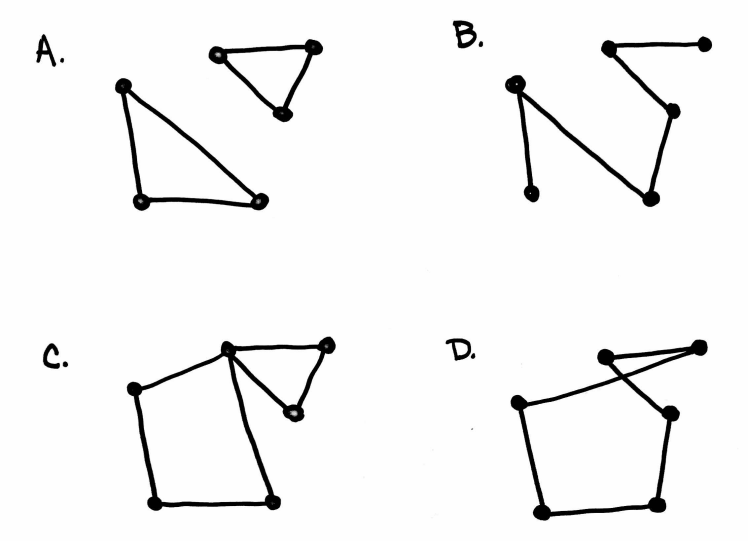
Lesson 11. Traveling Salesperson Problem (TSP)

1 Tours and TSP

A **tour** is a closed route that visits every location exactly once.

In graph terminology, a **tour** is a single cycle (collection of edges) that visits each node exactly once. This is also known as a Hamiltonian Cycle or Hamiltonian Circuit.

Problem 1. For each graph below, does the set of edges represent a tour through the 6 nodes? If not, explain why not.



Given a graph $G = (V, E)$ with edge weights (representing costs or distances), the **Traveling Salesperson Problem (TSP)** seeks a *minimum cost tour of* G.

1. If we were to formulate this problem like the MST problem, what would be the decisions (in words) for the integer programming formulation?

2 History of the TSP

- TSP was first studied in the 1930s! The first IP formulation of TSP (which we're studying mostly in these notes) was in 1954.
- Nowadays TSP has been solved on graphs with as many as 50,000 nodes such as a historic tour of the US.

3 Visiting Graduate Schools: IP Formulation of TSP

Problem 2. A college student is interested in visiting as many graduate schools as possible. She reasons that a single visit to each school is appropriate, and she wants to return to her own campus only after visiting all the schools. It is conceivable that she visits the schools in any order, but she would like to minimize the amount of driving she has to do. If the distance between schools i and j is $d_{i,j}$ $(i < j)$, where the matrix D of distances is given below, in which order should she visit the schools? Note that school 1 is her current school.

$$D = \left[\begin{array}{c|ccccc} & 2 & 3 & 4 & 5 & 6 \\ \hline 1 & 16 & 23 & 14 & 8 & 15 \\ 2 & - & 12 & 19 & 9 & 13 \\ 3 & - & - & 7 & 25 & 16 \\ 4 & - & - & - & 18 & 15 \\ 5 & - & - & - & - & 20 \end{array}\right]$$

2. Draw the graph $G = (V, E)$ of the network below. Include node labels and edge costs. Highlight a collection of edges that form a tour of the graph (doesn't have to be the minimum distance tour).

3. Define the Sets, Variables, and Parameters that are needed for the IP model.

4. Write the objective function for this model.

5. There are two types of constraints that should be included in this model:

   (a) A cycle on n nodes has exactly $\square$ edges

   (b) In order to have a tour, each node must be connected to exactly $\square$ edges.

   Write each of these constraints in both concrete/parameterized form.

6. Is there a solution for this graph that satisfies these two constraints but is not a tour? *Think back to MST.*

7. Write concrete constraint(s) that prevents the graph that you sketched above from being selected by the solver.

8. Write a parameterized constraint which prevents ANY graphs of this kind from being returned by the solver. Recall that there is one such constraint for every subset S of vertices in V such that $\boxed{}$ (restrict the number of vertices in S). These are called **subtour elimination constraints**.

4 Another Formulation of TSP

The formulation we saw here is a so-called edge based TSP formulation and dates back to 1954. Recall the major drawback was that there are an exponential number of subtour elimination constraints.

Another possible formulation, which dates to the 2000s, has an exponential number of variables. Here we will discuss it.

1. Define a set T which is the set of all possible tours of N.
    - If N has 4 nodes, how many possible tours/cycles are there?
    - For a general graph with n nodes, how many possible tours/cycles are there?
2. Define a variable x_i for all $i \in T$. $x_i = 1$ if tour i is chosen and 0 otherwise.
3. Let c_i be the cost of tour i for all $i \in T$.

4. Using the sets N, E, and T, and the variables/parameters x_i, c_i for all $i \in T$; write a parameterized formulation for the TSP problem that has an exponential number of variables.

Formulations like this can be more effective than the edge based formulations depending on the application area; however implementing them can be quite difficult. You can learn about how to solve a problem like this in either an advanced OR course and/or a capstone/honors project.