

Anomaly detection using surprisals

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28 October 2025

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Outline

- 1 Anomalies and surprisals
- 2 Extreme value theory and surprisals
- 3 Lookout algorithm
- 4 Conclusions

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Definitions of anomalies

an observation (or a subset of observations) which appears to be inconsistent with the remainder of that set of data.

(Barnett & Lewis, 1978)

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an observation which deviates so much from other observations as to arouse suspicion it was generated by a different mechanism.

(Hawkins, 1980)

Definitions of anomalies

Definition: Anomaly

Given a set of observations $\{y_1, \dots, y_n\}$ drawn from probability distribution F , y_i is an **anomaly** if

$$\mathbb{P}(f(Y) < f(y_i)) < \alpha$$

where $Y \sim F$, f is the generalized density of F , and $\alpha > 0$ is a chosen threshold.

Definitions of anomalies

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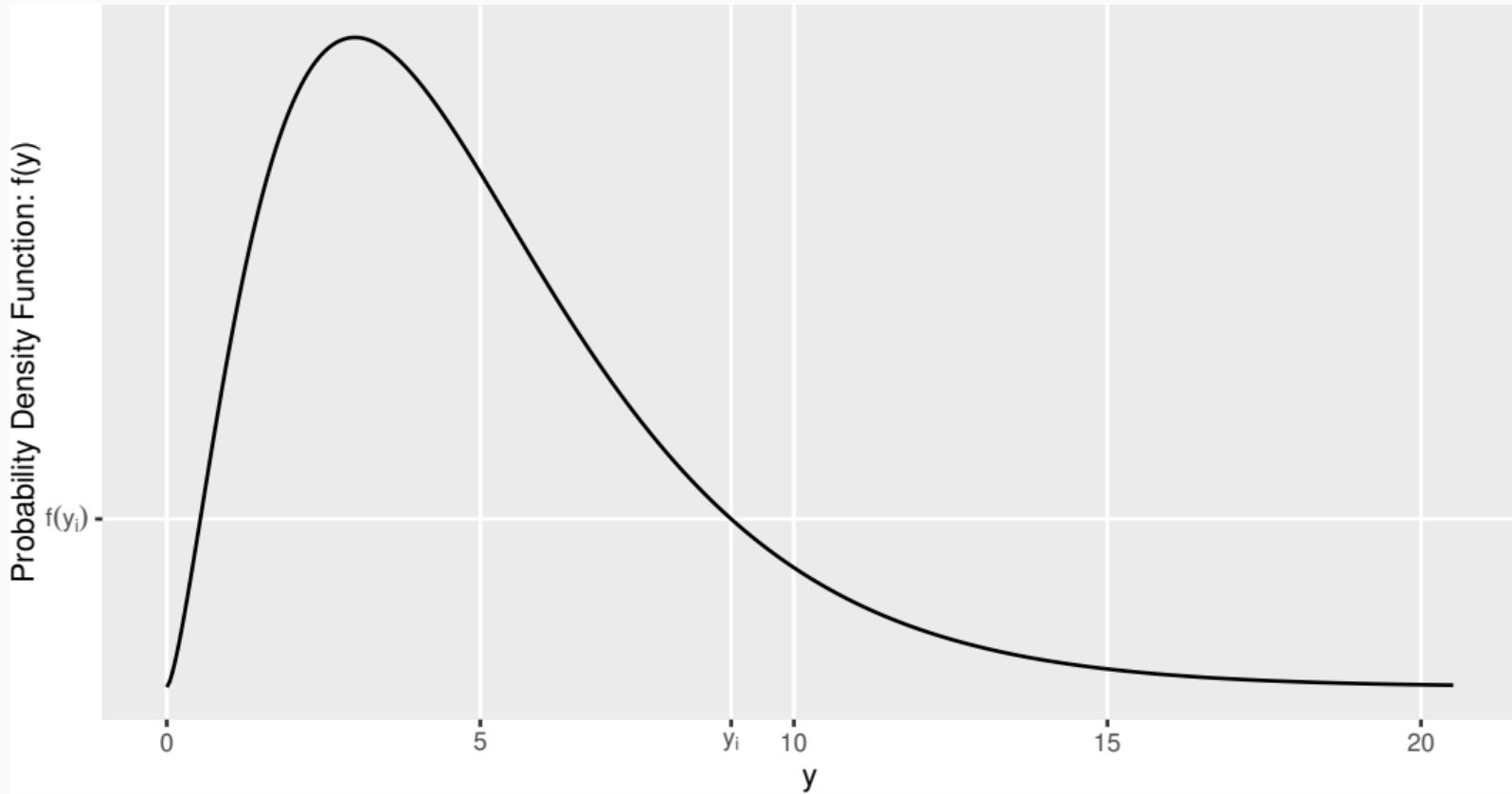
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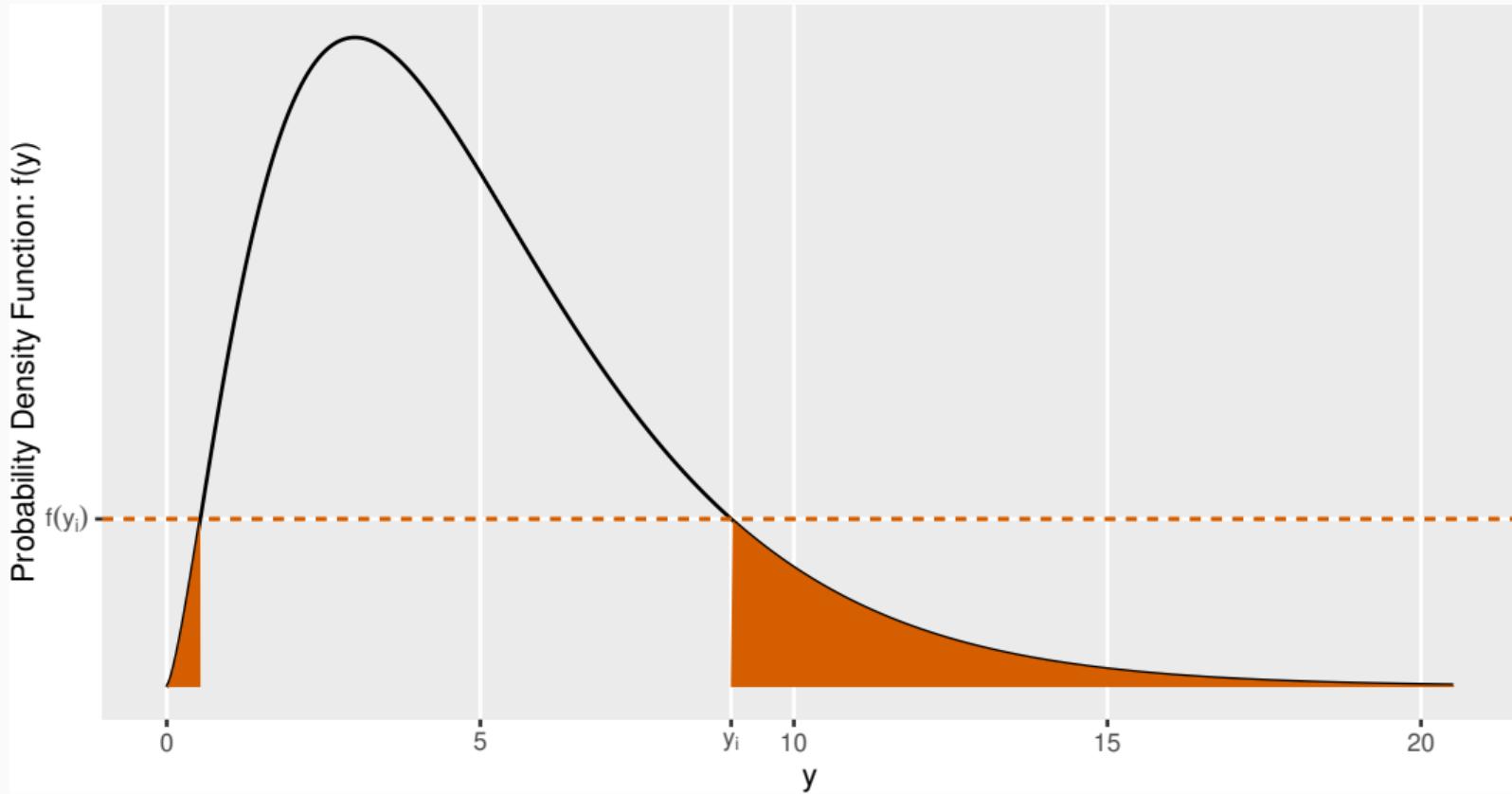
where $Y \sim F$, f is the generalized density of F , and $\alpha > 0$ is a chosen threshold.

- y_i can be a scalar, vector or a more complex object
- f can be a conditional density, and can be known or estimated

Definitions of anomalies



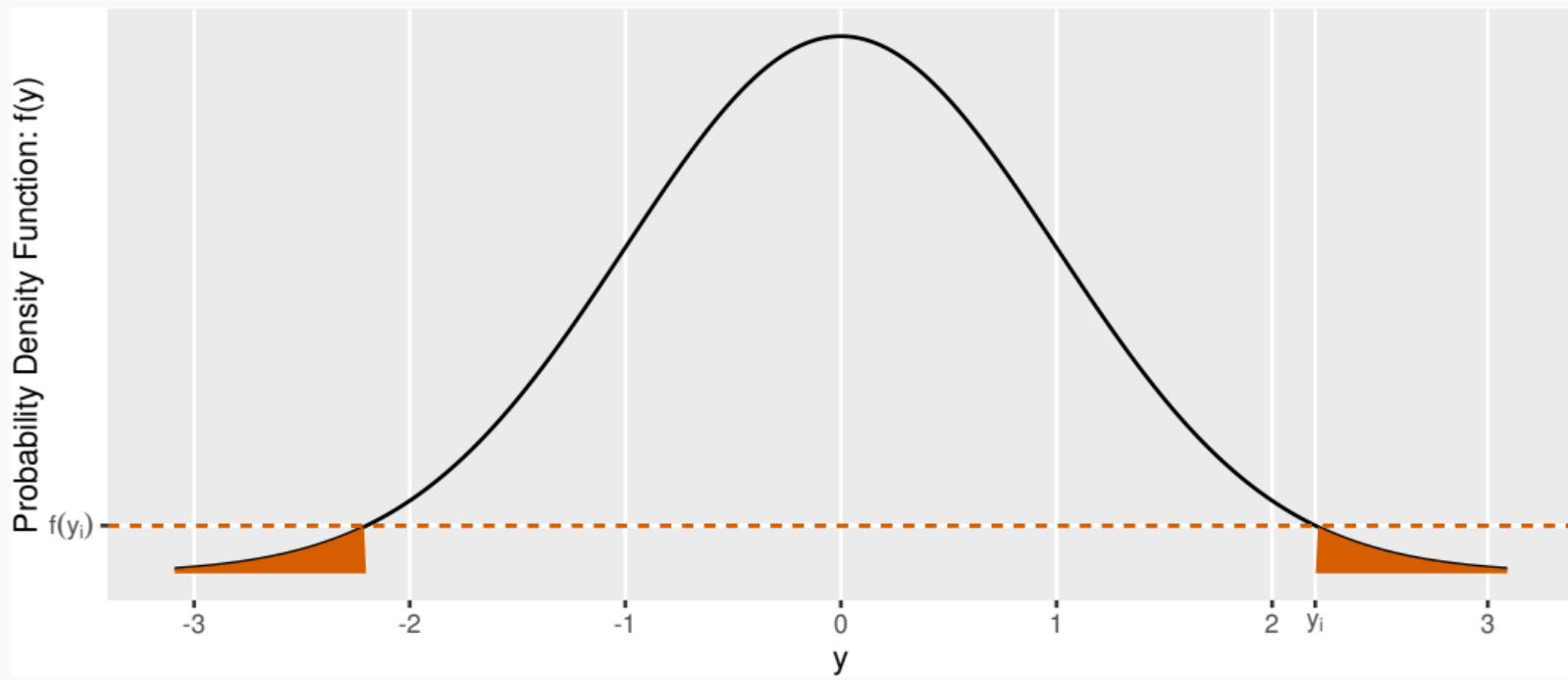
Definitions of anomalies



Anomaly detection: Normal distribution

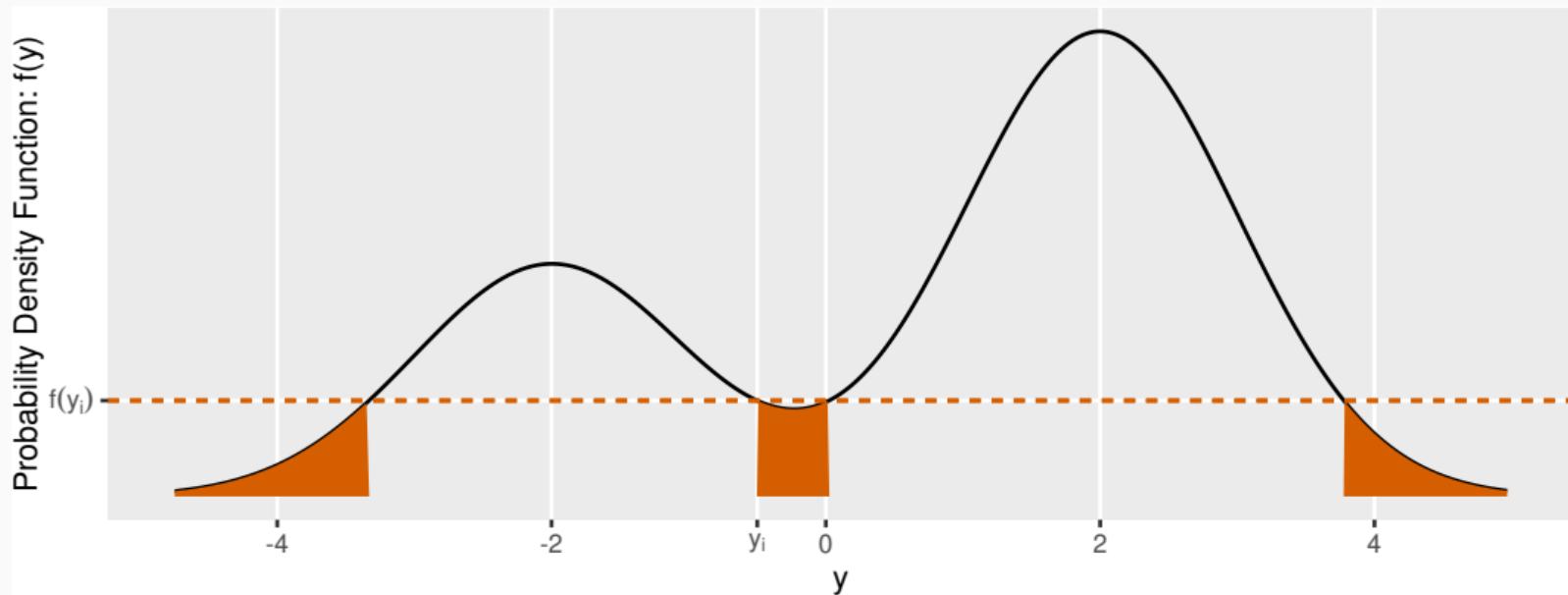
If $F \sim N(\mu, \sigma^2)$, then $p_i = 2 [1 - \Phi(|y_i - \mu|/\sigma)]$

Equivalent to a two-sided p-value from a z-score test.



Anomaly detection: Highest density regions

HDR with probability $1 - \alpha$ is $R_\alpha = \{y : f(y) \geq c_\alpha\}$ where c_α is largest constant s.t. $\mathbb{P}(Y \in R_\alpha) \geq 1 - \alpha$.
An observation is an anomaly if $y_i \notin R_\alpha$.



Surprises

Definition: Surprisal

The **surprisal** of an observation y_i drawn from probability distribution F with generalized density f is defined as

$$s_i = -\log f(y_i)$$

- Better known as “log scores” in statistics.
- “Surprisal” coined by Tribus (1961).
- Average surprisal = entropy of random variable
- Sum of surprisals = negative log likelihood

Anomaly detection using surprisals

Let $G(s) = \mathbb{P}(S \leq s)$ be the **surprisal distribution** where $S = -\log f(Y)$ and $Y \sim F$.

$$G(s) = \mathbb{P}(-\log f(Y) \leq s) = \mathbb{P}(f(Y) \geq e^{-s})$$

The **surprisal score** is

$$p_i = 1 - G(s_i)$$

and an observation is an **anomaly** if $p_i < \alpha$.

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Fisher-Tippett-Gnedenko theorem

Consider n iid rvs S_1, \dots, S_n with cdf G and $M_n = \max\{S_1, \dots, S_n\}$. If there exist sequences of constants $\{a_n > 0\}$ and $\{b_n\}$ such that

$$\mathbb{P}\left\{(M_n - b_n)/a_n \leq z\right\} \rightarrow H(z) \quad \text{as } n \rightarrow \infty,$$

for a non-degenerate cdf H , then

$$H(z) = \exp\left\{-\left[1 + \xi\left(\frac{z - \mu}{\sigma}\right)\right]^{-1/\xi}\right\}$$

- $\xi > 0$: Fréchet distribution (G heavy-tailed)
- $\xi \rightarrow 0$: Gumbel distribution (G light-tailed)
- $\xi < 0$: Weibull distribution (G bounded upper tail)

Pickands-Balkema-De Haan theorem

If G satisfies the FTG theorem, then the upper tail of G can be approximated by the Generalized Pareto Distribution (GPD):

$$\mathbb{P}(S \leq u + s \mid S > u) = 1 - \left(1 + \frac{\xi s}{\sigma_u}\right)^{-1/\xi}$$

for large enough u , where $\sigma_u = \sigma + \xi(u - \mu)$.

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So in practice, we can approximate the upper tail of many distributions by a GPD.

Surprises and EVT

- Suppose we have n iid observations Y_1, \dots, Y_n from distribution F with density f .
- Let $S_i = -\log f(Y_i)$ be the surprisal of Y_i .
- Then S_1, \dots, S_n are iid from the surprisal distribution $G(s) = \mathbb{P}(S \leq s)$.
- If G satisfies the FTG theorem, then we can approximate the upper tail of G by a GPD.

Three-type theorem for surprises

A1: Sub-Gaussian: $S = -\log f(Y)$ satisfies, for all $\lambda \in \mathbb{R}$, and some $\nu > 0$, $\mathbb{E} \exp\{\lambda(S - \mathbb{E}[S])\} \leq \exp\{\lambda^2 \nu^2 / 2\}$.

A2: Sub-exponential: S is sub-exponential with parameters ν and b , i.e., $\mathbb{E} \exp\{\lambda(S - \mathbb{E}[S])\} \leq \exp\{\lambda^2 \nu^2 / 2\}$ for all $|\lambda| < 1/b$.

A3: Polynomial: $|S|$ has polynomial moments of order $p \geq 1$; i.e., $\mathbb{E}[|S|^p] \leq C^p$ for some $C > 0$ such that $C^p - 1 > 0$.

- A1 satisfied when f has bounded support
- A2 satisfied when $\log f$ unbounded below, and light tails (e.g., Gaussian)
- A3 satisfied when f has heavy tails (e.g., t with df ≥ 3)

Three-type theorem for surprisals

Let y_1, \dots, y_n be an iid sequence from F , $S = -\log f(Y)$ where $Y \sim F$, $s_i = -\log f(y_i)$, and $M_n = \max\{s_1, \dots, s_n\}$.

1 Under A1:

$$\sup_{s:s>0} \left| \mathbb{P} \left\{ |M_n - \mathbb{E}[S]| \geq \sqrt{2\nu^2 s} + \sqrt{2\nu^2 \log(2n)} \right\} - e^{-s} \right| = o(1).$$

2 Under A2:

$$\sup_{s:s>1/b} \left| \mathbb{P} \left\{ |M_n - \mathbb{E}[S]| \geq (2b)s + (2b)\log(2n) \right\} - e^{-e^{-s}} \right| = o(1).$$

3 Under A3:

$$\sup_{s:s>c} \left| \mathbb{P} \left\{ |M_n - \mathbb{E}[S]| \geq (Csn^{1/p}) \right\} - e^{-s^{-p}} \right| = o(1).$$

Three-type theorem for surprisals

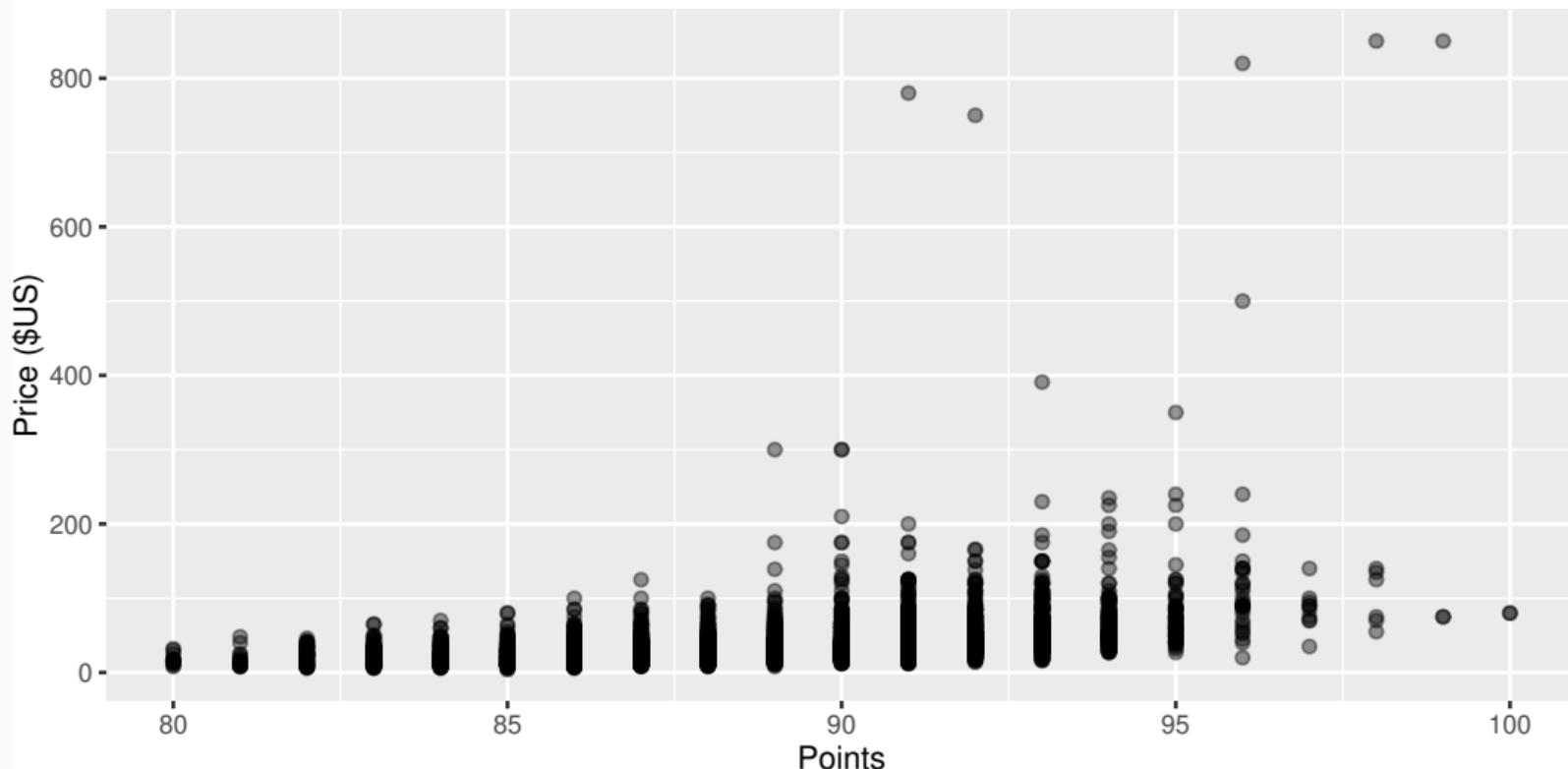
- If surprisal has Gaussian like tails, then maximum surprisal is a reversed Weibull;
- If surprisal only has an exponential moment, then maximum surprisal is Gumbel;
- If surprisal only has a polynomial moment, then maximum surprisal is Fréchet.

Corollary

upper tail of the surprisal distribution can be approximated by a GPD, even if the assumed distribution F is incorrect, provided one of A1–A3 is satisfied.

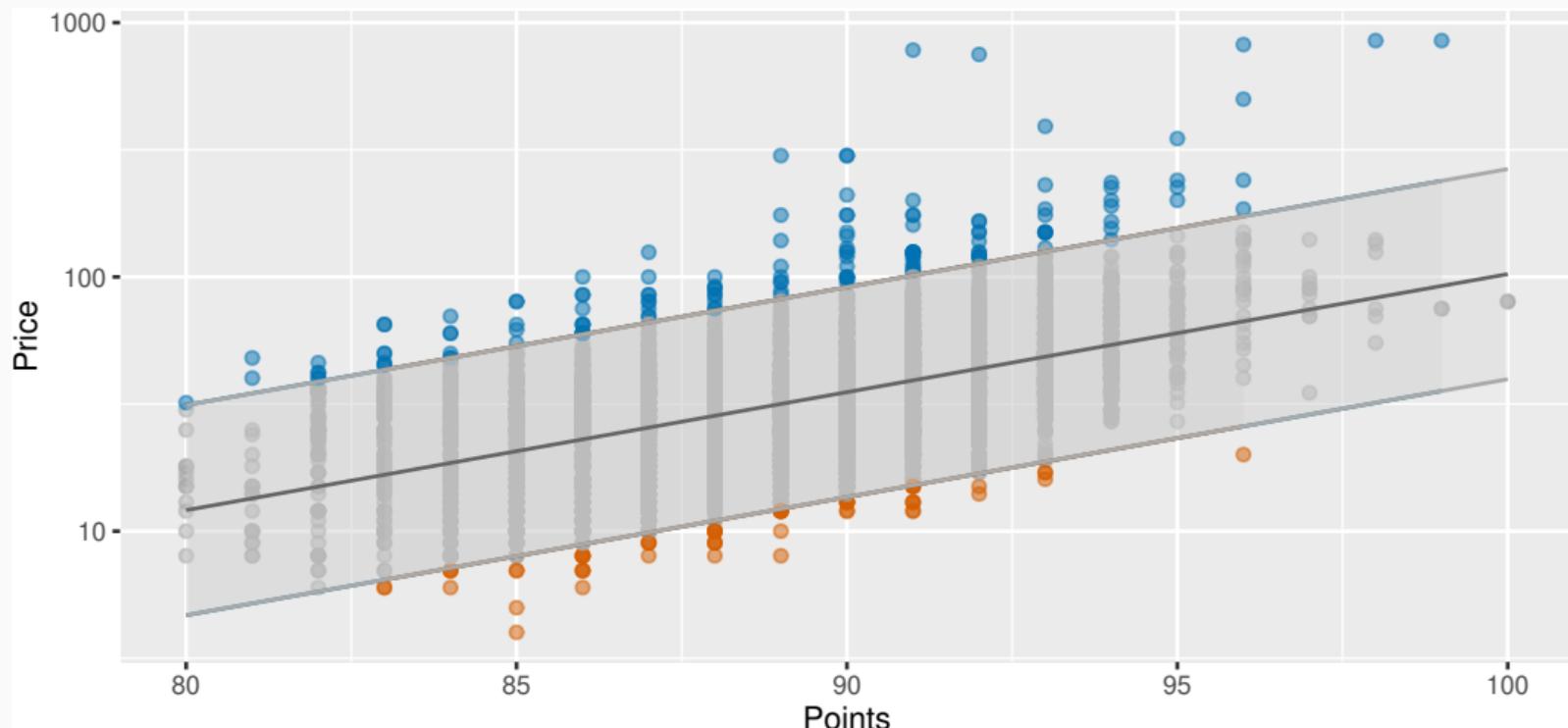
Application to wine quality and prices

Reviews of 4496 Shiraz/Syrah wines from 'Wine Enthusiast', 15 June 2017



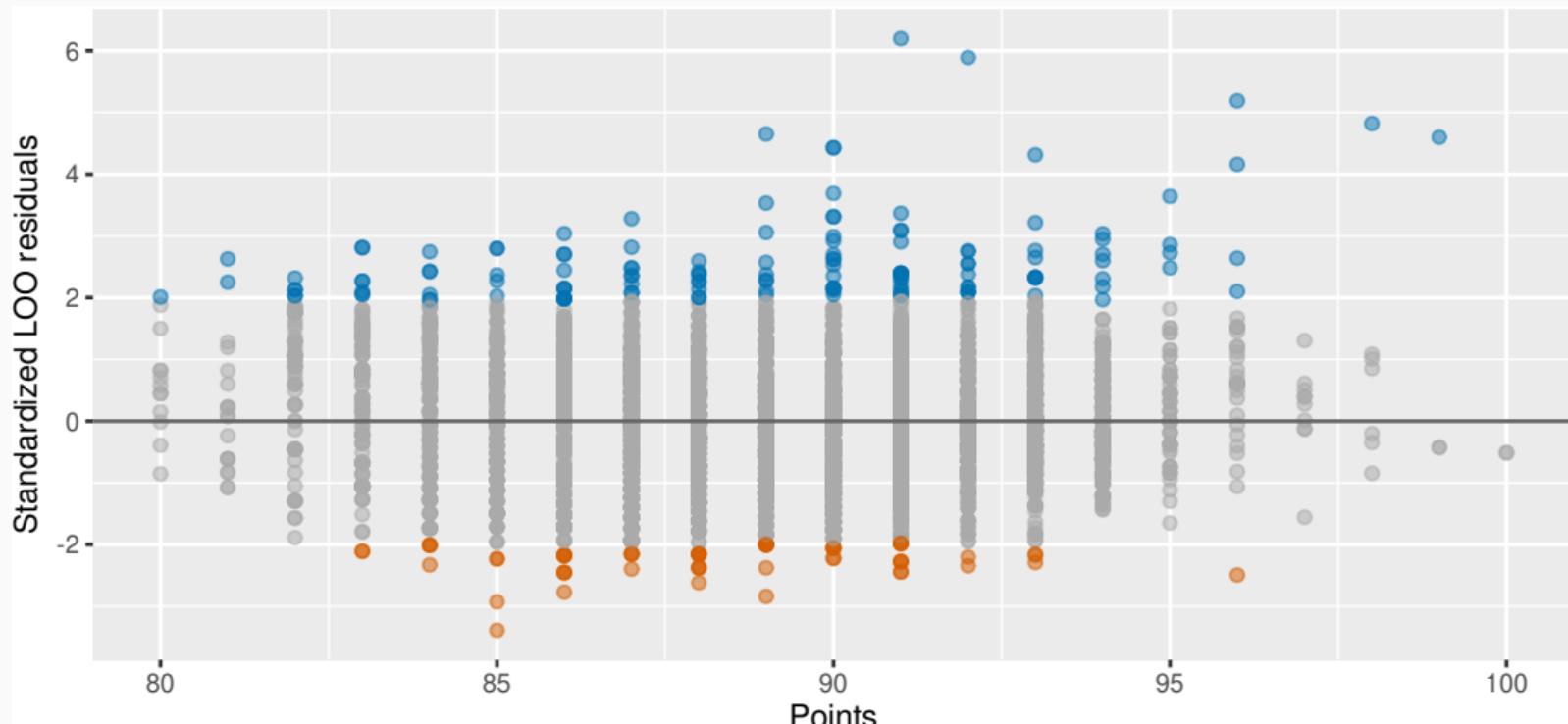
Application to wine quality and prices

Proposed model: $\log \text{Price} | \text{Points} \sim N(a + b\text{Points}, \sigma^2)$.



Application to wine quality and prices

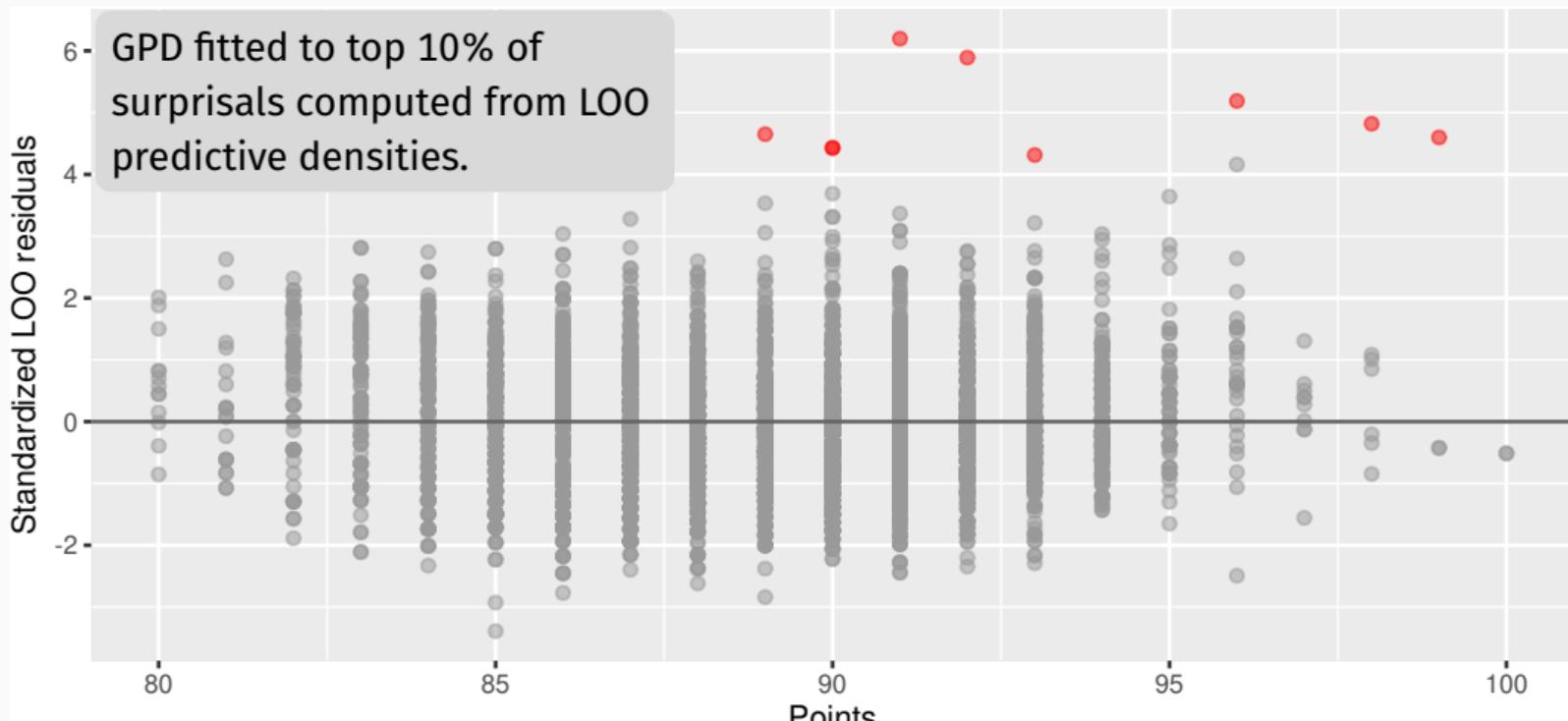
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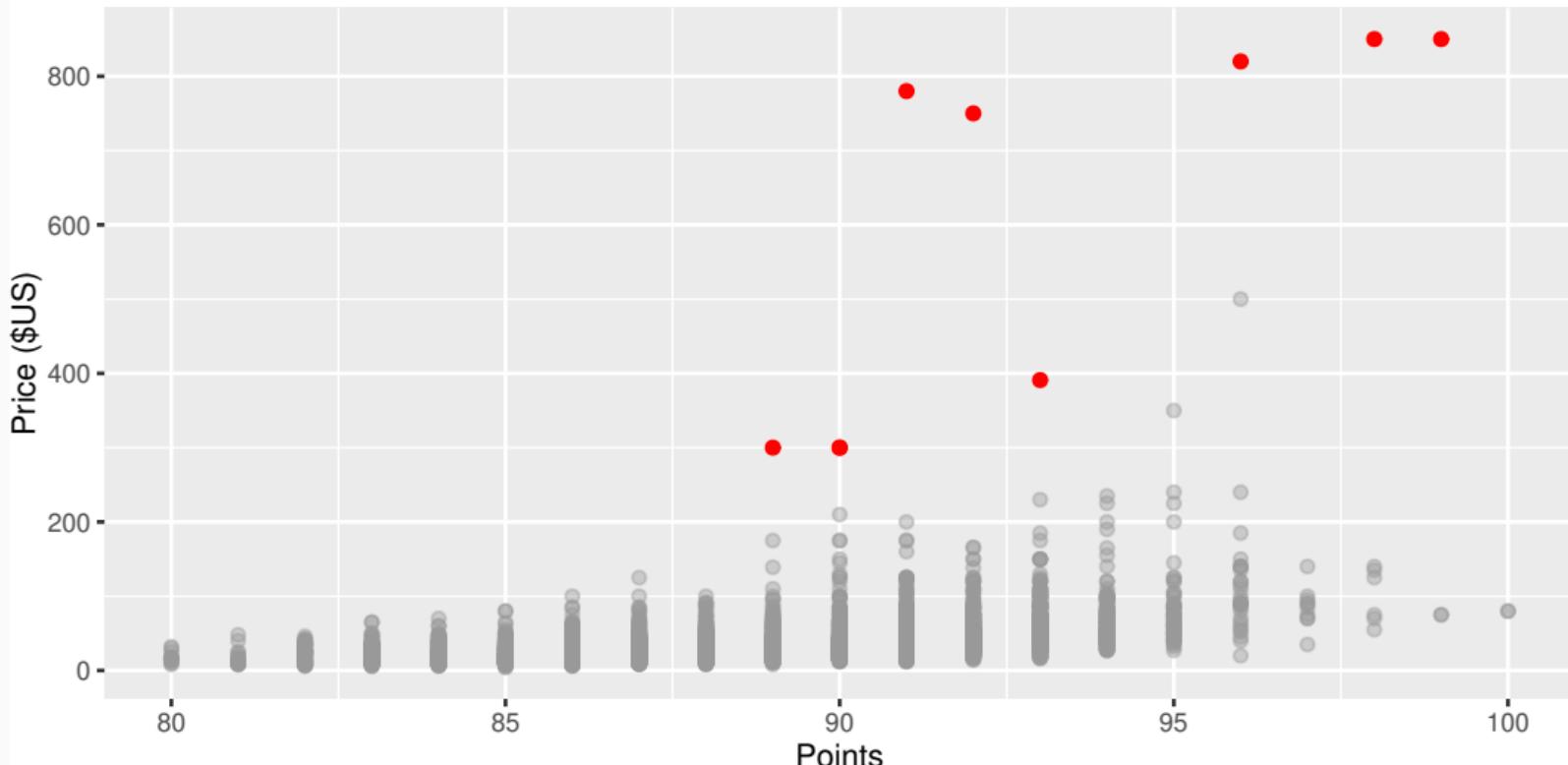
$\alpha = 0.001$



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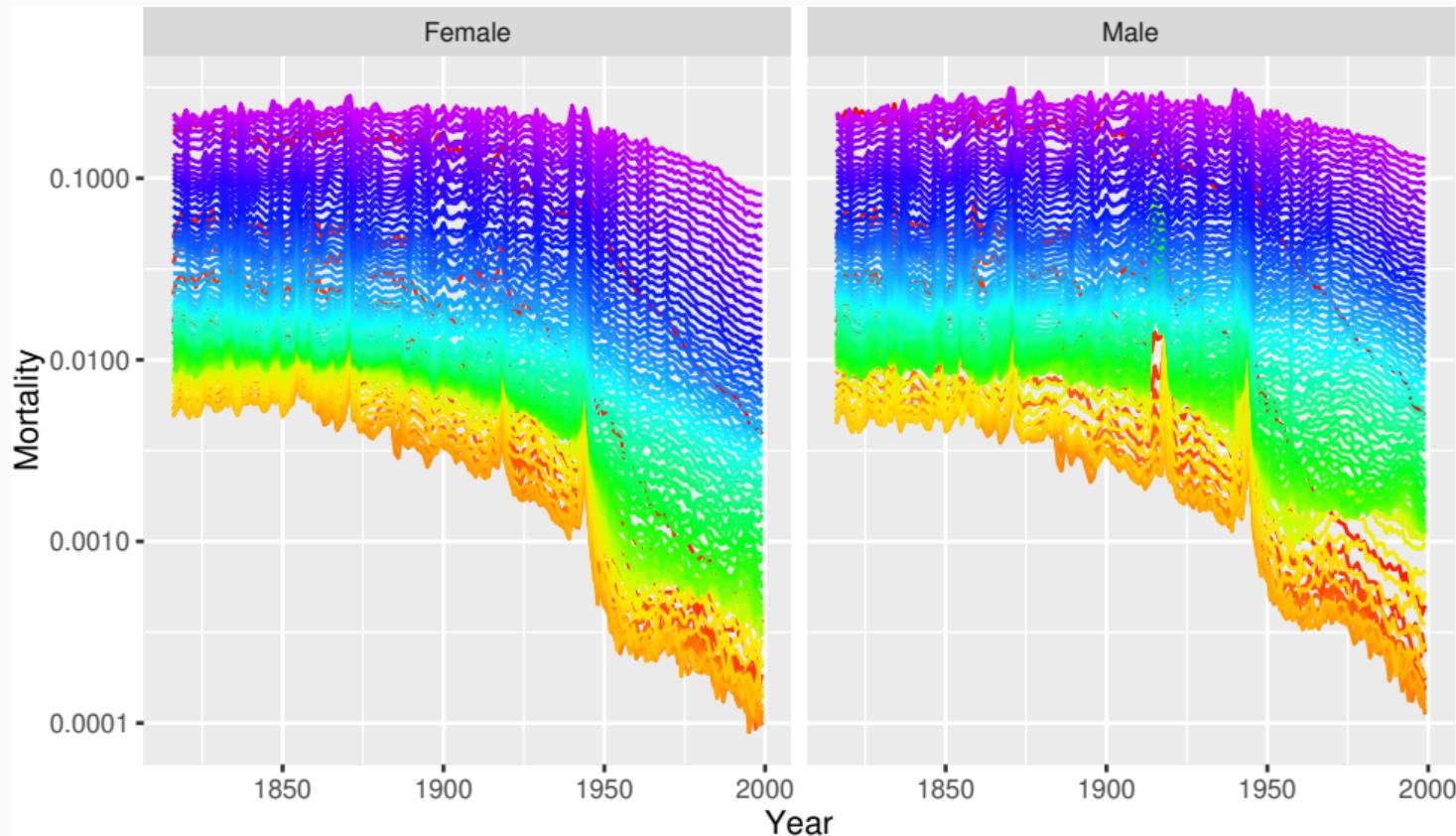


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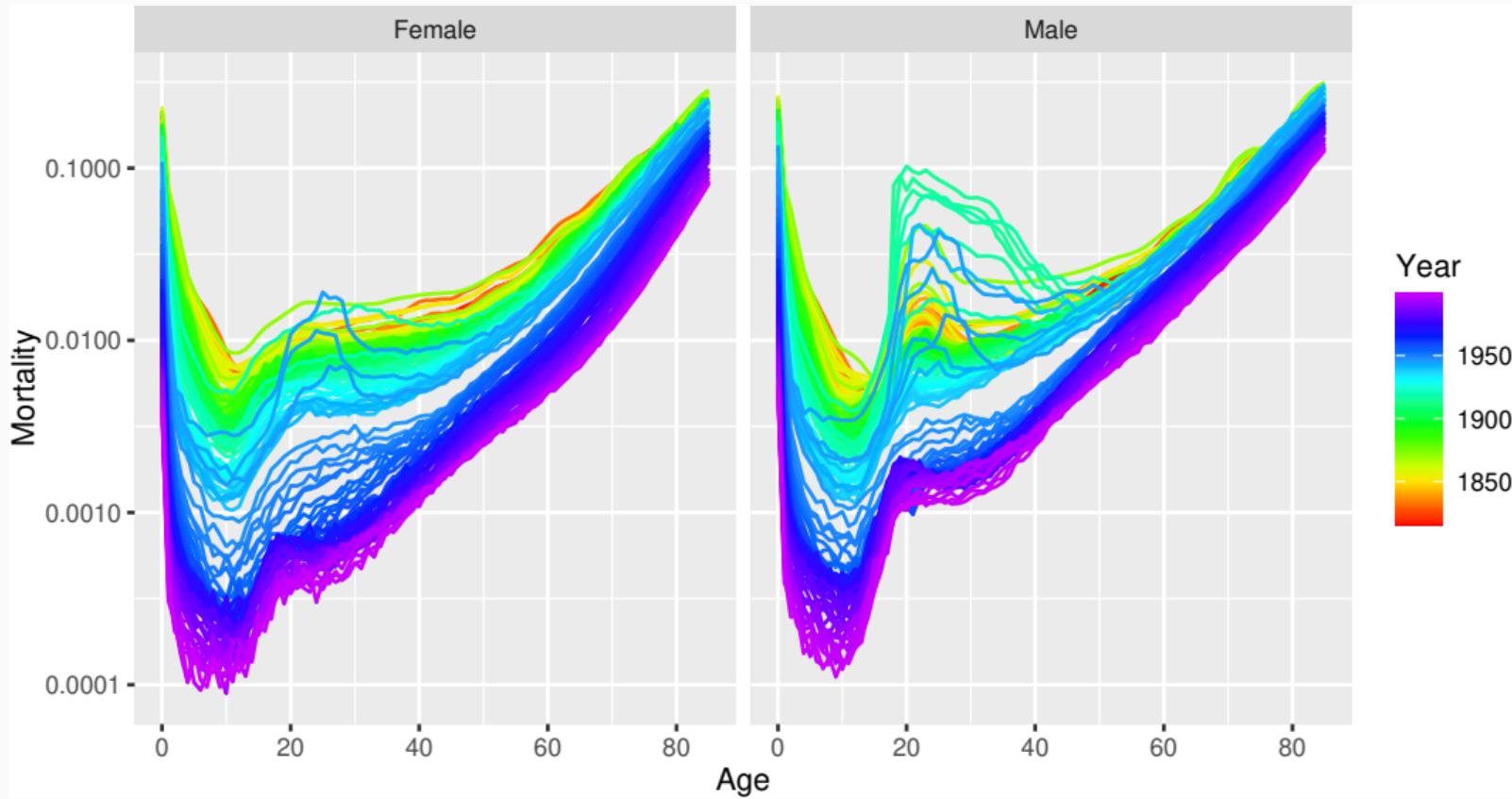
Anomalies detected ($\alpha = 0.001$):

Area	Winery	Year	Points	Price
South Australia	Henschke	2009	91	780
California	Law	2013	92	750
South Australia	Henschke	2010	96	820
South Australia	Penfolds	2008	98	850
Tuscany	Tua Rita	2011	89	300
South Australia	Penfolds	2010	99	850
Tuscany	Tua Rita	2013	90	300
Tuscany	Tua Rita	2012	90	300
Rhône Valley	Domaine Jean-Michel Gerin	2013	93	391

French mortality



French mortality



Application to French mortality

Model: $\log y_t \sim N(m_t, a_t)$ where m_t and a_t are locally and robustly estimated in a window of size $2h + 1$ around time t :

$$\hat{m}_t = \text{median}(\log y_{t-h}, \dots, \log y_{t+h})$$

$$\hat{a}_t = 1.4826 \times \text{median}(|\log y_{t-h} - \hat{m}_t|, \dots, |\log y_{t+h} - \hat{m}_t|)$$

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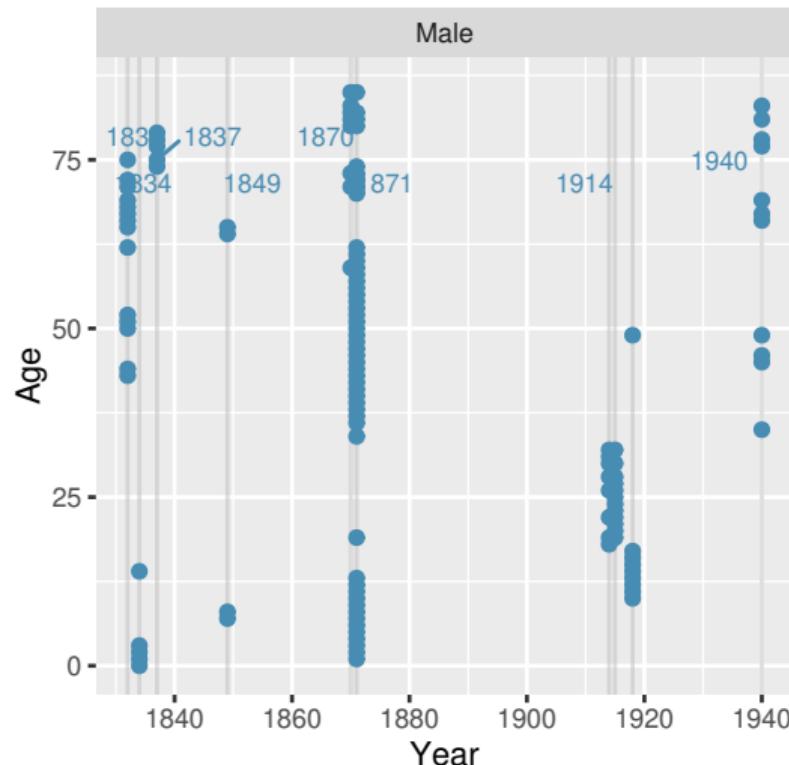
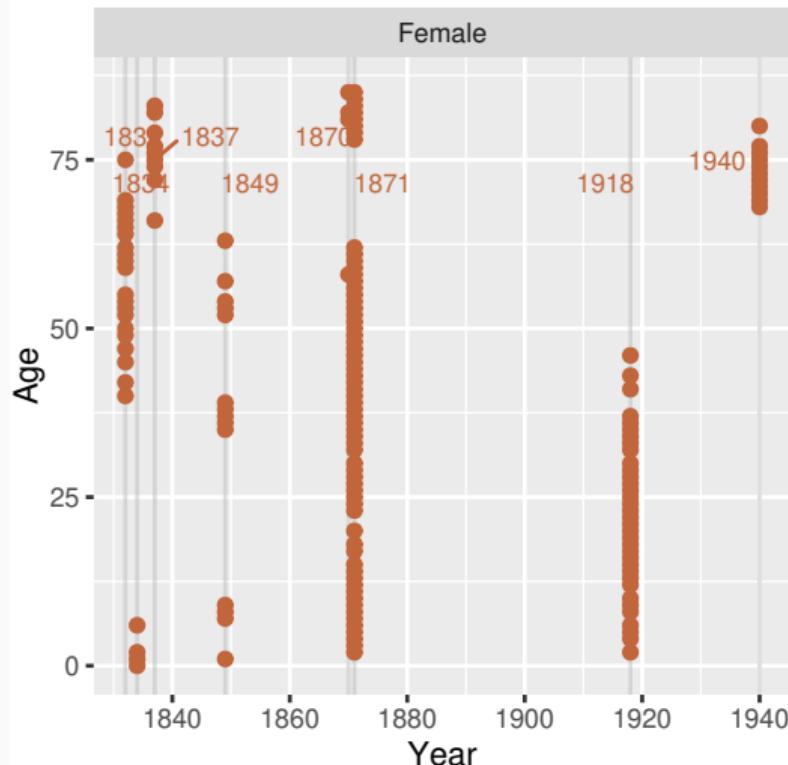
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- Male and female data from 1816–1999, over ages 0–85: 31648 observations.
- Compute surprisals under model, and surprisal probabilities under a GPD with $\alpha = 0.01$
- Identify when at least three age groups are anomalous in same year/sex.

Application to French mortality

French mortality anomalies



Application to French mortality

- 1832, 1849: Cholera outbreaks
- 1870: Franco–Prussian war
- 1871: Repression of the ‘Commune de Paris’
- 1914–1918: World War I
- 1918: Spanish flu outbreak
- 1940: World War II

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Lookout algorithm

- Compute Kernel Density Estimate (KDE) of the observations, with a bandwidth matrix chosen using persistent homology.
- Compute Leave-One-Out (LOO) KDE values
- Compute surprisals from LOO KDE values
- Fit a Generalized Pareto Distribution (GPD) to the largest surprisals
- Compute surprisal probabilities p_i from the fitted GPD
- An observation is an anomaly if $p_i < \alpha$.

Kernel density estimation

Observations: $\mathbf{y}_i \in \mathbb{R}^m$ for $i \in \{1, \dots, n\}$.

KDE

$$\hat{f}(\mathbf{y}) = \frac{1}{n} \sum_{i=1}^n |\mathbf{H}|^{-1/2} K(\mathbf{H}^{-1/2}(\mathbf{y} - \mathbf{y}_i)),$$

- K is a square-integrable spherically-symmetric function, bounded below by 0, with a finite second-order moment and unit integral.
- \mathbf{H} is a symmetric $m \times m$ positive-definite matrix.

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LOO KDE values

$$f_{-i} = \frac{1}{n-1} (n\hat{f}(\mathbf{y}_i) - \mathbf{H}^{-1/2}K(\mathbf{0}))$$

Kernel density estimation (consistency)

Let Y_1, \dots, Y_n be iid from square integrable density function f . Then KDE is a *consistent* estimator of f if

$$\lim_{n \rightarrow 0} E \left[\left(\hat{f}(\mathbf{u}) - f(\mathbf{u}) \right)^2 d\mathbf{u} \right] = 0$$

For \hat{f} to be a consistent estimator, \mathbf{H} must be a symmetric positive-definite matrix that satisfies the following conditions:

- All elements of \mathbf{H} approach zero as $n \rightarrow \infty$
- $n^{-1}|\mathbf{H}|^{-1/2} \rightarrow 0$ as $n \rightarrow \infty$

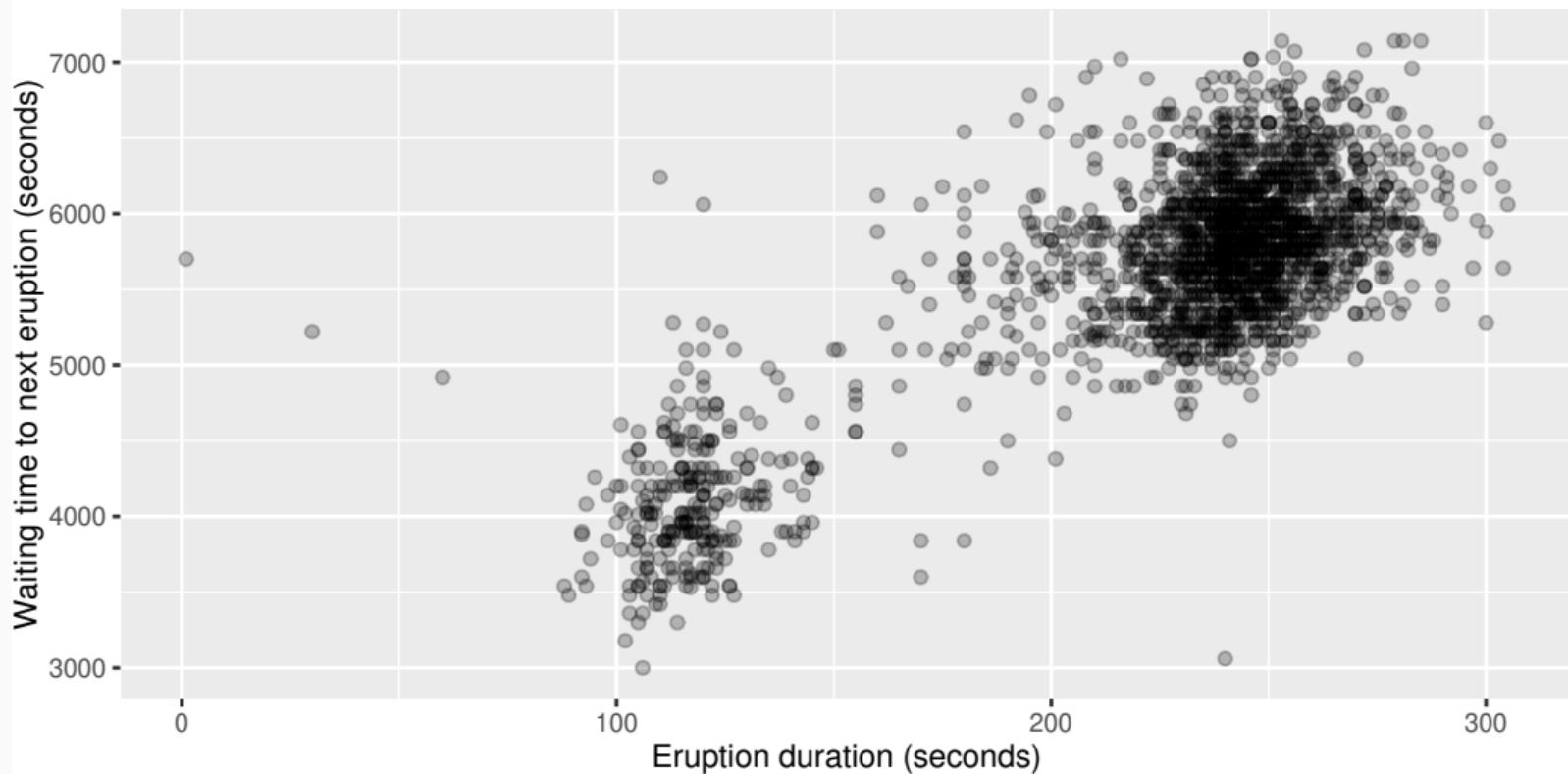
Persistent homology

Lookout algorithm

- 1 \mathbf{Y} = data matrix with rows $\mathbf{y}_1, \dots, \mathbf{y}_n$.
- 2 $\hat{\Sigma}$ = orthogonalized Gnanadesikan-Kettenring estimate of $\text{Cov}(\mathbf{Y})$, with eigendecomposition $\hat{\Sigma} = \mathbf{U}\mathbf{D}\mathbf{U}^\top$.
- 3 Rotate and scale the data: $\mathbf{Z} = \mathbf{U}\mathbf{Y}$.
- 4 Compute persistence homology barcode of \mathbf{Z} for dim zero using Vietoris-Rips diameter; obtain ordered death diameters $\{d_i\}_{i=1}^n$.
- 5 $d^* = \gamma$ sample quantile computed from $\{d_i\}_{i=1}^n$.
- 6 Compute kde: $f_i = \hat{f}(\mathbf{z}_i)$, $i = 1, \dots, n$, where $\mathbf{H} = (d^*)^{2/m} \mathbf{I}_m$.
- 7 Compute LOO kde values $f_{-i} = \frac{1}{n-1} (nf_i - \mathbf{H}^{-1/2} \mathbf{K}(\mathbf{0}))$, $i = 1, \dots, n$.
- 8 Fit GPD to largest $1 - \beta$ of surprisals $\{-\log f_i\}_{i=1}^n$, constraining shape parameter to be non-positive.
- 9 $p_i = (1 - \beta)P(-\log f_{-i} | \hat{\mu}, \hat{\sigma}, \hat{\xi})$, P = GPD cdf.

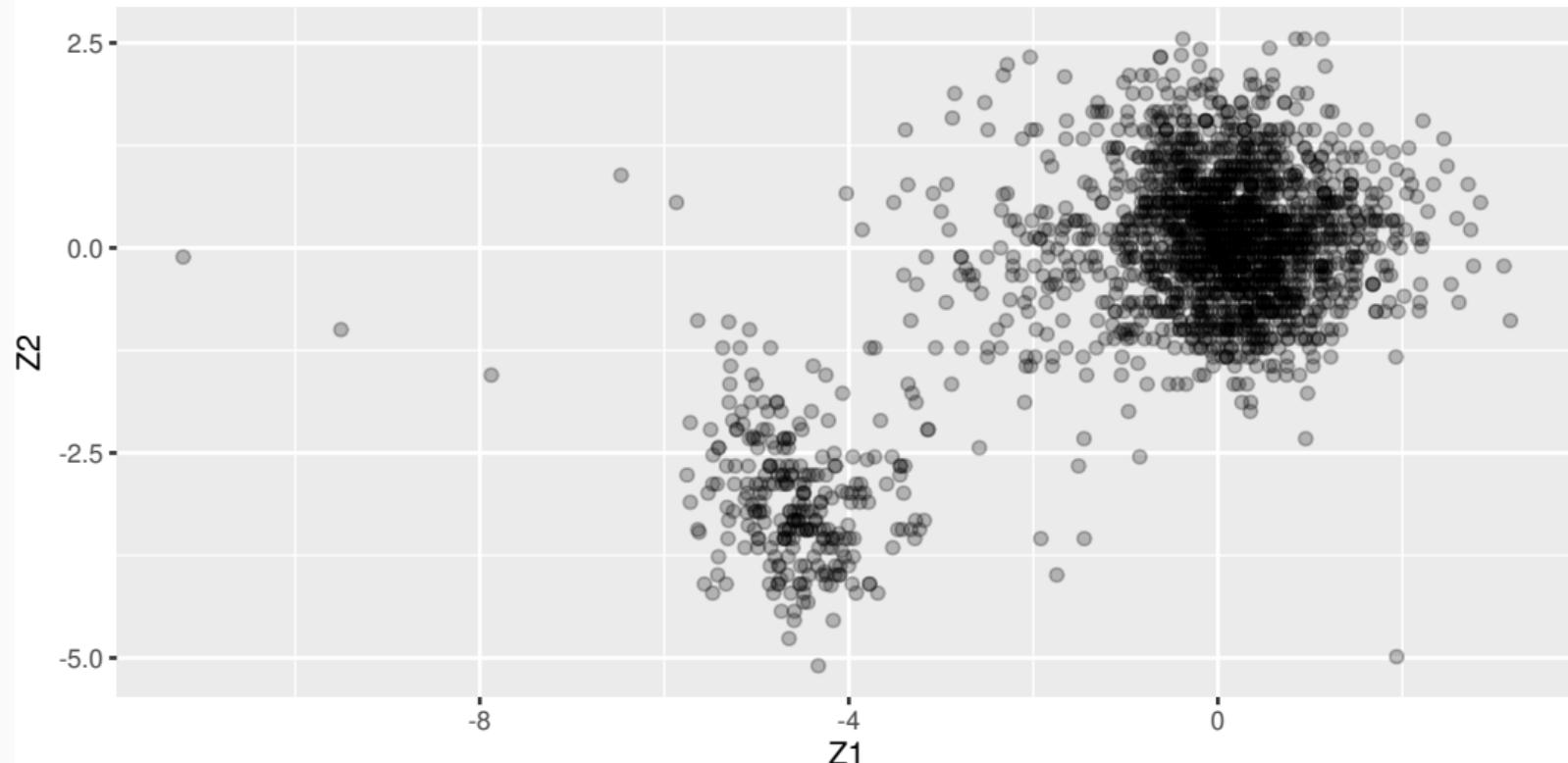
Old Faithful eruptions

Old Faithful eruptions from 14 January 2017 to 29 December 2023



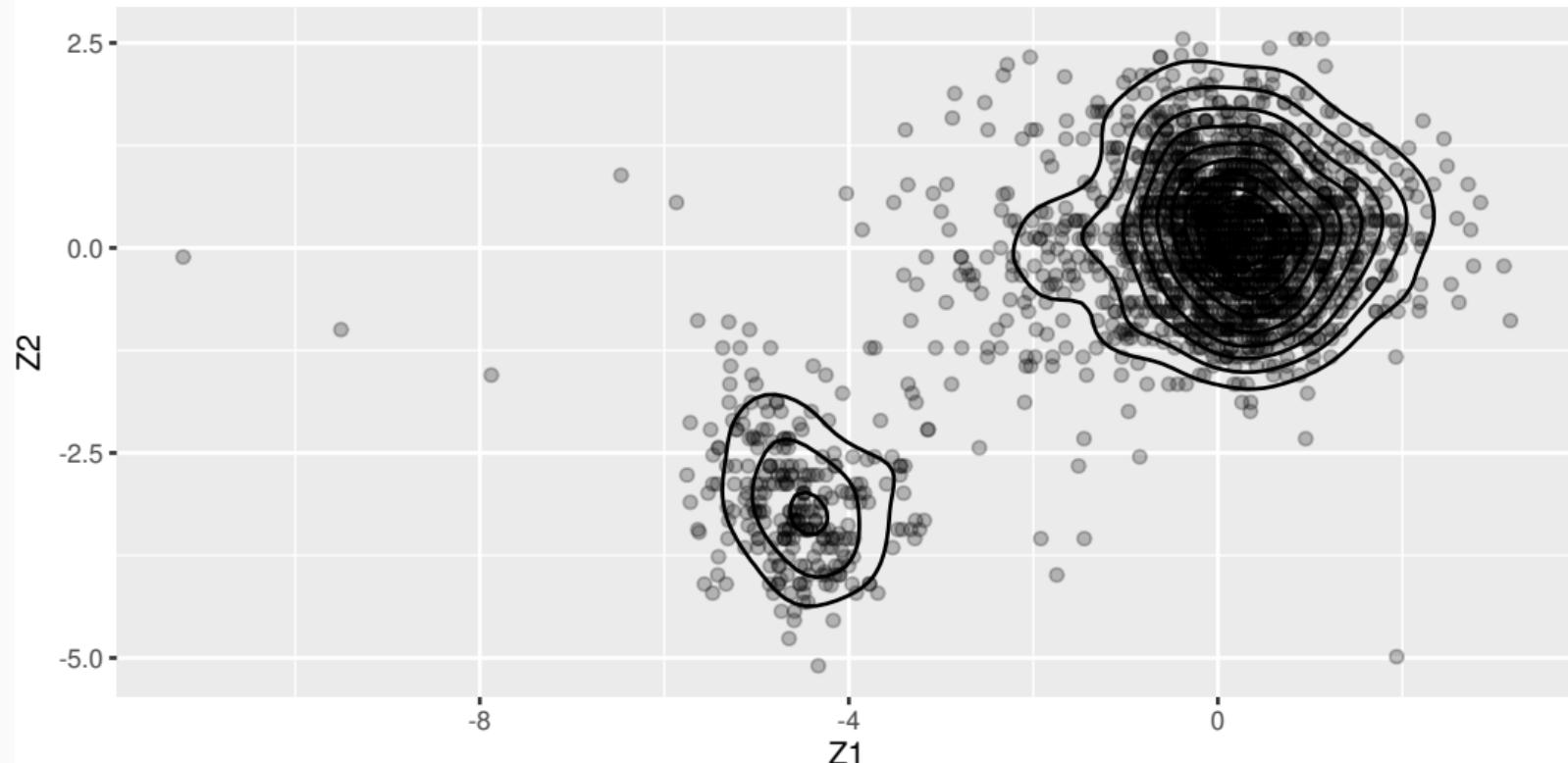
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Standardized Old Faithful eruptions from 14 January 2017 to 29 December 2023



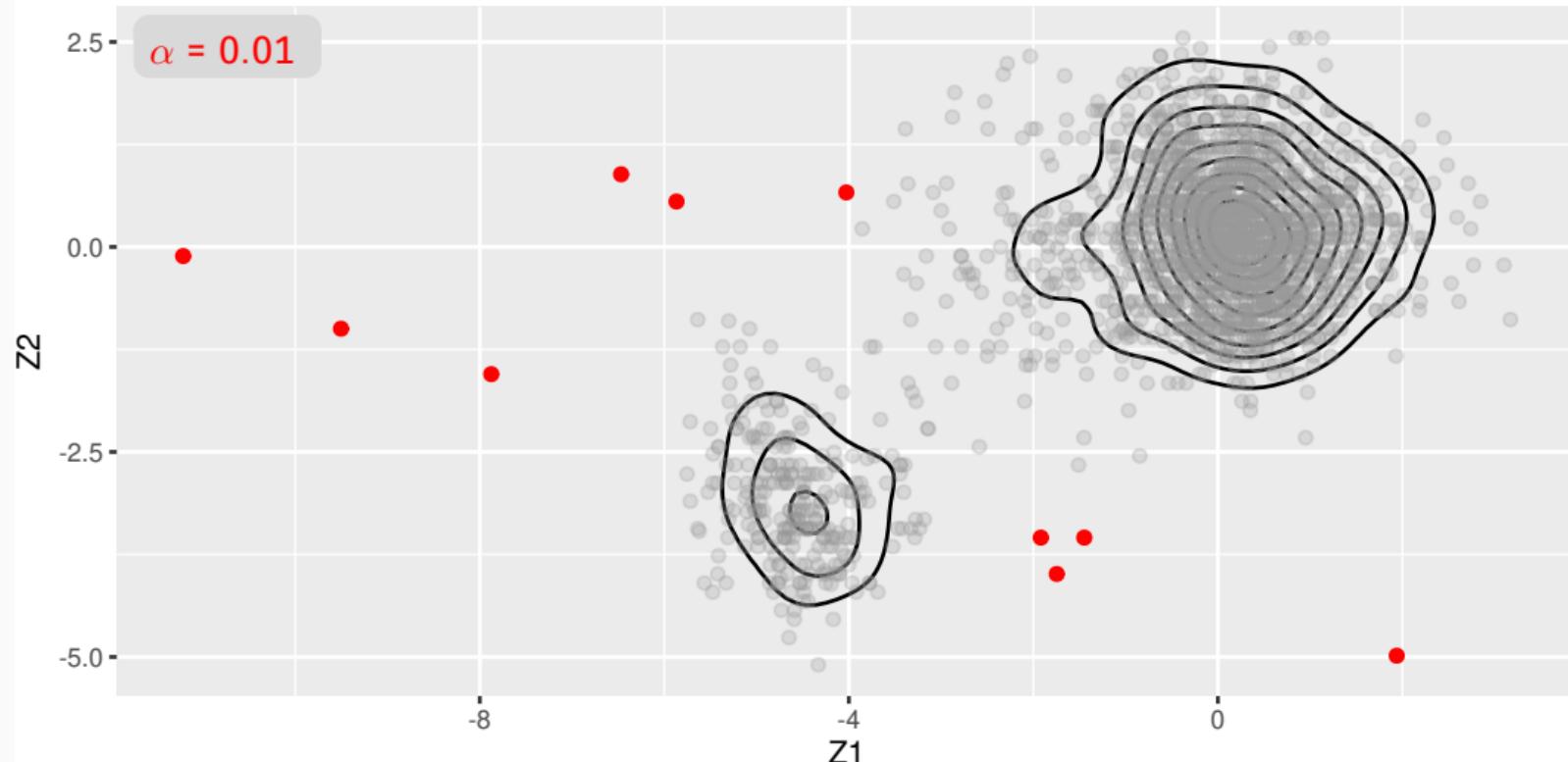
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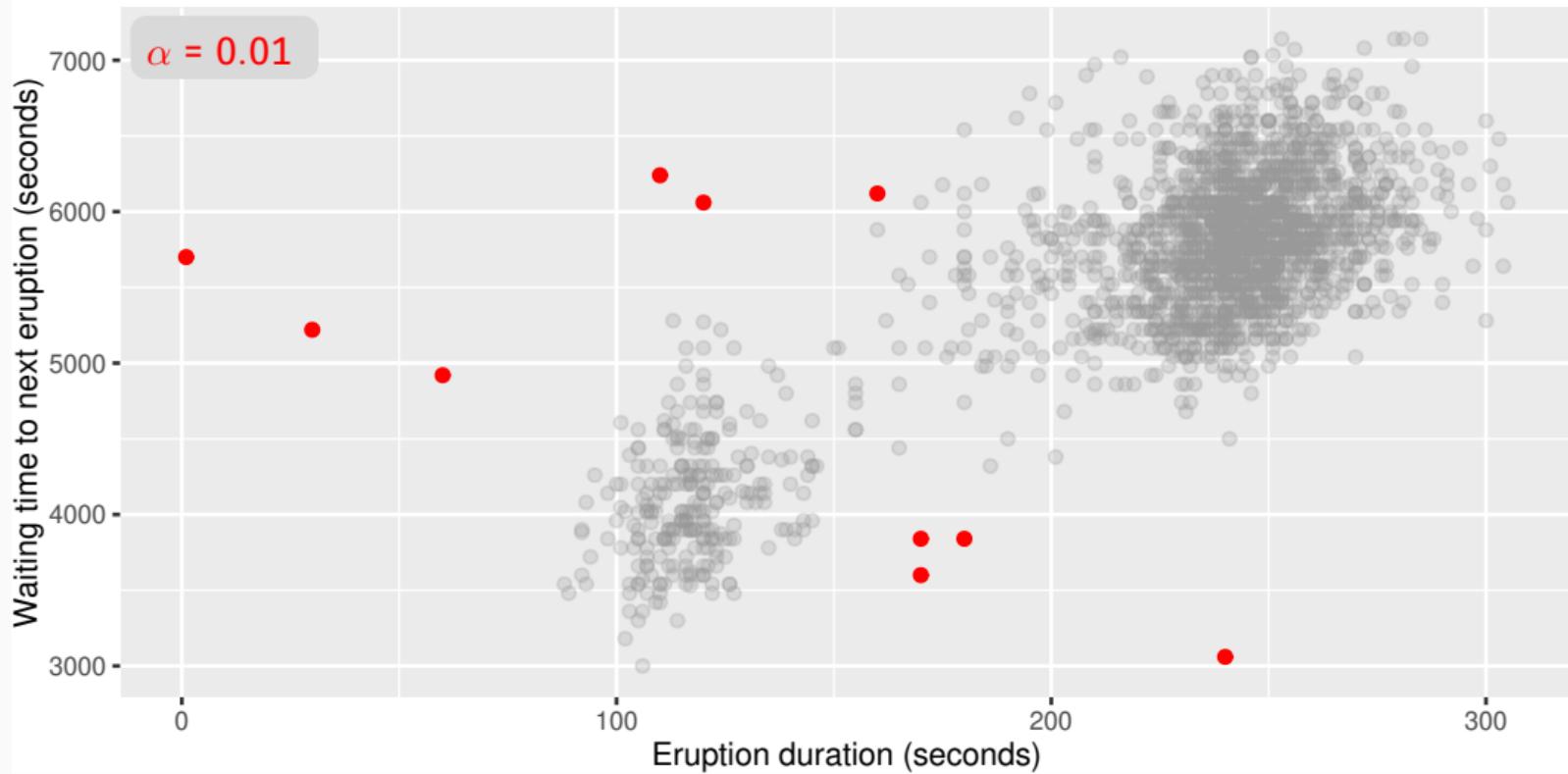
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Old Faithful eruptions

time	recorded_duration	duration	waiting	prob
2018-04-25 19:08:00	1s	1	5700	0.0000000
2022-12-07 17:19:00	~4 30s	30	5220	0.0000000
2023-07-04 12:03:00	~1 minute 55ish seconds	60	4920	0.0000612
2020-09-04 01:38:00	>1m 50s	110	6240	0.0000012
2020-06-01 21:04:00	2 minutes	120	6060	0.0001390
2020-09-16 14:44:00	>2m40s	160	6120	0.0078057
2020-08-31 09:56:00	~2m50s	170	3840	0.0076247
2021-01-22 18:35:00	2m50s	170	3600	0.0029265
2022-11-29 14:51:00	~3m	180	3840	0.0033112
2022-12-03 16:20:00	~4m	240	3060	0.0000000

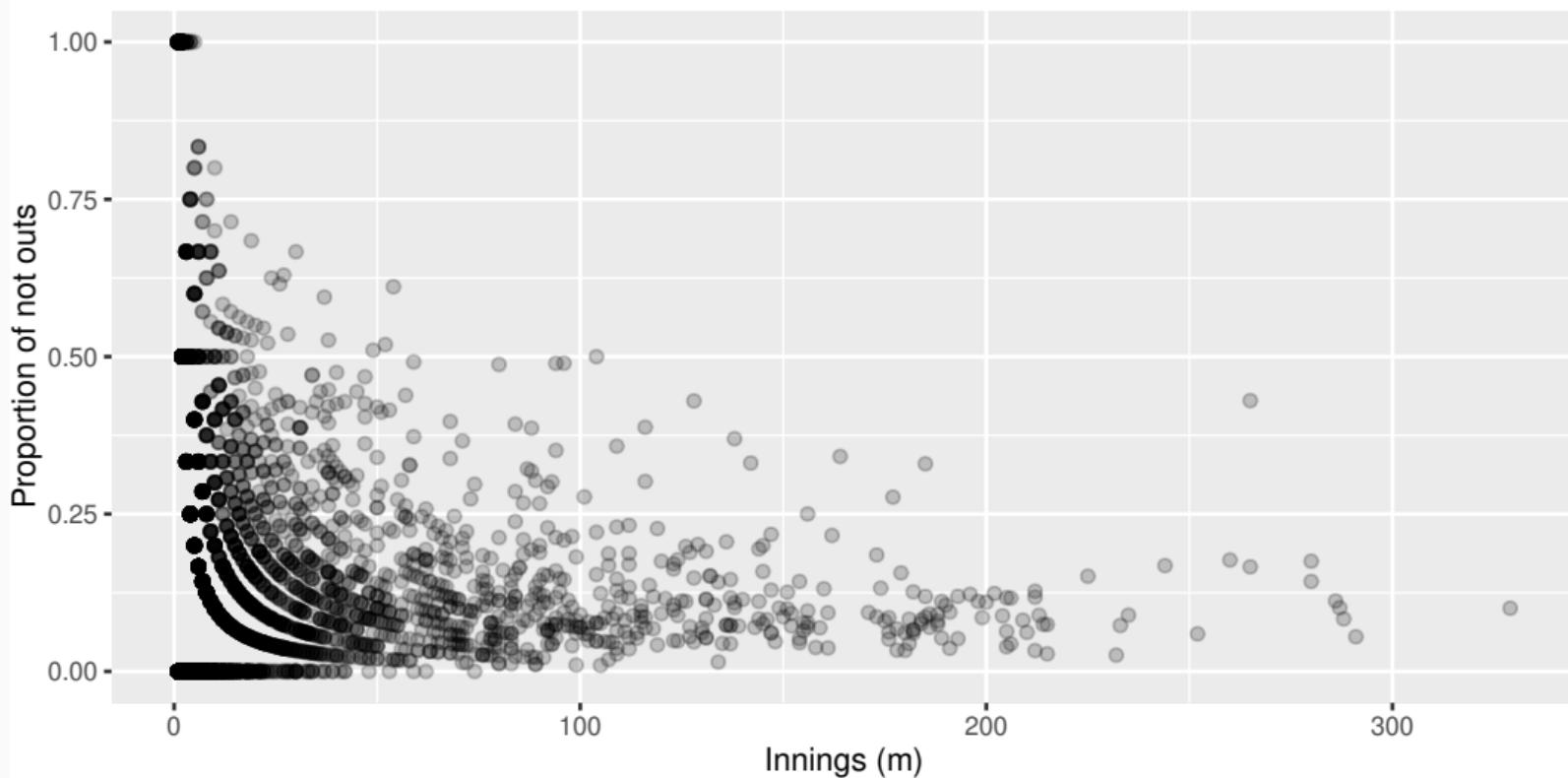
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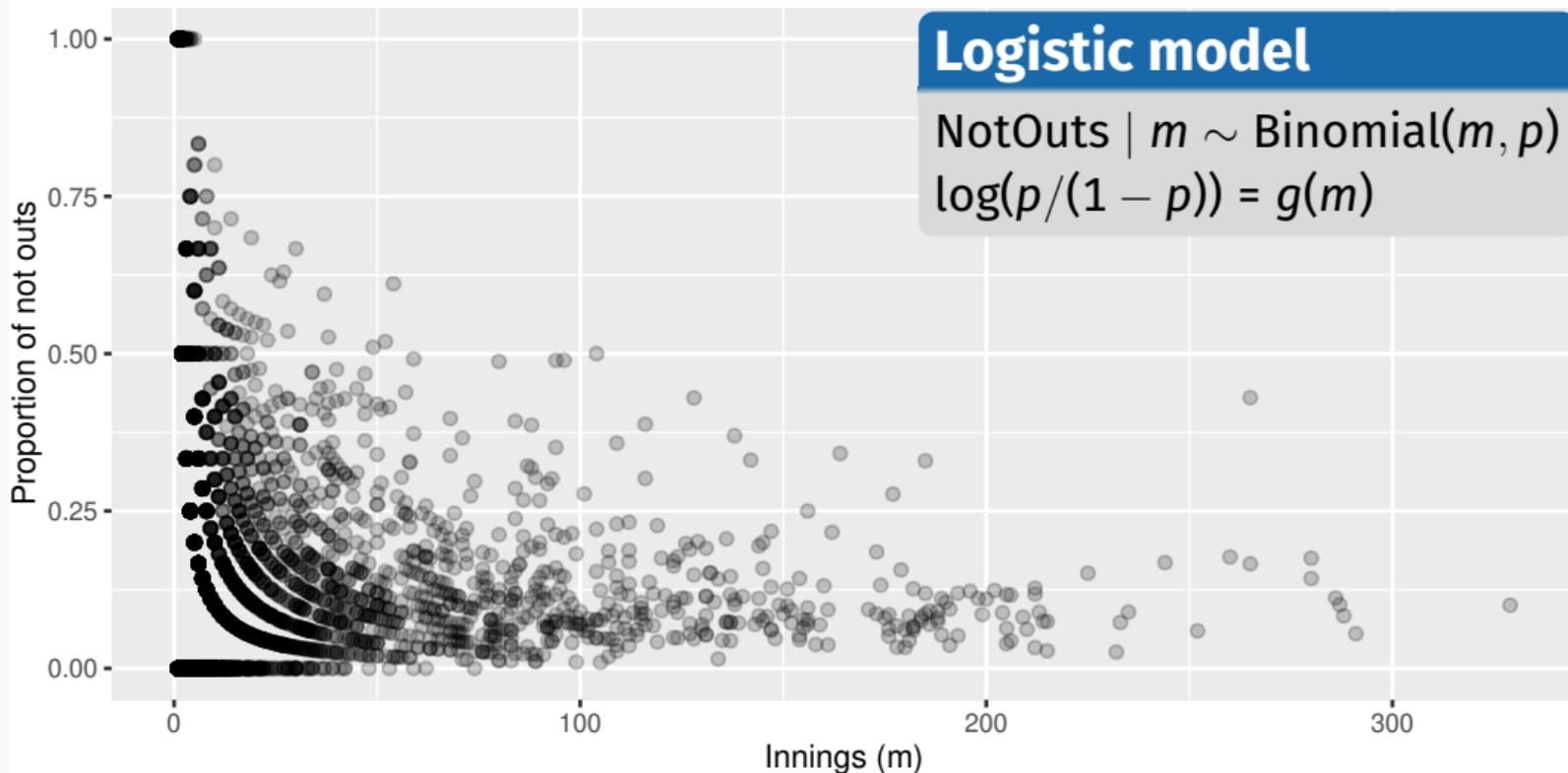
Application to test cricket not-outs

Career batting data for all test cricketers (M+W): 1834-2025



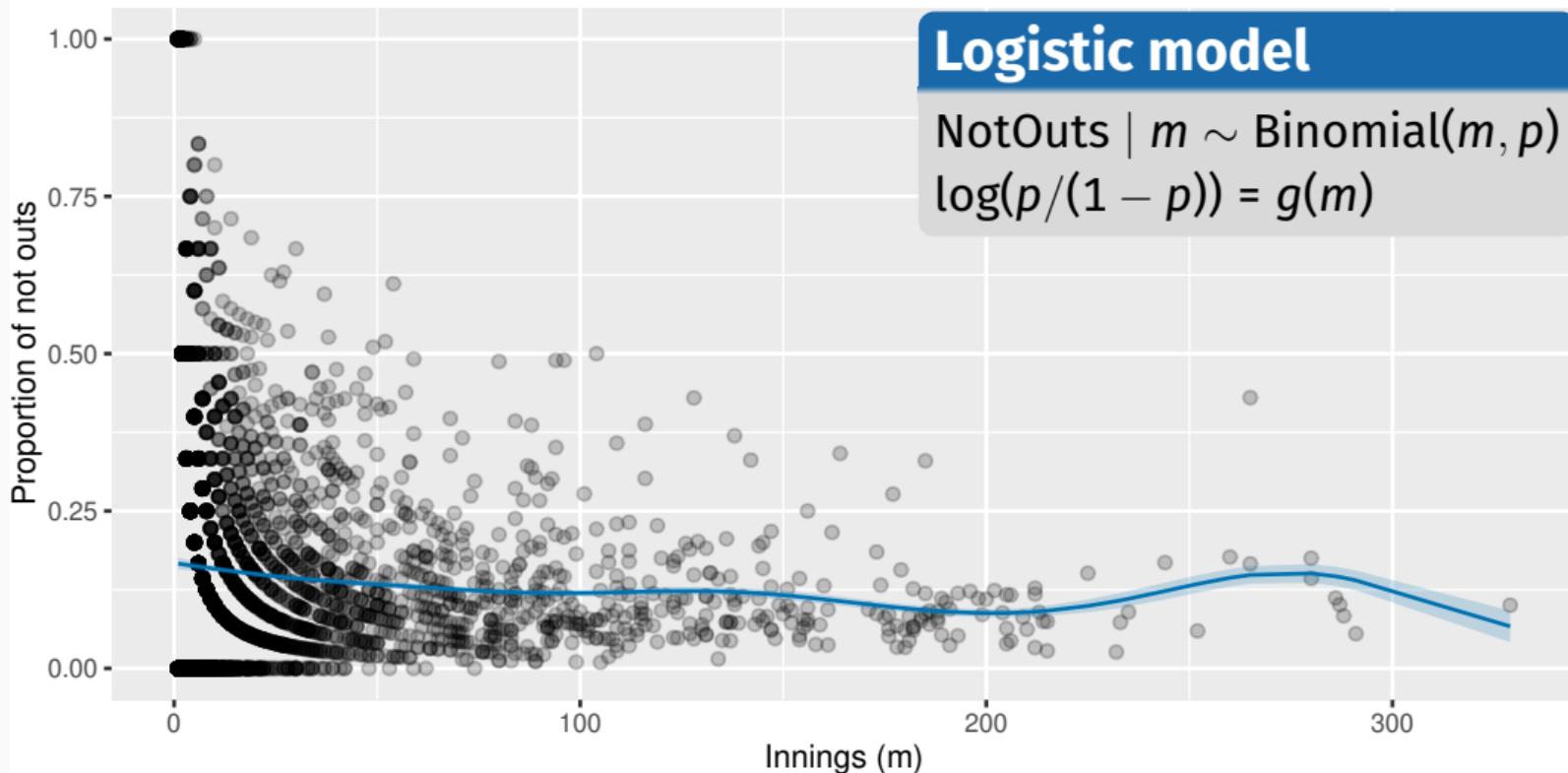
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