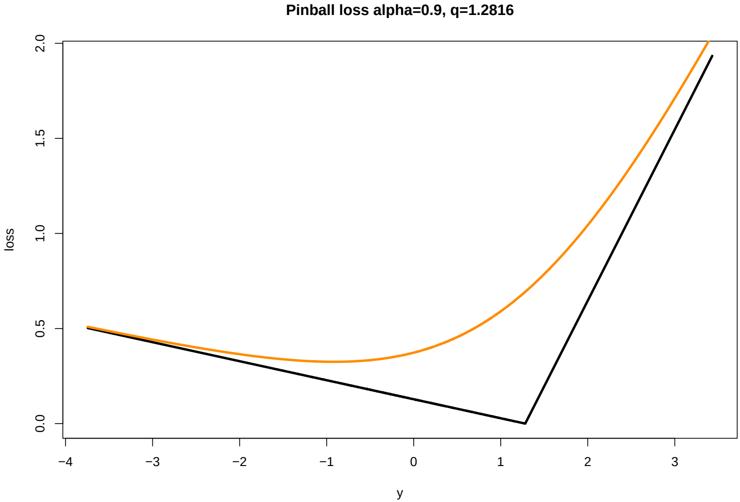
Smooth pinball loss

The following function approximates the pinball loss and converges to pinball loss as $\beta \to \infty$

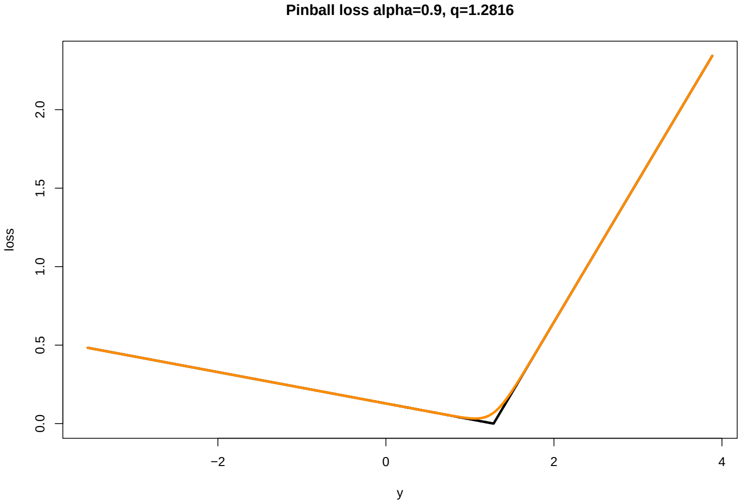
$$L_{\alpha}^{\beta}(y, q) = \frac{1}{\beta} \log \left(e^{\beta\alpha(y-q)} + e^{\beta(1-\alpha)(q-y)} \right)$$

Unlike the pinball function it is smooth.

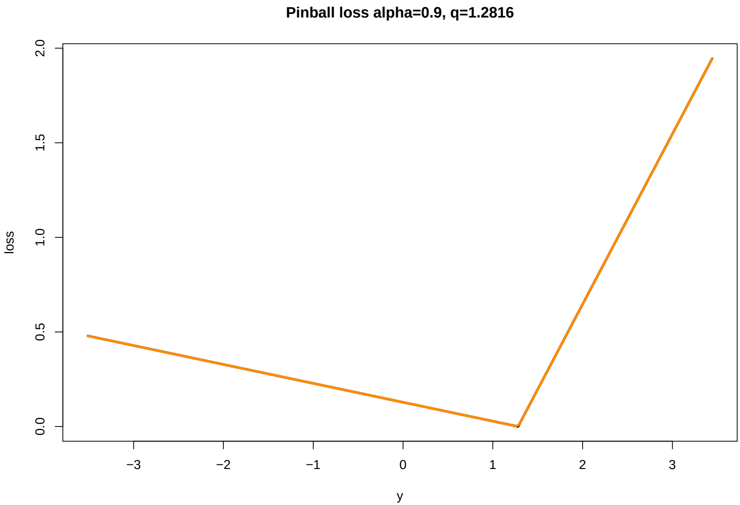
Smoothed pinball loss ($\beta = 1$)



Smoothed pinball loss ($\beta = 10$)



Smoothed pinball loss ($\beta = 100$)



Optimisation problem

Recall the problem is

$$\underset{G}{argmin} \sum_{i=1}^{n} \sum_{t \in \mathcal{T}_{\text{train}}} L_{\alpha}^{\beta}(y_{i,t}, \tilde{q}_{i,t|t-1})$$

where

$$\tilde{q}_{i,t|t-1} = \underset{q}{argmin} \sum_{j} L_{\alpha}^{\beta}(\tilde{y}_{i,t|t-1}^{(j)}, q)$$

and

$$\tilde{y}_{i,t|t-1}^{(j)} = \mathbf{SG}\hat{y}_{i,t|t-1}^{(j)}$$

A trick

To use gradient descent we need

$$\frac{\partial L_{\alpha}^{\beta}(y_{i,t},\tilde{q}_{i,t|t-1})}{\partial g_{l,k}}$$

This can be found using the chain rule, however the gradient must *pass through* the argmin in the lower level. A lemma by Gould et. al. (2016) can be used for this.

Empirical study

- Use Australian tourism data.
- Grouped hierarchy of states and purpose of travel.
- Dimension of $\mathbf{S}$ is 40×28.
- Seasonal ARIMA used for base forecasts.

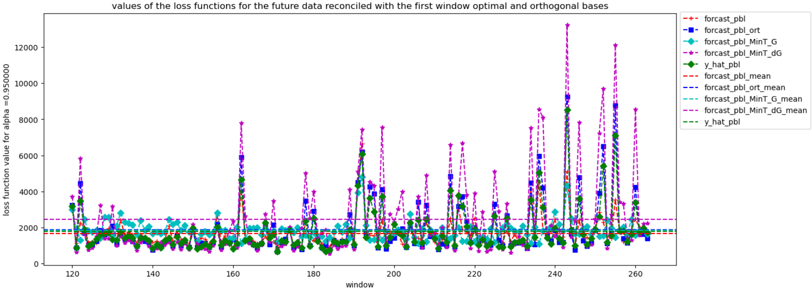
Results - In Sample

Method	Pinball Loss
Base	92278
OLS	96693
WLS	109926
MinT	112954
QOpt	78776

Results - Out of Sample

Method	Pinball Loss
Base	1768
OLS	1876
WLS	2440
MinT	1845
QOpt	1673

Results


values of the loss functions for the future data reconciled with the first window optimal and orthogonal bases
loss function value for alpha =0.950000
window
forcast_pbl
forcast_pbl_ort
forcast_pbl_MinT_G
forcast_pbl_MinT_dG
y_hat_pbl
forcast_pbl_mean
forcast_pbl_ort_mean
forcast_pbl_MinT_G_mean
forcast_pbl_MinT_dG_mean
y_hat_pbl

Work to do

- Determine whether these differences are significant
  - Use VaR backtesting literature
- Effect of the quantile
  - Conjecture is that results will differ for more extreme quantiles
- Consider regularisation
  - Shrink **SG** to an orthogonal projection or MinT

Thanks!

Questions...