SA403 – Networks

Lesson 1: Introducing Notation

Notes

Book acknowledgment:

> "Linear Programming and Network Flows" by Bazaraa, Jarvis, and Sherali. *Fourth Edition*

Goals

- Introduce (or re-introduce) network/graph notation

1 Undirected Graphs

Let $G = (V, E)$ be an **undirected graph**.

> V is a finite set of nodes (or vertices) with $n := |V|$.
>
> E is a collection of unordered pairs (edges) of elements of V with $m := |E|$.

Attributes

- G has *weights* on edges and/or nodes.
  - For edges, $c_{ij},\ \forall (i,j) \in E$.
  - For nodes, $w_i,\ \forall i \in V$.
- Let $V = \{1, 2, \ldots, n\}$.
- Let the edges in E have the form $(i,j), \forall i,j \in V$
  - For the sake of simplicity, we assume that edges (i,j) and (j,i) are equivalent. Thus, we assume that E only consists of edges (i,j) for which $i < j$.
- Two nodes i and j are said to be *adjacent* when there exists an edge (i,j) that connects them.
- Two edges are referred to as *adjacent* if they have an edge in common.
- The number of edges incident on a vertex v is called the *degree* of the vertex, and is usually denoted by $d_G(v)$.

- Graph G is *complete* if it contains all possible edges. (When $E = \{(i,j) : i,j \in V, i < j\}$.
- Graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. Additionally, for G' to be a subgraph of G, then for all edges $(i,j) \in E'$, then both vertices i and j must belong to V'.
- A *path* in G is a sequence of consecutive edges $e_1, e_2, \ldots, e_k \in E$, in which $e_1 = (v_1, v_2),\ e_2 = (v_2, v_3), \ldots, e_k = (v_k, v_{k+1})$. The path connects v_1 and v_{k+1}, visiting the intermediate vertices $v_2,\ v_3, \ldots, v_k$.
    - The path is a *cycle* if $v_1 = v_{k+1}$.
    - A path is *elementary* if no edge is used twice.
    - The path is referred to as *simple* if no node is visited twice.
    - An elementary path is defined as an *Eulerian* path if it visits every edge in E once and only once.
    - A simple path is said to be *Hamiltonian* if it visits every vertex in V once and only once.
- Vertex v is *connected* to vertex w if there exists a path connecting them.
- A graph G is referred to as *connected* if all vertices v and w in V are connected.
- A *cut* in G is a set of edges of the type:

$$\gamma_G(S) := \{(i,j) \in E : |S \cap \{i,j\}| = 1\}$$

  in which S is the subset of vertices that induces the cut (i.e., the cut contains all edges with one endpoint in S and the other in $V \setminus S$.
- A partial graph $G' = (V, E'$ of G is called a *forest* if it is *acyclic*–it does not contain a cycle.
    - A forest is *maximal* if every edge in $E \setminus E'$ forms a cycle with the edges in E'. Therefore, G' and G have the same connected components.
- A maximal connected forest, if it exists, is called a *spanning tree.* Every tree has exactly $|V| - 1$ edges. Graph G contains a tree if and only if G is connected.
- A graph is said to be *bipartite* if there exists a partition (V_1, V_2) of V such that each e3dge $(i,j) \in E$ connects a vertex $i \in V_1$ to a vertex $j \in V_2$.
    - Graph G is bipartite if and only if it does not contain any cycles visiting an odd number of vertices.
- A *clique* is a subgraph $G' = (V', E')$ of G in which every pair of vertices in V' is connected by an edge.
- A *stable* set of G is a subgraph $G' = (V', E')$ induced by V' such that $E' = \emptyset$.

2 Directed Graphs

A *directed graph* is a pair $G = (V, A)$.

- V is a finite set of vertices.
- A is a family of *arcs* $(i, j) \in A$.
- The order in which the nodes i and j appear is relevant; thus, $(i, j) \neq (j, i)$.
  - In this case, we say that arc (i, j) leaves node i and enters node j.
  - Nodes i and j are often referred to as the out-going and in-coming nodes, respectively.
- A *directed path* is a sequences of arcs $a_1, a_2, \ldots, a_k$ of consecutive arcs of the type $a_1 = (v_1, v_2),\ (v_2, v_3), \ldots, (v_k, v_{k+1}$.