Improving forecasts via subspace projections

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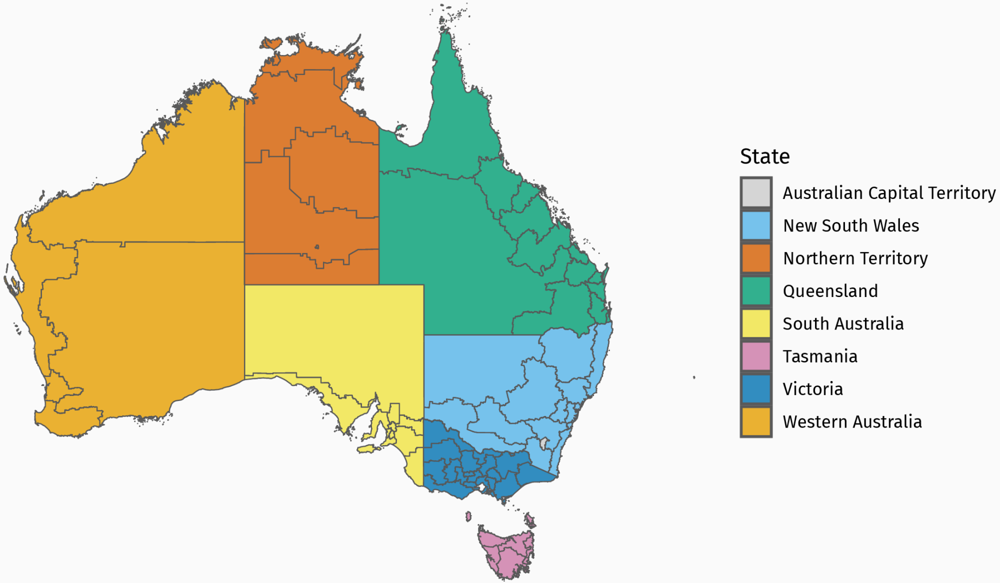
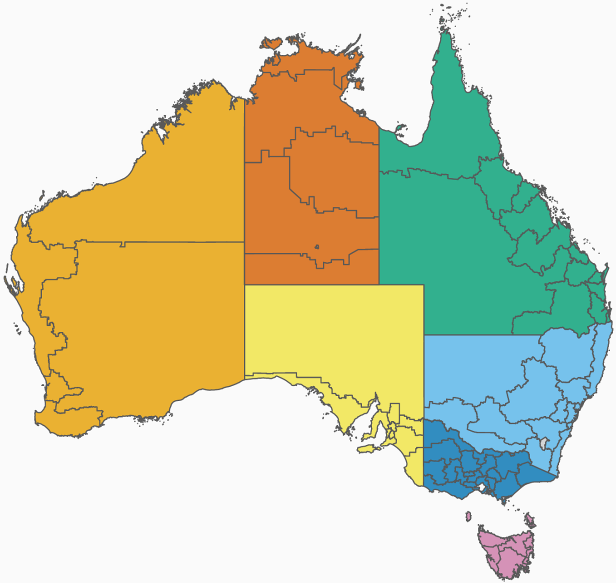
Outline

Outline

Australian tourism regions

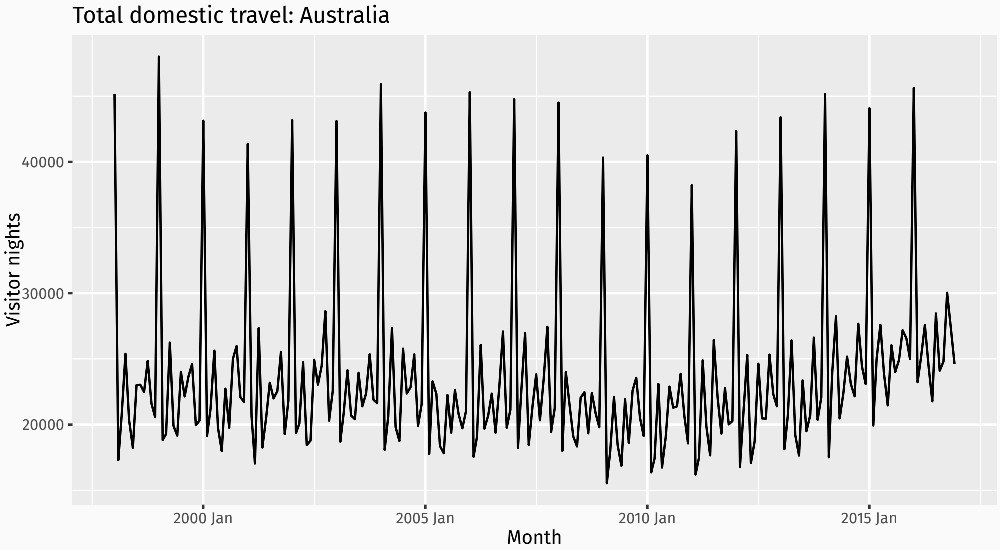


Australian tourism regions

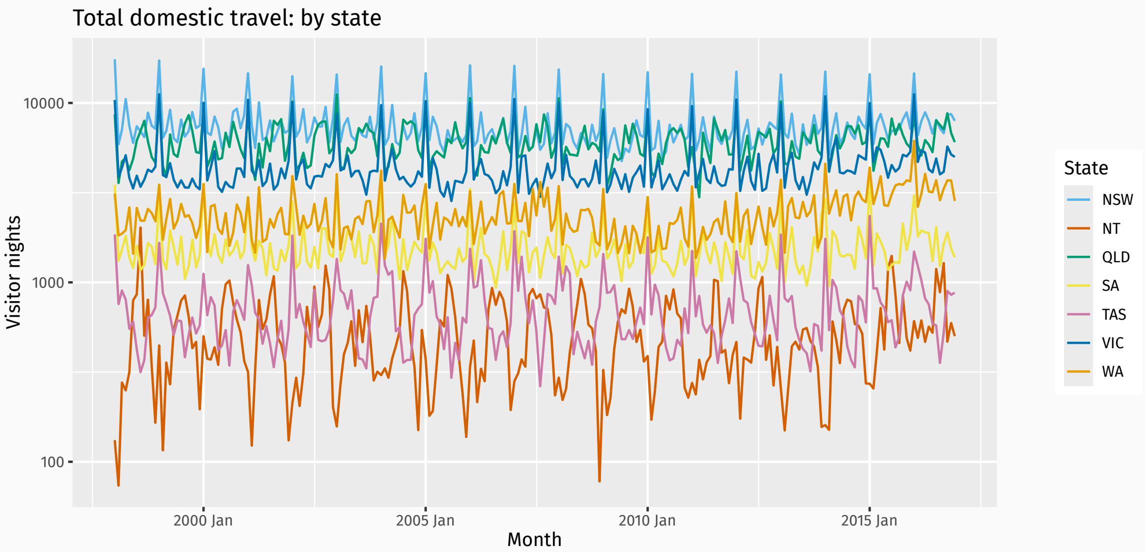


- Monthly data on visitor nights: 1998 – 2016
- 7 states
- 27 zones
- 76 regions

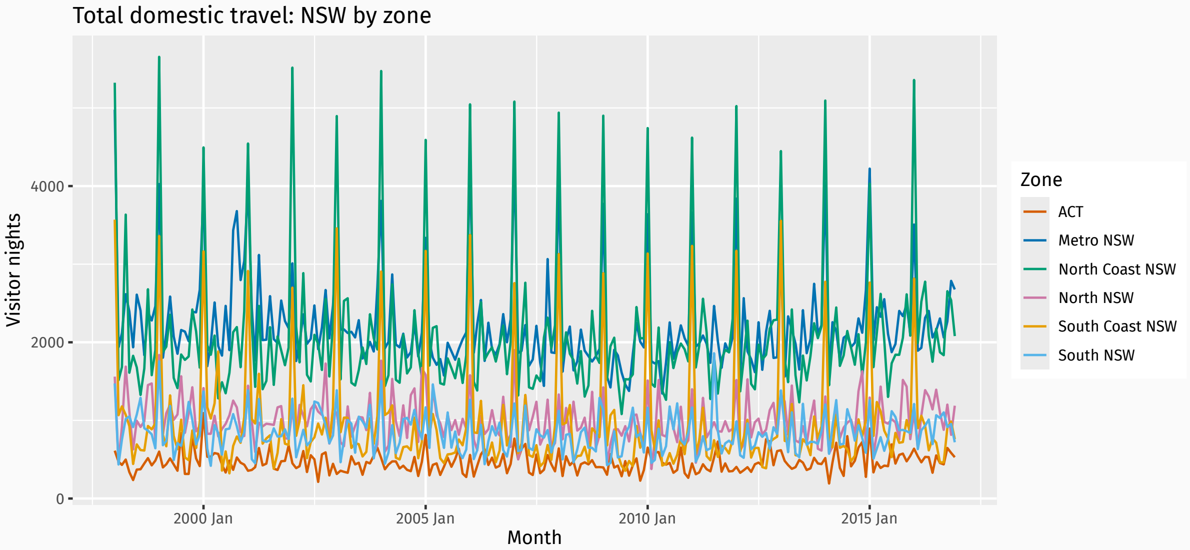
Australian tourism data



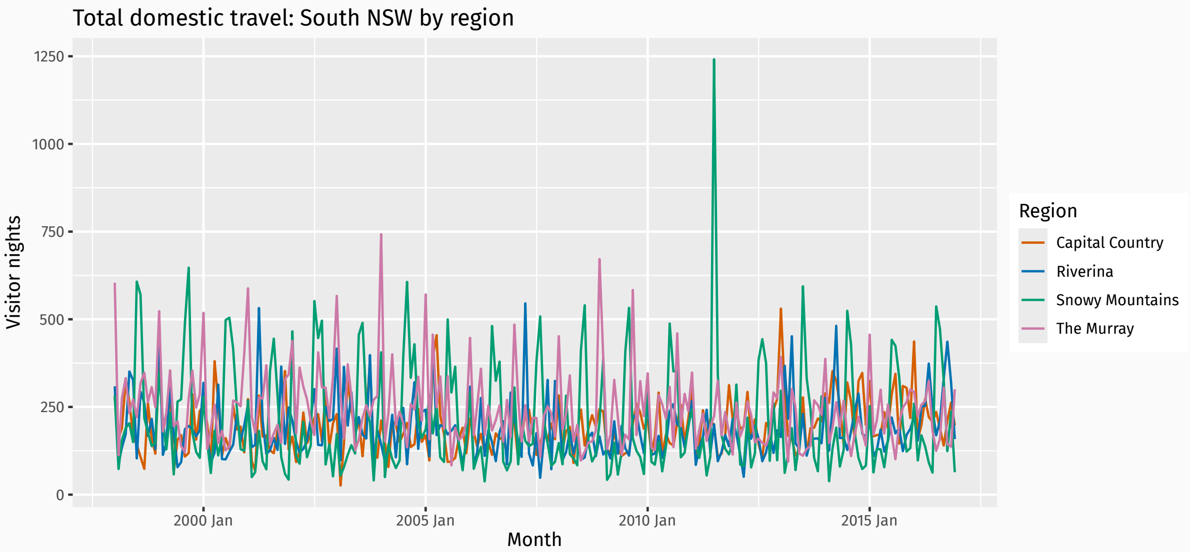
Australian tourism data



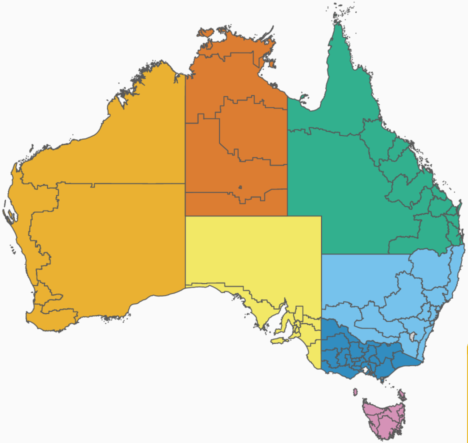
Australian tourism data



Australian tourism data



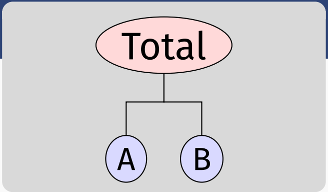
Australian tourism data



Aggregation level	# series
National	1
State	7
Zone	27
Region	76
Total	**111**

- Need forecasts at all levels of aggregation.
- Compute **base forecasts** using univariate models. These will not add up.
- Adjust base forecasts to ensure they are “coherent” giving **reconciled forecasts**.

Notation



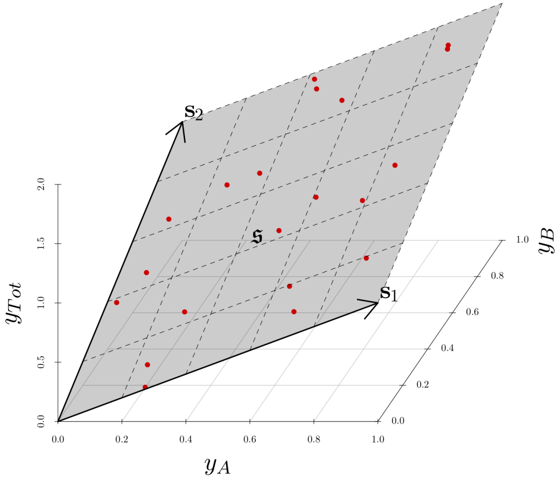
Almost all collections of time series with linear constraints can be written as

$$\boldsymbol{y}_t = \boldsymbol{S}\boldsymbol{b}_t$$

- $\boldsymbol{y}_t$ = vector of all series at time t
- $y_{\text{Total},t}$ = aggregate of all series at time t.
- $y_{X,t}$ = value of series X at time t.
- $\boldsymbol{b}_t$ = vector of most disaggregated series at time t
- $\boldsymbol{S}$ = "summing matrix" containing the linear constraints.

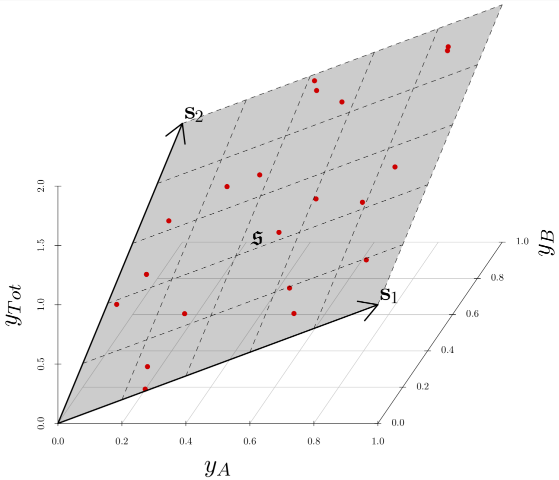
$$\boldsymbol{y}_t = \begin{pmatrix} y_{\text{Total},t} \\ y_{A,t} \\ y_{B,t} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{\boldsymbol{S}} \underbrace{\begin{pmatrix} y_{A,t} \\ y_{B,t} \end{pmatrix}}_{\boldsymbol{b}_t}$$

Projections onto the coherent subspace



$$y_{\text{Total}} = y_A + y_B$$

Projections onto the coherent subspace

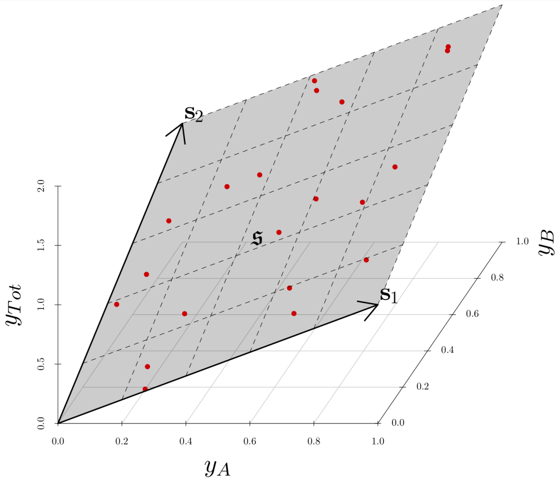


$y_{\text{Total}} = y_A + y_B$

Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial h-step forecasts.

Projections onto the coherent subspace



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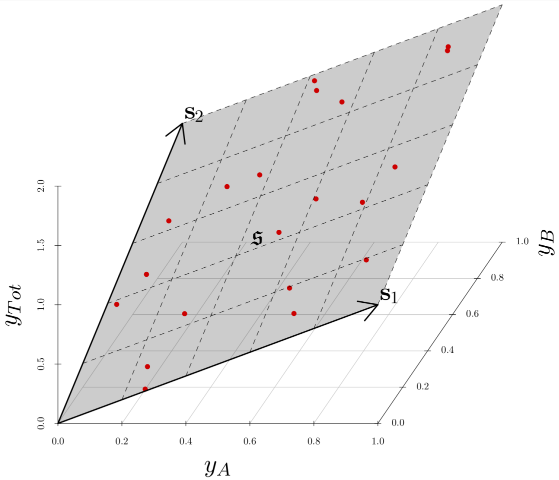
Base forecasts

Let $\hat{\boldsymbol{y}}_{t+h|t}$ be vector of *incoherent* initial *h*-step forecasts.

Reconciled forecasts

Let $\boldsymbol{M}$ be a projection matrix. $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

Projections onto the coherent subspace



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Let $\boldsymbol{M}$ be a projection matrix. $\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$ "reconciles" $\hat{\boldsymbol{y}}_{t+h|t}$.

- $\boldsymbol{S}$ forms a basis set for $\mathfrak{s}$
- All projections are of the form $\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\Psi\boldsymbol{S})^{-1}\boldsymbol{S}'\Psi$ where Ψ is a positive definite matrix.
- How to choose the best Ψ?

Minimum trace reconciliation

Wickramasuriya et al (2019)

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

Reconciled forecasts

Base forecasts

Minimum trace reconciliation

Wickramasuriya et al (2019)

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

Reconciled forecasts   Base forecasts

- Base forecast covariance:
  $\boldsymbol{W}_h = \text{Var}[\boldsymbol{y}_{T+h} - \hat{\boldsymbol{y}}_{t+h|t} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]$
- Reconciled forecast covariance:
  $\boldsymbol{V}_h = \text{Var}[\boldsymbol{y}_{T+h} - \tilde{\boldsymbol{y}}_{t+h|t} \mid \boldsymbol{y}_1, \dots, \boldsymbol{y}_T]) = \boldsymbol{M}\boldsymbol{W}_h\boldsymbol{M}'$

Minimum trace reconciliation

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- Reconciled forecast covariance:
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Minimum trace (MinT) reconciliation

If $\boldsymbol{M}$ is a projection, then trace of $\boldsymbol{V}_h$ is minimized when $\Psi = \boldsymbol{W}_h$

$$\boldsymbol{M} = \boldsymbol{S}(\boldsymbol{S}'\boldsymbol{W}_h^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\boldsymbol{W}_h^{-1}$$

Linear projections

$$\tilde{\boldsymbol{y}}_{t+h|t} = \boldsymbol{M}\hat{\boldsymbol{y}}_{t+h|t}$$

Reconciliation method	$\boldsymbol{M}$
OLS	$\boldsymbol{S}(\boldsymbol{S}'\boldsymbol{S})^{-1}\boldsymbol{S}'$
WLS(var)	$\boldsymbol{S}(\boldsymbol{S}'\Lambda_v\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_v$
WLS(struct)	$\boldsymbol{S}(\boldsymbol{S}'\Lambda_s\boldsymbol{S})^{-1}\boldsymbol{S}'\Lambda_s$
MinT(sample)	$\boldsymbol{S}(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{sam}}^{-1}$
MinT(shrink)	$\boldsymbol{S}(\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}\boldsymbol{S})^{-1}\boldsymbol{S}'\hat{\boldsymbol{W}}_{\text{shr}}^{-1}$

These approximate MinT by assuming $\boldsymbol{W}_h = k_h\boldsymbol{W}_1$.

- $\Lambda_v = \text{diag}(\boldsymbol{W}_1)^{-1}$
- $\Lambda_s = \text{diag}(\boldsymbol{S}\boldsymbol{1})^{-1}$
- $\hat{\boldsymbol{W}}_{\text{sam}}$ is sample estimate of the residual covariance matrix
- $\hat{\boldsymbol{W}}_{\text{shr}}$ is shrinkage estimator $\tau\text{diag}(\hat{\boldsymbol{W}}_{\text{sam}}) + (1-\tau)\hat{\boldsymbol{W}}_{\text{sam}}$ where τ selected optimally.

Example: Australian tourism

```
tourism
```

```
# A tsibble: 17,328 x 5 [1M]
# Key:       state, zone, region [76]
   state zone  region      month visitors
   <chr> <chr> <chr>       <mth>    <dbl>
 1 NSW   ACT   Canberra 1998 Jan     612.
 2 NSW   ACT   Canberra 1998 Feb     471.
 3 NSW   ACT   Canberra 1998 Mar     430.
 4 NSW   ACT   Canberra 1998 Apr     499.
 5 NSW   ACT   Canberra 1998 May     338.
 6 NSW   ACT   Canberra 1998 Jun     236.
 7 NSW   ACT   Canberra 1998 Jul     371.
 8 NSW   ACT   Canberra 1998 Aug     375.
 9 NSW   ACT   Canberra 1998 Sep     449.
10 NSW   ACT   Canberra 1998 Oct     517.
# i 17,318 more rows
```

Example: Australian tourism

```r
tourism_agg <- tourism |>
  aggregate_key(state / zone / region, visitors = sum(visitors))
```

```
# A tsibble: 25,308 x 5 [1M]
# Key:       state, zone, region [111]
      month state        zone         region       visitors
      <mth> <chr*>       <chr*>       <chr*>          <dbl>
 1 1998 Jan <aggregated> <aggregated> <aggregated>   45151.
 2 1998 Feb <aggregated> <aggregated> <aggregated>   17295.
 3 1998 Mar <aggregated> <aggregated> <aggregated>   20725.
 4 1998 Apr <aggregated> <aggregated> <aggregated>   25389.
 5 1998 May <aggregated> <aggregated> <aggregated>   20330.
 6 1998 Jun <aggregated> <aggregated> <aggregated>   18238.
 7 1998 Jul <aggregated> <aggregated> <aggregated>   23005.
 8 1998 Aug <aggregated> <aggregated> <aggregated>   23033.
 9 1998 Sep <aggregated> <aggregated> <aggregated>   22483.
10 1998 Oct <aggregated> <aggregated> <aggregated>   24845.
# i 25,298 more rows
```

Example: Australian tourism

```r
fit <- tourism_agg |>
  filter(year(month) <= 2015) |>
  model(ets = ETS(visitors))
```

```
# A mable: 111 x 4
# Key:     state, zone, region [111]
   state  zone            region                      ets
   <chr*> <chr*>          <chr*>                  <model>
 1 NSW    ACT             Canberra           <ETS(M,N,A)>
 2 NSW    ACT             <aggregated>       <ETS(M,N,A)>
 3 NSW    Metro NSW       Central Coast      <ETS(M,N,A)>
 4 NSW    Metro NSW       Sydney             <ETS(M,N,A)>
 5 NSW    Metro NSW       <aggregated>       <ETS(M,N,A)>
 6 NSW    North Coast NSW Hunter             <ETS(M,N,M)>
 7 NSW    North Coast NSW North Coast NSW    <ETS(M,N,M)>
 8 NSW    North Coast NSW <aggregated>       <ETS(M,N,M)>
 9 NSW    North NSW       Blue Mountains     <ETS(M,N,M)>
10 NSW    North NSW       Central NSW        <ETS(A,N,A)>
# i 101 more rows
```

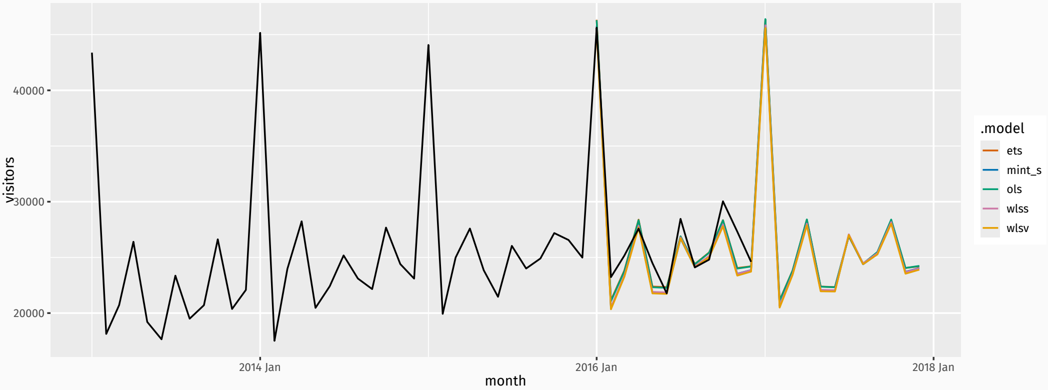
Example: Australian tourism

```r
fc <- fit |>
  reconcile(
    ols = min_trace(ets, method = "ols"),
    wlsv = min_trace(ets, method = "wls_var"),
    wlss = min_trace(ets, method = "wls_struct"),
    # mint_c = min_trace(ets, method="mint_cov"),
    mint_s = min_trace(ets, method = "mint_shrink"),
  ) |>
  forecast(h = "2 years")
```

```
# A fable: 13,320 x 7 [1M]
# Key:     state, zone, region, .model [555]
   state  zone   region   .model    month
   <chr*> <chr*> <chr*>   <chr>     <mth>
 1 NSW    ACT    Canberra ets    2016 Jan
 2 NSW    ACT    Canberra ets    2016 Feb
 3 NSW    ACT    Canberra ets    2016 Mar
 4 NSW    ACT    Canberra ets    2016 Apr
 5 NSW    ACT    Canberra ets    2016 May
```

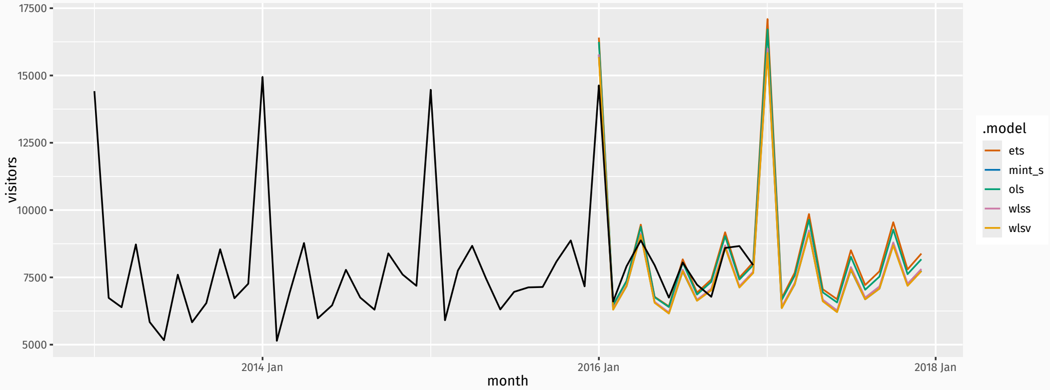
Example: Australian tourism

```
fc |>
  filter(is_aggregated(state)) |>
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



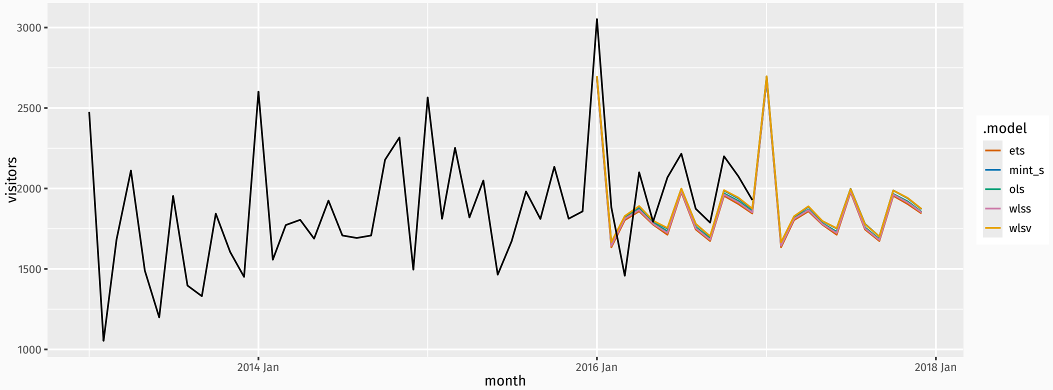
Example: Australian tourism

```r
fc |>
  filter(state == "NSW" & is_aggregated(zone)) |>
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



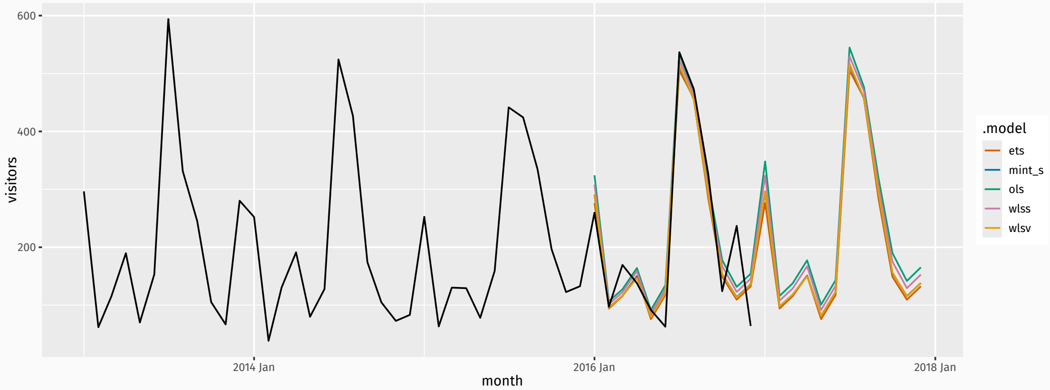
Example: Australian tourism

```r
fc |>
  filter(region == "Melbourne") |>
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



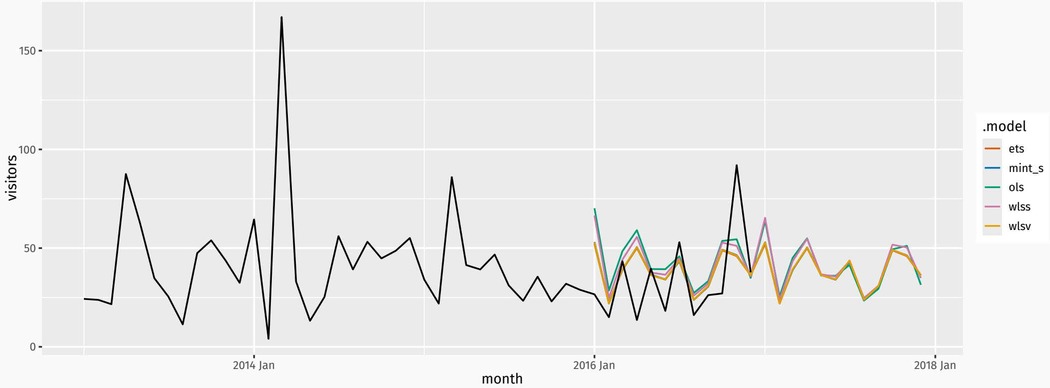
Example: Australian tourism

```
fc |>
  filter(region == "Snowy Mountains") |>
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Example: Australian tourism

```r
fc |>
  filter(region == "Barossa") |>
  autoplot(filter(tourism_agg, year(month) > 2012), level = NULL)
```



Performance evaluation

$$\text{MASE} = \text{mean}(|q_j|)$$

$$q_j = \frac{e_j}{\dfrac{1}{T-m}\displaystyle\sum_{t=m+1}^{T} |y_t - y_{t-m}|}$$

- y_t = observation for period t
- e_j = forecast error for forecast horizon j
- T = size of training set
- m = 12

Performance evaluation

$$\text{RMSSE} = \sqrt{\text{mean}(q_j^2)}$$

$$q_j^2 = \frac{e_j^2}{\dfrac{1}{T-m}\displaystyle\sum_{t=m+1}^{T}(y_t - y_{t-m})^2}$$

- y_t = observation for period t
- e_j = forecast error for forecast horizon j
- T = size of training set
- $m = 12$

Example: Australian tourism

```r
fc |>
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE))
```

```
# A tibble: 555 x 8
   .model state  zone             region           .type  mase rmsse level
   <chr>  <chr*> <chr*>           <chr*>           <chr> <dbl> <dbl> <fct>
 1 ets    NSW    ACT              Canberra         Test  0.546 0.513 Region
 2 ets    NSW    ACT              <aggregated>     Test  0.546 0.513 Zone
 3 ets    NSW    Metro NSW        Central Coast    Test  0.909 0.829 Region
 4 ets    NSW    Metro NSW        Sydney           Test  0.891 0.764 Region
 5 ets    NSW    Metro NSW        <aggregated>     Test  0.848 0.715 Zone
 6 ets    NSW    North Coast NSW Hunter           Test  0.804 0.696 Region
 7 ets    NSW    North Coast NSW North Coast NSW Test  1.21  1.17  Region
 8 ets    NSW    North Coast NSW <aggregated>     Test  1.10  0.986 Zone
 9 ets    NSW    North NSW        Blue Mountains   Test  0.932 1.13  Region
10 ets    NSW    North NSW        Central NSW      Test  1.02  0.805 Region
# i 545 more rows
```

Example: Australian tourism

```r
fc |>
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>
  group_by(.model) |>
  summarise(mase = mean(mase), rmsse = sqrt(mean(rmsse^2))) |>
  arrange(rmsse)
```

```
# A tibble: 5 x 3
  .model   mase rmsse
  <chr>   <dbl> <dbl>
1 ols     0.890 0.863
2 mint_s  0.878 0.866
3 wlss    0.886 0.871
4 wlsv    0.882 0.873
5 ets     0.886 0.880
```

Example: Australian tourism

```r
fc |>
  accuracy(tourism_agg, measures = list(mase = MASE, rmsse = RMSSE)) |>
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```

```
# A tibble: 5 x 3
  .model  mase rmsse
  <chr>  <dbl> <dbl>
1 ols    0.890 0.863
2 mint_s 0.878 0.866
3 wlss   0.886 0.871
4 wlsv   0.882 0.873
5 ets    0.886 0.880
```

- Overall, every reconciliation method is better than the base ETS forecasts.

Example: Australian tourism

```
# A tibble: 20 x 4
# Groups:   .model [5]
   .model level     mase rmsse
   <chr>  <fct>    <dbl> <dbl>
 1 ets    National 0.806 0.755
 2 ols    National 0.812 0.768
 3 wlss   National 0.846 0.889
 4 mint_s National 0.853 0.896
 5 wlsv   National 0.883 0.934
 6 ols    State    0.902 0.905
 7 ets    State    0.921 0.919
 8 mint_s State    0.956 0.953
 9 wlss   State    0.950 0.954
10 wlsv   State    0.966 0.971
11 ols    Zone     0.932 0.912
12 mint_s Zone     0.924 0.914
13 wlss   Zone     0.931 0.924
14 wlsv   Zone     0.933 0.925
15 ets    Zone     0.936 0.935
```

Example: Australian tourism

```
# A tibble: 20 x 4
# Groups:   .model [5]
   .model level     mase rmsse
   <chr>  <fct>    <dbl> <dbl>
 1 ets    National 0.806 0.755
 2 ols    National 0.812 0.768
 3 wlss   National 0.846 0.889
 4 mint_s National 0.853 0.896
 5 wlsv   National 0.883 0.934
 6 ols    State    0.902 0.905
 7 ets    State    0.921 0.919
 8 mint_s State    0.956 0.953
 9 wlss   State    0.950 0.954
10 wlsv   State    0.966 0.971
11 ols    Zone     0.932 0.912
12 mint_s Zone     0.924 0.914
13 wlss   Zone     0.931 0.924
14 wlsv   Zone     0.933 0.925
15 ets    Zone     0.936 0.935
```

- OLS is best for all levels except national.
- Improvements due to reconciliation are greater at lower levels.

Mean square error bounds

Panagiotelis, Gamakumara, Athanasopoulos, and Hyndman (2021)

Distance reducing property

Let $\|\boldsymbol{u}\|_\Psi = \boldsymbol{u}'\Psi\boldsymbol{u}$. Then

$$\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h|t}\|_\Psi \leq \|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}\|_\Psi$$

- Ψ-projection is guaranteed to improve forecast accuracy over base forecasts *using this distance measure*.
- Distance reduction holds for any realisation and any forecast.
- OLS reconciliation minimizes Euclidean distance.

Mean square error bounds

Wickramasuriya (2021)

$$
\begin{aligned}
\|\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h}\|_2^2 &= \|\boldsymbol{M}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h})\|_2^2 \\
&\leq \|\boldsymbol{M}\|_2^2\|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2 \\
&= \sigma_{\text{max}}^2\|\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}\|_2^2
\end{aligned}
$$

- σ_{max} is the largest eigenvalue of $\boldsymbol{M}$
- $\sigma_{\text{max}} \geq 1$ as $\boldsymbol{M}$ is a projection matrix.
- Every projection reconciliation is better than base forecasts using Euclidean distance.

Mean square error bounds

Wickramasuriya (2021)

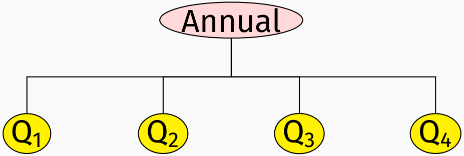
$$
\begin{aligned}
&\operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{MinT}}_{t+h|t}]\Big) \\
\le\ &\operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}^{\text{OLS}}_{t+h|t}]\Big) \\
\le\ &\operatorname{tr}\Big(\mathrm{E}[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]'[\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h|t}]\Big)
\end{aligned}
$$

Using sums of variances:

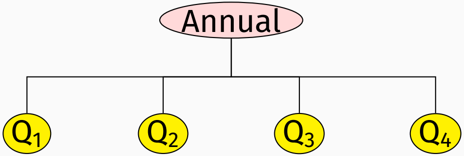
- MinT reconciliation is better than OLS reconciliation
- OLS reconciliation is better than base forecasts

Outline

Temporal reconciliation: quarterly data

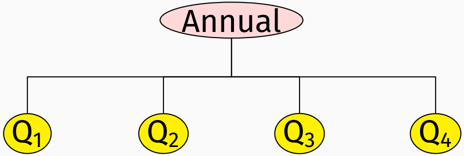


Temporal reconciliation: quarterly data



- ➥ Forecast series at each available frequency.
- ➥ Optimally combine forecasts within the same year.

Temporal reconciliation: quarterly data

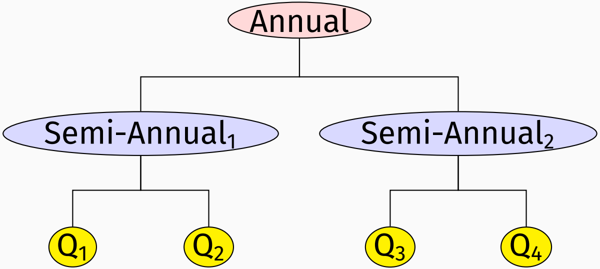


$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

τ = index of largest temporal aggregation level.

Temporal reconciliation: quarterly data

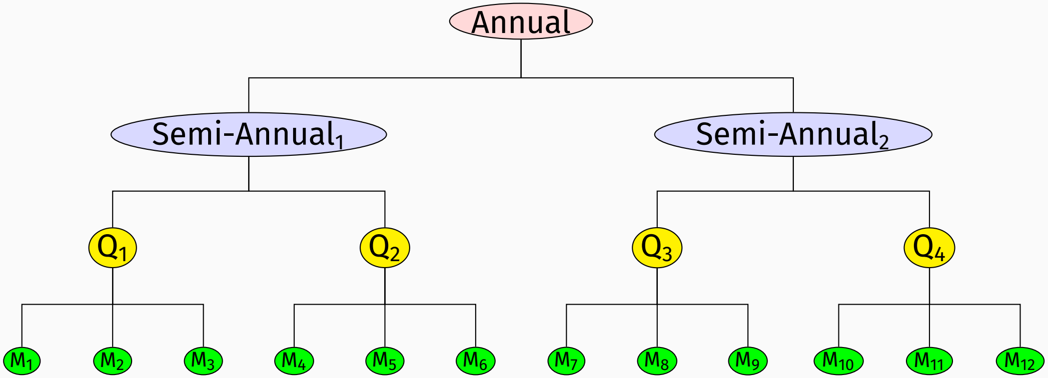


- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[4]} \\ x_{\tau,1}^{[2]} \\ x_{\tau,2}^{[2]} \\ x_{\tau,1}^{[1]} \\ x_{\tau,2}^{[1]} \\ x_{\tau,3}^{[1]} \\ x_{\tau,4}^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

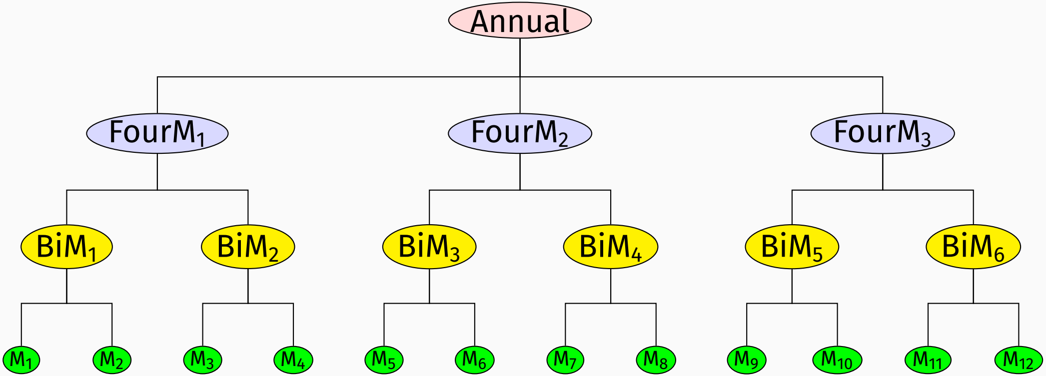
τ = index of largest temporal aggregation level.

Temporal reconciliation: monthly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: monthly data



- Forecast series at each available frequency.
- Optimally combine forecasts within the same year.

Temporal reconciliation: monthly data

$$\boldsymbol{y}_\tau = \begin{bmatrix} x_\tau^{[12]} \\ \boldsymbol{x}_\tau^{[6]} \\ \boldsymbol{x}_\tau^{[4]} \\ \boldsymbol{x}_\tau^{[3]} \\ \boldsymbol{x}_\tau^{[2]} \\ \boldsymbol{x}_\tau^{[1]} \end{bmatrix} \qquad \boldsymbol{S} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ & & & & & & & & & & & \\ & & & & & \boldsymbol{I}_{12} & & & & & & \end{bmatrix}$$

Temporal reconciliation

For a time series $y_1, \dots, y_T$, observed at frequency m:

$$x_j^{[k]} = \sum_{t=(j-1)k+1}^{jk} y_t \qquad \text{for } j = 1, \dots, \lfloor T/k \rfloor$$

- $k \in \mathcal{K} = \{k_1, \dots, k_p\}$ denote the p factors of m in ascending order, where $k_1 = 1$ and $k_p = m$
- $x_j^{[1]} = y_t$
- A single unique hierarchy is only possible when there are no coprime pairs in $\mathcal{K}$.
- $M_k = m/k$ is seasonal period of aggregated series.

Temporal reconciliation

$$\boldsymbol{x}_\tau = \boldsymbol{S}\boldsymbol{x}_\tau^{[1]}, \qquad \boldsymbol{S} = \begin{bmatrix}\boldsymbol{A}\\ \boldsymbol{I}\end{bmatrix}$$

where

$$\boldsymbol{x}_\tau = \begin{bmatrix} x_\tau^{[k_p]} \\ \boldsymbol{x}_\tau^{[k_{p-1}]} \\ \vdots \\ \boldsymbol{x}_\tau^{[k_1]} \end{bmatrix} \qquad \boldsymbol{x}_\tau^{[k]} = \begin{bmatrix} x_{M_k(\tau-1)+1}^{[k]} \\ x_{M_k(\tau-1)+2}^{[k]} \\ \vdots \\ x_{M_k\tau}^{[k]} \end{bmatrix} \qquad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{1}'_m \\ \boldsymbol{I}_{m/k_{p-1}} \otimes \boldsymbol{1}'_{k_{p-1}} \\ \vdots \\ \boldsymbol{I}_{m/k_2} \otimes \boldsymbol{1}'_{k_2} \end{bmatrix}$$

τ is time index for most aggregated series,

$k \in \mathcal{K} = \{k_1, \dots, k_p\}$, $\quad k_1 = 1$, $\quad k_p = m$, $\quad \tau = 1, \dots, T/m$.

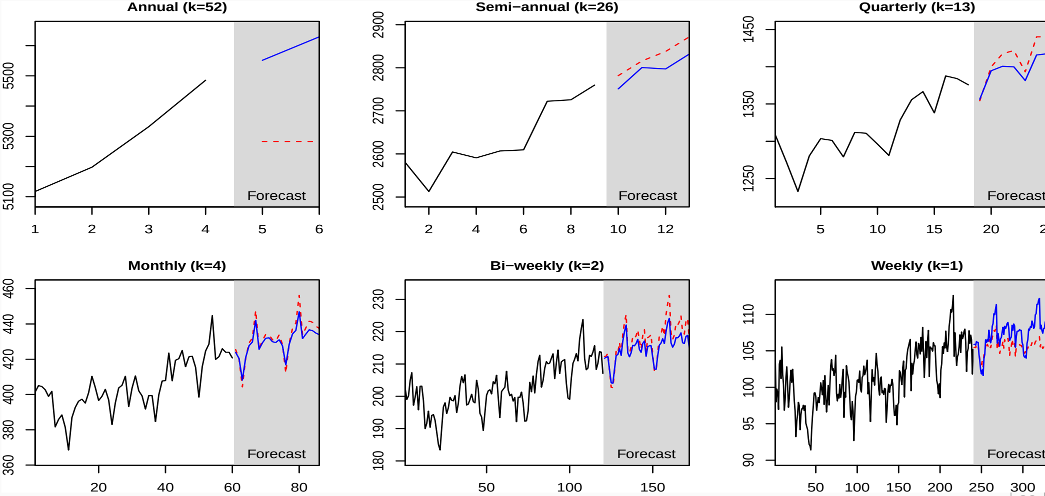
Example: Accident & emergency services demand

Weekly A&E demand data: 7 November 2010 to 7 June 2015.

Type 1 Departments — Major A&E
Type 2 Departments — Single Specialty
Type 3 Departments — Other A&E/Minor Injury Unit
Total Attendances
Type 1 Departments — Major A&E $>$ 2 hours
Type 2 Departments — Single Specialty $>$ 2 hours
Type 3 Departments — Other A&E/Minor Injury Unit $>$ 2 hours
Total Attendances $>$ 2 hours
Emergency Admissions via Type 1 A&E
Total Emergency Admissions via A&E
Other Emergency Admissions (i.e not via A&E)
Total Emergency Admissions
Number of patients spending $>$ 2 hours from decision to admit to admission

Example: Accident & emergency services demand

Total emergency admissions via A&E



Example: Accident & emergency services demand

Test set: last 52 weeks

MASE comparison (ARIMA models)

Aggregation Level	h	Base	Reconciled	Change
Annual	1	3.4	1.9	-42.9%
Weekly	1--52	2.0	1.9	-5.0%
Weekly	13	2.3	1.9	-16.2%
Weekly	4	1.9	1.5	-18.6%
Weekly	1	1.6	1.3	-17.2%

Temporal reconciliation: M3 monthly series

- Apply temporal reconciliation to all 1428 monthly series from M3 competition
- Forecast horizon h = 18 months
- ETS and ARIMA models
- Measure percentage difference to base forecasts
- Reconciliation methods:
  - WLS_H (diagonal)
  - WLS_V (diagonal with common variances for same frequency)
  - WLS_S (diagonal/structural)

Temporal reconciliation: M3 monthly series

Improvement in MASE relative to base forecasts

		ETS				ARIMA			
Aggregation level	*h*	BU	WLS_H	WLS_V	WLS_S	BU	WLS_H	WLS_V	WLS_S
Annual	1	-12.1	-17.9	-17.8	**-18.5**	-25.4	-29.9	-29.9	**-30.2**
Semi-annual	3	0.0	-6.3	-6.0	**-6.9**	-2.9	-8.1	-8.2	**-9.4**
Four-monthly	4	3.1	-3.2	-3.0	**-3.4**	-1.8	-6.2	-6.5	**-7.1**
Quarterly	6	3.2	-2.8	-2.7	**-3.4**	-2.6	-6.9	-7.4	**-8.1**
Bi-monthly	9	2.7	-2.9	-3.0	**-3.7**	-1.3	-5.0	-5.5	**-6.3**
Monthly	18	0.0	-3.7	-4.6	**-5.0**	0.0	-1.9	-3.2	**-3.7**
Average	NA	-0.5	-6.1	-6.2	**-6.8**	-5.7	-9.7	-10.1	**-10.8**

Outline

Forecast Linear Augmented Projection (FLAP)

- We want to forecast a multivariate series $\boldsymbol{y}_t$.
- Construct many linear combinations $\boldsymbol{c}_t = \Phi\boldsymbol{y}_t$ of the multivariate series (e.g., principal components or random combinations)
- Produce univariate forecasts of all series $\hat{\boldsymbol{y}}_t$ and all linear combinations $\hat{\boldsymbol{c}}_t$.
- Reconcile forecasts so they are coherent ($\tilde{\boldsymbol{c}}_t = \Phi\tilde{\boldsymbol{y}}_t$)

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- Reconcile forecasts so they are coherent ($\tilde{\boldsymbol{c}}_t = \Phi\tilde{\boldsymbol{y}}_t$)

$$\boldsymbol{z}_t = \begin{bmatrix}\boldsymbol{y}_t \\ \boldsymbol{c}_t\end{bmatrix} \qquad \tilde{\boldsymbol{z}}_{t+h} = \boldsymbol{M}\hat{\boldsymbol{z}}_{t+h}$$

where $\boldsymbol{M}$ is a projection matrix onto the coherent subspace.

Forecast error variance reduction

- The variance reduction $\text{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \text{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h})$ is **positive semi-definite**.
- The diagonal elements of $\text{Var}(\boldsymbol{y}_{t+h} - \hat{\boldsymbol{y}}_{t+h}) - \text{Var}(\boldsymbol{y}_{t+h} - \tilde{\boldsymbol{y}}_{t+h})$ are non-decreasing as the number of components increases.

Minimum variance of individual series

The projection is equivalent to the mapping

$$\tilde{\boldsymbol{y}}_{t+h} = \boldsymbol{G}\hat{\boldsymbol{z}}_{t+h},$$

where $\boldsymbol{G} = \begin{bmatrix}\boldsymbol{g}_1 & \boldsymbol{g}_2 & \dots & \boldsymbol{g}_m\end{bmatrix}' \in \mathbb{R}^{m\times(m+p)}$ is the solution to

$$\underset{\boldsymbol{G}}{\arg\min}\ \boldsymbol{G}\boldsymbol{W}_h\boldsymbol{G}' \qquad \text{s.t. } \boldsymbol{G}\boldsymbol{S} = \boldsymbol{I}$$

or

$$\underset{\boldsymbol{g}_i}{\arg\min}\ \boldsymbol{g}_i'\boldsymbol{W}_h\boldsymbol{g}_i \qquad \text{s.t. } \boldsymbol{g}_i'\boldsymbol{s}_j = \boldsymbol{1}(i=j),$$

where $\boldsymbol{S} = \begin{bmatrix}\boldsymbol{I}_m \\ \boldsymbol{\Phi}\end{bmatrix} = \begin{bmatrix}\boldsymbol{s}_1 \cdots \boldsymbol{s}_m\end{bmatrix}$.

Key results

1. The forecast error variance is **reduced** with FLAP
2. The forecast error variance **monotonically** decreases with increasing number of components
3. The forecast projection is **optimal** to achieve minimum forecast error variance of each series

Key results

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2. The forecast error variance **monotonically** decreases with increasing number of components
3. The forecast projection is **optimal** to achieve minimum forecast error variance of each series

In practice, we need to:

- Estimate $\boldsymbol{W}_h = \text{Var}(\boldsymbol{z}_{t+h} - \hat{\boldsymbol{z}}_{t+h})$. (We can use the MinT shrinkage estimator.)
- Construct the components, Φ.

Construction of Φ

Principal component analysis (PCA)

Finding the weights matrix Φ so that the resulting components **maximise variance**

Simulation

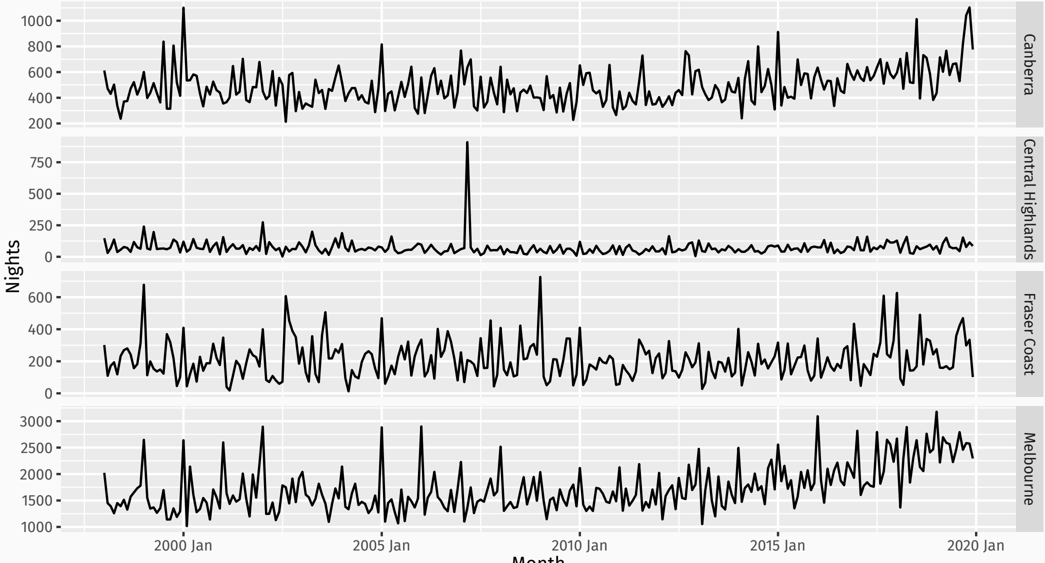
Generating values of Φ from a random distribution and normalising them to unit vectors

- Normal distribution
- Uniform distribution
- Orthonormal matrix

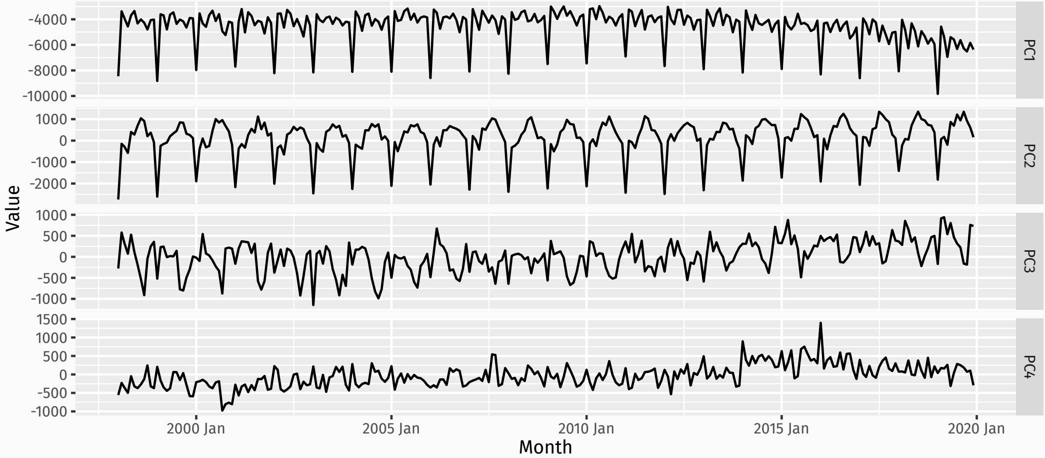
Monthly Australian regional tourism

- Monthly Australian tourism data set aggregated by region giving 77 series, from Jan 1998 to Dec 2019
- Use expanding window time series cross-validation with $T = 84$ observations in first training set, and forecast horizons $h = 1, 2, \ldots, 12$.

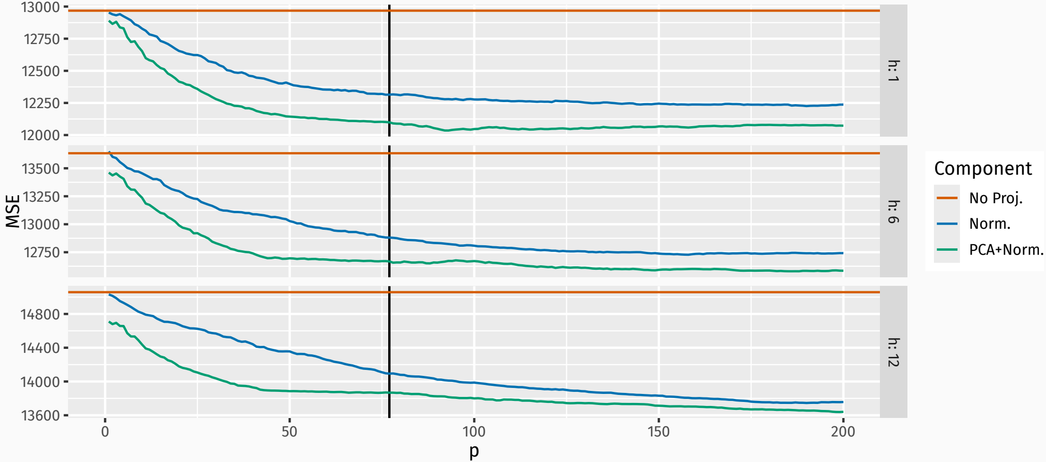
Monthly Australian regional tourism



Monthly Australian regional tourism



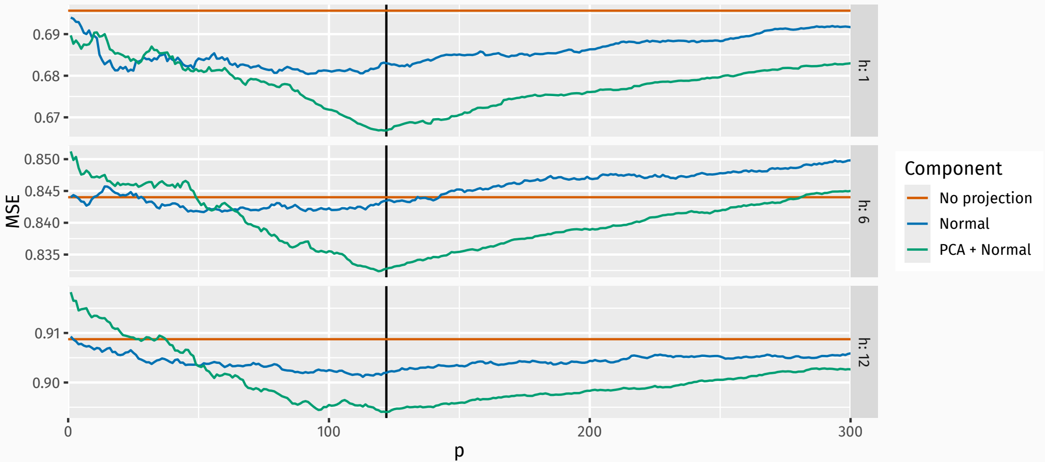
Monthly Australian regional tourism



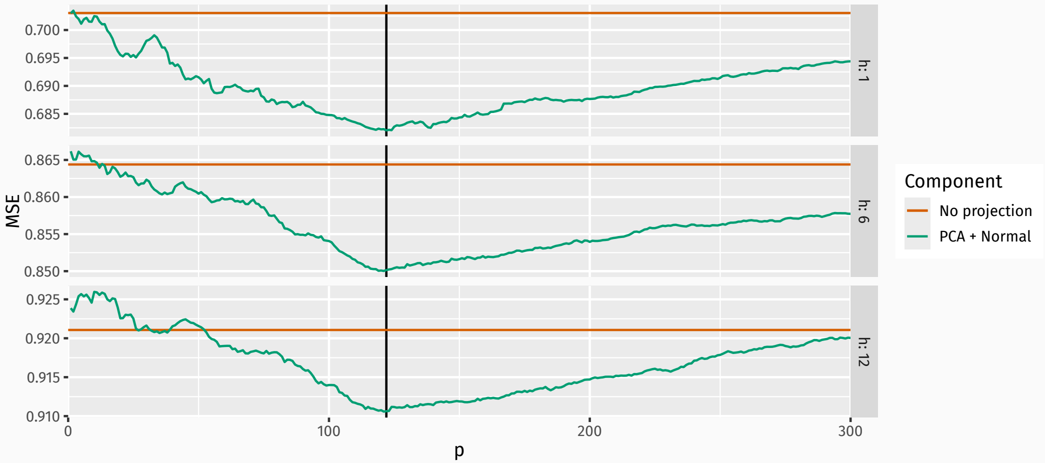
FRED-MD

- Monthly data of macroeconomic variables (McCracken and Ng, 2016).
- Data from Jan 1959 – Sep 2023. 777 observations on 122 series.
- Same cleaning process as per McCracken and Ng (2016).
- All series scaled to have mean 0 and variance 1.
- Expanding time series cross-validation with initial size of 25 years and forecast horizon 12 months.

FRED-MD (ARIMA)



FRED-MD (DFM)

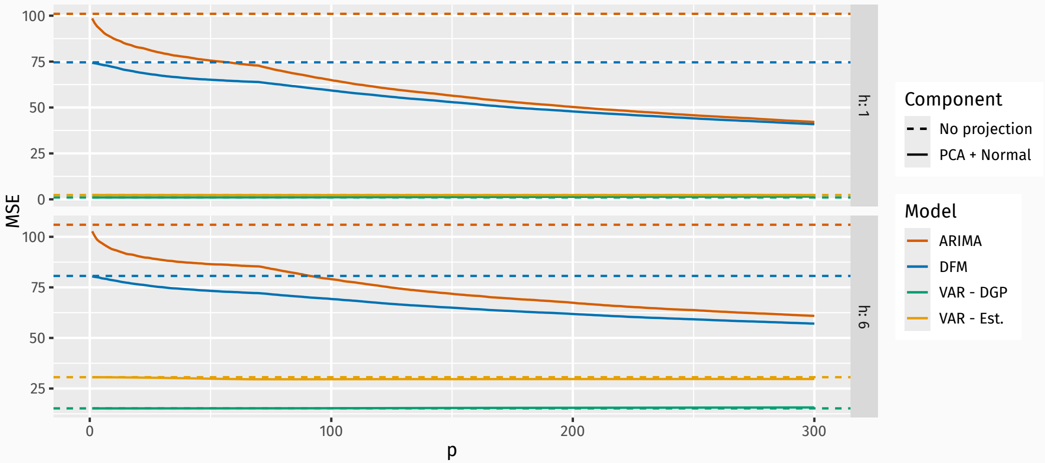


MSE
0.700
0.695
0.690
0.685
0.865
0.860
0.855
0.850
0.925
0.920
0.915
0.910
0
100
200
300
p
h: 1
h: 6
h: 12
Component
No projection
PCA + Normal

Simulation

- Data generating process: VAR(3) with 70 variables
- Sample size: $T = 400$
- Number of repeated samples: 220
- Base models:
  - automatic ARIMA (based on AICc)
  - DFM (structure chosen using BIC, different model for each horizon)

Simulation



MSE
100
75
50
25
0
h: 1
100
75
50
25
h: 6
0
100
200
300
p
Component
No projection
PCA + Normal
Model
ARIMA
DFM
VAR - DGP
VAR - Est.

Future research directions

- Investigate why PCA performs better than random weights
- Find other components that are better than PCA
- Find optimal components by minimising forecast error variance with respect to Φ
- Use forecast projection and forecast reconciliation together

Outline

Software

Package	Language	Cross-sectional	Temporal	Cross-temporal	Probabilistic	Multivariate
`hts`	R	✓				
`thief`	R		✓			
`fable`	R	✓			✓	
`FoReco`	R	✓	✓	✓	✓	
`flap`	R					✓
`pyhts`	Python	✓	✓			
`hierarchicalforecast`	Python	✓			✓	

- `hts`, `thief`, and `FoReco` use `ts` objects
- `fable` uses `tsibble` objects
- `flap` uses matrices of base forecasts
- `fable` has plans to implement temporal and cross-temporal reconciliation

Thanks!

More information

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