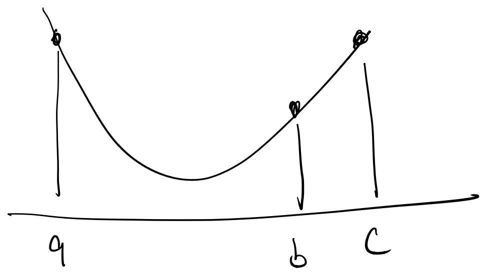
Univariate

Bracketing for optimization is more complicated than for roots



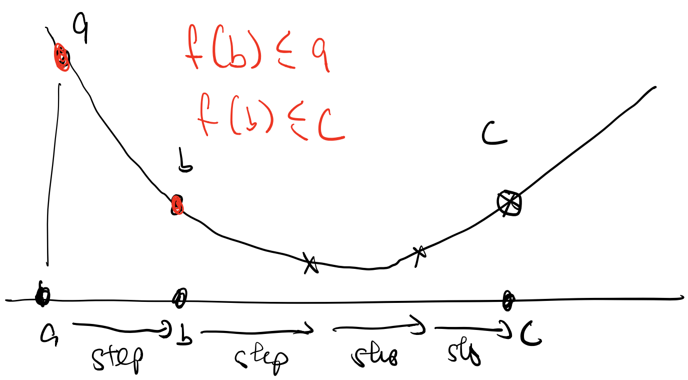
$a < b < c$

$f(b) \leq f(a)$

$f(b) \leq f(c)$

$\underbrace{\text{Bisection}}_{\text{Root finding}} \longrightarrow \underbrace{\text{Interval elimination}}_{\text{optimization}}$

How do we find a bracket?



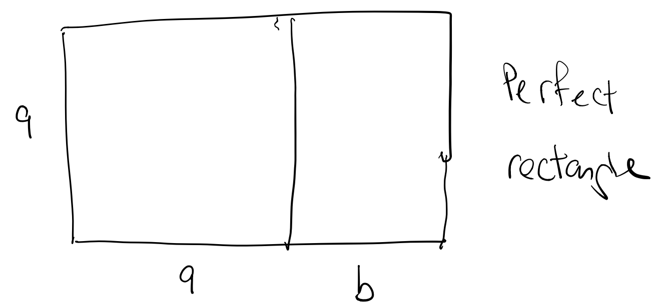
Choose some "big" step s.t. $f(b)$

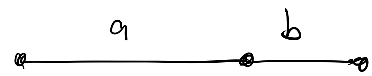
Go downhill $\rightarrow$ b $\rightarrow$ Check $f(b) < f(a)$

Continue in same direction until $f(c) > f(b)$

Golden section φ

Bisection





$$\frac{a+b}{a} = \frac{a}{b}$$

$$\frac{a+b}{a} = \boxed{\frac{a}{b}} = \varphi \text{ (Golden section)}$$

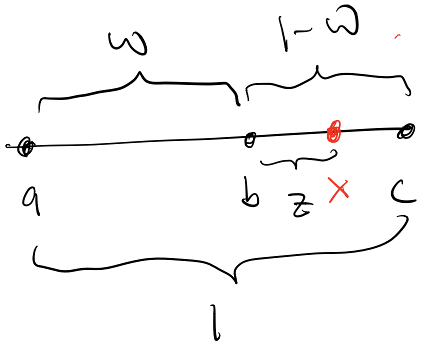
What is value of φ?

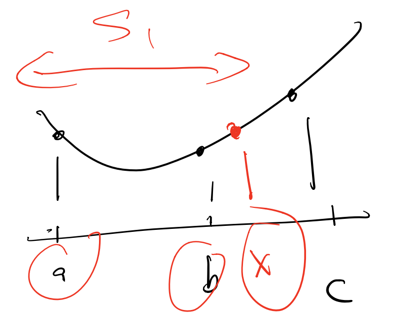
$$\varphi = \frac{a+b}{a} = \frac{a}{a} + \frac{b}{a} = 1 + \frac{b}{a} = 1 - \frac{1}{\varphi}$$

$$\varphi = 1 - \frac{1}{\varphi} \qquad \varphi = \frac{1+\sqrt{5}}{2}$$

$\underline{1.608.}$

Golden section search





$f(x) > f(b)$

① (a, b, x)

① $s_1 = w + z$

② $s_2 = 1 - w$

No information, so let

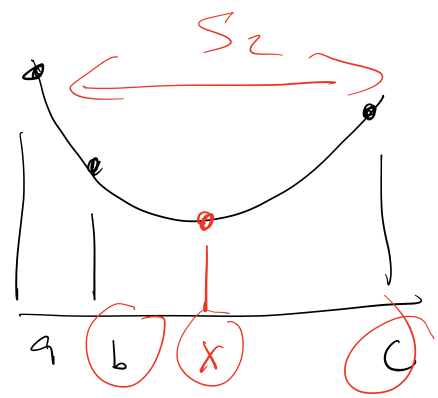
$s_1 = s_2$

(A) $\boxed{z = 1 - 2w}$

(B) self similarity

$$\boxed{\frac{z}{1-w} = w}$$

$$\boxed{z = w - w^2}$$



② $f(x) < f(b)$

(b, x, c)

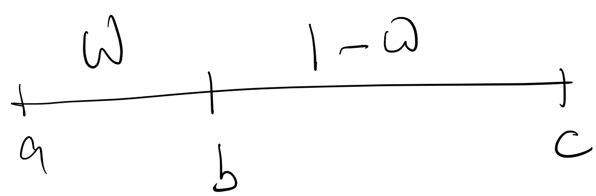
$$1-2w = w - w^2$$

$$w^2 - 3w + 1 = 0$$

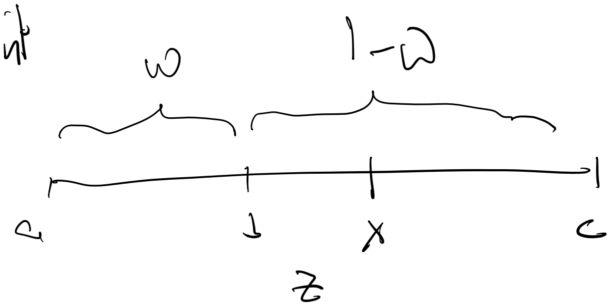
$$w = \frac{3 \pm \sqrt{9-4}}{2}$$

$$= \boxed{\frac{3-\sqrt{5}}{2}}$$

So optimal bracketing interval is



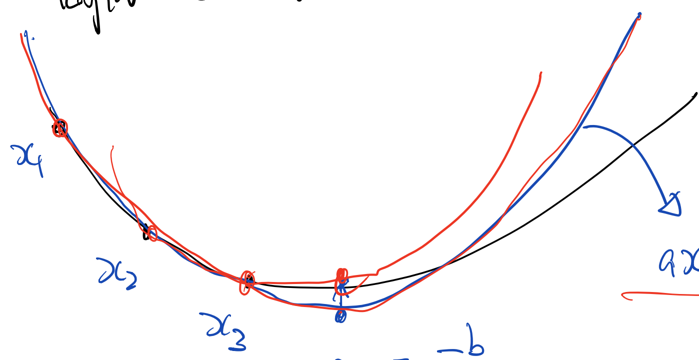
Next point



$$z = \frac{w}{1-w} \quad \text{from } b$$

Interpolation

Taylor → near minimum function is ≈ quadratic



$$x_4 = \frac{-b}{2a}$$

$ax^2 + bx + c$

a, b, c

Now interpolate using x_2, x_3, x_4. and so on

Note How to do quadratic interpolation?

Recall from linear algebra

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} c \\ b \\ a \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

$$-\frac{b}{2a}$$

Golden section

Quadratic interpolation

Newton method

Newton Method Linear Approx from Taylor

$$f(x+h) = f(x) + h f'(x)$$

Assume $x+h = x^*$

Differentiate, $f'(x+h) = f'(x) + h f''(x)$

$\overset{\shortparallel}{0}$

$$\boxed{h = -\frac{f'(x)}{f''(x)}}$$

But $h = x_{k+1} - x_k$

So $\boxed{x_{k+1} = x_k - \dfrac{f'(x_k)}{f''(x_k)}}$

Alternative

Newton-Raphson $x_{k+1} = x_k - \dfrac{g(x)}{g'(x)}$ } zeros of g.

But we want to find zeros of the derivative

$\Rightarrow$ $g(x) = f'(x)$

$g'(x) = f''(x)$

So $x_{k+1} = x_k - \dfrac{f'(x)}{f''(x)}$