

1. 4.1: NOTE: Error in book data; distances $c_{25} = c_{52}$ should be 35 instead of 40. We have a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{E} = \{(i, j) | i < j, i, j \in \mathcal{V}\}$. Note that the symmetry of the graph means we can treat the graph as being undirected so we assume arc (i, j) always has $i < j$. Let c_{ij} be the distance given in the table for arc $(i, j) \in \mathcal{E}$ and d_j the delivery requirements for the customers, for $j = 2, \dots, 7$. For a subset of customer nodes $U \subset \{2, \dots, 7\}$, let

$$h(U) = \text{minimum number of trucks needed to serve demand in } U = \left\lceil \frac{\sum_{i \in U} d_i}{120} \right\rceil.$$

Note that in this instance, $h(U) = 3$. Let

$$x_{ij} = \begin{cases} 1 & \text{if } (i, j) \text{ is on a route} \\ 0 & \text{otherwise.} \end{cases}$$

The initial formulation is:

$$\begin{aligned} \min \quad & \sum_{(i,j) \in \mathcal{E}} c_{ij} x_{ij} \\ \text{s.t.} \quad & \sum_{j=i+1}^7 x_{ij} + \sum_{j=1}^{i-1} x_{ji} = 2, \quad i = 2, \dots, 7 \quad (\text{Each non-depot visited by a tour}) \\ & \sum_{j=2, \dots, 7} x_{1j} = 6 \quad (\text{There are three trucks}) \\ & x_{ij} \in \{0, 1\}. \end{aligned}$$

After solving this integer problem, the solution must be examined for infeasible routes. Routes are infeasible if (a) the route is a cycle on a set of nodes, U , where $1 \notin U$, or (b) the route is $U \cup \{1\}$ but

$$\sum_{i \in U} d_i > 120.$$

Then, for the U that satisfies (a) or (b), add the constraint

$$\sum_{\{(i,j) \in \mathcal{E} | i,j \in U\}} x_{ij} \leq |U| - h(U)$$

which assures that there is no subtour on U and that the number of trucks used for U is sufficient. The initial solution provides the tours 1-2-3-1, 1-4-5-1, and 1-6-7-1. We note that demand on route 1-2-3-1 is 125, which exceeds the truck capacity. We add the subtour elimination constraint $x_{23} \leq 2 - 2 = 0$ and resolve. The new solution provides the tours 1-2-4-1, 1-3-5-1, and 1-6-7-1. We note that demand on route 1-2-4-1 exceeds the truck capacity. We add the subtour elimination constraint $x_{24} \leq 0$ and resolve. The new solution provides the tours 1-2-7-1, 1-3-5-1, and 1-4-6-1. None of these tours are infeasible using conditions (a) and (b) above. This means the solution is feasible, and has a length of 175 miles.

2. 4.2: To solve this, we add another condition to (a) and (b) when introducing constraints, namely: (c) the route is $U \cup \{1\}$ and

$$\sum_{(i,j) \in \mathcal{E} | i,j \in U \cup \{1\}} c_{ij} x_{ij} > 75.$$

If (a) or (b) is satisfied by U , or (c) is satisfied by U and $h(U) \geq 2$, then add the constraint

$$\sum_{\{(i,j) \in \mathcal{E} | i,j \in U\}} x_{ij} \leq |U| - h(U).$$

If (c) is satisfied and $h(U) = 1$ then add the constraint

$$\sum_{\{(i,j) \in \mathcal{E} | i,j \in U\}} x_{ij} \leq |U| - h(U) - 1 = |U| - 2.$$

We find that our solution to 4.1 is now infeasible because it violates condition (c); tour 1-2-7-1 is 80 miles in length. We note that $h(U) = 1$ for tour 1-2-7-1, so we add the constraint $x_{27} \leq 2 - 1 - 1 = 0$ and resolve. The new solution provides the tours 1-2-5-1, 1-3-4-1, and 1-6-7-1. We note that demand on route 1-3-4-1 exceeds the truck capacity. We add the subtour elimination constraint $x_{34} \leq 2 - 2 = 0$ and resolve. The new solution provides the tours 1-4-7-1, 1-3-5-1, and 1-2-6-1. We note that demand on route 1-4-7-1 exceeds the truck capacity. We add the subtour elimination constraint $x_{47} \leq 2 - 2 = 0$. We note that tour 1-2-6-1 is 100 miles in length. We note that $h(U) = 1$ for tour 1-2-6-1, so we add the constraint $x_{26} \leq 2 - 1 - 1 = 0$ and resolve. The new solution provides the tours 1-3-7-1, 1-2-5-1, and 1-4-6-1. We note that demand on route 1-3-7-1 exceeds the truck capacity. We add the subtour elimination constraint $x_{37} \leq 2 - 2 = 0$ and resolve. We now find that the solution is infeasible. This means the company must either: relax the travel distance requirement, increase the carrying capacity of their trucks, or add another delivery vehicle.

Not to be handed in.

1. 4.17: Note that, currently, the constraint

$$\sum_{j|i < j} x_{ij} + \sum_{j|j < i} x_{ji} = 2, \text{ for } i = 2, \dots, 7,$$

ensures that each node has two edges connected to it. In particular, a node could not be visited from the depot and then have the route return immediately to the depot. Here are two ways to deal with this (only one needs to be performed):

- (a) (the clever way) Allow the edge variables which touch the depot to be zero, one, or two, i.e., modify the binary constraints so that $x_{0j} \in \{0, 1, 2\}$ for all $(0, j) \in \mathcal{E}$.
- (b) (the longer, but still correct way) Node split so that each node now has a copy with the same exact neighbors. Between each node and its copy introduce an arc with zero cost.