

## Lesson: Introducing Notation

Book acknowledgment:

“Linear Programming and Network Flows” by Bazaraa, Jarvis, and Sherali. *Fourth Edition*

### Goals

- Introduce (or re-introduce) network/graph notation.
- Make sure that we are on the same page with definitions and notation.

### 1 Undirected Graphs

Let  $G = (V, E)$  be an **undirected graph**.

$V$  is a finite set of nodes (or vertices) with  $n := |V|$ .

$E$  is a collection of unordered pairs (edges) of elements of  $V$  with  $m := |E|$ .

*Attributes*

- $G$  has *weights* on edges and/or nodes.
  - For edges,  $c_{ij}$ ,  $\forall (i, j) \in E$ .
  - For nodes,  $w_i$ ,  $\forall i \in V$ .
- Let  $V = \{1, 2, \dots, n\}$ .
- Let the edges in  $E$  have the form  $(i, j)$ ,  $\forall i, j \in V$ 
  - For the sake of simplicity, we assume that edges  $(i, j)$  and  $(j, i)$  are equivalent. Thus, we assume that  $E$  only consists of edges  $(i, j)$  for which  $i < j$ .
- Two nodes  $i$  and  $j$  are said to be *adjacent* when there exists an edge  $(i, j)$  that connects them.
- Two edges are referred to as *adjacent* if they have a node in common.
- The number of edges incident on a vertex  $v$  is called the *degree* of the vertex, and is usually denoted by  $d_G(v)$ .

- Graph  $G$  is *complete* if it contains all possible edges. (When  $E = \{(i, j) : i, j \in V, i < j\}$ ).
- Graph  $G' = (V', E')$  is a *subgraph* of  $G$  if  $V' \subseteq V$  and  $E' \subseteq E$ . Additionally, for  $G'$  to be a subgraph of  $G$ , then for all edges  $(i, j) \in E'$ , then both vertices  $i$  and  $j$  must belong to  $V'$ .
- A *path* in  $G$  is a sequence of consecutive edges  $e_1, e_2, \dots, e_k \in E$ , in which  $e_1 = (v_1, v_2)$ ,  $e_2 = (v_2, v_3), \dots, e_k = (v_k, v_{k+1})$ . The path connects  $v_1$  and  $v_{k+1}$ , visiting the intermediate vertices  $v_2, v_3, \dots, v_k$ .
  - The path is a *cycle* if  $v_1 = v_{k+1}$ .
  - A path is *elementary* if no edge is used twice.
  - The path is referred to as *simple* if no node is visited twice.
  - An elementary path is defined as an *Eulerian* path if it visits every edge in  $E$  once and only once.
  - A simple path is said to be *Hamiltonian* if it visits every vertex in  $V$  once and only once.
- Vertex  $v$  is *connected* to vertex  $w$  if there exists a path connecting them.
- A graph  $G$  is referred to as *connected* if all vertices  $v$  and  $w$  in  $V$  are connected.
- A *cut* in  $G$  is a set of edges of the type:

$$\gamma_G(S) := \{(i, j) \in E : |S \cap \{i, j\}| = 1\}$$

in which  $S$  is the subset of vertices that induces the cut (i.e., the cut contains all edges with one endpoint in  $S$  and the other in  $V \setminus S$ ).

- A partial graph  $G' = (V, E')$  of  $G$  is called a *forest* if it is *acyclic*—it does not contain a cycle.
  - A forest is *maximal* if every edge in  $E \setminus E'$  forms a cycle with the edges in  $E'$ . Therefore,  $G'$  and  $G$  have the same connected components.
- A maximal connected forest, if it exists, is called a *spanning tree*. Every tree has exactly  $|V| - 1$  edges. Graph  $G$  contains a tree if and only if  $G$  is connected.
- A graph is said to be *bipartite* if there exists a partition  $(V_1, V_2)$  of  $V$  such that each edge  $(i, j) \in E$  connects a vertex  $i \in V_1$  to a vertex  $j \in V_2$ .
  - Graph  $G$  is bipartite if and only if it does not contain any cycles visiting an odd number of vertices.

## 2 Directed Graphs

A *directed graph* is a pair  $G = (V, A)$ .

- In this case, we say that arc  $(i, j)$  leaves node  $i$  and enters node  $j$ .
- Nodes  $i$  and  $j$  are often referred to as the out-going and in-coming nodes, respectively.