Homework 1 – Solutions
SA405, Fall 2018
Instructor: Foraker

You will receive the most benefit from the homework if you only read the solutions after making a sincere effort to solve the problems.

We assume that the linear program under consideration is

$$\begin{array}{ll} \max & c^\top x \\ \text{s.t.} & Ax = b \\ & x \geq 0, \end{array}$$

where $A \in \mathbb{R}^{m \times n}$ is full row rank and that $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$. We also assume the column vectors of A are A_i for $i = 1, \ldots, n$.

1. 8.23 (*Hint: consider the basis matrix associated with a solution with both x^+ and x^-.*): Following the hint, we can see that the basic matrix is singular. Suppose the original column for x was the vector A_x. Then, after the substitution there will be the columns A_x for x^+ and $-A_x$ for x^-. But because $A_x + (-A_x) = 0$, if x^+ and x^- were basic, then the basis would be associated with linearly dependent columns of the constraint matrix, a contradiction.

   Discussion: This problem highlights the importance of the definition of the basic solution and Lemma 7.1. In particular, the argument requires that the columns are linearly independent.

2. 8.24: As stated, the problem allows for the following assumptions. Suppose in a given simplex iteration, the current basic feasible solution is x. Also, suppose that for a nonbasic variable x_k, the simplex direction d^k is improving so that $c^\top d^k > 0$. Finally, suppose that the stepsize is calculated according to the min. ratio test so that x_j is the leaving variable, which implies $d^k_j < 0$. After the pivot, let the new basis be denoted by $\mathcal{B}$.

   Consider the new basic solution after pivoting in x_k and pivoting out x_j. As x_j is now nonbasic, we can construct an associated simplex direction, d^j, which will have:

   $$d^j_j = 1, \quad d^j_i = 0 \text{ for all nonbasic components } i \neq j.$$

   Then, consider the vector $d^k + (-d^k_j)d^j$. The jth component is zero. Moreover, if $d^k + (-d^k_j)d^j \neq 0$, then because $Ad^k = Ad^j = 0$,

   $$0 = Ad^k + (-d^k_j)Ad^j = \sum_i A_i(d^k - d^k_j d^j)_i = \sum_{i \in \mathcal{B}} A_i(d^k_i - d^k_j d^j_i),$$

   a contradiction, as the vectors $\{A_i | i \in \mathcal{B}\}$ are linearly independent. Now, if $d^k + (-d^k_j)d^j = 0$, then

   $$c^\top d^k = d^k_j(c^\top d^j) > 0,$$

   which implies $c^\top d^j < 0$ since $d^k_j < 0$. Thus, entering x_j would not be associated with an **improving** simplex direction.

   Discussion: Note that the definition of the basic solution and Lemma 7.1 is again necessary for this argument, as it relies on the linear independence of the columns of A associated with the basic solution, x^k. The argument also requires a knowledge of how the simplex directions are constructed.

3. 8.27: We must show that $x + \lambda_{\max} d^k$ is a basic feasible solution and can assume $\lambda_{\max} > 0$ and exists, as x is nondegenerate and d^k is not an unbounded improving direction. Using Lemma 7.1, we must show: (a) $A(x + \lambda_{\max} d^k) = b$, $x + \lambda_{\max} d^k \geq 0$; (b) at least $n - m$ components of $x + \lambda_{\max} d^k$ are zero; and (c) there is a basis associated with $x + \lambda_{\max} d^k$ which corresponds to m linearly independent columns of A.

   To see (a), note that the definition of d^k indicates that $Ad^k = 0$. Thus, for all $\lambda \in \mathbb{R}$,

   $$A(x + \lambda d^k) = Ax + \lambda A d^k = b + 0 = b.$$

Also, as $\lambda_{\max} = \min\{-x_i/d_i | d_i < 0, x_i \text{ basic}\}$, for any component $i \in \{1, \ldots, n\}$,

$$x_i + \lambda_{\max} d_i^k \geq x_i - (x_i/d_i)d_i = 0.$$

Note that the inequality holds if $d_i > 0$ as $\lambda > 0$ (this is the case when $i = k$. Also, if x_i is nonbasic but $i \neq k$, then $x_i = d_i = 0$. Therefore, (a) is true.

To see (b), note that $d_i^k = 0$ for all nonbasic components except one, namely $d_k^k = 1$. Also, there is a basic component i such that $-x_i/d_i^k = \lambda_{\max}$, so $x_i + \lambda_{\max} d_i^k = 0$. Thus, the number of nonzero components of $x + \lambda_{\max} d^k$ is at most the same as x. Thus, the number of zero components is at least as many as x, namely, at least $n - m$ as x is a basic feasible solution. Thus, (b) is true.

To see (c), let $\mathcal{B}$ be the basis associated with x and $\widehat{\mathcal{B}}$ the basis associated with $x + \lambda_{\max} d^k$. Note that $k \in \widehat{\mathcal{B}}$ by the definition of d^k. Also, let the indice of the leaving variable be j, so that $x_j > 0$ but $x_j + \lambda_{\max} d_j^k = 0$ and $j \in \mathcal{B}$ but $j \notin \widehat{\mathcal{B}}$. For a contradiction, suppose that the column vectors, $\{A_i | i \in \widehat{\mathcal{B}}\}$ were linearly dependent. Then, there is a nonzero vector v such that $v_i = 0$ for all $i \notin \widehat{\mathcal{B}}$ and

$$Av = \sum_{i \in \widehat{\mathcal{B}}} A_i v_i = 0.$$

If $v_k = 0$ then $\sum_{i \in \mathcal{B}} A_i v_i = 0$, a contradiction. But then, because $d_j^k < 0$ and $v_j = 0$, we have $d^k - v \neq 0$ and so

$$0 = Av + A(-v_k)d^k = \sum_{i \in \mathcal{B}} A_i(v_i - v_k d_i),$$

implying that the vectors $\{A_i | i \in \mathcal{B}\}$ are linearly dependent, a contradiction.

Discussion: The problem emphasizes the importance of linear dependence and independence in the definition of basic solution, as well as the role the min. ratio test plays in maintaining nonnegativity.