Unit 5 Progress Check: FRQ

1. Show all your work. Indicate clearly the methods you use, because you will be scored on the correctness of your methods as well as on the accuracy and completeness of your results and explanations.

Researchers are studying two populations of sea turtles. In population D, 30 percent of the turtles have a shell length greater than 2 feet. In population E, 20 percent of the turtles have a shell length greater than 2 feet. From a random sample of 40 turtles selected from D, 15 had a shell length greater than 2 feet. From a random sample of 60 turtles selected from E, 11 had a shell length greater than 2 feet. Let $\hat{p}_{\text{D}}$ represent the sample proportion for D, and let $\hat{p}_{\text{E}}$ represent the sample proportion for E.

(a) What is the value of the difference $\hat{p}_{\text{D}} - \hat{p}_{\text{E}}$? Show your work.

(b) What are the mean and standard deviation of the sampling distribution of the difference in sample proportions $\hat{p}_{\text{D}} - \hat{p}_{\text{E}}$? Show your work and label each value.

(c) Can it be assumed that the sampling distribution of the difference of the sample proportions $\hat{p}_{\text{D}} - \hat{p}_{\text{E}}$ is approximately normal? Justify your answer.

(d) Consider your answer in part (a). What is the probability that $\hat{p}_{\text{D}} - \hat{p}_{\text{E}}$ is greater than the value found in part (a)? Show your work.

Part A, B, C, and D

The primary goals of this question are to assess a student's ability to (1) find the difference between two sample proportions; (2) identify the sampling distribution for the difference between sample proportions; (3) check conditions for normality; and (4) use the normal model to calculate the probability that an observed difference will be greater than a certain value.

Scoring

Parts (a), (b), (c), and (d) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Each essentially correct (E) part counts as 1 point.

Each partially correct (P) part counts as ½ point.

If a response is between two scores (for example, $2\frac{1}{2}$ points), use a holistic approach to decide whether to score up or down, depending on the overall strength of the response and communication.

Reasons to score up:

· All notation is correct and clearly marked

· All explanations are clear

· No wrong information is included that was not part of the scoring (for example, saying sample size must be greater than 30 when that has nothing to do with the problem)

· No minor calculation errors are made, if they are not part of the scoring

Unit 5 Progress Check: FRQ

· Interpretation parts are especially strong

Reasons to score down:

· Notation is not wrong, but is spotty and not clearly marked

· Explanations are not wrong, but are hard to follow

· Wrong or extraneous information is included but not part of scoring

· Minor calculation errors that are not part of the scoring are made

· Interpretation parts are scored an E but are considered a weak E

0	1	2	3	4 ✓

Parts (a) through (d) sum to 4 points

OR

Parts (a) through (d) sum to $3\frac{1}{2}$ points AND a holistic approach is used to decide to score up

- ☐ Part (a) essentially correct
- ☐ Part (a) partially correct
- ☐ Part (a) incorrect
- ☐ Part (b) essentially correct
- ☐ Part (b) partially correct
- ☐ Part (b) incorrect
- ☐ Part (c) essentially correct
- ☐ Part (c) partially correct
- ☐ Part (c) incorrect
- ☐ Part (d) essentially correct
- ☐ Part (d) partially correct
- ☐ Part (d) incorrect

Solution

Part (a)

$\hat{p}_{\text{D}} = \frac{15}{40} = 0.375$ and $\hat{p}_{\text{E}} = \frac{11}{60} \approx 0.183$.

The difference is $0.375 - 0.183 = 0.192$.

Unit 5 Progress Check: FRQ

Scoring

Part (a) is scored as follows:

Essentially correct (E) if the response calculates a difference of 0.192 and shows the work.

Partially correct (P) if the response shows the difference of 0.192 without work;

OR

if the response shows a calculation error in work shown for the individual sample proportions.

Incorrect (I) if the response does not meet the criteria for E or P.

Solution

Part (b)

The mean of the sampling distribution is $\mu_{\hat{p}_{\mathrm{D}}-\hat{p}_{\mathrm{E}}} = p_{\mathrm{D}} - p_{\mathrm{E}} = 0.3 - 0.2 = 0.1.$

The standard deviation of the sampling distribution is $\sigma_{\hat{p}_{\mathrm{D}}-\hat{p}_{\mathrm{E}}} = \sqrt{\frac{(0.3)(0.7)}{40} + \frac{(0.2)(0.8)}{60}} \approx 0.089.$

Scoring

Part (b) is scored as follows:

Essentially correct (E) if the response satisfies the following two components.

· The correct mean is clearly labeled AND there is supporting work.

· The correct standard deviation is clearly labeled AND there is supporting work.

Partially correct (P) if the response satisfies only one of the two components.

Incorrect (I) if the response does not meet the criteria for E or P.

Solution

Part (c)

The distribution can be assumed approximately normal because $n_D p_D = (40)(0.3) = 12$,
$n_D(1 - p_D) = (40)(0.7) = 28$, $n_E p_E = (60)(0.2) = 12$, and $n_E(1 - p_E) = (60)(0.8) = 48$ are each greater than or equal to 10.

Scoring

Part (c) is scored as follows:

Essentially correct (E) if the response checks the four conditions for normality:

· $n_D p_D \geq 10$

Unit 5 Progress Check: FRQ

· $n_D(1 - p_D) \geq 10$

· $n_E p_E \geq 10$

· $n_E(1 - p_E) \geq 10$

Partially correct (P) if the response includes only two or three of the four conditions.

Incorrect (I) if the response does not meet the criteria for E or P.

Solution

Part (d)

The difference between the sample proportions can be assumed to be approximately normal with mean 0.1 and standard deviation 0.089. The observed difference of 0.192 found in part (a) has a *z*-score of $z = \frac{(\hat{p}_{\text{D}} - \hat{p}_{\text{E}}) - \mu_{\hat{p}_{\text{D}} - \hat{p}_{\text{E}}}}{\sigma_{\hat{p}_{\text{D}} - \hat{p}_{\text{E}}}} = \frac{0.192 - 0.1}{0.089} \approx 1.03.$ The area under the standard normal curve to the right of 1.03 is about 0.15. The probability that $\hat{p}_{\text{D}} - \hat{p}_{\text{E}}$ is greater than 0.192 is approximately 0.15.

Scoring

Part (d) is scored as follows:

Essentially correct (E) if the response calculates the correct probability AND shows supporting work.

Partially correct (P) if the response calculates an incorrect value for the probability but demonstrates a reasonable approach.

Incorrect (I) if the response does not meet the criteria for E or P.

Notes:

· The response in part (d) can be scored E if it follows from incorrect, but reasonable, answers parts (a) and (b). A reasonable answer in part (a) is between 0 and 1 and does not equal the population difference of 0.1. A reasonable answer in part (b) has a positive standard deviation.

· Reasonable approaches include

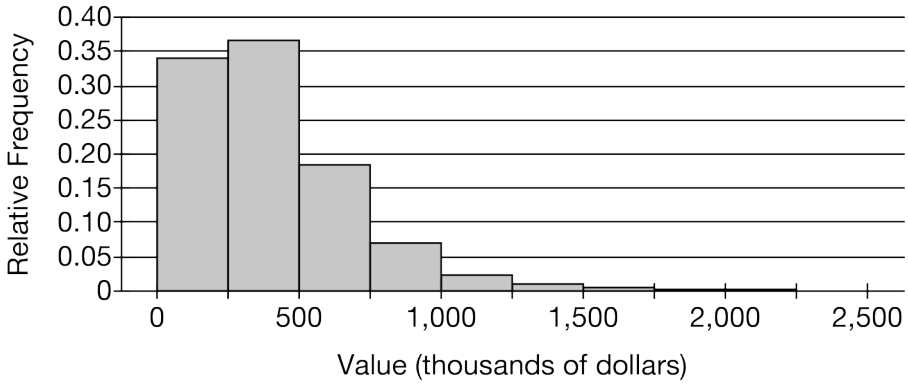
· Finding the probability less than the difference found in part (a)

· Reversing the sample estimate and mean of the sampling distribution in the calculation for the Alt text: z-score.

Unit 5 Progress Check: FRQ

2. Show all your work. Indicate clearly the methods you use, because you will be scored on the correctness of your methods as well as on the accuracy and completeness of your results and explanations.

The following histogram shows the distribution of house values in a certain city. The mean of the distribution is $\$403{,}000$ and the standard deviation is $\$278{,}000$.



(a) Suppose one house from the city will be selected at random. Use the histogram to estimate the probability that the selected house is valued at less than $\$500{,}000$. Show your work.

(b) Suppose a random sample of 40 houses are selected from the city. Estimate the probability that the mean value of the 40 houses is less than $\$500{,}000$. Show your work.

To estimate a population mean μ, the sample mean $\bar{x}$ is often used as an estimator. However, a different estimator is called the sample midrange, given by the formula $\frac{\text{sample minimum} + \text{sample maximum}}{2}$.

(c) The following table shows the values, in thousands of dollars, of 40 randomly selected houses in the city.

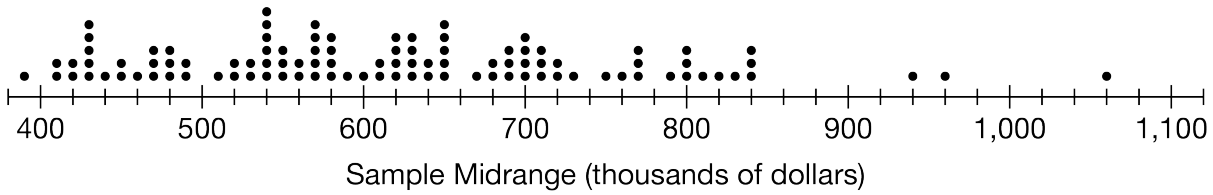
34	38	95	110	137	140	155	155	169	207
209	217	274	314	314	323	343	347	349	363
369	373	388	389	397	416	448	450	483	487
488	516	571	595	600	738	762	769	863	1,084

(i) Calculate the sample midrange for the data.

(ii) Explain why the sample midrange might be preferred to the sample mean as an estimator of the population mean.

(d) To investigate the sampling distribution of the sample midrange, a simulation is performed in which 100 random samples of size $n = 40$ were selected from the population of house values. For each sample, the sample midrange was calculated and recorded on the following dotplot. The mean of the distribution of sample midranges is $\$617{,}000$ with standard deviation $\$136{,}000$.

Unit 5 Progress Check: FRQ



Based on the results of the simulation, explain why the sample mean might be preferred to the sample midrange as an estimator of the population mean.

Part A, B, C, and D

The primary goals of this question are to assess a student's ability to (1) calculate the probability of a random variable; (2) calculate and interpret an expected value; (3) use probability to decide on one of two choices; and (4) determine probabilities for which one of the choices will have a more favorable outcome.

Scoring

Parts (a), (b), (c), and (d) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Each essentially correct (E) part counts as 1 point.

Each partially correct (P) part counts as ½ point.

If a response is between two scores (for example, 2½ points), use a holistic approach to decide whether to score up or down, depending on the overall strength of the response and communication.

Reasons to score up:

· *All notation is correct and clearly marked*

· *All explanations are clear*

· *No wrong information is included that was not part of the scoring (for example, saying sample size must be greater than 30 when that has nothing to do with the problem)*

· *No minor calculation errors are made, if they are not part of the scoring*

· *Interpretation parts are especially strong*

Reasons to score down:

· *Notation is not wrong, but is spotty and not clearly marked*

· *Explanations are not wrong, but are hard to follow*

· *Wrong or extraneous information is included but not part of scoring*

· *Minor calculation errors that are not part of the scoring are made*

· *Interpretation parts are scored an E but are considered a weak E*

Unit 5 Progress Check: FRQ

0	1	2	3	4 ✓

Parts (a) through (d) sum to 4 points

OR

Parts (a) through (d) sum to $3\frac{1}{2}$ points AND a holistic approach is used to decide to score up

- [ ] Part (a) essentially correct
- [ ] Part (a) partially correct
- [ ] Part (a) incorrect
- [ ] Part (b) essentially correct
- [ ] Part (b) partially correct
- [ ] Part (b) incorrect
- [ ] Part (c) essentially correct
- [ ] Part (c) partially correct
- [ ] Part (c) incorrect
- [ ] Part (d) essentially correct
- [ ] Part (d) partially correct
- [ ] Part (d) incorrect

Solution

Part (a)

From the histogram, $P(\text{value} < 500,000) \approx 0.34 + 0.37 = 0.71.$

Scoring

Part (a) is scored as follows:

Essentially correct (E) if the response gives a probability between 0.68 and 0.74 and shows work that includes the sum of the heights of the first two bars in the histogram.

Partially correct (P) if the response gives a probability between 0.68 and 0.74 but does not show work.

OR

if the response shows correct work but gives a probability that is not between 0.68 and 0.74 AND the probability given is

between 0 and 1.

OR

If the response shows correct work but does not give a probability.

Incorrect (I) if the response does not meet the criteria for E or P, including if the response uses a normal distribution to calculate a probability.

Solution

Part (b)

Because the sample size $n = 40$ is greater than or equal to 30, the sampling distribution of $\bar{x}$ is approximately normal with mean $\mu_{\bar{x}} = \mu = 403,000$ and standard deviation $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}} = \frac{278{,}000}{\sqrt{40}} \approx 43,956$.

$P(\bar{x} < 500,000) = P\left(z < \frac{500{,}000 - 403{,}000}{43{,}956}\right) = P(z < 2.21) \approx 0.986$

Scoring

Part (b) is scored as follows:

Essentially correct (E) if the response satisfies the following three components.

· Indicates use of a normal distribution with the correct mean ($403,000) and the correct standard deviation ($43,956).

· Indicates the correct boundary value and direction.

· Includes a probability consistent with components 1 and 2.

Partially correct (P) if the response satisfies only two of the three components.

Incorrect (I) if the response does not meet the criteria for E or P.

Notes:

· Components 1 and 2 are satisfied with a well-labeled calculator command such as
$\text{normalcdf}(\text{lower: } -1,000,000,\ \text{upper: } 500,000,\ \text{mean: } 403,000,\ \text{SD: } 43,956)$

· Components 1 and 2 are satisfied with a well-labeled normal sketch, provided that labels include the mean, the values one standard deviation from the mean in each direction $(\pm 278,000)$ and the boundary value, and that the shading is in the correct direction.

· Component 1 is satisfied if work is shown for a *z*-score calculation. A *z*-score calculation alone does not satisfy component 3 because it lacks direction.

· If the response indicates that the use of the normal model is appropriate due to the sample size being large enough, this should be considered a plus for the purposes of holistic scoring.

Solution

Part (c):

Unit 5 Progress Check: FRQ

Part (c)(i) The sample midrange is $\frac{34{,}000+1{,}084{,}000}{2} = \$559,000.$

Part (c)(ii) The sample midrange is much easier to calculate than the sample mean because it uses only 2 values.

Scoring

Part (c) is scored as follows:

Essentially correct (E) if the response satisfies the following two components.

· In part (c-i), provides the correct midrange (either 559,000 or 559).

· In part (c-ii), states that the sample midrange is easier to calculate than the sample mean or gives another plausible reason for why the midrange is preferable.

Partially correct (P) if the response satisfies only one of the components.

Incorrect (I) if the response does not meet the criteria for E or P.

Note: The answer to part (c-i) does not require units (dollars or thousands of dollars), but the correct inclusion of units should be considered a plus for the purposes of holistic scoring.

Solution

Part (d)

Because the mean of the sampling distribution of the sample midrange ($617,000) is much larger than the population mean ($403,000), the sample midrange is a biased estimator of the population mean. Also, the standard deviation of the sampling distribution of the sample midrange ($136,000) is much larger than the standard deviation of the sampling distribution of the sample mean ($43,956), meaning that the sample midrange is a less precise estimator.

Scoring

Part (d) is scored as follows:

Essentially correct (E) if the response states that the sample midrange is a biased estimator of the population mean and provides numerical evidence from the simulation (the center argument);

OR

the sample midrange is more variable (less precise) than the sample mean and provides numerical evidence from the simulation (the variability argument);

OR

the distribution of the sample midrange is not approximately normal, while the sampling distribution of the sample mean is approximately normal for samples of size 40, making it easier to perform probability calculations (the shape argument).

Partially correct (P) if the response states that the sample midrange is a biased estimator of the population mean but does not include numerical evidence from the simulation;

OR

Unit 5 Progress Check: FRQ

the sample midrange is more variable than the sample mean but does not include numerical evidence from the simulation;

OR

the distribution of the sample midrange is not normal or is less normal than the distribution of the sample mean, but does not indicate why this is a disadvantage.

Incorrect (I) if the response does not meet the criteria for E or P.

Note: If a response addresses more than one argument (center, variability, shape), this should be considered a plus for the purposes of holistic scoring.