

# Improving forecasts via subspace projections

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OPTiMA

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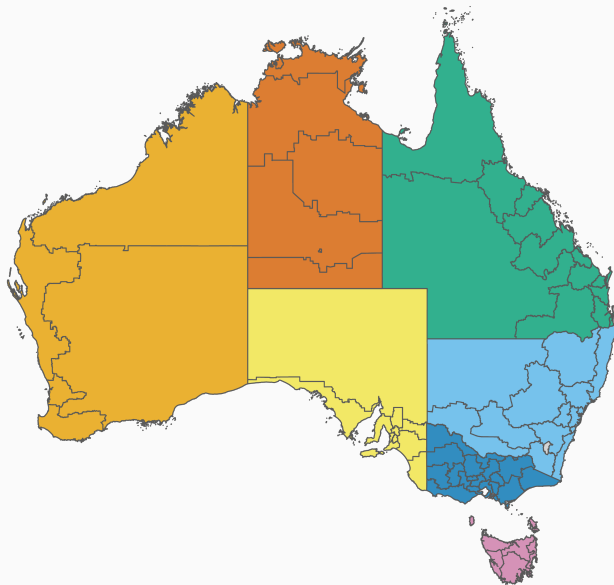
# Outline

- 1 Improving hierarchical forecasts
- 2 Improving univariate forecasts
- 3 Improving multivariate forecasts
- 4 Final comments

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- 1 Improving hierarchical forecasts
- 2 Improving univariate forecasts
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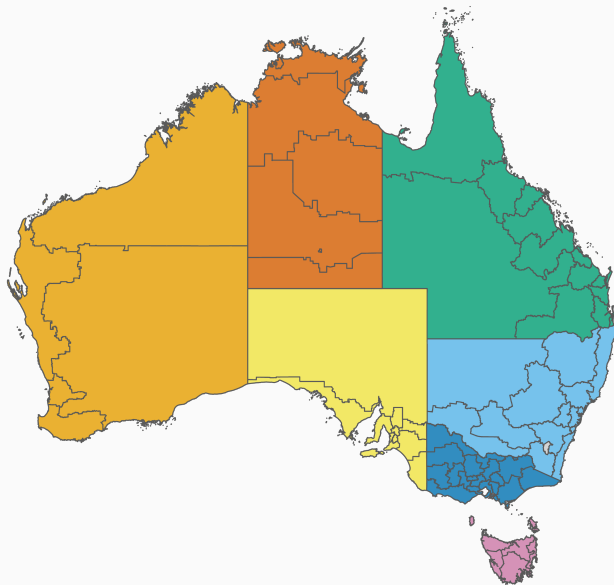
# Australian tourism regions



## State

	Australian Capital Territory
	New South Wales
	Northern Territory
	Queensland
	South Australia
	Tasmania
	Victoria
	Western Australia

# Australian tourism regions



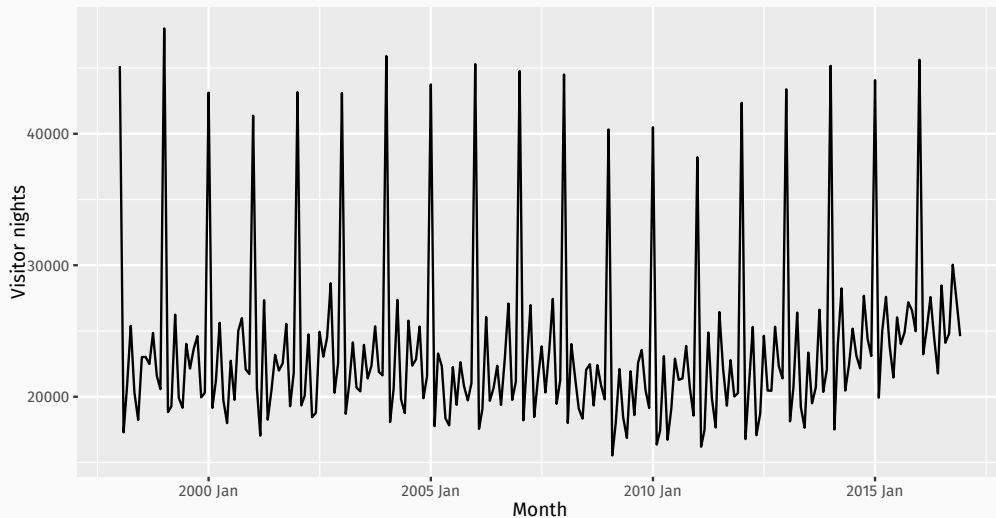
- Monthly data on visitor nights: 1998 – 2016
- 7 states
- 27 zones
- 76 regions

## State

	Australian Capital Territory
	New South Wales
	Northern Territory
	Queensland
	South Australia
	Tasmania
	Victoria
	Western Australia

# Australian tourism data

Total domestic travel: Australia



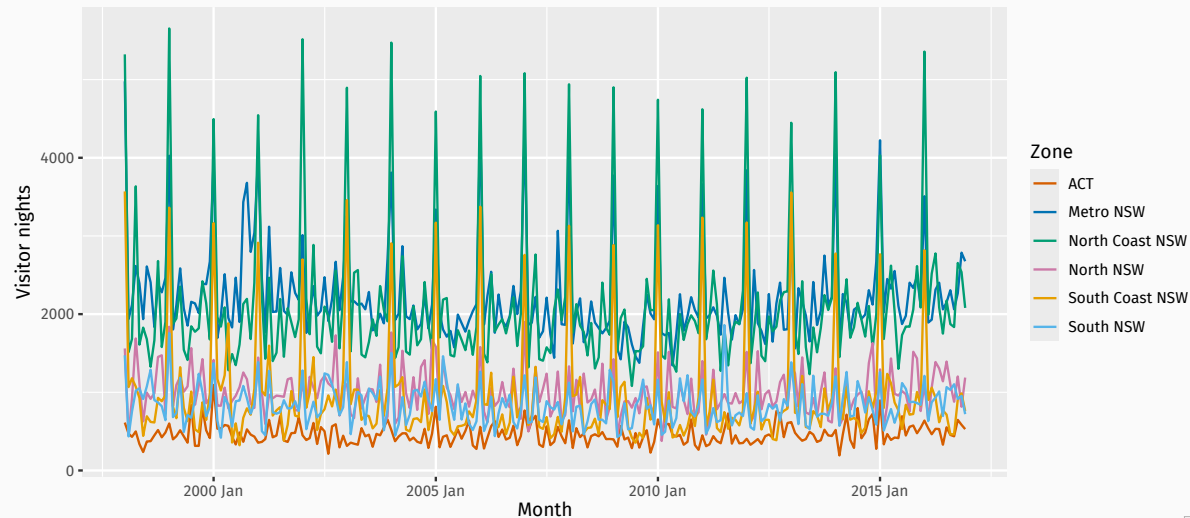
# Australian tourism data

Total domestic travel: by state



# Australian tourism data

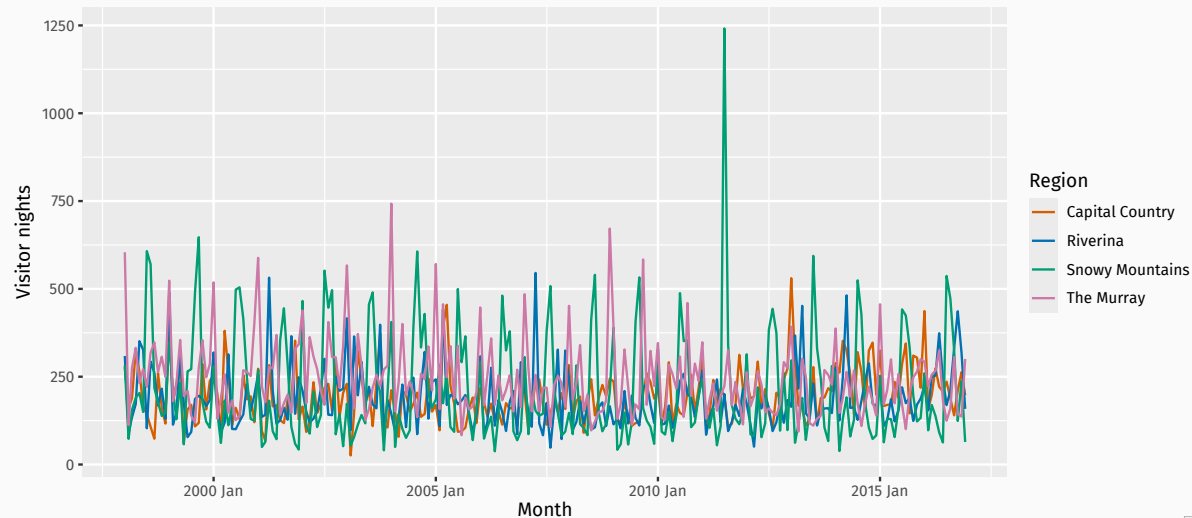
Total domestic travel: NSW by zone



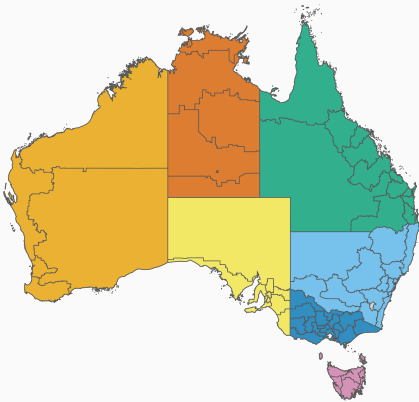


# Australian tourism data

Total domestic travel: South NSW by region



# Australian tourism data



State

	Australian Capital Territory
	New South Wales
	Northern Territory
	Queensland
	South Australia
	Tasmania
	Victoria
	Western Australia

Aggregation level	# series
-------------------	----------

National	1
----------	---

State	7
-------	---

Zone	27
------	----

Region	76
--------	----

<b>Total</b>	<b>111</b>
--------------	------------

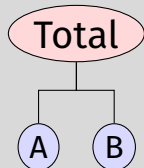
- Need forecasts at all levels of aggregation.
- Compute **base forecasts** using univariate models. These will not add up.
- Adjust base forecasts to ensure they are “coherent” giving **reconciled forecasts**.

# Notation

Almost all collections of time series with linear constraints can be written as

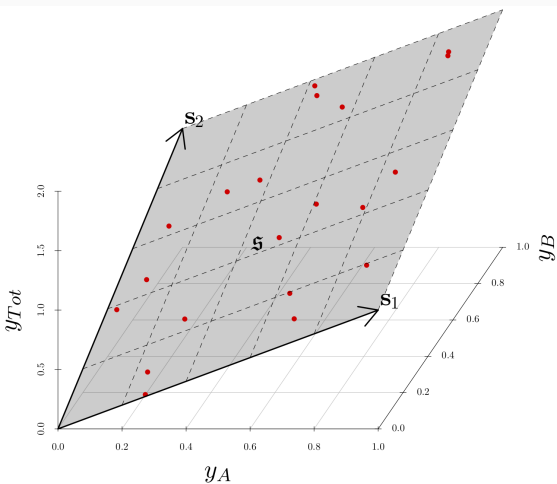
$$\mathbf{y}_t = \mathbf{S} \mathbf{b}_t$$

- $\mathbf{y}_t$  = vector of all series at time  $t$
- $y_{\text{Total},t}$  = aggregate of all series at time  $t$ .
- $y_{X,t}$  = value of series  $X$  at time  $t$ .
- $\mathbf{b}_t$  = vector of most disaggregated series at time  $t$
- $\mathbf{S}$  = “summing matrix” containing the linear constraints.



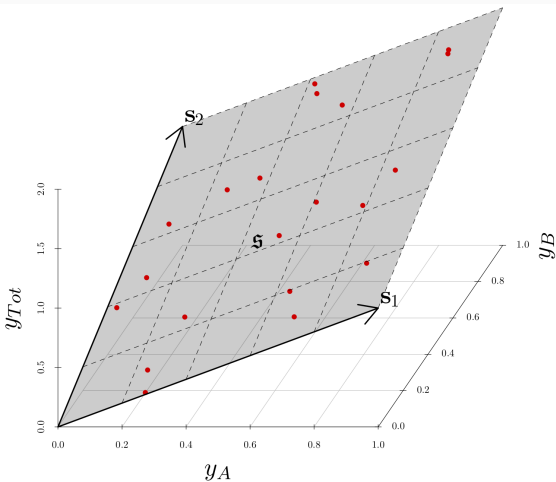
$$\begin{aligned} \mathbf{y}_t &= \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \end{pmatrix} \\ &= \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\mathbf{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \end{pmatrix}}_{\mathbf{b}_t} \end{aligned}$$

# Projections onto the coherent subspace



$$y_{Total} = y_A + y_B$$

# Projections onto the coherent subspace

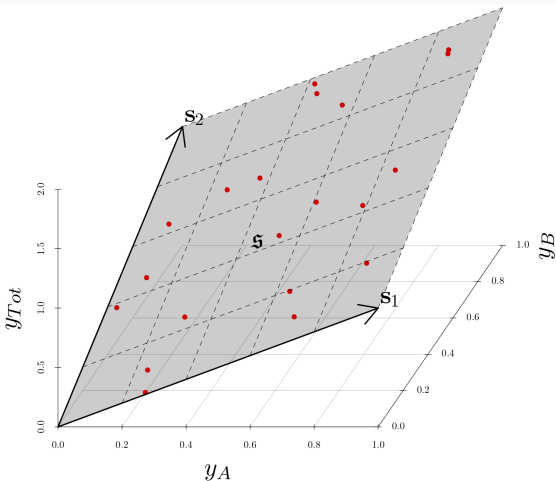


$$y_{Total} = y_A + y_B$$

## Base forecasts

Let  $\hat{y}_{t+h|t}$  be vector of *incoherent* initial  $h$ -step forecasts.

# Projections onto the coherent subspace



$$y_{Total} = y_A + y_B$$

## Base forecasts

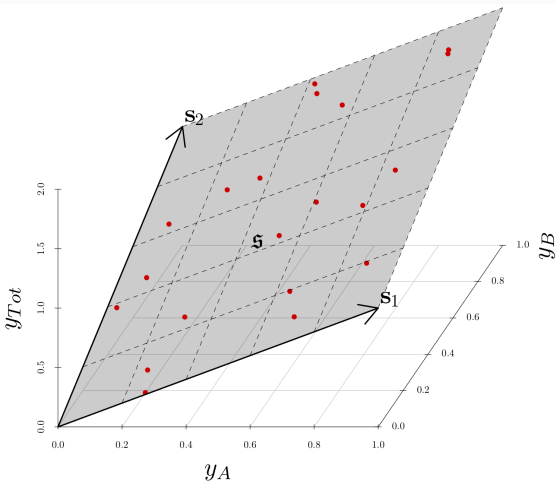
Let  $\hat{y}_{t+h|t}$  be vector of *incoherent* initial  $h$ -step forecasts.

## Reconciled forecasts

Let  $M$  be a projection matrix.

$\tilde{y}_{t+h|t} = M\hat{y}_{t+h|t}$  “reconciles”  $\hat{y}_{t+h|t}$ .

# Projections onto the coherent subspace



$$y_{\text{Total}} = y_A + y_B$$

## Base forecasts

Let  $\hat{\mathbf{y}}_{t+h|t}$  be vector of *incoherent* initial  $h$ -step forecasts.

## Reconciled forecasts

Let  $\mathbf{M}$  be a projection matrix.

$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$  “reconciles”  $\hat{\mathbf{y}}_{t+h|t}$ .

- $\mathbf{S}$  forms a basis set for  $\mathfrak{s}$
- All projections are of the form  $\mathbf{M} = \mathbf{S}(\mathbf{S}'\Psi\mathbf{S})^{-1}\mathbf{S}'\Psi$  where  $\Psi$  is a positive definite matrix.
- How to choose the best  $\Psi$ ?

$$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

Reconciled forecasts

Base forecasts



$$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

Reconciled forecasts

Base forecasts

- Base forecast covariance:

$$\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{t+h|t} \mid \mathbf{y}_1, \dots, \mathbf{y}_T]$$

- Reconciled forecast covariance:

$$\mathbf{V}_h = \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{t+h|t} \mid \mathbf{y}_1, \dots, \mathbf{y}_T] = \mathbf{M}\mathbf{W}_h\mathbf{M}'$$

$$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{M}\hat{\mathbf{y}}_{t+h|t}$$

Reconciled forecasts

Base forecasts

- Base forecast covariance:

$$\mathbf{W}_h = \text{Var}[\mathbf{y}_{T+h} - \hat{\mathbf{y}}_{t+h|t} \mid \mathbf{y}_1, \dots, \mathbf{y}_T]$$

- Reconciled forecast covariance:

$$\mathbf{V}_h = \text{Var}[\mathbf{y}_{T+h} - \tilde{\mathbf{y}}_{t+h|t} \mid \mathbf{y}_1, \dots, \mathbf{y}_T] = \mathbf{M}\mathbf{W}_h\mathbf{M}'$$

## Minimum trace (MinT) reconciliation

If  $\mathbf{M}$  is a projection, then trace of  $\mathbf{V}_h$  is minimized when  $\Psi = \mathbf{W}_h$

$$\mathbf{M} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}$$

$$\tilde{\mathbf{y}}_{t+h|t} = \mathbf{S}(\mathbf{S}'\mathbf{W}_h^{-1}\mathbf{S})^{-1}\mathbf{S}'\mathbf{W}_h^{-1}\hat{\mathbf{y}}_{t+h|t}$$

Reconciliation method	$\mathbf{W}_h$
OLS	$\mathbf{I}$
WLS(var)	$\text{diag}(\mathbf{W}_1)$
WLS(struct)	$\text{diag}(\mathbf{S}\mathbf{1})$
MinT(sample)	$\hat{\mathbf{W}}_{\text{sam}}$
MinT(shrink)	$\hat{\mathbf{W}}_{\text{shr}}$

- All approximate MinT by assuming  $\mathbf{W}_h = k_h \mathbf{W}_1$ .
- $\hat{\mathbf{W}}_{\text{sam}}$  is sample estimate of the residual covariance matrix
- $\hat{\mathbf{W}}_{\text{shr}}$  is shrinkage estimator  $\tau \text{diag}(\hat{\mathbf{W}}_{\text{sam}}) + (1 - \tau) \hat{\mathbf{W}}_{\text{sam}}$  where  $\tau$  selected optimally.

# Example: Australian tourism

```
tourism
```

```
# A tsibble: 17,328 x 5 [1M]
# Key:      state, zone, region [76]
   state zone  region      month visitors
   <chr> <chr> <chr>      <mth>      <dbl>
1 NSW   ACT    Canberra 1998 Jan      612.
2 NSW   ACT    Canberra 1998 Feb      471.
3 NSW   ACT    Canberra 1998 Mar      430.
4 NSW   ACT    Canberra 1998 Apr      499.
5 NSW   ACT    Canberra 1998 May      338.
6 NSW   ACT    Canberra 1998 Jun      236.
7 NSW   ACT    Canberra 1998 Jul      371.
8 NSW   ACT    Canberra 1998 Aug      375.
9 NSW   ACT    Canberra 1998 Sep      449.
10 NSW  ACT    Canberra 1998 Oct      517.
# i 17,318 more rows
```

# Example: Australian tourism

```
tourism_agg <- tourism |>  
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
# A tsibble: 25,308 x 5 [1M]
```

```
# Key:      state, zone, region [111]
```

	month	state	zone	region	visitors
	<mth>	<chr*>	<chr*>	<chr*>	<dbl>
1	1998 Jan	<aggregated>	<aggregated>	<aggregated>	45151.
2	1998 Feb	<aggregated>	<aggregated>	<aggregated>	17295.
3	1998 Mar	<aggregated>	<aggregated>	<aggregated>	20725.
4	1998 Apr	<aggregated>	<aggregated>	<aggregated>	25389.
5	1998 May	<aggregated>	<aggregated>	<aggregated>	20330.
6	1998 Jun	<aggregated>	<aggregated>	<aggregated>	18238.
7	1998 Jul	<aggregated>	<aggregated>	<aggregated>	23005.
8	1998 Aug	<aggregated>	<aggregated>	<aggregated>	23033.
9	1998 Sep	<aggregated>	<aggregated>	<aggregated>	22483.
10	1998 Oct	<aggregated>	<aggregated>	<aggregated>	24845.

```
# i 25,298 more rows
```

# Example: Australian tourism

```
fit <- tourism_agg |>
  filter(year(month) <= 2015) |>
  model(ets = ETS(visitors))
```

```
# A mable: 111 x 4
```

```
# Key:      state, zone, region [111]
```

	state	zone	region	ets
	<chr*>	<chr*>	<chr*>	<model>
1	NSW	ACT	Canberra	<ETS(M,N,A)>
2	NSW	ACT	<aggregated>	<ETS(M,N,A)>
3	NSW	Metro NSW	Central Coast	<ETS(M,N,A)>
4	NSW	Metro NSW	Sydney	<ETS(M,N,A)>
5	NSW	Metro NSW	<aggregated>	<ETS(M,N,A)>
6	NSW	North Coast NSW	Hunter	<ETS(M,N,M)>
7	NSW	North Coast NSW	North Coast NSW	<ETS(M,N,M)>
8	NSW	North Coast NSW	<aggregated>	<ETS(M,N,M)>
9	NSW	North NSW	Blue Mountains	<ETS(M,N,M)>
10	NSW	North NSW	Central NSW	<ETS(A,N,A)>

```
# i 101 more rows
```

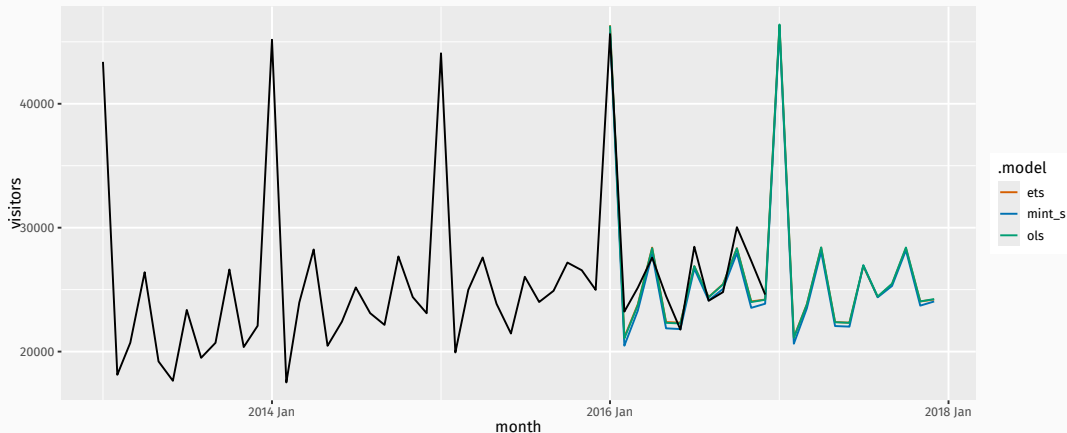
# Example: Australian tourism

```
fc <- fit |>
  reconcile(
    ols = min_trace(ets, method = "ols"),
    mint_s = min_trace(ets, method = "mint_shrink"),
  ) |>
  forecast(h = "2 years")
```

```
# A fable: 7,992 x 7 [1M]
# Key:      state, zone, region, .model [333]
  state zone  region  .model  month
  <chr*> <chr*> <chr*>   <chr>   <mth>
1 NSW    ACT    Canberra ets     2016 Jan
2 NSW    ACT    Canberra ets     2016 Feb
3 NSW    ACT    Canberra ets     2016 Mar
4 NSW    ACT    Canberra ets     2016 Apr
5 NSW    ACT    Canberra ets     2016 May
# i 7,987 more rows
# i 2 more variables: visitors <dist>, .mean <dbl>
```

# Example: Australian tourism

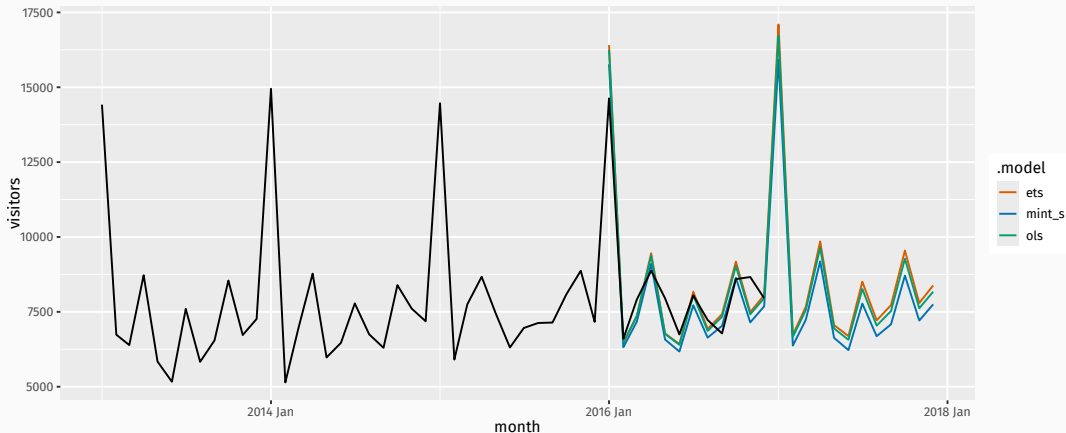
```
fc |>  
  filter(is_aggregated(state)) |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```





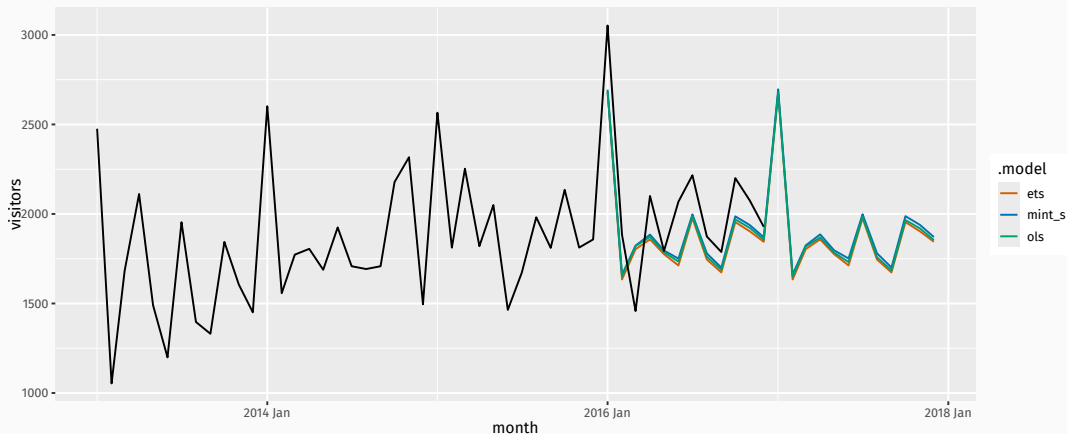
# Example: Australian tourism

```
fc |>  
  filter(state == "NSW" & is_aggregated(zone)) |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



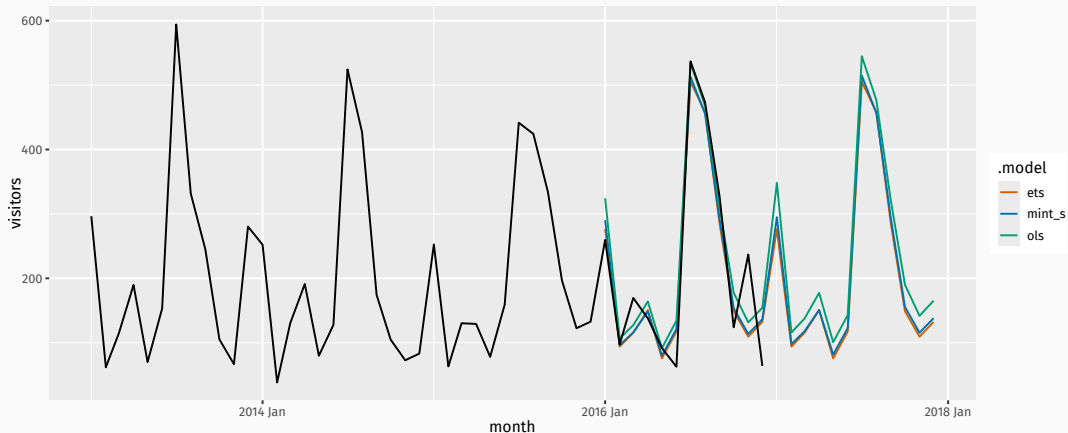
# Example: Australian tourism

```
fc |>  
  filter(region == "Melbourne") |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



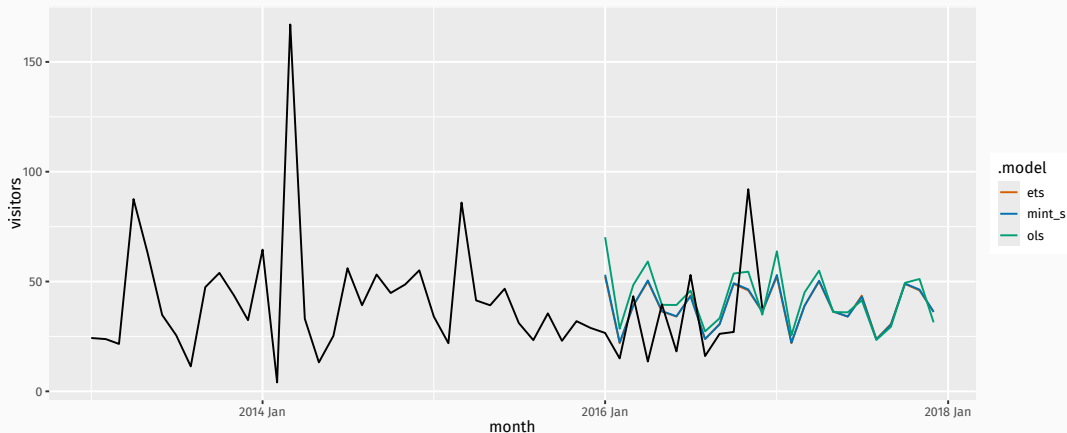
# Example: Australian tourism

```
fc |>  
  filter(region == "Snowy Mountains") |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



# Example: Australian tourism

```
fc |>  
  filter(region == "Barossa") |>  
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



# Performance evaluation

$$\text{RMSSE} = \sqrt{\text{mean}(q_j^2)}$$

$$q_j^2 = \frac{e_j^2}{\frac{1}{T-m} \sum_{t=m+1}^T (y_t - y_{t-m})^2}$$

- $y_t$  = observation for period  $t$
- $e_j$  = forecast error for forecast horizon  $j$
- $T$  = size of training set
- $m = 12$

# Example: Australian tourism

```
accuracy(fc, tourism_agg, measures = list(rmsse = RMSSE))
```

```
# A tibble: 333 x 7
```

	.model	state	zone	region	.type	rmsse	level
	<chr>	<chr*>	<chr*>	<chr*>	<chr>	<dbl>	<fct>
1	ets	<aggregated>	<aggregated>	<aggregated>	Test	0.755	National
2	mint_s	<aggregated>	<aggregated>	<aggregated>	Test	0.896	National
3	ols	<aggregated>	<aggregated>	<aggregated>	Test	0.768	National
4	ets	NSW	<aggregated>	<aggregated>	Test	0.921	State
5	mint_s	NSW	<aggregated>	<aggregated>	Test	0.893	State
6	ols	NSW	<aggregated>	<aggregated>	Test	0.881	State
7	ets	NT	<aggregated>	<aggregated>	Test	1.24	State
8	mint_s	NT	<aggregated>	<aggregated>	Test	1.22	State
9	ols	NT	<aggregated>	<aggregated>	Test	1.18	State
10	ets	QLD	<aggregated>	<aggregated>	Test	0.860	State

```
# i 323 more rows
```

# Example: Australian tourism

```
accuracy(fc, tourism_agg, measures = list(rmsse = RMSSE)) |>  
  summarise(rmsse = sqrt(mean(rmsse^2)), .by = .model)
```

```
# A tibble: 3 x 2
```

```
  .model rmsse
```

```
  <chr>   <dbl>
```

```
1 ols     0.863
```

```
2 mint_s  0.866
```

```
3 ets     0.880
```

# Example: Australian tourism

```
accuracy(fc, tourism_agg, measures = list(rmsse = RMSSE)) |>  
  summarise(rmsse = sqrt(mean(rmsse^2)), .by = .model)
```

```
# A tibble: 3 x 2
```

	.model	rmsse
	<chr>	<dbl>
1	ols	0.863
2	mint_s	0.866
3	ets	0.880

- Overall, both reconciliation methods are more accurate than the base ETS forecasts.



# Example: Australian tourism

```
accuracy(fc, tourism_agg, measures = list(rmsse = RMSSE)) |>  
  summarise(rmsse = sqrt(mean(rmsse^2)), .by = c(.model, level)) |>
```

```
# A tibble: 12 x 3
```

	.model	level	rmsse
	<chr>	<fct>	<dbl>
1	ets	National	0.755
2	ols	National	0.768
3	mint_s	National	0.896
4	ols	State	0.905
5	ets	State	0.919
6	mint_s	State	0.953
7	ols	Zone	0.912
8	mint_s	Zone	0.914
9	ets	Zone	0.935
10	mint_s	Region	0.839
11	ols	Region	0.843
12	ets	Region	0.858

# Example: Australian tourism

```
accuracy(fc, tourism_agg, measures = list(rmsse = RMSSE)) |>  
  summarise(rmsse = sqrt(mean(rmsse^2)), .by = c(.model, level)) |>
```

```
# A tibble: 12 x 3
```

	.model	level	rmsse
	<chr>	<fct>	<dbl>
1	ets	National	0.755
2	ols	National	0.768
3	mint_s	National	0.896
4	ols	State	0.905
5	ets	State	0.919
6	mint_s	State	0.953
7	ols	Zone	0.912
8	mint_s	Zone	0.914
9	ets	Zone	0.935
10	mint_s	Region	0.839
11	ols	Region	0.843
12	ets	Region	0.858

- Reconciliation is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

## Distance reducing property

Let  $\|\mathbf{u}\|_{\Psi} = \mathbf{u}'\Psi\mathbf{u}$ . Then

$$\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}\|_{\Psi} \leq \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|_{\Psi}$$

- $\Psi$ -projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure*.
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.

$$\begin{aligned}\|\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}\|_2^2 &= \|\mathbf{M}(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t})\|_2^2 \\ &\leq \|\mathbf{M}\|_2^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|_2^2 \\ &= \sigma_{\max}^2 \|\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}\|_2^2\end{aligned}$$

- $\sigma_{\max}$  is the largest eigenvalue of  $\mathbf{M}$
- $\sigma_{\max} \geq 1$  as  $\mathbf{M}$  is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

$$\begin{aligned} & \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{MinT}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]'[\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t}^{\text{OLS}}]\right) \\ & \leq \text{tr}\left(\mathbb{E}[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]'[\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}]\right) \end{aligned}$$

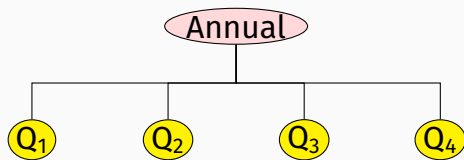
Using sums of variances:

- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

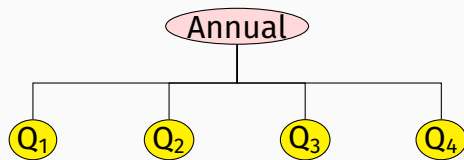
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# Temporal reconciliation: quarterly data



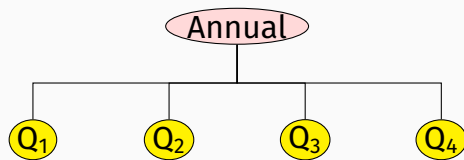
# Temporal reconciliation: quarterly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.



# Temporal reconciliation: quarterly data



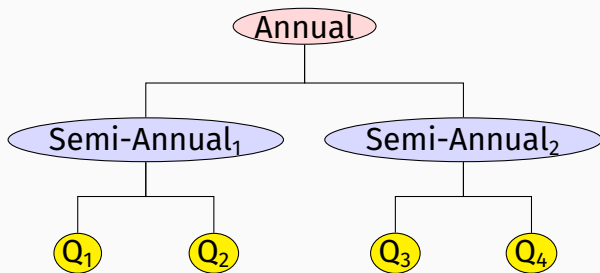
$$\mathbf{y}_{\tau} = \begin{bmatrix} x_{\tau}^{[4]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- ➔ Forecast series at each available frequency.
- ➔ Optimally combine forecasts within the same year.

$\tau = \text{year}$

# Temporal reconciliation: quarterly data

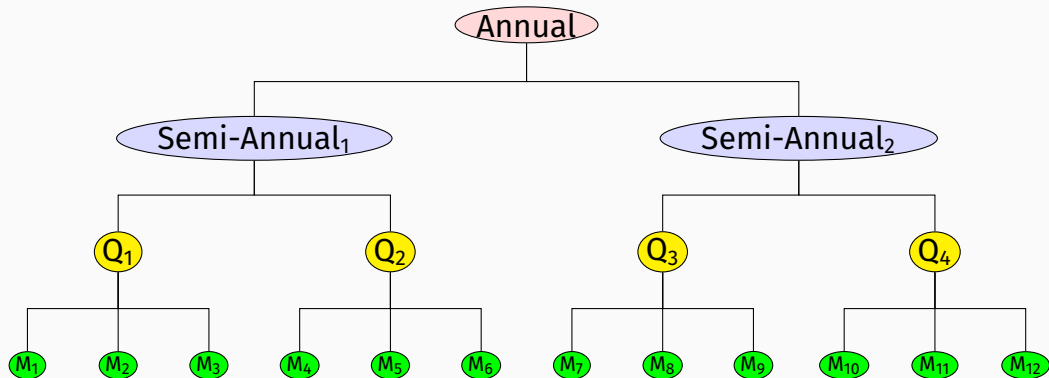


- ➡ Forecast series at each available frequency.
- ➡ Optimally combine forecasts within the same year.

$$\mathbf{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[2]} \\ x_{\tau,2}^{[2]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

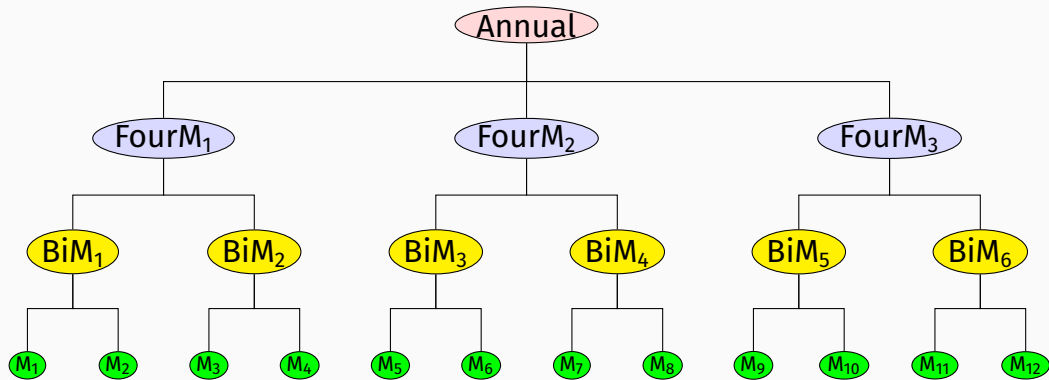
$\tau = \text{year}$

# Temporal reconciliation: monthly data



- ➡ Forecast series at each available frequency.
- ➡ Optimally combine forecasts within the same year.

# Temporal reconciliation: monthly data



- ➡ Forecast series at each available frequency.
- ➡ Optimally combine forecasts within the same year.

# Temporal reconciliation: monthly data

$$\mathbf{y}_\tau = \begin{bmatrix} \mathbf{x}_\tau^{[12]} \\ \mathbf{x}_\tau^{[6]} \\ \mathbf{x}_\tau^{[4]} \\ \mathbf{x}_\tau^{[3]} \\ \mathbf{x}_\tau^{[2]} \\ \mathbf{x}_\tau^{[1]} \end{bmatrix} \quad \mathbf{S} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & \mathbf{I}_{12} & & & & & \end{bmatrix}$$

# Temporal reconciliation

For a time series  $y_1, \dots, y_T$ , observed at frequency  $m$ :

$$x_j^{[k]} = \sum_{t=(j-1)k+1}^{jk} y_t \quad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in \mathcal{K} = \{k_1, \dots, k_p\}$  denote the  $p$  factors of  $m$  in ascending order, where  $k_1 = 1$  and  $k_p = m$
- $x_j^{[1]} = y_t$
- A single unique hierarchy is only possible when there are no coprime pairs in  $\mathcal{K}$ .
- $M_k = m/k$  is seasonal period of aggregated series.

# Temporal reconciliation

$$\mathbf{x}_\tau = \mathbf{S} \mathbf{x}_\tau^{[1]}, \quad \mathbf{S} = \begin{bmatrix} \mathbf{A} \\ \mathbf{I} \end{bmatrix}$$

where

$$\mathbf{x}_\tau = \begin{bmatrix} x_\tau^{[k_p]} \\ \mathbf{x}_\tau^{[k_{p-1}]} \\ \vdots \\ \mathbf{x}_\tau^{[k_1]} \end{bmatrix} \quad \mathbf{x}_\tau^{[k]} = \begin{bmatrix} x_{M_k(\tau-1)+1}^{[k]} \\ x_{M_k(\tau-1)+2}^{[k]} \\ \vdots \\ x_{M_k\tau}^{[k]} \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} & \mathbf{1}'_m & \\ \mathbf{I}_{m/k_{p-1}} \otimes \mathbf{1}'_{k_{p-1}} & & \\ & \vdots & \\ \mathbf{I}_{m/k_2} \otimes \mathbf{1}'_{k_2} & & \end{bmatrix}$$

$\tau$  is time index for most aggregated series,

$k \in \mathcal{K} = \{k_1, \dots, k_p\}, \quad k_1 = 1, \quad k_p = m, \quad \tau = 1, \dots, T/m.$

# Example: Accident & emergency services demand

Weekly A&E demand data: 7 November 2010 to 7 June 2015.

---

Type 1 Departments — Major A&E

Type 2 Departments — Single Specialty

Type 3 Departments — Other A&E/Minor Injury Unit

Total Attendances

Type 1 Departments — Major A&E > 2 hours

Type 2 Departments — Single Specialty > 2 hours

Type 3 Departments — Other A&E/Minor Injury Unit > 2 hours

Total Attendances > 2 hours

Emergency Admissions via Type 1 A&E

Total Emergency Admissions via A&E

Other Emergency Admissions (i.e not via A&E)

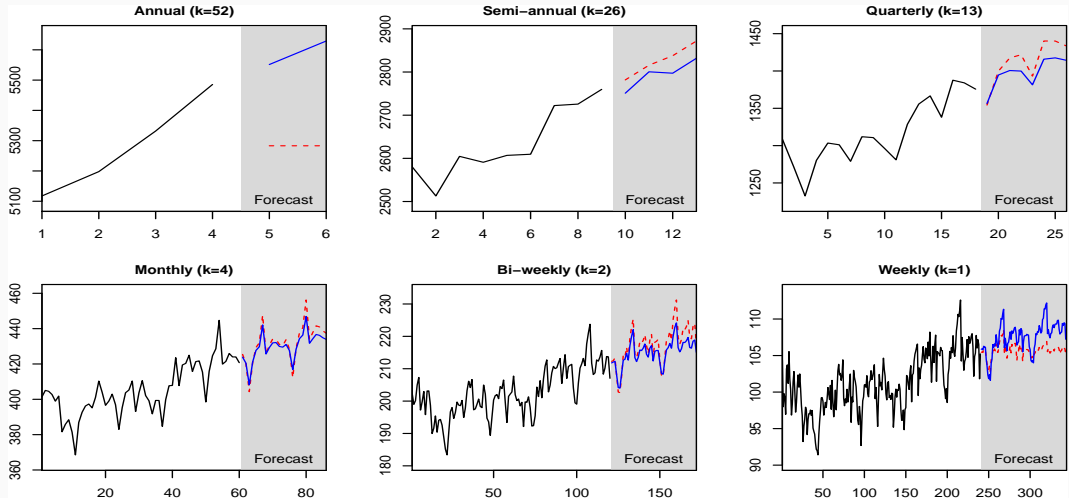
Total Emergency Admissions

Number of patients spending > 2 hours from decision to admit to admission



# Example: Accident & emergency services demand

## Total emergency admissions via A&E



# Example: Accident & emergency services demand

Test set: last 52 weeks

**MASE comparison** (ARIMA models)

Aggregation Level	h	Base	Reconciled	Change
Annual	1	3.4	1.9	-42.9%
Weekly	1-52	2.0	1.9	-5.0%
Weekly	13	2.3	1.9	-16.2%
Weekly	4	1.9	1.5	-18.6%
Weekly	1	1.6	1.3	-17.2%

# Temporal reconciliation: M3 monthly series

- Apply temporal reconciliation to all 1428 monthly series from M3 competition
- Forecast horizon  $h = 18$  months
- ETS and ARIMA models
- Measure percentage difference in MASE to base forecasts
- Reconciliation methods:
  - ▶  $WLS_H$  (diagonal)
  - ▶  $WLS_V$  (diagonal with common variances for same frequency)
  - ▶  $WLS_S$  (diagonal/structural)

# Temporal reconciliation: M3 monthly series

## Improvement in MASE relative to base forecasts

Aggregation level	$h$	ETS				ARIMA			
		BU	$WLS_H$	$WLS_V$	$WLS_S$	BU	$WLS_H$	$WLS_V$	$WLS_S$
Annual	1	-12.1	-17.9	-17.8	<b>-18.5</b>	-25.4	-29.9	-29.9	<b>-30.2</b>
Semi-annual	3	0.0	-6.3	-6.0	<b>-6.9</b>	-2.9	-8.1	-8.2	<b>-9.4</b>
Four-monthly	4	3.1	-3.2	-3.0	<b>-3.4</b>	-1.8	-6.2	-6.5	<b>-7.1</b>
Quarterly	6	3.2	-2.8	-2.7	<b>-3.4</b>	-2.6	-6.9	-7.4	<b>-8.1</b>
Bi-monthly	9	2.7	-2.9	-3.0	<b>-3.7</b>	-1.3	-5.0	-5.5	<b>-6.3</b>
Monthly	18	0.0	-3.7	-4.6	<b>-5.0</b>	0.0	-1.9	-3.2	<b>-3.7</b>
Average		-0.5	-6.1	-6.2	<b>-6.8</b>	-5.7	-9.7	-10.1	<b>-10.8</b>

# Outline

- 1 Improving hierarchical forecasts
- 2 Improving univariate forecasts
- 3 Improving multivariate forecasts
- 4 Final comments

# Forecast Linear Augmented Projection (FLAP)

- We want to forecast multivariate series  $\mathbf{y}_t$ .
- Linear combinations of  $\mathbf{y}_t$  may have better signal-noise ratio
- Construct many linear combinations  $\mathbf{c}_t = \Phi \mathbf{y}_t$  (e.g., principal components or random combinations)
- Produce univariate forecasts of all series  $\hat{\mathbf{y}}_{t+h|t}$  and all linear combinations  $\hat{\mathbf{c}}_{t+h|t}$ .
- Reconcile forecasts so they are coherent ( $\tilde{\mathbf{c}}_{t+h|t} = \Phi \tilde{\mathbf{y}}_{t+h|t}$ )

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$$\mathbf{z}_t = \begin{bmatrix} \mathbf{y}_t \\ \mathbf{c}_t \end{bmatrix} \quad \tilde{\mathbf{z}}_{t+h} = \mathbf{M} \hat{\mathbf{z}}_{t+h|t}$$

# Forecast error variance reduction

If we know the covariance matrix  $\mathbf{W}_h = \text{Var}(\mathbf{z}_{t+h} - \hat{\mathbf{z}}_{t+h|t})$ , then

1 The forecast error variance is reduced:

▶  $\text{Var}(\mathbf{y}_{t+h} - \hat{\mathbf{y}}_{t+h|t}) - \text{Var}(\mathbf{y}_{t+h} - \tilde{\mathbf{y}}_{t+h|t})$  is positive semi-definite.

2 The forecast error variance of each series monotonically decreases with increasing number of components.



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- 2 The forecast error variance of each series monotonically decreases with increasing number of components.

In practice, we need to:

- Estimate  $\mathbf{W}_h$  (use the MinT shrinkage estimator).
- Construct the components,  $\Phi$ .

# Construction of $\Phi$

## Principal component analysis (PCA)

Finding the weights matrix  $\Phi$  so that the resulting components **maximise variance**

## Simulation

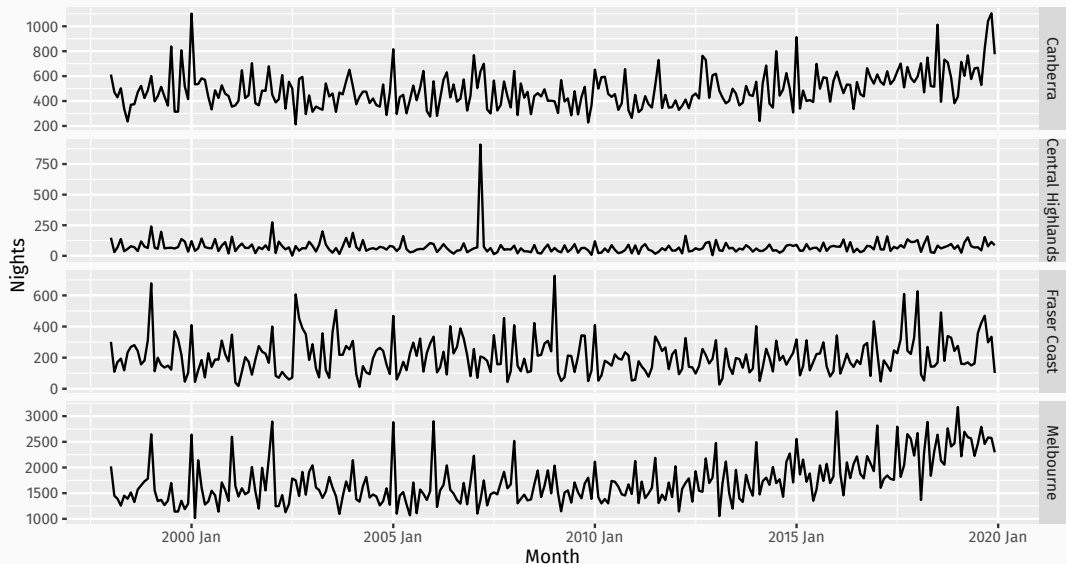
Generating values of  $\Phi$  from a random distribution and normalising them to unit vectors

- Normal distribution
- Uniform distribution
- Orthonormal matrix

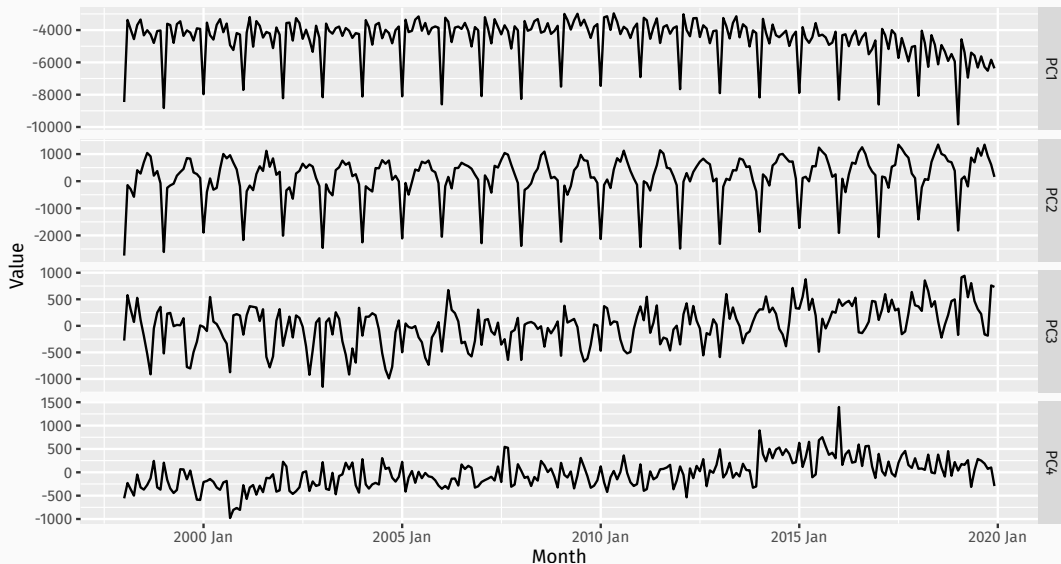
# Monthly Australian regional tourism

- Monthly Australian tourism data set aggregated by region giving 77 series, from Jan 1998 to Dec 2019.
- Use expanding window time series cross-validation with  $T = 84$  observations in first training set, and forecast horizons  $h = 1, 2, \dots, 12$ .
- Fit univariate ETS models to each series.

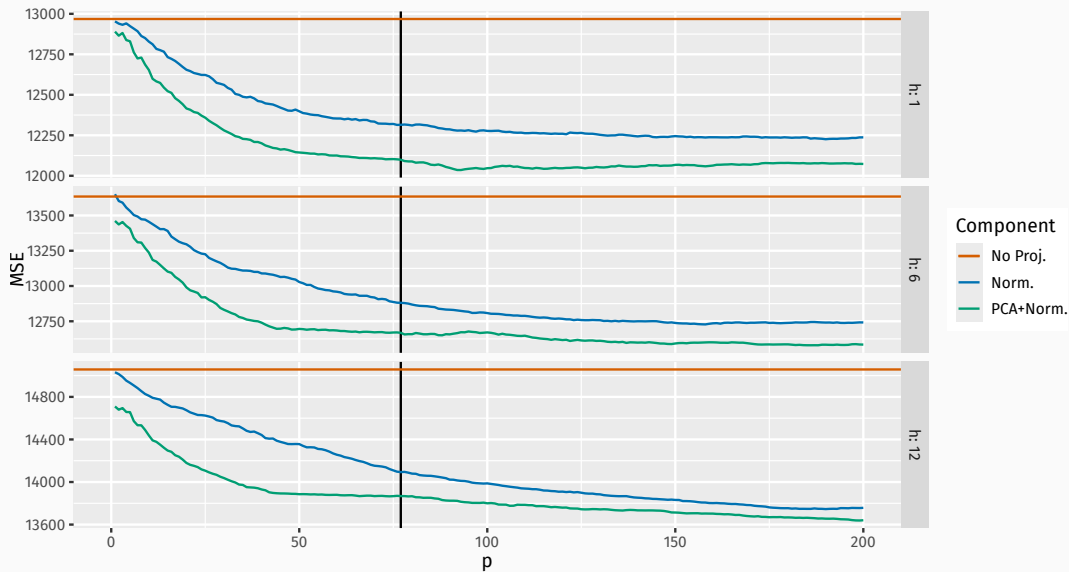
# Monthly Australian regional tourism



# Monthly Australian regional tourism

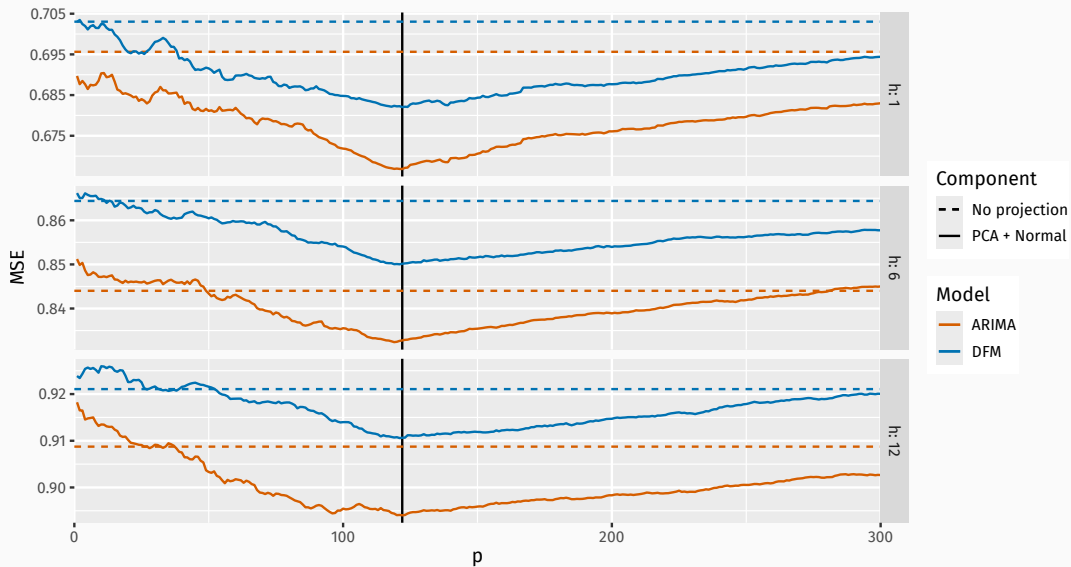


# Monthly Australian regional tourism



- Monthly data of macroeconomic variables (McCracken and Ng, 2016).
- Data from Jan 1959 – Sep 2023. 777 observations on 122 series.
- Same cleaning process as per McCracken and Ng (2016).
- All series scaled to have mean 0 and variance 1.
- Expanding time series cross-validation with initial size of 25 years and forecast horizon 12 months.

# FRED-MD

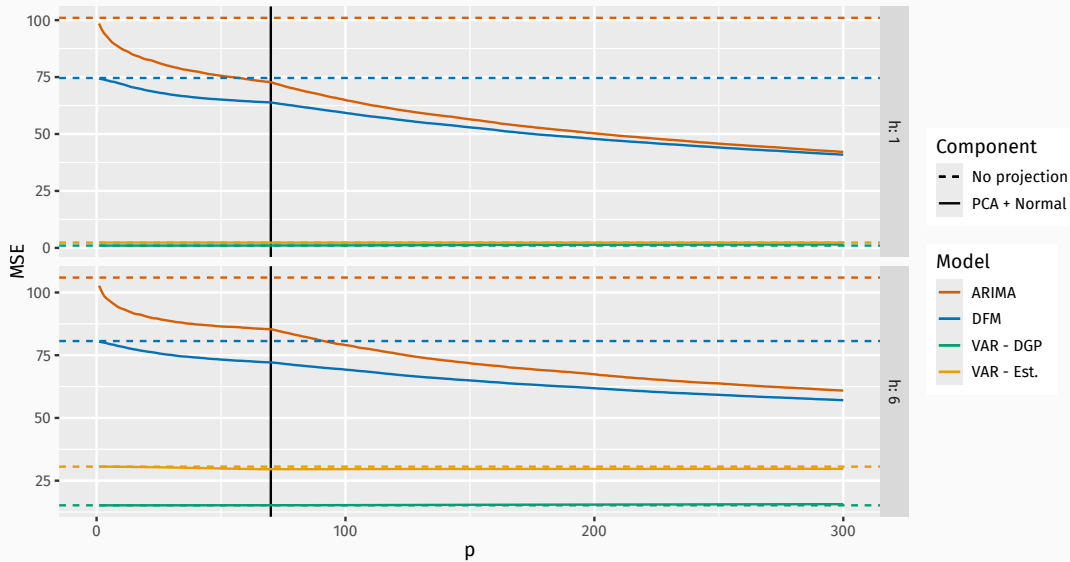




# Simulation

- Data generating process: VAR(3) with 70 variables
- Sample size:  $T = 400$
- Number of repeated samples: 220
- Base models:
  - ▶ automatic ARIMA (based on AICc)
  - ▶ DFM (structure chosen using BIC, different model for each horizon)

# Simulation



# Outline

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# Software

Package	Language	Cross-sectional	Temporal	Cross-temporal	Probabilistic	Multivariate
hts	R	✓				
thief	R		✓			
fable	R	✓			✓	
FoReco	R	✓	✓	✓	✓	
flap	R					✓
pyhts	Python	✓	✓			
hierarchicalforecast	Python	✓			✓	

- hts, thief, and FoReco use ts objects
- fable uses tsibble objects
- flap uses matrices of base forecasts
- fable has plans to implement temporal and cross-temporal reconciliation






# Thanks!



## More information

[robjhyndman.com/vienna2025](http://robjhyndman.com/vienna2025)

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-  Di Fonzo, T & D Girolimetto (2023). Cross-temporal forecast reconciliation: Optimal combination method and heuristic alternatives. *International J Forecasting* **39**(1), 39–57.
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-  Hyndman, RJ, RA Ahmed, G Athanasopoulos & HL Shang (2011). Optimal combination forecasts for hierarchical time series. *Computational Statistics & Data Analysis* **55**(9), 2579–2589.

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-  Wickramasuriya, SL, G Athanasopoulos & RJ Hyndman (2019). Optimal forecast reconciliation for hierarchical and grouped time series through trace minimization. *J American Statistical Association* **114**(526), 804–819.
-  Yang, YF, G Athanasopoulos, RJ Hyndman & A Panagiotelis (2024). Forecast Linear Augmented Projection (FLAP): A free lunch to reduce forecast error variance. *arXiv:2407.01868*.