Question 1

Intent of Question

The primary goals of this question are to assess a student's ability to (1) interpret a type of bar chart called a population pyramid; and (2) answer questions about age distributions within and between two countries using population pyramids.

Solution

Part (a):

No, the proportion of the female population age 60 or older in Country A is not greater than the corresponding proportion in Country B. This is clear without doing any calculations, because the bars in all of the age categories from 60 and above are longer for Country B than the corresponding bars for Country A, and both graphs use the same scale on the horizontal axis. Therefore, the sum of the proportions represented by those bars must be higher for Country B than for Country A.

Part (b):

Country B is the one that experienced an increase in the birth rate from 1946 to 1955 and again from about 1966 to 1975. The people born in 1946 to 1955 would be in the 60 to 69 age group in 2015, and those born about 20 years later would be in the 40 to 49 age group in 2015. The bars representing the proportions of the population in those age groups for Country B for both males and females are longer than the bars for the age groups above and below them, indicating that in Country B more people were born in the years in question than in the years immediately before and after those intervals. That pattern is not indicated for Country A.

Part (c):

The median age for the males in Country A in 2015 is in the 30 to 39 age group. The median age group corresponds to the bar such that at least 50% of the population is in that age group or higher, and at least 50% of the population is in that age group or lower. Adding the lengths of the bars either above or below the 30 to 39 age group shows that this group satisfies the condition.

Scoring

Parts (a), (b) and (c) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the response:

- (1) savs no:
- (2) justifies the choice by either estimating the sum of the lengths of the bars or by comparing their lengths in each of the relevant age groups for the two countries; and
- (3) includes an explicit comparison between Countries A and B for the bar lengths, such as by noting that the sum of the bars for Country A is less than the sum for Country B.

Question 1 (continued)

Partially correct (P) if the response says no and justifies the choice by adding or discussing the lengths of bars for age groups 60 and above in Country A or Country B, but does not explicitly compare them to the equivalent measure for the other Country. An example is "No, because in Country A, the bars for the age groups 60 and above are shorter;"

OR

if the response meets the three criteria for an E, but bases it on comparing the bars in the two graphs for the age group of 60 to 69 only.

Incorrect (I) if the response does not meet the criteria for E or P.

Part (b) is scored as follows:

Essentially correct (E) if the response:

- (1) correctly identifies Country B;
- (2) identifies the age groups that the people born in the designated years would be 20 years later (in 2015); and
- (3) explains (or demonstrates on the graph) that the bars for the surrounding age groups in Country B are shorter than for the age groups in question, or at least that the bar for the 50-59 age group is shorter.

Partially correct (P) if the response correctly identifies Country B and includes one of the remaining two components required for an E, for instance by noting that the age groups of 40 to 49 and 60 to 69 have higher proportions than the age groups surrounding them, but not explicitly identifying that those would be the age groups in 2015 that satisfy the stated increases in birth rate.

Incorrect if the response does not meet the criteria for E or P.

Part (c) is scored as follows:

Essentially correct (E) if the response:

- (1) correctly identifies the age group of 30 to 39;
- (2) provides correct supporting work for the choice of the median age group; and
- (3) describes why this age group contains the median.

Note: The response does not need to include numerical values verifying the calculation.

Partially correct (P) if the response correctly includes component (1) and one of the other two components.

Incorrect if the response includes at most one of the components required for E.

Question 1 (continued)

4 Complete Response

Three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and no parts partially correct

OR

One part essentially correct and one or two parts partially correct

OR

Three parts partially correct

1 Minimal Response

One part essentially correct

OR

No parts essentially correct and two parts partially correct

Question 2

Intent of Question

The primary goals of this question are to assess a student's ability to (1) describe how to randomly assign participants to treatment groups; (2) identify the appropriate test procedure and hypotheses to answer a question of interest and (3) provide information about how to increase the power of a test in an experiment.

Solution

Part (a):

Number the participants from 1 to 60, then using a random number generator on a calculator, statistical software, or a random number table, choose 40 numbers out of 1 to 60 without replacement. Use the first 20 of those numbers to choose the 20 individuals to assign to the driving game, and the next 20 to choose the individuals to assign to the sports game. The remaining 20 individuals are assigned to the puzzle game.

Part (b):

The appropriate test is a two sample t-test for the difference in means.

Define μ_D to be the mean improvement time if everyone in the population were to play the driving game, and μ_S to be the mean improvement time if everyone in the population were to play the sports game.

The null hypothesis is $H_0: \mu_D = \mu_S$ and the alternative hypothesis is $H_a: \mu_D \neq \mu_S$.

Part (c):

To increase power, the researchers should use a larger sample size and/or increase the significance level α . Using a larger sample size reduces the standard error of the sampling distribution, which increases the value of the test statistic, making it easier to detect a difference in the population means if it exists. Using a larger significance level makes it easier to reject a false null hypothesis, which also increases the power of the test.

Question 2 (continued)

Scoring

Parts (a), (b) and (c) are scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the response includes a method that:

- (1) Uses a random process;
- (2) Guarantees equal probability of assignment;
- (3) Results in 20 volunteers per group;
- (4) Specifies which game is assigned to each group/volunteer.

Partially correct (P) if the response includes two or three of the components required for E.

Note: A response that simply states "Randomly Assign" is not providing a method, so it is scored I.

Incorrect if the response does not meet the criteria for E or P.

Part (b) is scored as follows:

Essentially correct (E) if the response:

- (1) Identifies the correct test;
- (2) Defines appropriate parameter(s) in context for the stated test;
- (3) Specifies the correct null and alternative hypotheses consistent with the stated test.

Partially correct (P) if the response includes two of the three components required for E.

Incorrect if the response does not meet the criteria for E or P.

Note: If the response describes a block design in part (a), with blocks of 3 people of similar skill, then the appropriate test in part (b) is a paired *t*-test.

Part (c) is scored as follows:

Essentially correct (E) if the response:

- Identifies increasing the sample size or increasing alpha;
- Provides a reasonable statistical explanation that illustrates how their choice relates to power.

Partially correct (P) if the response includes the first component required for E OR

Displays an understanding of the concept of power.

Incorrect if the response does not meet the criteria for E or P.

Question 2 (continued)

4 Complete Response

Three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and no parts partially correct

OR

One part essentially correct and one or two parts partially correct

OR

Three parts partially correct

1 Minimal Response

One part essentially correct

OR

No parts essentially correct and two parts partially correct

Question 3

Intent of Question

The primary goals of this question are to assess a student's ability to (1) list possible outcomes of a random event; (2) calculate probabilities for compound events; (3) calculate a conditional probability given the individual and joint probabilities; and (4) construct a probability distribution and use it to find an expected value.

Solution

Part (a):

The possible outcomes are listed below, organized by who wins the match. Within each match winner category, who wins each set is shown.

i) Player V wins: V V V M V M V V

ii) Player M wins: M M M V M V M M

Part (b):

The ways in which Player V can win a match against Player M and the corresponding probabilities are shown below. Adding the probabilities for the various ways Player V wins the match yields the overall probability of 0.4575.

<u>Outcome</u>	<u>Probability</u>
VV	(0.5)(0.6) = 0.3
VMV	(0.5)(1 - 0.6)(0.45) = 0.09
MVV	(0.5)(1 - 0.7)(0.45) = 0.0675

Total:
$$0.3 + 0.09 + 0.0675 = 0.4575$$

Part (c):

$$P(3 \text{ sets} \mid V \text{ wins}) = \frac{P(3 \text{ sets and } V \text{ wins})}{P(V \text{ wins})} = \frac{(0.09 + 0.0675)}{0.4575} = \frac{0.1575}{0.4575} \approx 0.344$$

Part (d):

The number of sets played must be either two or three. The probability of exactly two sets is

$$P(VV) + P(MM) = (0.5)(0.6) + (0.5)(0.7) = 0.3 + 0.35 = 0.65.$$

Therefore, the probability of three sets is 1 - 0.65 = 0.035.

The expected value is (2)(0.65) + (3)(0.35) = 1.3 + 1.05 = 2.35 sets.

Question 3 (continued)

Scoring

The response is scored in three sections. Section 1 consists of part (a), section 2 consists of parts (b) and (c) and section 3 consists of part (d). Each section is scored as essentially correct (E), partially correct (P), or incorrect (I).

Section 1 is scored as follows:

Essentially correct (E) if the response includes the three outcomes that result in a Match win for each player (parts a) i and a) ii), including the fact that the outcomes in which the same player wins the first two sets do not have a third set played. The outcomes can be displayed in a list, in a table, in a tree diagram, or in another reasonable format.

Partially correct (P) if the response includes the correct 3-set outcomes in parts a) i and a) ii, but expresses the 2-set outcomes as 3-set outcomes. For instance, the response might list the outcomes under Player V as VVV, VVM, VMV and MVV;

OR

if the response includes all correct outcomes in parts a) i and a) ii, but includes extras, such as VVV and/or VVM.

Incorrect (I) if the response does not meet the criteria for E or P.

Section 2 is scored as follows:

Essentially correct (E) if the response provides:

- (1) the correct answer in part (b);
- (2) links calculation to outcomes in (a):
- (3) the correct answer in part (c);
- (4) sufficient work shown to understand how the conditional probability in part (c) was calculated.

Partially correct (P) if the response provides two or three of the components for E.

Note: If component (1) is incorrect, a response can still receive credit for components (3) and (4) if both are consistent with component (1).

Incorrect if the response does not meet the criteria for E or P.

Question 3 (continued)

Section 3 is scored as follows:

Essentially correct (E) if the response:

- (1) Correctly determines the probability distribution for the number of sets played or provides a probability distribution for the number of sets played consistent with part (b);
- (2) Correctly computes the expected value of the probability distribution in component (1).

Partially correct (P) if the response provides one of the components for E.

Incorrect (I) if the response does not meet the criteria for E or P.

Note: An expected value that is rounded to an integer cannot receive E.

4 Complete Response

Three sections essentially correct

3 Substantial Response

Two sections essentially correct and one section partially correct

2 Developing Response

Two sections essentially correct and no sections partially correct OROne section essentially correct and one section or two sections partially correct ORThree sections partially correct

1 Minimal Response

One section essentially correct

OR

No sections essentially correct and two sections partially correct

Question 4

Intent of Question

The primary goals of this question were to assess a student's ability to (1) identify and compute an appropriate confidence interval after checking the necessary conditions, (2) interpret the confidence interval in context, and (3) use data to determine an appropriate estimate to answer a non-standard question.

Solution

Part (a):

Step 1: Identify the appropriate confidence interval (by name or formula) and check appropriate conditions.

The appropriate procedure is a paired t-interval for a population mean difference.

Conditions:

- 1. The sample is randomly selected from the population.
- 2. The population has a normal distribution, or the sample size is large.

Condition 1 is met because a random sample of students was used.

For condition 2, the sample size of 10 is not large, so we need to examine the sample data to assess whether there are any major outliers or skewness. The appropriate data to examine are the differences in ages (grandmother – grandfather). A stem and leaf plot is shown below. It does not exhibit substantial skewness or major outliers, so we will conclude that the second condition is not violated.

- -1 0
- -0 855
- -0 43
- 0 144
- 0 8

Step 2: Correct mechanics

A 95% confidence interval for the population mean difference is given by $\bar{d} \pm t^* \left(\frac{s}{\sqrt{n}} \right)$. The critical

value for 95% confidence, based on 10-1=9 degrees of freedom, is $t^*=2.262$. The 95% confidence interval for the population mean difference in ages is

$$-1.8 \pm 2.262 \left(\frac{5.81}{\sqrt{10}}\right)$$

-1.8 ± 4.16
-5.96 to 2.36 years.

Step 3: Interpretation

We can be 95% confident that for the population of high school students the mean difference in the ages of their maternal grandparents (grandmother's age – grandfather's age) is between -5.96 and 2.36 years.

Question 4 (continued)

Part (b):

The difference in age without taking into account which grandparent is older is the absolute value of the differences reported in the table. Therefore, the point estimate for the mean difference is

$$\frac{(1+5+10+8+5+8+3+4+4+4)}{10} = \frac{52}{10} = 5.2 \text{ years.}$$

Scoring

The question is scored in four sections. Section 1 consists of part (a) step 1; section 2 consists of part (a) step 2; section 3 consists of part (a) step 3, and section 4 consists of part (b). Each section is scored as essentially correct (E), partially correct (P), or incorrect (I).

Section 1 is scored as follows:

Essentially correct (E) if the response

- 1. identifies a paired t-interval for a population difference (either by name or by formula),
- 2. mentions the random sample condition, and
- 3. mentions and checks the normality condition with a visual display.

Notes:

- The random sample condition does not need to be explicitly checked because it is noted in the stem.
- A boxplot is acceptable for checking the normality condition, as long as a reasonable statement is made about the result.
- It is acceptable to name the procedure as a one-sample *t*-interval without calling it a paired *t*-interval as long as the response treats the differences as a single sample.

Partially correct (P) if the response includes component (1) and one of components (2) or (3), but not both OR names a two-sample t-interval, mentions the random sample condition, and checks the normality condition appropriate for that procedure by checking each set of ages separately.

Incorrect (I) if the response does not meet the criteria for E or P.

Section 2 is scored as follows:

Essentially correct (E) if the response gives the correct confidence interval. Supporting work is not required, but if included, it must be correct.

Partially correct (P) if the response gives an incorrect but reasonable confidence interval with appropriate supporting work shown, for instance by using the wrong degrees of freedom for the t multiplier;

OR

if the response gives a correct confidence interval with incorrect (but appropriate) supporting work shown.

Incorrect if the response does not meet the criteria for E or P.

Note: If the response in section 1 names a two-sample t-interval, then section 2 is scored E if a correct two-sample interval is computed from the data in section 2.

Question 4 (continued)

Section 3 is scored as follows:

Essentially correct (E) if the response gives a reasonable interpretation of the interval that includes three components:

- 1. Estimating a population mean difference;
- 2. 95% confidence; and
- 3. context (grandparents' ages).

Partially correct (P) if the response gives a reasonable interpretation of the interval that includes component (1) and one of the other two components;

OR

if the response gives a correct interpretation of the confidence <u>level</u> in context, which includes component (1), but does not attempt to interpret the confidence interval.

Incorrect (I) if the response does not meet the criteria for E or P.

Note: If a two-sample *t*-interval is computed in section 2 then section 3 can be scored as E if the interpretation is consistent with the two-sample interval.

Section 4 is scored as follows:

Essentially correct (E) if the response provides a correct numerical answer with supporting work.

Partially correct (P) if the response correctly identifies the mean of the absolute values of the differences as the appropriate point estimate, but does not carry out a correct calculation;

OR

if the response provides correct supporting work, but does not provide a correct numerical answer;

if the response provides a correct numerical answer with no supporting work.

Incorrect (I) if the response does not meet the criteria for E or P.

Question 4 (continued)

Each essentially correct (E) section counts as 1 point, and a partially correct (P) section counts as ½ point.

- 4 Complete Response
- 3 Substantial Response
- 2 Developing Response
- 1 Minimal Response

If a response is between two scores (for example, 2½ points), use a holistic approach to decide whether to score up or down, depending on the strength of the response and communication.

Question 5

Intent of Question

The primary goals of this question are to assess a student's ability to (1) describe a method of obtaining a simple random sample from a population; (2) distinguish between a conditional percent and an unconditional percent; and (3) use the results of a random sample to estimate the number of individuals in a population that share a particular characteristic.

Solution

Part (a):

Number the customers from 1 to 30,000 and then use a calculator or computer to generate 1,000 random numbers between 1 and 30,000 without replacement. If a random number generator is used that generates non-unique numbers, the repeated numbers are ignored until 1,000 unique numbers are obtained. The customers whose numbers correspond to the randomly generated numbers are then selected for the sample.

Part (b):

32.5% should not be used to estimate the proportion of the entire population with power door lock problems because it represents the percent of cars that had door lock problems *given* that the car had some sort of problem. But only $\frac{40}{1000}$ or 4% of the cars in the sample had any sort of problem.

Part (c):

There were 13 customers out of 1,000 in the sample who had a problem with the power door locks, representing 1.3% of the sample. Therefore, a point estimate for the number of cars out of the 30,000 sold that experienced door lock problems is 1.3% of 30,000, which is 390.

Question 5 (continued)

Scoring

Parts (a) and (b) and (c) are each scored as essentially correct (E), partially correct (P), or incorrect (I).

Part (a) is scored as follows:

Essentially correct (E) if the response describes a sampling plan that:

- (1) Uses a random process;
- (2) Results in a sample size of 1,000;
- (3) Guarantees that every possible sample is equally likely;
- (4) Specifies the sample consists of customers.

Partially correct (P) if the response includes two or three of the components required for E.

Note: A response that simply states "Randomly Select" is not providing a plan, so it is scored I.

Incorrect if the response does not meet the criteria for E or P.

Part (b) is scored as follows:

Essentially correct (E) if the response states:

- (1) 32.5% should not be used because it is a conditional percent;
- (2) That the percent is conditioned on the subset of the sample that had experienced a problem.

Note: If a response that is otherwise E implies that the proportion that should be used as the estimate is based on something other than 13 out of 1,000, then the score is reduced to P.

Partially correct (P) if the response only states 32.5% should not be used because it is a conditional percent;

OR

States only that 32.5% is not the percent of the entire sample that had power door lock problems.

Incorrect if the response does not meet the criteria for E or P.

Part (c) is scored as follows:

Essentially correct (E) if the response:

- (1) states that the point estimate is 390 cars;
- (2) provides sufficient details to justify how that number was calculated.

Partially correct (P) if the response states that the point estimate is 390 cars;

OR

Calculates the percent of cars with power door lock problems as 1.3%, or a proportion of 0.013, but does not provide the point estimate for the *number* of cars with the problem.

Incorrect (I) if the response does not meet the criteria for E or P.

Question 5 (continued)

4 Complete Response

Three parts essentially correct

3 Substantial Response

Two parts essentially correct and one part partially correct

2 Developing Response

Two parts essentially correct and no parts partially correct OROne part essentially correct and one or two parts partially correct ORThree parts partially correct

1 Minimal Response

One part essentially correct

OR

No parts essentially correct and two parts partially correct

Question 6

Intent of Question

The primary goals of this question are to assess a student's ability to (1) determine whether a value is an outlier using the definition that an outlier falls more than 1.5 IORs above the upper quartile; (2) find the mean and standard deviation of a difference of two random variables; (3) use data to estimate the mean and standard deviation of a difference of two random variables; (4) calculate a test statistic for a non-standard situation; and (5) use a p-value to make a conclusion in a non-standard situation.

Solution

Part (a):

An outlier at the upper end is defined as a value greater than $O3 + (1.5) \times (O3 - O1)$. For Emily's data, an outlier at the upper end is a value greater than $3.8 + (1.5) \times (3.8 - 2.6) = 3.8 + 1.8 = 5.6$. Because 7.5 is greater than 5.6, 7.5 is identified as an outlier.

Part (b):

(i)
$$E(Y - \overline{X}) = E(Y) - E(\overline{X}) = \mu_Y - \mu_X$$
.

(ii)
$$Var(Y - \overline{X}) = Var(Y) + Var(\overline{X}) = \sigma^2 + \frac{\sigma^2}{n-1} = \sigma^2 \left(1 + \frac{1}{n-1}\right)$$
, so that the standard deviation is $SD(Y - \overline{X}) = \sqrt{\sigma^2 \left(1 + \frac{1}{n-1}\right)}$.

Part (c):

(i) Estimate
$$E(Y - \overline{X}) = \mu_Y - \mu_X$$
 by $y - \overline{x} = 7.5 - 3.171 = 4.329$.

(ii) To estimate
$$SD(Y - \overline{X})$$
, first estimate $Var(Y - \overline{X}) = \sigma^2 \left(1 + \frac{1}{n-1}\right)$ by $S_{Y - \overline{X}}^2 = S^2 \left(1 + \frac{1}{n-1}\right) = (0.821)^2 \left(1 + \frac{1}{n-1}\right) \approx 0.7021$. Then the estimate of the standard definition

$$s_{Y-\bar{X}}^2 = s^2 \left(1 + \frac{1}{n-1}\right) = (0.821)^2 \left(1 + \frac{1}{24}\right) \approx 0.7021$$
. Then the estimate of the standard deviation is $s_{Y-\bar{X}} = \sqrt{0.7021} \approx 0.838$.

Part (d):

The appropriate test statistic has the point estimate of $\mu_Y - \mu_X$ in the numerator, and the point estimate of $SD(Y - \bar{X}) = \sqrt{\sigma^2 \left(1 + \frac{1}{n-1}\right)}$ in the denominator. The test statistic is

$$\frac{4.329}{\sqrt{0.7021}} = \frac{4.329}{0.838} = 5.17.$$

Part (e):

Because the p-value is so small, there is convincing statistical evidence that $\mu_Y \neq \mu_X$. Therefore, the value of 7.5 appears to come from a population with a different mean than the mean of the population from which the remaining 24 observations came. There is very strong evidence that the value of 7.5 is an outlier using the new method for determining outliers.

Question 6 (continued)

Scoring

This question is scored in four sections. Section 1 consists of part (a), section 2 consists of parts (b) and (c), section 3 consists of part (d) and section 4 consists of part (e). Each section is scored as essentially correct (E), partially correct (P), or incorrect (I).

Section 1 is scored as follows:

Essentially correct (E) if the response includes:

- 1. work supporting the calculation of the correct upper endpoint using the given method;
- 2. a comparison of 7.5 to the calculated upper endpoint; and
- 3. a consistent conclusion regarding whether or not 7.5 is an outlier.

Partially correct (P) if the response includes two of the three components required for E.

Incorrect (I) if the response does not meet the criteria for E or P.

Section 2 is scored as follows:

Essentially correct (E) if the response includes the following four components:

- 1. The correct formula for the mean in part (b);
- 2. The correct formula for the standard deviation in part (b);
- 3. The point estimate for the mean in part (c) is correct, or consistent with the formula in part (b); and
- 4. The point estimate for the standard deviation in part (c) is correct, or consistent with the formula in part (b).

Partially correct (P) if the response includes two or three of the four components required for E.

Incorrect if the response does not meet the criteria for E or P.

Section 3 is scored as follows:

Essentially correct (E) if the response uses the results from part (c) correctly to calculate a test statistic and provides appropriate supporting work.

Partially correct (P) if the response provides a value of the test statistic that is consistent with part (c), but without appropriate supporting work shown;

OR

if the response gives a consistent numerator for the test statistic, and the denominator makes use of the point estimate of the standard deviation from part (c), but with a mistake (e.g., dividing the denominator by $\sqrt{24}$ or $\sqrt{25}$).

Incorrect if the response does not meet the criteria for E or P.

Note: In sections 1, 2, and 3, minor arithmetic or transcription errors should be ignored.

Question 6 (continued)

Section 4 is scored as follows:

Essentially correct (E) if the response: (1) provides the correct conclusion about the population means by stating that there is convincing evidence that they differ; (2) links the conclusion to the *p*-value provided; and (3) concludes that there is strong evidence that 7.5 is an outlier.

Notes:

- If an incorrect value is computed for the test statistic in part (d) the response must still base the conclusion about population means on the *p*-value provided in part (e), and not on the test statistic computed in part (d).
- The response must base the conclusion on whether or not 7.5 is an outlier on the information in part (e), not on the result from part (a).

Partially correct (P) if the response:

provides the correct conclusion about the population means by stating there is convincing evidence that they differ, justified by correct linkage to the *p*-value, but does not make a conclusion about 7.5 being an outlier;

OR

provides a statement as to whether or not 7.5 is an outlier consistent with a conclusion about the population means, but does not link the conclusion concerning the population means to the p-value provided.

Incorrect (I) if the response does not meet the criteria for E or P.

Each essentially correct (E) section counts as 1 point, and a partially correct (P) section counts as ½ point.

- 4 Complete Response
- 3 Substantial Response
- 2 Developing Response
- 1 Minimal Response

If a response is between two scores (for example, 2½ points), use a holistic approach to decide whether to score up or down, depending on the strength of the response and communication.