Linear Algebra Study Guide - Day 1

Axler Sections 1.A and 1.B

Study Notes

Reading Summary

Section 1.A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Key Definitions

Definition 0.1 (Complex Numbers)**.** The set of complex numbers $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ with addition and multiplication defined by:

- $(a + bi) + (c + di) = (a + c) + (b + d)i$
- $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

Definition 0.2 (List)**.** A **list** of length n is an ordered collection of n elements, written $(x_1, \ldots, x_n)$. Two lists are equal if and only if they have the same length and the same elements in the same order.

Definition 0.3 ($\mathbb{F}^n$)**.** For $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, we define $\mathbb{F}^n$ as the set of all lists of length n of elements of $\mathbb{F}$:

$$\mathbb{F}^n = \{(x_1, \ldots, x_n) : x_j \in \mathbb{F} \text{ for } j = 1, \ldots, n\}$$

Definition 0.4 (Addition in $\mathbb{F}^n$)**.** Addition in $\mathbb{F}^n$ is defined by adding corresponding coordinates:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n)$$

Definition 0.5 (Scalar Multiplication in $\mathbb{F}^n$)**.** For $\lambda \in \mathbb{F}$ and $(x_1, \ldots, x_n) \in \mathbb{F}^n$:

$$\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

Key Properties

- Complex arithmetic satisfies commutativity, associativity, existence of identities (0 for addition, 1 for multiplication), existence of inverses, and distributivity
- The zero vector in $\mathbb{F}^n$ is $0 = (0, \ldots, 0)$
- The additive inverse of $x = (x_1, \ldots, x_n)$ is $-x = (-x_1, \ldots, -x_n)$
- Addition in $\mathbb{F}^n$ is commutative and associative
- Scalar multiplication satisfies: $(ab)x = a(bx)$, $1x = x$, $\lambda(x + y) = \lambda x + \lambda y$, and $(a + b)x = ax + bx$

Section 1.B: Definition of Vector Space

Key Definitions

Definition 0.6 (Vector Space)**.** A **vector space** is a set V along with an addition on V and a scalar multiplication on V such that the following properties hold:

- **Commutativity:** $u + v = v + u$ for all $u, v \in V$
- **Associativity:** $(u + v) + w = u + (v + w)$ and $(ab)v = a(bv)$ for all $u, v, w \in V$ and all $a, b \in \mathbb{F}$
- **Additive identity:** There exists an element $0 \in V$ such that $v + 0 = v$ for all $v \in V$
- **Additive inverse:** For every $v \in V$, there exists $w \in V$ such that $v + w = 0$
- **Multiplicative identity:** $1v = v$ for all $v \in V$
- **Distributive properties:** $a(u + v) = au + av$ and $(a + b)v = av + bv$ for all $a, b \in \mathbb{F}$ and all $u, v \in V$

Definition 0.7 (Vectors and Points)**.** Elements of a vector space are called **vectors** or **points**.

Key Theorems and Results

Theorem 0.1 (Uniqueness of Additive Identity)**.** A vector space has a unique additive identity.

Theorem 0.2 (Uniqueness of Additive Inverse)**.** Every element in a vector space has a unique additive inverse.

Theorem 0.3. For any $v \in V$ (where V is a vector space):

- $0v = 0$ (where the first 0 is the scalar and the second is the zero vector)
- $(-1)v = -v$ (the additive inverse of v)

Important Examples

- $\mathbb{F}^n$ with componentwise addition and scalar multiplication
- $\mathbb{F}^\infty$ (infinite sequences) with componentwise operations
- $\mathbb{F}^S$ (functions from a set S to $\mathbb{F}$) with pointwise operations
- The set $\mathcal{P}_m(\mathbb{F})$ of polynomials with coefficients in $\mathbb{F}$ and degree at most m

Exercises

Exercise 1.A.3

Exercise 1. Find two distinct square roots of i.

Solution. □

Exercise 1.A.9

Exercise 2. Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$ for all $\lambda, \alpha, \beta \in \mathbb{C}$.

Solution. □

Exercise 1.B.2

Exercise 3. Suppose $a \in \mathbb{F}$, $v \in V$, and $av = 0$. Prove that $a = 0$ or $v = 0$.

Solution. □