Lesson 15. IP Formulations

1 Today

- Convex Hull Formulations (*Ideal* Formulations)
- IP Bounds

2 Convex Hull Formulations

2.1 Convex sets review

Recall from SA305:

> A region in n-dimensions is **convex** if for *any 2 points*, $\mathbf{x}$ and $\mathbf{w}$, in the region, the line connecting $\mathbf{x}$ and $\mathbf{w}$ is *completely* contained inside the region.

Problem 1. Draw a 2-dimensional example for each.

(a) A region that is convex.

(b) A region is not convex. Illustrate a pair of points $\mathbf{x}$ and $\mathbf{w}$ in the region that demonstrate that the region is not convex.

This is an important property for the simplex algorithm:

> The feasible region of an LP is a convex set.

Question: Is the feasible region of any IP convex?

2.2 Convex Hull Formulations

> The **convex hull** of a set of integer feasible solutions is the **smallest convex set** that contains all of the points.

For example, we saw two different formulations for the same problem in the last lesson. The inner formulation (below), is the convex hull formulation of the integer feasible region to the IP problem described by both formulations.

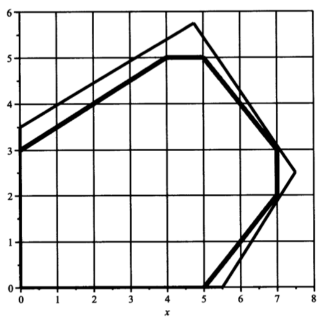
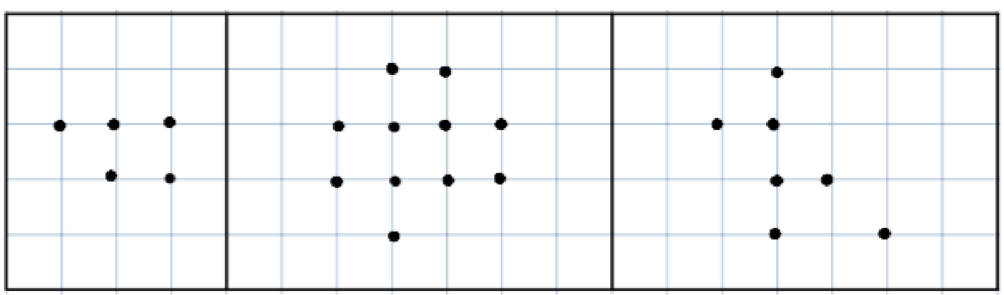


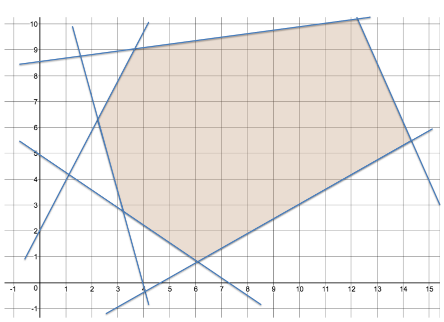
FIGURE 13.2 Feasible region for integer program (13.4).

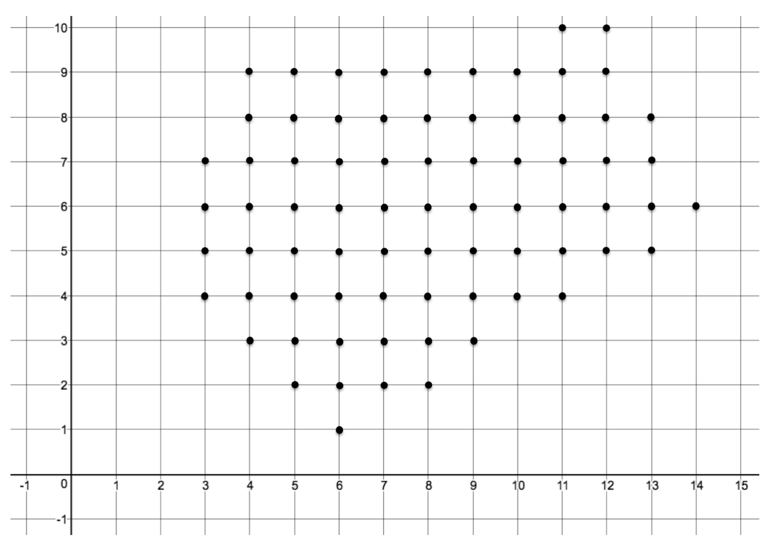
- Note that *all* linear formulations are necessarily convex.
- It is the *smallness* of the inner formulation that makes it the *convex hull*.
- Notice that all corner points are integer points in the convex hull formulation.

Problem 2. Sketch the convex hull of the each of the following three collections of points.



Problem 3. A formulation for a set of feasible integer solutions is pictured on the left. The integer solutions are highlighted on the right. Sketch the **convex hull formulation** of this set of solutions.





The **convex hull formulation** of a finite set of integer feasible solutions is considered to be the **"ideal"** formulation.

Why?

2.3 Practical considerations

HOWEVER, for most problems we can't use the ideal, convex hull formulation because the number of ______ required to describe the convex hull is often very, very ______, i.e., *exponential* in the number of variables.

What are we to do?...

When choosing which constraints to include in an IP formulation, there is a **tradeoff**:

– use **enough** constraints to make a reasonably tight "container" for the feasible points,

– but **few enough** constraints so the resulting problem is of manageable size.

One strategy is to iteratively add constraints as we need them, to ______ *fractional* solutions obtained by solving LP ______. We discussed this **separation** strategy in the context of both the

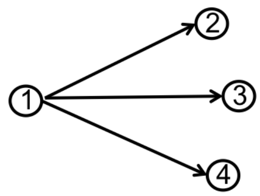
– ______ ______ problem, and

– ______ ______ problems.

2.4 Example: Fixed-Charge Weak Vs. Strong Formulations

Many common IP problems have been studied extensively to determine effective modeling strategies. One such problem type is the **fixed-charge facility location problem** that we modeled earlier in the semester.

Problem 4. Suppose there is a possible warehouse at location 1 with maximum capacity C_1, and customers at locations 2, 3, and 4. The binary variable z_1 indicates whether or not facility 1 is used. Integer variables x_{12}, x_{13}, and x_{14} represent the amount of flow on the edges leaving facility 1.



(a) Find two different ways to enforce the requirement that if facility 1 is closed, there is no flow out of facility 1: one that uses 1 constraint, and one that uses 3 constraints.

________ formulation   ________ formulation

According to our discussion, one of these is the "weak formulation" and one is the "strong formulation". Which is which? Label accordingly.

(b) This bears out in practice. The ________ formulation is has better performance in IP solvers on large problems. (Although, professional-quality solvers will take care of this during pre-processing.)

In summary:

> Finding the convex hull of an integer program is the gold standard of IP formulations. That said, there are several issues with this:
>
> 1. Exponential number of constraints
> 2. Potential numerical issues with tons of constraints
>
> In general, we do not look for the convex hull. We do, however, use this idea to generate **cuts** when solving IPs.

3 Bounds for IPs

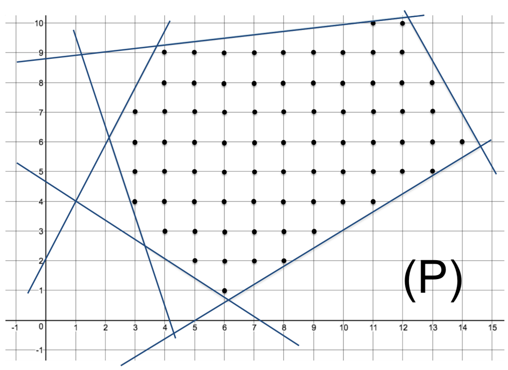
In the next two class periods, we will look at "branch-and-bound", the algorithmic framework that most MIP (mixed-integer *linear* programming) solvers use. A critical component of this algorithm is producing bounds on the integer optimal solution.

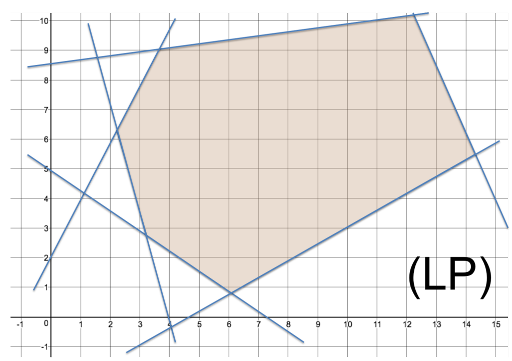
3.1 Upper and lower bounds for IPs

Problem 5. Suppose (P) is an IP with a maximizing objective function,

$$\text{maximize } f(\mathbf{x}) = c_1x_1 + c_2x_2,$$

where c_1 and c_2 are integers. The feasible regions of (P) and (LP), the LP relaxation of (P), are pictured below.





Let z^* be the optimal objective value of (P), which we want to find upper and lower bounds for as part of the branch and bound algorithm.

(a) Suppose we solve the LP relaxation (LP) and get an optimal objective value of 83.9. What can we say about z^* relative to 83.9? Explain.

(b) We already stated that the c_1 and c_2 are integers. What does that tell us about z^*? Explain. *Hint: Suppose* (x_1^*, x_2^*) *is an optimal solution to (P). What do we know about* x_1^* *and* x_2^**?*

(c) Combining parts (a) and (b), find a better bound for z^*. Explain.

(d) Now suppose that $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)$, is some *feasible* solution to (P) (not necessarily optimal). What can we say about $f(\hat{x}) = c_1\hat{x}_1 + c_2\hat{x}_2$ relative to z^*? Explain.

3.2 Better formulation leads to better (LP) bounds

The quality of the bound obtained by solving the LP relaxation depends on the formulation:

A **tighter** formulation provides a ____ bound via its LP relaxation.

3.3 Summary of IP bounds

If (P) is a **maximizing** IP with integer objective coefficients and optimal objective value z^*,

- If z^*_{LP} is the optimal objective value to the LP relaxation of (P), then ____ is a/an ____ bound on z^*.
- The objective value for *any feasible* solution to (P) provides a/an ____ bound on z^*.

If (P) is a **minimizing** IP with integer objective coefficients and optimal objective value z^*,

- If z^*_{LP} is the optimal objective value to the LP relaxation of (P), then ____ is a/an ____ bound on z^*.
- The objective value for *any feasible* solution to (P) provides a/an ____ bound on z^*.