ESSENTIAL QUESTION

How can you graph a rational function?

EXAMPLE 1 Rewrite a Rational Function to Identify Asymptotes

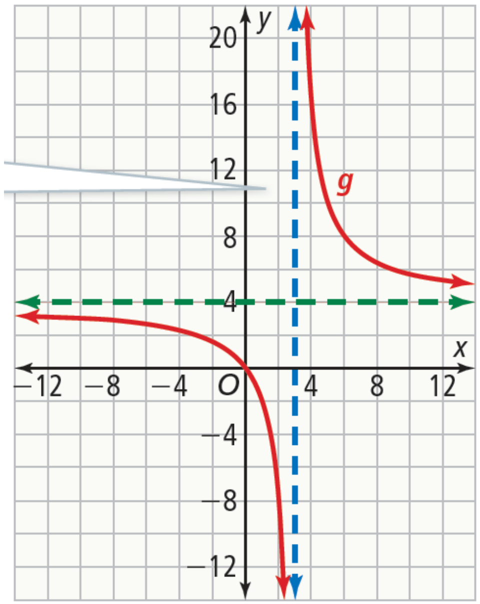
How is the quotient $g(x) = \frac{4x}{x-3}$ related to the reciprocal function, $f(x) = \frac{1}{x}$? Sketch the graph.

Use long division to write the rational expression in the form $\frac{a}{x-h} + k$.

$$\begin{array}{r} 4 \\ x-3\overline{)4x} \\ -(4x-12) \\ \hline 12 \end{array}$$

Therefore, $g(x) = 4 + \frac{12}{x-3}$.

In terms of $f(x)$, $g(x) = 12 \bullet f(x-3) + 4$.
So, the graph of g will be the graph of f translated and stretched vertically.



EXAMPLE 1 Rewrite a Rational Function to Identify Asymptotes

Try It!

1. Use long division to rewrite each rational function. Find the asymptotes of *f* and sketch the graph.

a. $f(x) = \frac{6x}{2x + 1}$

EXAMPLE 1 Rewrite a Rational Function to Identify Asymptotes

Try It!

1. Use long division to rewrite each rational function. Find the asymptotes of *f* and sketch the graph.

b. $f(x) = \frac{x}{x - 6}$

CONCEPT Rational Functions

Just as a rational number is a number that can be expressed as the ratio of two integers, a **rational expression** is an expression that can be expressed as the ratio of two polynomials, such as $\frac{P(x)}{Q(x)}$.

A **rational function** is any function defined by a rational expression, such as $R(x) = \frac{P(X)}{Q(X)}$. The domain of $R(x)$ is all values of x for which $Q(x) \neq 0$.

The function $g(x) = \frac{4x}{x - 3}$ is a rational function.

CONCEPTUAL UNDERSTANDING

EXAMPLE 2 Find Multiple Vertical Asymptotes of a Rational Function

What are the vertical asymptotes for the graph of $f(x) = \dfrac{3x - 2}{x^2 + 7x + 12}$?

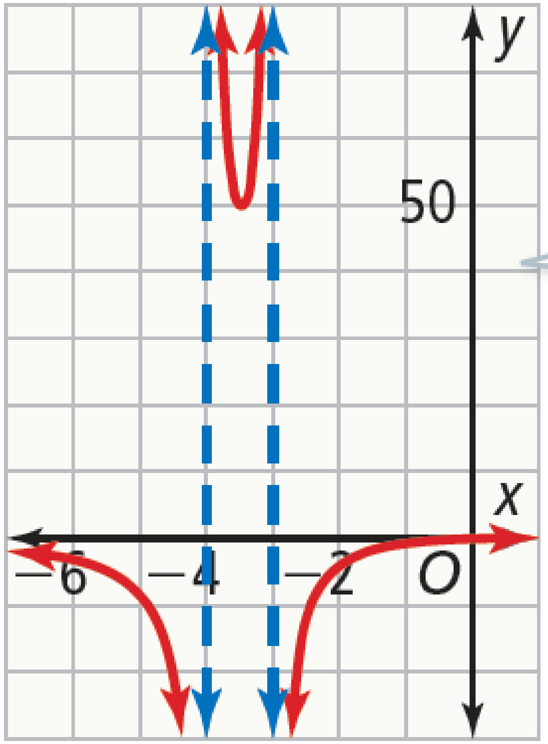
Vertical asymptotes can occur at the *x*-values where the function is undefined.
Determine where the denominator of the rational function is equal to 0.

$$x^2 + 7x + 12 = 0$$

$$(x + 3)(x + 4) = 0$$

$$x + 3 = 0 \text{ or } x + 4 = 0$$

$$x = -3 \quad \text{or} \quad x = -4$$



EXAMPLE 2 Find Multiple Vertical Asymptotes of a Rational Function

Try It!

2. Find the vertical asymptotes for each function. Graph the function to check your work.

a. $g(x) = \dfrac{5x}{x^2 - x - 6}$

EXAMPLE 2 Find Multiple Vertical Asymptotes of a Rational Function

Try It!

2. Find the vertical asymptotes for each function. Graph the function to check your work.

b. $h(x) = \dfrac{7 - x}{(x - 5)(x + 1)(x + 3)}$

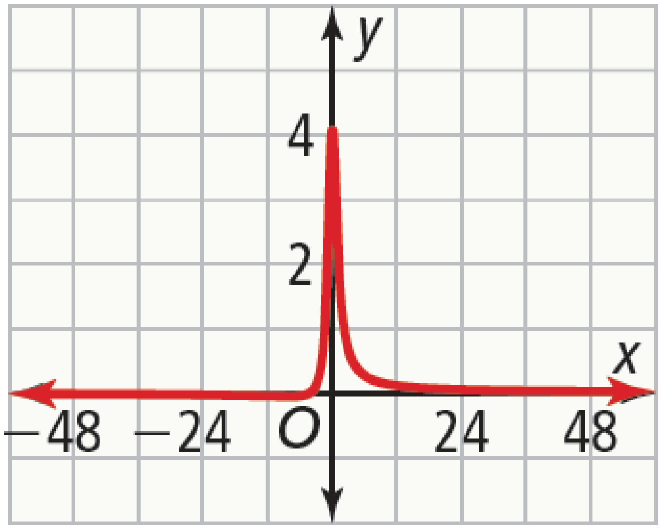
CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Find Types of Horizontal Asymptotes

There are 3 cases to consider, below is case #1

Consider $g(x) = \dfrac{x+4}{x^2+1}$.

As $|x| \to \infty$ the value of the denominator gets very large in relation to the numerator. The value of the function gets closer and closer to 0.



CONCEPTUAL UNDERSTANDING

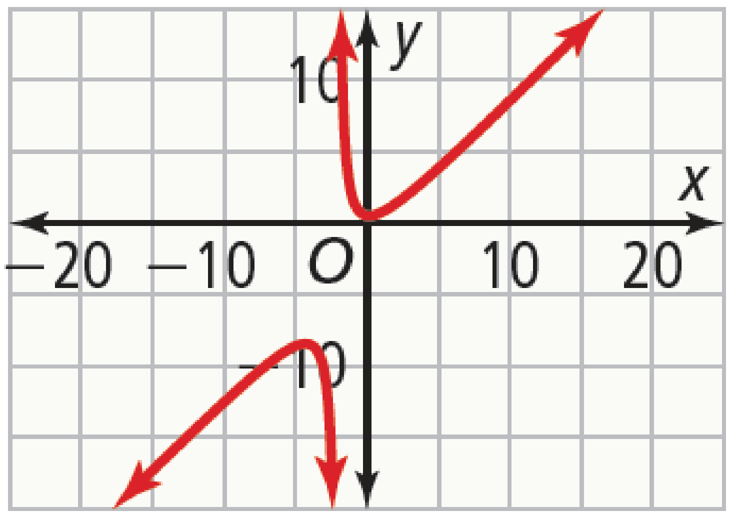
EXAMPLE 3 Find Types of Horizontal Asymptotes

What are the horizontal asymptotes for the graph

Case 2: When the degree of the numerator is greater than the degree of the denominator,there is no horizontal asymptote.

Consider $h(x) = \dfrac{x^2 + 1}{x + 2}$.

As $|x| \to \infty$, the value of the numerator gets very large in relation to the denominator, so $y \to \pm\infty$.



CONCEPTUAL UNDERSTANDING

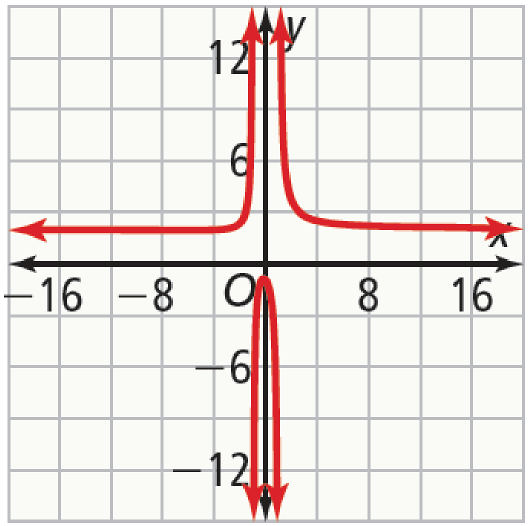
EXAMPLE 3 Find Types of Horizontal Asymptotes

Case 3; When the degree of the numerator is equal to the degree of the denominator, the horizontal asymptote is at $y = \frac{p}{q}$ where $\frac{p}{q}$ is the ratio of the leading coefficients.

Consider $k(x) = \frac{2x^2 + x + 1}{x^2 - 1}$.

Using long division, you can rewrite this as $k(x) = 2 + \frac{x + 3}{x^2 - 1}$.

As $|x| \to \infty$, the value of the rational expression approaches 0, so the value of the function approaches 2.



$k(x)$ has a horizontal asymptote at $y = \frac{2}{1}$.

CONCEPTUAL UNDERSTANDING

EXAMPLE 3 Find Types of Horizontal Asymptotes

What are the horizontal asymptotes for the graph $f(x) = \dfrac{3x - 2}{x^2 + 7x + 12}$?

For the function $k(x) = \dfrac{3x - 2}{x^2 + 7x + 12}$, the degree of the numerator is less than the degree of the denominator.

It has a horizontal asymptote at $y = 0$.

EXAMPLE 3 Find Types of Horizontal Asymptotes

Try It!

3. What are the horizontal asymptotes of the graph of each function?

a. $g(x) = \dfrac{2x^2 + x - 9}{2x - 8}$

EXAMPLE 3 Find Types of Horizontal Asymptotes

Try It!

3. What are the horizontal asymptotes of the graph of each function?

b. $h(x) = \dfrac{x^2 + 5x + 4}{3x^2 - 12}$

EXAMPLE 3 Find Types of Horizontal Asymptotes

Try It!

3. What are the horizontal asymptotes of the graph of each function?

c. $k(x) = \dfrac{x}{(2x-1)(x+6)}$

EXAMPLE 4 Graph a Function of the Form $\frac{ax+b}{cx+d}$

What is the graph of the function $f(x) = \frac{2x+1}{3x-4}$?

Step 1 Determine if there is a vertical asymptote.

$$3x - 4 = 0$$
$$3x = 4$$
$$x = \frac{4}{3}$$

By setting the denominator equal to zero, you can find the values of x for which the function is undefined.

At $x = \frac{4}{3}$, the value of the denominator is 0.

There is a vertical asymptote at $x = \frac{4}{3}$.

Step 2 Determine if there is a horizontal asymptote.

$$y = \frac{2}{3}$$

Since the degrees of the numerator and denominator are the same, use the ratio of the leading coefficients to find the horizontal asymptote.

As $x \to \pm\infty$, $y \to \frac{2}{3}$.

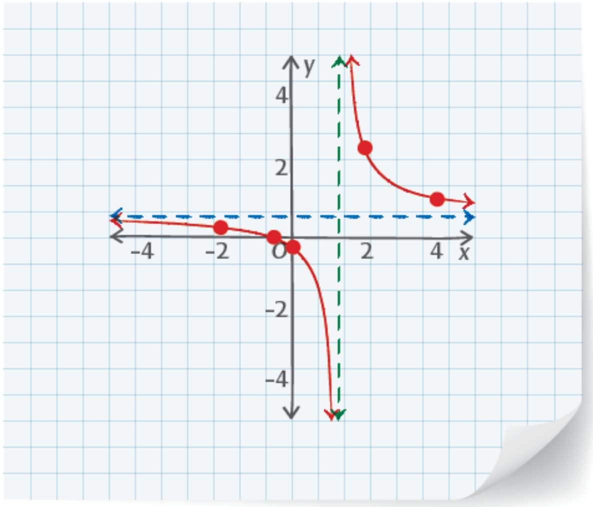
There is a horizontal asymptote at $y = \frac{2}{3}$.

EXAMPLE 4 Graph a Function of the Form $\frac{ax+b}{cx+d}$

What is the graph of the function $f(x) = \frac{2x+1}{3x-4}$?

Step 3 Graph the function.

- Indicate the asymptotes.
- Choose x-values on either side of the vertical asymptote, and evaluate the function for those x-values to create coordinate points to determine the shape near the asymptotes. For example, $f(2) = 2.5$, $f(4) = 1.125$, $f(0) = -0.25$, $f(-0.5) = 0$, and $f(-2) = 0.3$.
- Plot the points and sketch the graph through the points.



EXAMPLE 4 Graph a Function of the Form $\frac{ax+b}{cx+d}$

Try It!

4. Graph each function.

a. $f(x) = \dfrac{4x-3}{x+8}$

b. $g(x) = \dfrac{3x+2}{x-1}$

APPLICATION

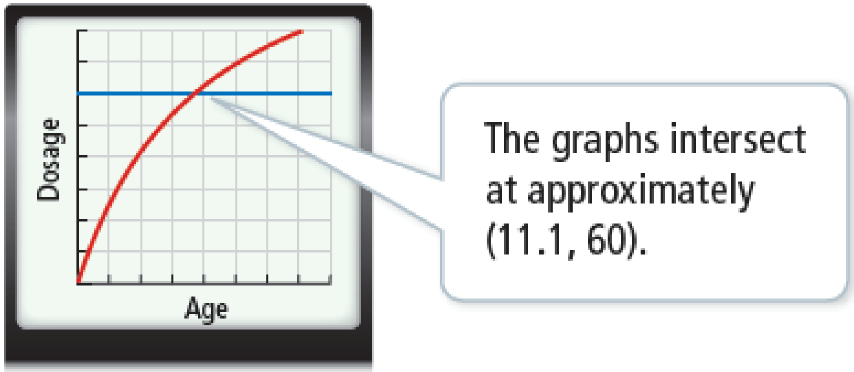
EXAMPLE 5 Use a Rational Function Model

Formulate

The adult dosage is 125 mcg, so the dosage for a child a years old is $D(a) = \dfrac{125a}{a + 12}$.
A child can receive the medication if $D(a) \geq 60$.

Compute

Graph the function using technology. Then graph the line $y = 60$. You can solve the inequality $D(a) \geq 60$ by finding the intersection point of the two graphs.



x scale: 3 *y* scale: 10

The solution to the inequality $D(a) \geq 60$ is approximately $a \geq 11.1$.

Interpret

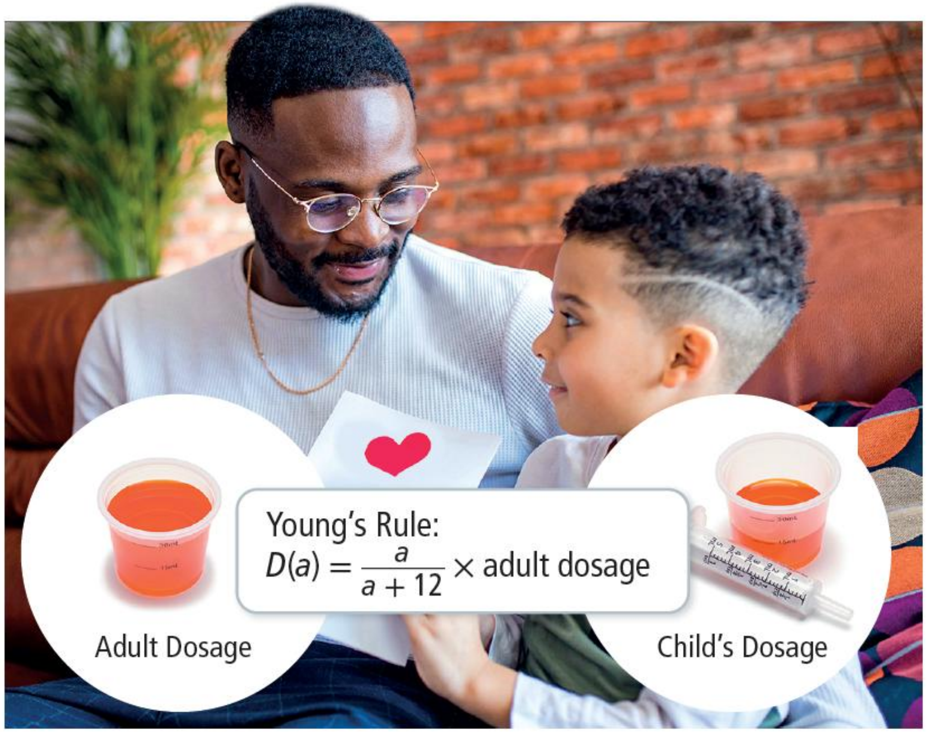
A child 11.1 years old or older will be able to receive the medication.

APPLICATION

EXAMPLE 5 Use a Rational Function Model

A pediatric doctor may need to administer medication without knowing a child's weight. Young's Rule can be used to calculate a child's dosage *D*(*a*) given their age a and the adult dosage.

A doctor has 60 mcg of a medication. What is the youngest a child can be to receive this dose of medication if the adult dosage is 125 mcg?



EXAMPLE 5 Use a Rational Function Model

Try It!

5. Use Young's Rule to calculate the minimum age a child can be if a doctor has 85 mcg of a medication, and the adult dosage is 200 mcg.

EXAMPLE 6 Graph a Rational Function that Crosses a Horizontal Asymptote

What is the graph of $f(x) = \dfrac{4x^2 - 9}{x^2 + 2x - 15}$?

Step 1 Determine if there are any vertical asymptotes.

$$x^2 + 2x - 15 = 0$$

$$(x + 5)(x - 3) = 0$$

$$x + 5 = 0 \text{ or } x - 3 = 0$$

$$x = -5 \quad \text{or} \quad x = 3$$

Neither of these values makes the numerator equal to 0, but they each make the denominator equal to 0.

There are vertical asymptotes at $x = -5$ and $x = 3$.

EXAMPLE 6 Graph a Rational Function that Crosses a Horizontal Asymptote

Step 2 Determine if there is a horizontal asymptote.

The degree of the numerator equals the degree of the denominator. Using long division you can write $f(x)$ as $f(x) = 4 - \frac{8x + 51}{x^2 + 2x - 15}$. As $x \to \infty$ or $x \to -\infty$, the rational expression goes to 0, and $f(x) \to 4$.

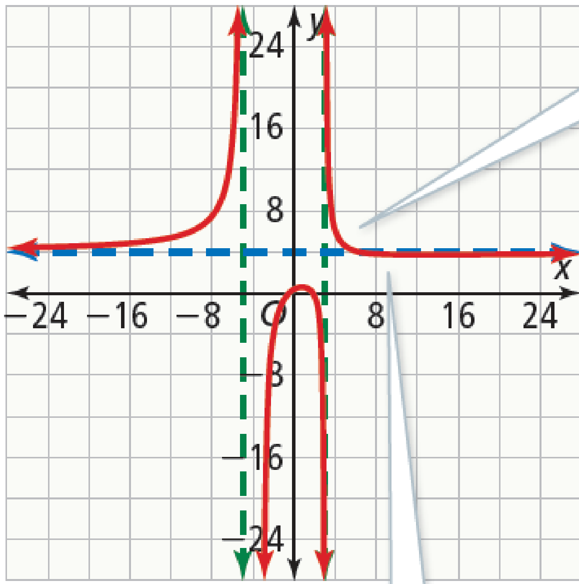
There is a horizontal asymptote at $y = 4$.

COMMUNICATE PRECISELY

When graphing, it is important to include both horizontal and vertical asymptotes to clearly identify the graph's behavior near them.

EXAMPLE 6 Graph a Rational Function that Crosses a Horizontal Asymptote

Step 3 Sketch the graph by first graphing the asymptotes.



Unlike a vertical asymptote, which indicates a domain restriction, horizontal asymptotes may be crossed. The graph crosses the asymptote at (6.375, 4).

When $x > 6.375$, the graph drops below the horizontal asymptote $y = 4$, and then, as $x \to \infty$, approaches $y = 4$ from below. As $x \to -\infty$, the graph approaches $y = 4$, from above.

EXAMPLE 6 Graph a Rational Function that Crosses a Horizontal Asymptote

Try It!

6. Identify the asymptotes and sketch the graph of

$$g(x) = \frac{x^2 - 5x + 6}{2x^2 - 10}.$$

CONCEPT SUMMARY

Graphing Rational Functions

RATIONAL FUNCTION	A function that is expressible as a fraction with polynomials in the numerator and the denominator	
ASYMPTOTES	**Vertical** Vertical asymptotes are guides for the behavior of a graph as it approaches a vertical line. • The line $x = a$ is a vertical asymptote of $\frac{P(x)}{Q(x)}$, if $Q(a) = 0$ and $P(a) \neq 0$. • The up or down behavior of the function as it approaches the asymptote can be determined by substituting values close to a on either side of the asymptote.	**Horizontal** Horizontal asymptotes are guides for the end behavior of a graph as it approaches a horizontal line. If the degree of the numerator is • less than the degree of the denominator, the horizontal asymptote is at $y = 0$. • greater than the denominator, there is no horizontal asymptote. • equal to the degree of the denominator, set y equal to the ratio of the leading coefficients. The graph of this line is the horizontal asymptote.

CONCEPT SUMMARY

Graphing Rational Functions

ALGEBRA

$$f(x) = \frac{8x - 3}{4x + 1}$$

Vertical Asymptote: Let $4x + 1 = 0$ and solve.

$$x = -\frac{1}{4}$$

Horizontal Asymptote: Find the ratio of the leading coefficients $\left(\frac{8}{4}\right)$.

$$y = 2$$

GRAPH