① Column space $C(A)$

$3x_1 + 2x_2 + x_3 = 7$

$x_1 + 4x_2 + x_3 = 10$

A is $m \times n$

		# of typical solutions
$m = n$	0, 1	
$m > n$	0	
$\boxed{n > m}$	$\boxed{\infty}$	

$$\underbrace{\begin{bmatrix} \begin{pmatrix}3\\1\end{pmatrix}, \begin{pmatrix}2\\4\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix} \end{bmatrix}}_{\substack{A \\ X \\ \text{column vectors}}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\substack{x \\ \beta \\ \text{weights}}} = \underbrace{\begin{bmatrix} 7 \\ 10 \end{bmatrix}}_{\substack{b \\ y \\ \text{vector that we want to approximate}}}$$

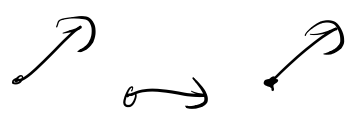
$Ax = b$

column vector → A ; weights → x ; vector we want to approximate → b

$$\underline{x_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

Linear combination of column vectors of A

redundant



$\mathbb{R}^2$

$\{0\}$ / $\mathbb{R}^2$

$\{ a_1, a_2, a_3 \}$

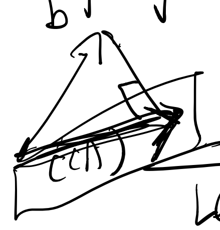
$c_1 a_1 + c_2 a_2 + c_3 a_3$

↓

Column space of A

$C(A)$

$$\begin{bmatrix} 1 & 7 \\ 3 & 11 \\ 2 & 9 \\ 7 & f \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$



Least squares

(2) Null space $N(A)$ (kernel)

$n > m$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$n = 3 \quad m = 2$

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$\mathbb{R}^2$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow$ solution x_p

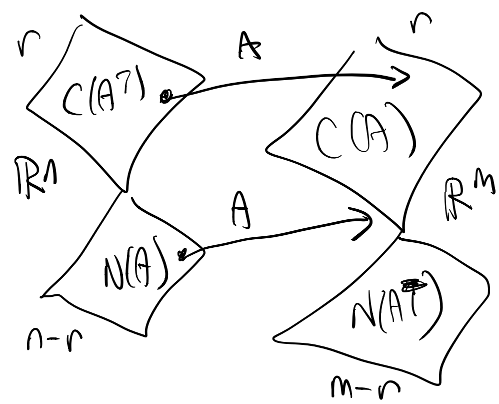
$C(A)$

$N(A)$

$\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$

$C(A^T)$

$N(A^T)$



$Ax = 0 \rightarrow$ null space

$\downarrow$

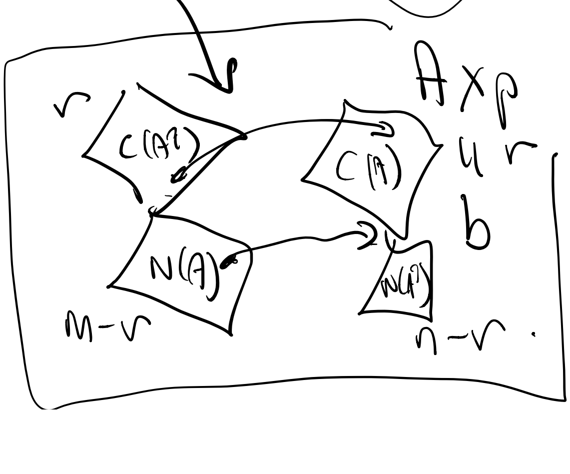
$x_{ns} \in N(A)$ / kernel.

Ax $\quad \mathbb{R}^m$

$m \times n \;\; n \times 1$

$$A(x_p + c\,x_{ns}) = b \Big\} \infty$$

SVD



$A x_p + c\,\underline{A x_{ns}} \qquad \{x : Ax = 0\}$

$\downarrow \qquad\quad 0 \qquad\qquad n \times 1 \qquad \mathbb{R}^n.$

$C(A) \qquad C(A^T)$

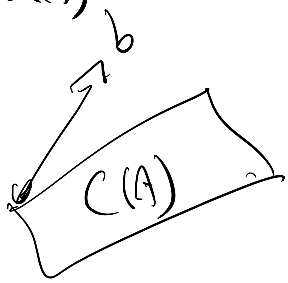
$N(A) \qquad N(A^T)$

(3) Particular & special solutions $(n > m)$

$C(A) \rightarrow$ Basis $\{v_1 \;\; v_2\}$

$N(A) \rightarrow$

$\rightarrow$ Gaussian elimination



linear combination

linear independence

$Ax \rightarrow \underbrace{x_1 a_1 + x_2 a_2 + \dots}$

(4) Gaussian elimination   REF
RREF $\boxed{\text{Backsubstitution}}$

$$\text{REF} \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} \qquad A = \left\{ \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \right\}$$

reduced echelon form

pivots

$$2x_2 = b \qquad \begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\text{RREF} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \text{basis vectors for } N(A)$$

$$x_1 + 3x_2 = a$$

$$x_1 = a - 3x_2 = a - 3b$$

Gaussian elimination $\rightarrow$ elementary row operations

$\downarrow$

Don't change row space

$C(A^T)$

$$c_1 \begin{bmatrix} a \\ b \end{bmatrix} + c_2 \begin{bmatrix} c \\ d \end{bmatrix}$$

$$d_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + d_2 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$[\ \ ] \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

(5) Elementary operations preserve row space

$\boxed{\triangle \text{ matrices}}$

$$\underbrace{L_1^{-1} L_2^{-1}}_{} L_2 L_1 A = U \quad (L_1^{-1} L_2^{-1}) \rightarrow L$$

$$L = L_2^{-1} L_1^{-1}$$

$$A = LU$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 5 \\ 4 & 6 & 8 \end{bmatrix}$$

Right $\triangle$

L is unit lower triangular

$A = LDL^T$

$L_2 L_1 A = U$

unit left triangle $\begin{bmatrix} 1 & 0 \\ ** & 1 \end{bmatrix}$

lower unit $\triangle$

$$\underbrace{A = LL^T}_{\text{Cholesky}} \left(\text{if symmetric PD}\right)$$

$A = LU$ — upper $\triangle$; unit lower triangular

$$\begin{bmatrix} 5 & 3 \\ 0 & 4 \end{bmatrix} \rightarrow D, \; U \text{ (unit)}$$

LU
LDU
CCT

$A = LDU$ — symmetric PD

$$L \underbrace{\begin{bmatrix} \sqrt{5} & 0 \\ 0 & \sqrt{4} \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} & \\ & \end{bmatrix}}_{U}$$

$$\underbrace{L D^{1/2}}_{C} \underbrace{D^{1/2} U}_{C^T}$$

$$\underbrace{A = CC^T}_{}$$

Cholesky decomposition

Topics

- Solving equations
- Fundamental subspaces
- Gaussian elimination
- LU
  LDL^T
  LL^T