Homework 9 – solutions
SA405, Fall 2018
Instructor: Foraker

1. 13.1: One possible formulation is

$$\widehat{X} = \left\{ \begin{aligned} \mathbf{x} \in \{0,1\}^5 : 3x_1 + 2x_2 + 5x_3 + 4x_4 + 2x_5 &\leq 13 \\ x_1 + x_2 + x_3 + x_4 &\leq 3 \end{aligned} \right\}$$

Note that $\widehat{X} \subset X$, which means $\widehat{X}$ is a proper subset of X, by definition and because we have $(1,1,1,0.75,0) \in X$, but $(1,1,1,0.75,0) \notin \widehat{X}$.

2. 13.5: If we add the constraints in P_2 together we get the single constraint in P_1, which implies that $P_2 \subseteq P_1$. To show that $P_2 \subset P_1$, note that the solution $(x_1,x_2,x_3,y) = (1,0,0,\frac{1}{3}) \in P_1$, but $(x_1,x_2,x_3,y) = (1,0,0,\frac{1}{3}) \notin P_2$.

3. 13.9: If we multiply the constraint by λ and place it into the objective function, we get

$$\begin{aligned} L(\lambda) &= \max_{\mathbf{x}\in\{0,1\}^5} 20x_1 + 16x_2 + 25x_3 + 14x_4 + 9x_5 + \lambda(13 - (3x_1 + 2x_2 + 5x_3 + 4x_4 + 2x_5)) \\ &= 13\lambda + \max_{\mathbf{x}\in\{0,1\}^5}((20-3\lambda)x_1 + (16-2\lambda)x_2 + (25-5\lambda)x_3 + (14-4\lambda)x_4 + (9-2\lambda)x_5)) \\ &= \begin{cases} 13\lambda + (84 - 16\lambda), & 0 \leq \lambda \leq \frac{14}{4} \\ 13\lambda + (70 - 12\lambda), & \frac{14}{4} \leq \lambda \leq \frac{9}{2} \\ 13\lambda + (61 - 10\lambda), & \frac{9}{2} \leq \lambda \leq 5 \\ 13\lambda + (36 - 5\lambda), & 5 \leq \lambda \leq \frac{20}{3} \\ 13\lambda + (16 - 2\lambda), & \frac{20}{3} \leq \lambda \leq 8 \\ 13\lambda, & \lambda \geq 8. \end{cases} \end{aligned}$$

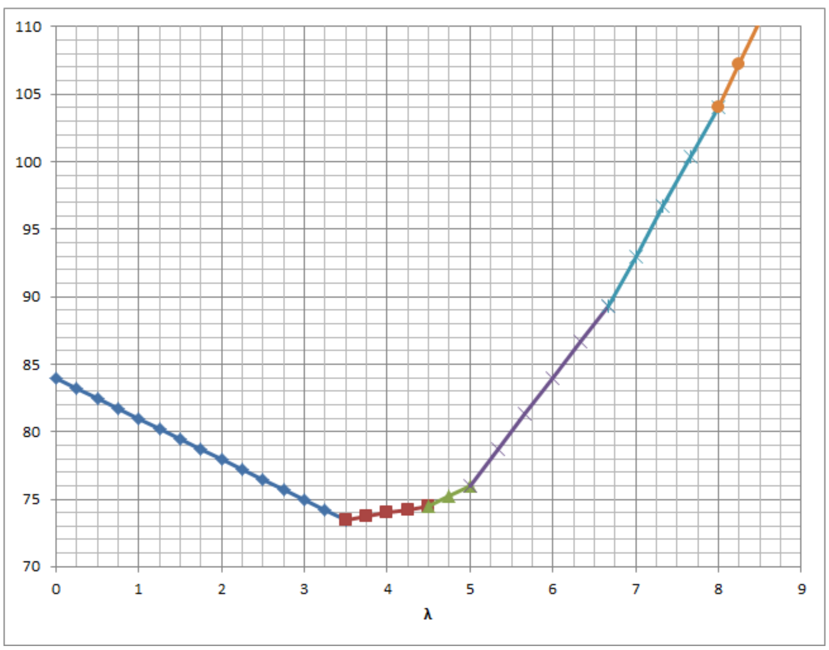

Figure 1: Graph of $L(\lambda)$

It is clear from Figure 1 that the optimal solution occurs at $\lambda = \frac{14}{4}$ with value $L(\frac{14}{4}) = 73.5$.

4. 14.6: We start by considering how to solve knapsack LPs. To use the results of Exercise 9.17, we must first re-order the variables so that

$$\frac{c_1}{a_1} \geq \frac{c_2}{a_2} \geq \cdots \geq \frac{c_n}{a_n}.$$

The formulation of the LP-relaxation is then given by

$$\begin{aligned} \max \quad & 16y_1 + 20y_2 + 25y_3 + 9y_4 + 14y_5 \\ \text{s.t.} \quad & 2y_1 + 3y_2 + 5y_3 + 2y_4 + 4y_5 \leq 13 \\ & 0 \leq y_i \leq 1, i \in \{1, 2, ..., 5\} \end{aligned}$$

where $y_1 = x_2, y_2 = x_1, y_3 = x_3, y_4 = x_5$, and $y_5 = x_4$. This corresponds to the node labeled P1 below.

Problem P1 : Using the results from Exercise 9.17, $r = 5$ and we find the solution to the LP-relaxation as follows

$$(y_1, y_2, y_3, y_4, y_5) = \left(1, 1, 1, 1, \frac{13 - (2 + 3 + 5 + 2)}{4}\right) = (1, 1, 1, 1, \frac{1}{4}),$$

which means $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, \frac{1}{4}, 1)$ with relaxed value 73.5. We then form two subproblems, P2a, which corresponds to $x_4 \leq 0$, i.e., $x_4 = 0$, and P2b, which corresponds to $x_4 \geq 1$, i.e., $x_4 = 1$.

Problem P2a : Branching on $x_4 = 0$ means that we can include all other items in the knapsack, yielding a relaxed solution of $(x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 0, 1)$ with value 70.

Problem P2b : Branching on $x_4 = 1$ gives the following formulation for the LP-relaxation (using the same re-ordering as above)

$$\begin{aligned} \max \quad & 16y_1 + 20y_2 + 25y_3 + 9y_4 \\ \text{s.t.} \quad & 2y_1 + 3y_2 + 5y_3 + 2y_4 \leq 9 \\ & 0 \leq y_i \leq 1, i \in \{1, 2, ..., 4\}. \end{aligned}$$

Then, using the results from Exercise 9.17, $r = 3$ and we find the solution to the LP-relaxation as follows

$$(y_1, y_2, y_3, y_4) = \left(1, 1, \frac{9 - (2 + 3)}{5}, 0\right) = (1, 1, \frac{4}{5}, 0),$$

which means $(x_1, x_2, x_3, x_4, x_5) = (1, 1, \frac{4}{5}, 1, 0)$ with relaxed value 70. However, because the solution from P2a is a candidate integer solution with objective value also 70, we do not need to branch any further.