Lesson 1: Introducing Notation

Notes

Book acknowledgment:

Goals

- Learn (or remember) new notation

1 Undirected Graphs

Let $G = (V, E)$ be an **undirected graph** in which V is a finite set of nodes (or vertices) and E is a collection of unordered pairs (edges) of elements of V.

We will typically let

- $n := |V|$
- $m := |E|$

Examples?

In some cases, G has *weights* on edges and/or nodes. For edges, $c_{ij},\ \forall (i,j) \in E$. For nodes, $w_i,\ \forall i \in V$.

We will let $V = \{1, 2, \dots, n\}$ and the edges in E have the form $(i, j),\ foralli, j \in V$. For the sake of simplicity, we assume that edges (i, j) and (j, i) are equivalent. Thus, we assume that E only consists of edges (i, j) for which $i < j$.

Two nodes i and j are said to be *adjacent* when there exists an edge (i, j) that connects them. Two edges are referred to as *adjacent* if they have an edge in common.

The number of edges incident on a vertex v is called the degree of the vertex, and is usually denoted by $d_G(v)$.

Graph G is *complete* if it contains all possible edges. (When $E = \{(i, j) : i, j \in V, i < j\}$.

Graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. Additionally, for G' to be a subgraph of G, then for all edges $(i, j) \in E'$, then both vertices i and j must belong to V'.

A *path* in G is a sequence of consecutive edges $e_1, e_2, \dots, e_k \in E$, in which $e_1 = (v_1, v_2),\ e_2 = (v_2, v_3), \dots, e_k = (v_k, v_{k+1})$. The path connects v_1 and v_{k+1}, visiting the intermediate vertices $v_2,\ v_3, \dots, v_k$. The path is a *cycle* if $v_1 = v_{k+1}$. A path is *elementary* if no edge is used twice, and the path is referred to as *simple* if no node is visited twice.

Vertex v is *connected* to vertex w if there exists a path connecting them. In undirected graphs, this definition is symmetrical. If v is connected to w, then w is also connected to v. A graph G is referred to as *connected* if all vertices v and w in V are connected.

A *cut* in G is a set of edges of the type:

$$\gamma_G(S) := \{(i,j) \in E : |S \cap \{i,j\}| = 1\}$$

in which S is the subset of vertices that induces the cute (i.e., the cut contains all ecdges with one endpoint in S and the other in $V \setminus S$.

When can easily verify that G is connected if and only if $\gamma(S) \neq \emptyset$ for all $\emptyset \subset S \subset V$. Given two vertices s and t, there exist k edge-disjoint paths connecting them if and only if $|\gamma(S)| \geq k$ for all $S \subset V$ such that $s \in S$, $t \notin S$.

A partial graph $G' = (V, E'$ of G is called a *forest* if it is *acyclic*–it does not contain a cycle. A forest is *maximal* if every edge in $E \setminus E'$ forms a cycle with the edges in E'. Therefore, G' and G have the same connected components.

A maximal connected forest, if it exists, is called a *spanning tree*. Every tree has exactly $|V| - 1$ edges. Graph G contains a tree if and only if G is connected.

A graph is said to be *bipartite* if there exists a partition (V_1, V_2) of V such that each e3dge $(i,j) \in E$ connects a vertex $i \in V_1$ to a vertex $j \in V_2$. Graph G is bipartite if and only if it does not contain any cycles visiting an odd number of vertices.

An elementary path is defined as an *Eulerian* path if it visits every edge in E once and only once.

A simple path is said to be *Hamiltonian* if it visits every vertex in V once and only once.

A *clique* is a subgraph $G' = (V', E')$ of G in which every pair of vertices in V' is connected by an edge.

A *stable* set of G is a subgraph $G' = (V', E')$ induced by V' such that $E' = \emptyset$.

2 Directed Graphs

A *directed graph* is a pair $G = (V, A)$ in which V is a finite set of vertices and A is a family of *arcs* $(i,j) \in A$. The order in which the nodes i and j appear is relevant; thus, $(i,j) \neq (j,i)$. In this case, we say that arc (i,j) leaves node i and enters node j. Nodes i and j are often referred to as the out-going and in-coming nodes, respectively.

Similar to a general path, a *directed path* is a sequences of arcs $a_1, a_2, \ldots, a_k$ of consecutive arcs of the type $a_1 = (v_1, v_2),\ (v_2, v_3), \ldots, (v_k, v_{k+1}$.