Practice Problem #18

1 Problems

1. Exercises 13.11 and 13.12 in Rader Textbook.

Lemma 13-6. A matrix A is totally unimodular if:

(1) $a_{ij} \in \{0, 1, -1\}$ $\forall i, j$

(2) Each column contains at most 2 non-zero elements

(3) $\exists$ a partition (M_1, M_2) of the rows of A s.t. for each column j containing exactly 2 non-zero elements,

$$\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0 \quad \forall j \in A$$

13.11 Show A is totally unimodular by lemma 13-6.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(1) is satisfied by inspection, only 0 or 1 exist in A.

(2) is satisfied by inspection, each column has two nonzero entries

(3) Choose partition M_1 = rows 1 and 2. M_2 = rows 3, 4, 5

$$M_1 = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

Check $\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0 \ \forall j$

column 1: $(1+0) - (1+0+0) = 0$ ✓

column 2: $(1+0) - (0+1+0) = 0$ ✓

column 3: $(1+0) - (0+0+1) = 0$ ✓

column 4: $(0+1) - (1+0+0) = 0$ ✓

column 5: $(0+1) - (0+1+0) = 0$ ✓

column 6: $(0+1) - (0+0+1) = 0$ ✓

$\boxed{\text{A is totally unimodular}}$

13.12 show matrix A is totally unimodular by Lemma 13.6

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix}$$

① by inspection, all elements are $\{0, 1, -1\}$

② by inspection, each column has at most 2 nonzero elements

③ choose partition M_1 = rows 1 and 3, M_2 = row 2.

$$M_1 = \begin{bmatrix} 1 & 0 & 1 & -1 & -1 \\ 0 & -1 & -1 & 0 & 1 \end{bmatrix} \quad M_2 = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

check $\sum_{i \in M_1} a_{ij} - \sum_{i \in M_2} a_{ij} = 0 \quad \forall$ colums $j \in A$

column 1: $(1+0) - 1 = 0$ ✓

column 2: $(0-1) - (-1) = 0$ ✓

column 3: $(1-1) - 0 = 0$ ✓

column 4: does not have exactly 2 non-zero elements

column 5: $(-1+1) - 0 = 0$ ✓

$\boxed{\text{A is totally unimodular}}$