Lesson 6. Introduction to Binary Variables

1 Today...

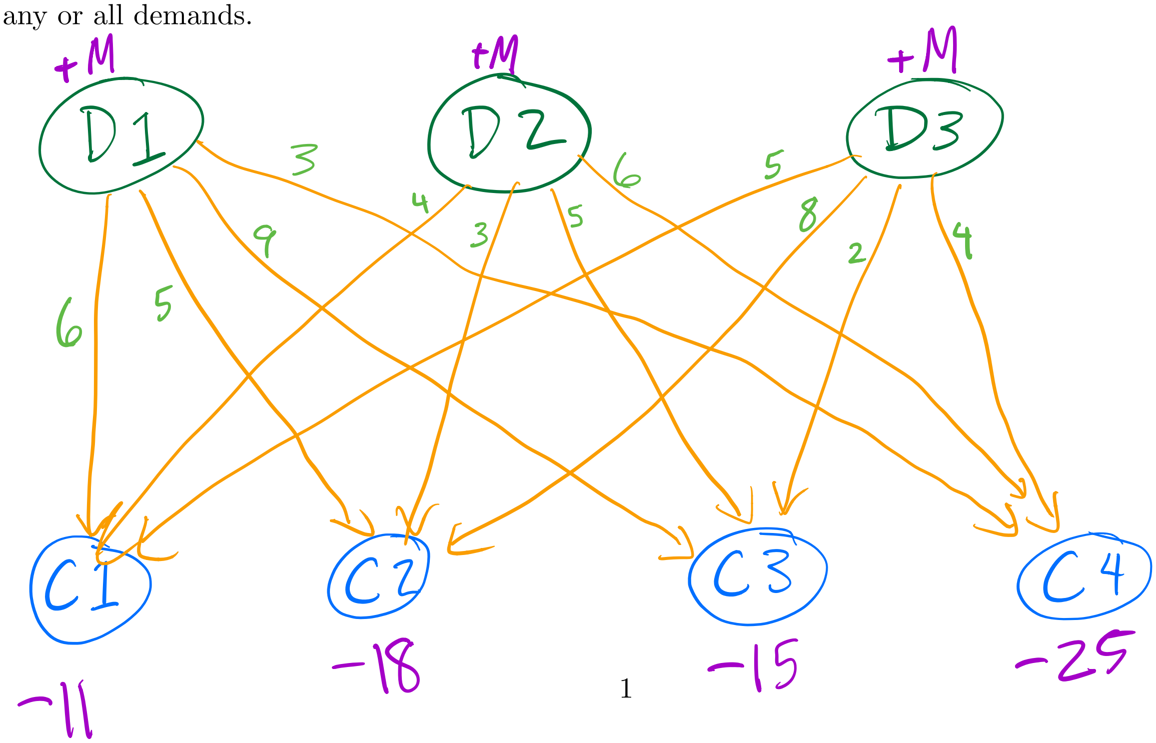
- We extend a min-cost network flow model to a *fixed-charge facility location* model.
- This will require the use of *binary decision variables*.
- There are two well-known formulations for modeling the fixed-charge forcing constraints: the so-called *weak* and *strong* formulations.

2 Cloud Nine Department Store

Cloud Nine is considering 3 locations for new distribution centers to serve its customers in 4 nearby cities. The following table shows the fixed cost (in millions of dollars) of opening each potential center, the number (in thousands) of truckloads forecasted to be demanded at each city over the next 5 years, and the transporation cost (in millions of dollars) per thousand truckloads moved from each center location to each city.

	Fixed Cost	Transport Costs City 1	City 2	City 3	City 4
Distribution center 1	200	6	5	9	3
Distribution center 2	400	4	3	5	6
Distribution center 3	225	5	8	2	4
Demand	—	11	18	15	25

Cloud Nine seeks a minimum cost distribution system assuming any distribution center can meet any or all demands.



3 Concrete transportation model, ignoring fixed cost for opening centers

Problem 1. *Ignoring the facility opening costs*, write a concrete model (objective function and constraints only) for this transportation problem. Let x_{ij} represent the number of truckloads to move from center i to city j, and assume that each center is capable of supplying enough truckloads to meet all demand.

Minimize $6x_{11} + 5x_{12} + 9x_{13} + 3x_{14} + \dots + 4x_{34}$

s.t.

$x_{11} + x_{12} + x_{13} + x_{14} \leq M$

$x_{21} + x_{22} + x_{23} + x_{24} \leq M$

$x_{31} + x_{32} + x_{33} + x_{34} \leq M$

$x_{11} + x_{21} + x_{31} \geq 11$

$x_{12} + x_{22} + x_{32} \geq 18$

$x_{13} + x_{23} + x_{33} \geq 15$

$x_{14} + x_{24} + x_{34} \geq 25$

$x_{12}, x_{13}, \dots, x_{34}$

4 Binary Decision Variables for Distribution Centers

The use of binary $\{0, 1\}$ variables to model yes/no decisions is very common. In this model, we use a binary decision variable for each potential distribution center (DC) that indicates whether or not it is used in the solution:

$$z_1 = \begin{cases} 1 & \text{if DC 1 is opened} \\ 0 & \text{if DC 1 is not opened} \end{cases} \qquad z_2 = \begin{cases} 1 & \text{if DC 2 is opened} \\ 0 & \text{if DC 2 is not opened} \end{cases} \qquad z_3 = \begin{cases} 1 & \text{if DC 3 is opened} \\ 0 & \text{if DC 3 is not opened} \end{cases}$$

5 Objective function with fixed costs

Problem 2. Modify the concrete objective function in the model above to incorporate the fixed costs to open distribution centers.

$6x_{11} + 5x_{12} + 9x_{13} + 3x_{14} + \dots + 4x_{34} + 200z_1 + 400z_2 + 225z_3$

6 Fixed-charge constraints

- Whenever we use binary decision variables, we must include constraints that enforce the correct behavior of the variables in the context of the model.
- In this problem, we need constraints that enforce the logic: **If a DC is closed, then we can't ship out of that DC.**
- Without this logic, we wouldn't be forced to pay the fixed cost to open a warehouse.
- There are two options for fixed-charge constraints: "single-variable" **OR** "multiple-variable".

Problem 3. (*"Single-variable" fixed-charge constraints*)

a. Write an inequality that enforces the logic: **If** $z_1 = 0$**, then** $x_{11} = 0$**.** Include a coefficient on z_1 so that the constraint allows x_{11} to be large enough to ship all of the demand required by city 1 out of ~~the~~ DC 1 if $z_1 = 1$ (DC 1 is open).

$$x_{11} \le M z_1 \sim x_{11} \le 11 z_1 \qquad M > 0$$

Demand for Customer #1.

These are often referred to as *strong* constraints.

b. Write all of the "strong" fixed-charge constraints required in the concrete model.

$$x_{11} \le 11 z_1 \quad x_{12} \le 18 z_1 \quad x_{13} \le 15 z_1 \quad x_{14} \le 25 z_1$$
$$x_{21} \le 11 z_2 \quad x_{22} \le 18 z_2 \quad x_{23} \le 15 z_2 \quad x_{24} \le 25 z_2$$
$$x_{31} \le 11 z_3 \quad x_{32} \le 18 z_3 \quad x_{33} \le 15 z_3 \quad x_{34} \le 25 z_3$$

Problem 4. (*"Multiple-variable" fixed-charge constraints*)

a. Write an inequality that enforces the logic: **If** $z_1 = 0$**, then** $x_{11} = x_{12} = x_{13} = 0$. Include a coefficient on z_1 to allow x_{11}, x_{12}, and x_{13} to be large enough to ship all of the demand out of DC 1 if $z_1 = 1$ (DC 1 is open).

$x_{14} = 0$

DC

$$x_{11} + x_{12} + x_{13} + x_{14} \leq 69 z_1$$

These are often referred to ask *weak* constraints.

b. Write all of the "weak" fixed-charge constraints required in the concrete model.

$x_{11} + x_{12} + x_{13} + x_{14} \leq 69 z_1$ | $x_{21} + x_{22} + x_{23} + x_{24} \leq 69 z_2$ | $x_{31} + x_{32} + x_{33} + x_{34} \leq 69 z_3$

7 Abstract Model: Fixed-Charge Facility Location

The basic min-cost (transportation) network flow portion of the model is provided below. Extend this model to incorporate the facility location opening fee, or the "fixed charge". The new collection of variables are **binary** $\{0,1\}$ **decision variables**.

Sets:

$D :=$ the set of distribution centers

$C :=$ the set of cities

Parameters:

$c_{ij} :=$ the cost (in millions) to deliver 1000 truckloads from center i to city j, for $i \in D$ and $j \in C$

$d_j :=$ the demand (in thousands) of truckloads forecasted for city j, for $j \in C$

$f_i :=$ fixed cost to open DC $i \in D$.

$M :=$ LARGE #. The maximum # of truckloads that could leave a DC. $M = \sum_{j \in C} d_j$

Decision Variables:

$x_{ij} :=$ the number (in thousands) of **whole** truckloads to deliver from center i to city j, for $i \in D$ and $j \in C$

$z_i :=$ $\in \{0,1\}$ $= 1$ if DC $i \in D$ is opened, and $\emptyset$, otherwise.

Model: *(Add comments to describe the objective function and each constraint.)*

$$\text{maximize} \quad \sum_{i \in D} \sum_{j \in C} c_{ij} x_{ij} + \sum_{i \in D} f_i z_i$$

$$\text{subject to} \quad \sum_{i \in D} x_{ij} \geq d_j, \text{ for all } j \in C$$

strong fixed-charge constraints:

$$x_{ij} \leq d_j z_i \quad \forall i \in D, j \in C$$

OR

weak fixed-charge constraints:

$$\sum_{j \in C} x_{ij} \leq M z_i \quad \forall i \in D$$

$x_{i,j} \in \mathbb{Z}^{\geq 0}$, for all $i \in D, j \in C$ (We don't get integer solutions for free! Why not?)

$$z_i \in \{0,1\} \ \forall i \in D$$