SA 405 | **Final Exam**
Fall 2018 | **Dec 14, 2018**

Midshipmen are persons of integrity. **Name:** ______________________

Time Limit: 180 Minutes

- Do **not** write your name on each page, only write your name above.
- No books or notes are allowed.
- Show all work clearly. (Little or no credit will be given for a numerical answer without the correct accompanying work. Partial credit is given where appropriate.)
- Please read the question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.
- All formulations must have descriptions of any indices, parameters, and decision variables used. All constraints must be described.

Grade Table (for teacher use only)

Question	Points	Score
1	0	
2	20	
3	20	
4	0	
5	0	
6	0	
7	10	
8	10	
9	0	
Total:	60	

1. The Superintendent asks you and your OR classmate to give him directions from the Yard to Naval Base San Diego, the principal homeport of the Pacific Fleet. Of course, he wants to find the path that minimizes his total travel time, but he has a few more requirements: he only wants to pass through cities that have Navy Lodges, and he never wants to drive more than eight hours at a stretch between cities.

Your classmate defines the following sets and parameters:

$\mathcal{C}$	:=	the set of cities in the United States that have Navy Lodges;
$\mathcal{A}$	:=	the set of routes (i, j) from city i to city j, for i, j in $\mathcal{C}$, such that the drive time from i to j is no more than 8 hours;
$t_{i,j}$	:=	the drive time from city i to city j, for all (i, j) in $\mathcal{A}$;
a	:=	the element of $\mathcal{C}$ representing Annapolis;
s	:=	the element of $\mathcal{C}$ representing San Diego.

(a) Picking up where your partner left off, write a mathematical program in **abstract** form whose solution solves the Superintendent's routing problem. Clearly define any additional notation that you use.

D.V.

$$x_{ij} = \begin{cases} 1 & \text{if route } (i,j) \text{ is used} \\ 0 & \text{otherwise} \end{cases}$$

$$\min \sum_{(i,j) \in A} t_{i,j} \, x_{i,j}$$

s.t.

$$\sum_{(a,j) \in A} x_{a,j} = 1$$

$$\sum x_{i,j}$$

(b) Additional research reveals that due to traffic, traveling through many of the cities $i \in \mathcal{C}$ incur delays, and your classmate defines the parameters,

d_i := the extra time required if driving through city i, for all $i \in C$.

Adjust your model to incorporate city delays. Again, clearly define any additional notation that you use.

(c) After you brief the Superintendent on the optimal route, he makes a few requests due to the locations of various friends and relatives. Use the variables you have already defined to add constraints to the model that will enforce the Superintendent's latest requests.

i. He definitely wants to visit city 10, but not city 11.

ii. He wishes to visit at least one of cities 3, 4 and 5.

iii. If he visits cities 3 and 4, he definitely does not want to visit city 5.

2. Your regional manager at Dunder Mifflin, Michael Scott, has asked you to solve a **Vehicle Routing Problem (VRP)** to determine an optimal set of 3 paper-delivery routes for delivering paper to their 9 customers. He assumes that each route begins and ends at their warehouse in Scranton, PA. Mr. Scott's goal is to minimize the total distance traveled among all vehicles.

   Each delivery route must be used, must visit at least two customers, and can transport no more than 425 cases of paper. Assuming that all distances are symmetrical, the table below gives the distances between the warehouse (WH) and each of the 9 customers.

Table 1: Distances between the warehouse and all customers.

	WH	1	2	3	4	5	6	7	8	9
WH	-	10	12	13	40	25	16	37	23	19
1	-	-	12	23	14	17	11	7	34	21
2	-	-	-	13	24	28	16	9	8	19
3	-	-	-	-	24	26	14	13	12	5
4	-	-	-	-	-	5	37	17	8	9
5	-	-	-	-	-	-	16	4	18	21
6	-	-	-	-	-	-	-	27	11	13
7	-	-	-	-	-	-	-	-	18	9
8	-	-	-	-	-	-	-	-	-	19
9	-	-	-	-	-	-	-	-	-	-

Table 2: Demand (in number of cases of paper) for each customer

Customer	Demand
1	20
2	80
3	90
4	120
5	110
6	150
7	100
8	75
9	85

Answer the following questions related to Michael Scott's request.

(a) (10 points) Please provide an *abbreviated* **concrete** integer programming model for solving Mr. Scott's problem above. Clearly define your variables. Provide your objective function and all constraints required to solve Mr. Scott's problem. You can omit any subtour-elimination and route-splitting constraints in this formulation.

(b) (5 points) You solve the problem, and your solution thankfully contains no cycles. In this solution, vehicle 1 visits the sequence of customers $(1, 6)$, vehicle 2 visits the sequence of customers $(2, 4, 5, 7, 8)$ and vehicle 3 visits the sequence of customers $(3, 9)$. Is this solution feasible? If so, please provide a brief explanation why. If not, provide the appropriate set of constraints to eliminate this infeasible solution.

Subtours

Do we want to give them a solution to interpret?
(This would require defining variables for them.)

(c) (5 points) Michael Scott tells you that each vehicle can now only travel up to 80 miles along an entire delivery route. Compared to the original problem (without this travel distance maximum), do you expect the optimal objective function value for this new problem to increase, decrease, or remain the same? Give a brief explanation with your answer.

3. There is currently an outbreak of the Ebola virus in the Democratic Republic of Congo (DRC). Given a limited budget, the World Health Organization (WHO) is looking to build temporary testing facilities that will be assigned to one or more cities to identify who has been infected in those cities. WHO's task is determine which of the testing facilities should be built to maximize the total number of people tested for Ebola in the DRC.

Should we include a sentence of explanation?

$\mathcal{C}$:= the set of cities in the DRC;
$\mathcal{C}$:= the set of possible testing facilities in the DRC;

p_i := the population of each city $c \in C$;
m_f := the maximum number of people facility $f \in F$ can test for Ebola;
c_f := the cost of building testing facility $f \in F$ in American dollars;
B := the budget of the WHO in American dollars.

(a) (12 points) Formulate an **abstract** mathematical programming model to maximize the total number of people tested for Ebola in the DRC without exceeding the WHO's budget. Clearly define all variables, present an objective function, and present all appropriate constraints.

Is there some definition of neighborhoods?

(b) (4 points) To be more fiscally-minded, WHO decided to change their objective. Instead of maximizing the total number of people tested, WHO has decided to minimize the total cost of building testing facilities as long as D number of people are tested for Ebola. Provide an updated objective function and provide any additional sets of constraint(s) required to appropriately model this problem.

(c) (4 points) Finally, the WHO changed their strategy again. They are now seeking to minimize the total cost of building testing facilities as along as **all** people in the DRC are test for the Ebola virus. Please provide any additional constraint(s) to model this change.

4. (a) Recall that a linear program (LP) in canonical form is as follows:

$$\begin{aligned} \max\ & \mathbf{c}^T \mathbf{x} \\ \text{s.t.}\ & \mathbf{A}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned}$$

List two requirements that together guarantee that all basic feasible solutions of a linear program in canonical form are integer-valued.

(b) Describe a practical problem that can be modeled by an integer program (IP) whose continuous LP relaxation is guaranteed to provide an integer-valued optimal solution.

5. (a) Is the following matrix totally unimodular? Explain.

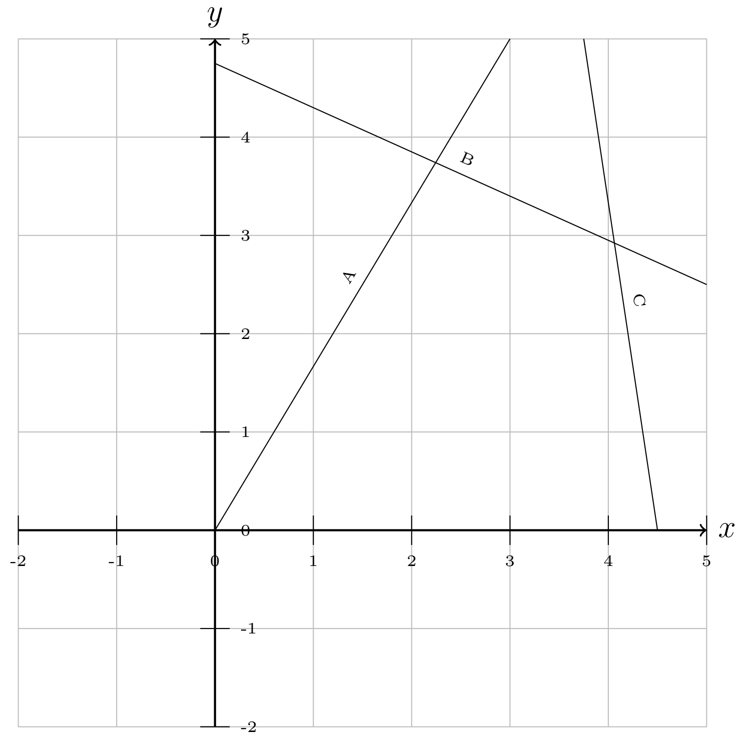
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

(b) Describe a problem that can be modeled by a linear program with continuous variables that is guaranteed to have integer-valued solutions.

6. Consider the integer program formulation provided and graphed below.

Formulation (I)

$$\begin{array}{rcll} -5x+3y & \leq & 0 & \text{(A)} \\ 9x+20y & \leq & 95 & \text{(B)} \\ 20x+3y & \leq & 90 & \text{(C)} \\ x,\ y & \geq & 0, \text{ integer} & \end{array}$$



(a) On the same graph, sketch the convex hull formulation, which we will call Formulation (II), of the integer feasible region described by Formulation (I). List the inequalities that describe Formulation (II).

(b) Which formulation is "better" for solving the associated integer program, Formulation (I) or Formulation (II)? Explain.

7. (10 points) Given the graph G in Figure 7 below, execute Prim's algorithm to calculate a minimum spanning tree starting at vertex A. Show your steps in the table below.

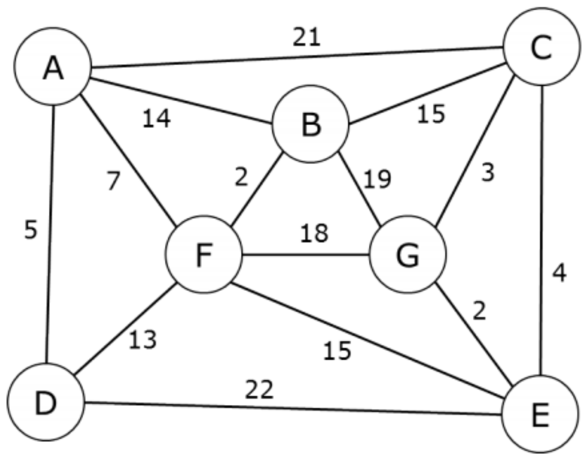


Figure 1: Graph G

(a) For each iteration, ~~As the algorithm iterates~~, provide all nodes currently in S, all edges in the cut-set $C_{S,\bar{S}}$, and the edge that should be added to the minimum spanning tree T. Break any ties by adding to S the vertex indexed by ~~choosing~~ the letter coming first in the alphabet. At conclusion of the algorithm, bold all the edges forming T in the graph above.

Iteration #	Set S	Cut-set $C_{S,\bar{S}}$	Edge to add to T
1			
2			
3			
4			
5			
6			

(b) Indicate the minimum cost spanning tree produced by the algorithm on the graph G above, and provide it's cost.

8. Given the graph G_2 H in Figure 8 below, execute the Bellman-Ford algorithm to determine a shortest path from node 1 to node 6.

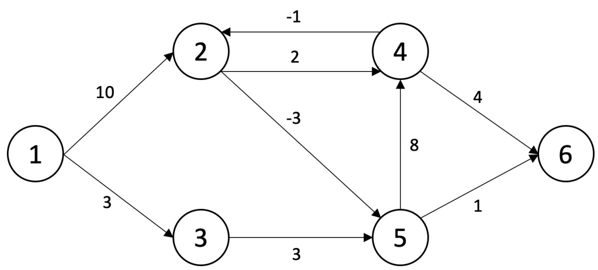


Figure 2: Graph G_2 H

To implement the Bellman-Ford algorithm, use the following ordering of edges:

$$(1,2),\ (1,3),\ (2,4),\ (2,5),\ (3,5),\ (4,2),\ (4,6),\ (5,4),\ (5,6)$$

(a) (8 points) Initialize each node with an appropriate distance label. In the table below, cross out old distance labels and predecessors and update with the new values as the algorithm iterates. Break any ties by choosing the minimum node label.

Vertex	Distance Label	Predecessor
1		
2		
3		
4		
5		
6		

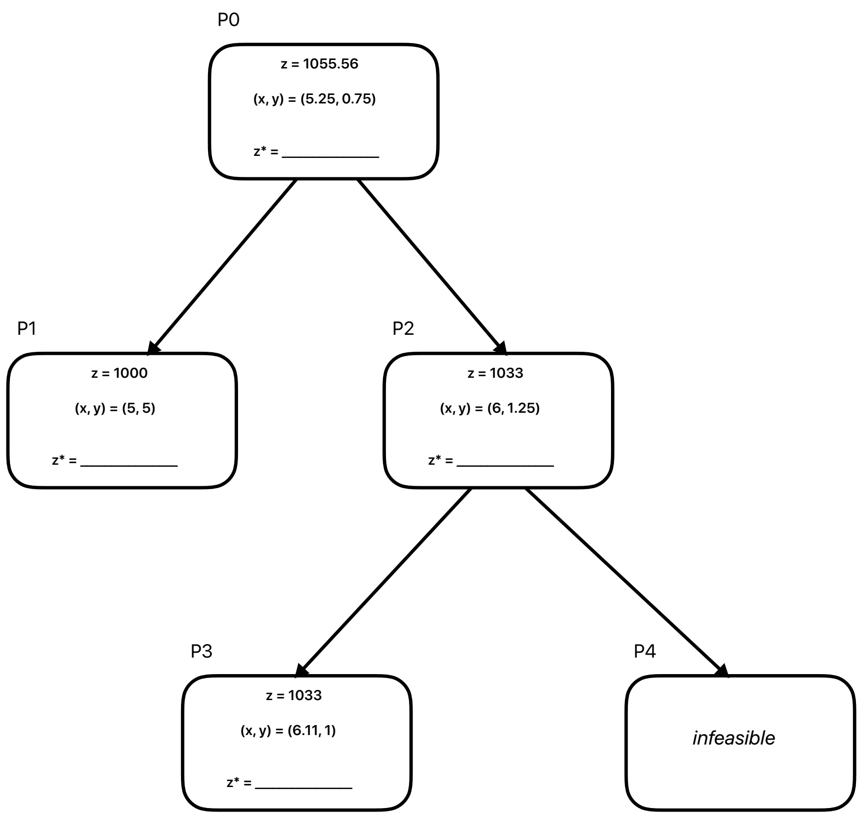
(b) (2 points) Provide the final vector of distance labels d and the final vector of predecessor labels $pred$.

$d = \{ \quad , \quad , \quad , \quad , \quad , \quad \}$

$pred = \{ \quad , \quad , \quad , \quad , \quad , \quad \}$

(c) Provide a list of the shortest paths you have found.

9. Consider the ~~partial~~ branch and bound tree provided below, as applied to a *maximizing*, 2-variable integer program with integer objective coefficients. For each subproblem, the optimal objective value, z, and optimal solution, (x, y), of its continuous (non-integer) relaxation is provided. Also, let z^* be the current upper bound on the integer subproblem implied by the solution to its relaxation.



(a) Label each arrow with the constraint that is added to form the next subproblem.

(b) Fill in z^* for each subproblem, the upper bound on the integer subproblem implied by the solution to its relaxation.

(c) Identify the current incumbent solution, if one exists, as well as the best known upper bound and lower bound for the optimal objective value of the original problem, $P0$.

Incumbent solution	Global upper bound	Global lower bound

(d) Is the branch and bound algorithm complete?_______ If so, provide the optimal solution. If not, describe the next step of the algorithm.