Lesson 16. IP Formulations, Part 2

1 Today

- Convex Hull Formulations (*Ideal* Formulations)
- IP Bounds

2 Convex Hull Formulations

2.1 Convex sets review

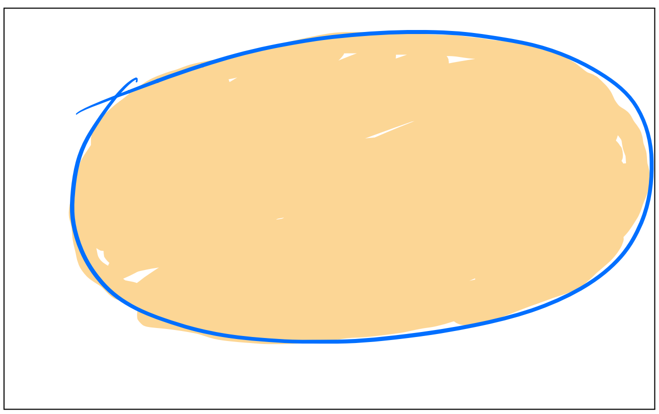
Recall from SA305:

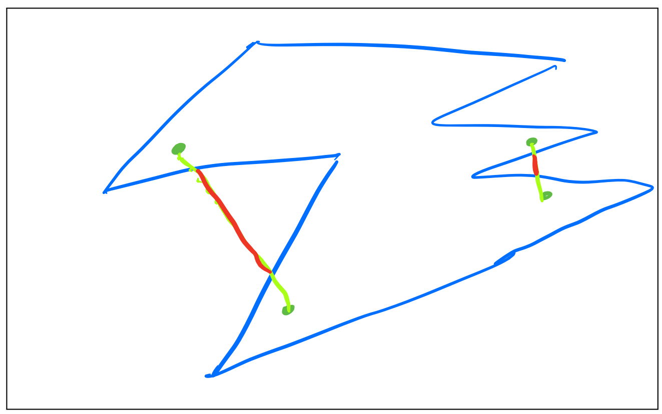
> A region in n-dimensions is **convex** if for *any 2 points*, $\mathbf{x}$ and $\mathbf{w}$, in the region, the line connecting $\mathbf{x}$ and $\mathbf{w}$ is *completely* contained inside the region.

Problem 1. Draw a 2-dimensional example for each.

(a) A region that is convex.

(b) A region is not convex. Illustrate a pair of points $\mathbf{x}$ and $\mathbf{w}$ in the region that demonstrate that the region is not convex.





This is an important property for the simplex algorithm:

> The feasible region of an LP is a convex set.

Question: Is the feasible region of any IP convex?

No! Cannot draw a line between two integer points and remain in the feasible region

2.2 Convex Hull Formulations

> The **convex hull** of a set of integer feasible solutions is the **smallest convex set** that contains all of the points.

For example, we saw two different formulations for the same problem in the last lesson. The inner formulation (below), is the convex hull formulation of the integer feasible region to the IP problem described by both formulations.

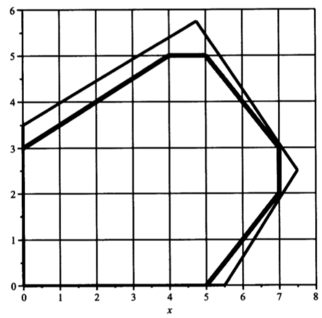
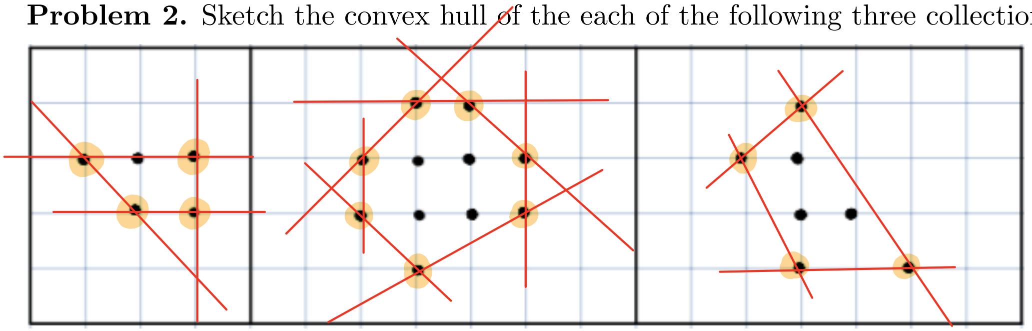


FIGURE 13.2 Feasible region for integer program (13.4).

- Note that *all* linear formulations are necessarily convex.
- It is the *smallness* of the inner formulation that makes it the *convex hull*.
- Notice that all corner points are integer points in the convex hull formulation.

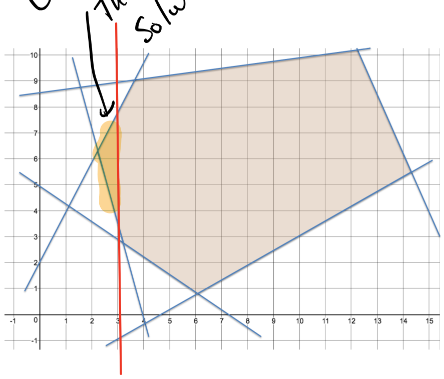
Problem 2. Sketch the convex hull of the each of the following three collections of points.

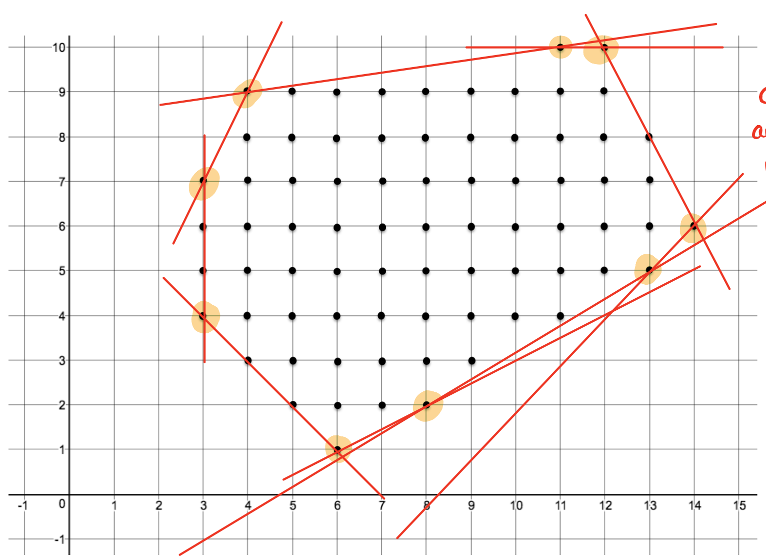


The smallest convex set that contains all points.

Problem 3. A formulation for a set of feasible integer solutions is pictured on the left. The integer solutions are highlighted on the right. Sketch the **convex hull formulation** of this set of solutions.

Cuts off the fractional solutions! :)





All the corner points are integer-valued.

The **convex hull formulation** of a finite set of integer feasible solutions is considered to be the **"ideal"** formulation.

Why?

Because all the corner points are integer-valued.
It's as small as possible.
If we solve it as an LP, then we will be guaranteebly return an integer solution

2.3 Practical considerations

HOWEVER, for most problems we can't use the ideal, convex hull formulation because the number of CONSTRAINTS required to describe the convex hull is often very, very LARGE!, i.e., *exponential* in the number of variables.

What are we to do?...

When choosing which constraints to include in an IP formulation, there is a **tradeoff**:

– use **enough** constraints to make a reasonably tight "container" for the feasible points,

– but **few enough** constraints so the resulting problem is of manageable size.

One strategy is to iteratively add constraints as we need them, to separate *fractional* solutions obtained by solving LP relaxation. We discussed this **separation** strategy in the context of both the

– Min spanning tree problem, and

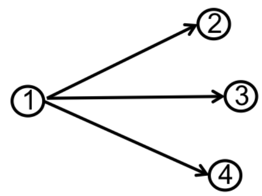
– Vehicle routing problems.

–Travelling Salesperson Problem
(TSP is a special version of the VRP w/ a single driver.)

2.4 Example: Fixed-Charge Weak Vs. Strong Formulations

Many common IP problems have been studied extensively to determine effective modeling strategies. One such problem type is the **fixed-charge facility location problem** that we modeled earlier in the semester.

Problem 4. Suppose there is a possible warehouse at location 1 with maximum capacity C_1, and customers at locations 2, 3, and 4. The binary variable z_1 indicates whether or not facility 1 is used. Integer variables x_{12}, x_{13}, and x_{14} represent the amount of flow on the edges leaving facility 1.



(a) Find two different ways to enforce the requirement that if facility 1 is closed, there is no flow out of facility 1: one that uses 1 constraint, and one that uses 3 constraints.

$X_{12} + X_{13} + X_{14} \leq C_1 Z_1$

If $z_1 = 1$ $\quad X_{12} + X_{13} + X_{14} \leq C_1$

$z_1 = 0$ $\quad X_{12} + X_{13} + X_{14} \leq 0$

WEAK formulation

$X_{12} \leq C_1 Z_1 \quad X_{13} \leq C_1 Z_1 \quad X_{14} \leq C_1 Z_1$

STRONG formulation

Forms a tighter polyhedron are the integer feasible points.

According to our discussion, one of these is the "weak formulation" and one is the "strong formulation". Which is which? Label accordingly.

(b) This bears out in practice. The STRONG formulation is has better performance in IP solvers on large problems. (Although, professional-quality solvers will take care of this during pre-processing.)

In summary:

> Finding the convex hull of an integer program is the gold standard of IP formulations. That said, there are several issues with this:
>
> 1. Exponential number of constraints
> 2. Potential numerical issues with tons of constraints
>
> In general, we do not look for the convex hull. We do, however, use this idea to generate **cuts** when solving IPs.

3 Bounds for IPs

In the next two class periods, we will look at "branch-and-bound", the algorithmic framework that most MIP (mixed-integer *linear* programming) solvers use. A critical component of this algorithm is producing bounds on the integer optimal solution.

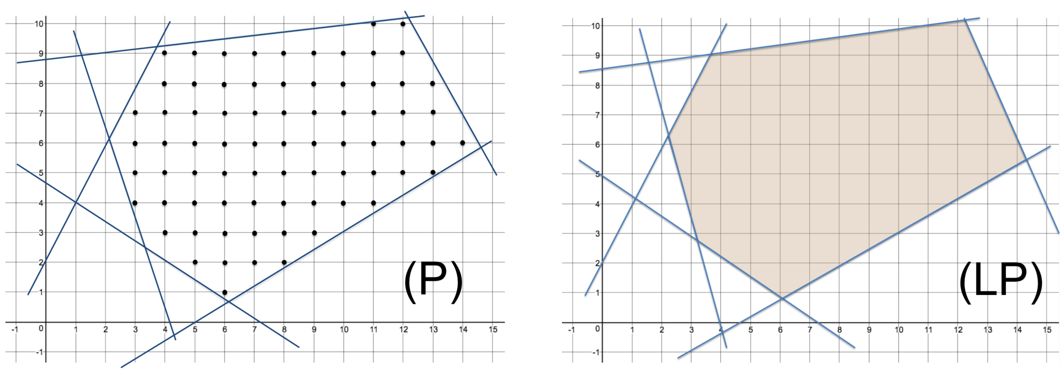
3.1 Upper and lower bounds for IPs

Problem 5. Suppose (P) is an IP with a maximizing objective function,

$$\text{maximize } f(\mathbf{x}) = c_1 x_1 + c_2 x_2,$$

where c_1 and c_2 are integers. The feasible regions of (P) and (LP), the LP relaxation of (P), are pictured below.

$LB \leq z^* \leq UB$



Let z^* be the optimal objective value of (P), which we want to find upper and lower bounds for as part of the branch and bound algorithm.

(a) Suppose we solve the LP relaxation (LP) and get an optimal objective value of 83.9. What can we say about z^* relative to 83.9? Explain.

$z^* \leq 83.9$

The LP has all the feasible solutions that (P) has (and more!), so we can get a better solution.

(b) We already stated that the c_1 and c_2 are integers. What does that tell us about z^*? Explain. *Hint: Suppose* (x_1^*, x_2^*) *is an optimal solution to (P). What do we know about* x_1^* *and* x_2^**?*

x_1^*, x_2^* are integers since (x_1^*, x_2^*) is feasible to (P).

THUS, $z^* = c_1 x_1^* + c_2 x_2^*$ is also an integer.

(c) Combining parts (a) and (b), find a better bound for z^*. Explain.

$z^* \leq 83.9$ & z^* integer

$\hookrightarrow \therefore z^* \leq \lfloor 83.9 \rfloor = 83$ $\quad z^* \leq 83$

(d) Now suppose that $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$, is some *feasible* solution to (P) (not necessarily optimal). What can we say about $f(\hat{x}) = c_1\hat{x}_1 + c_2\hat{x}_2$ relative to z^*? Explain.

(x_1^*, x_2^*) is the best feasible solution, so that means

$f(\hat{x}) = c_1\hat{x}_1 + c_2\hat{x}_2 \leq c_1 x_1^* + c_2 x_2^* = z^*$ $\quad f(\hat{x}) \leq z^*$

3.2 Better formulation leads to better (LP) bounds

The quality of the bound obtained by solving the LP relaxation depends on the formulation:

A **tighter** formulation provides a better bound via its LP relaxation.

3.3 Summary of IP bounds

If (P) is a **maximizing** IP with integer objective coefficients and optimal objective value z^*,

- If z^*_{LP} is the optimal objective value to the LP relaxation of (P), then $\lfloor z^*_{LP} \rfloor$ is a/an upper bound on z^*.
- The objective value for *any feasible* solution to (P) provides a/an lower bound on z^*.

If (P) is a **minimizing** IP with integer objective coefficients and optimal objective value z^*,

- If z^*_{LP} is the optimal objective value to the LP relaxation of (P), then $\lceil z^*_{LP} \rceil$ is a/an lower bound on z^*.
- The objective value for *any feasible* solution to (P) provides a/an upper bound on z^*.

$z^*_{LP} \leq \lceil z^*_{LP} \rceil \leq z^* \leq$ The obj. function value of any feasible solution