

Lesson 14. IP Formulations

1 Today

- LP review
- IP formulations

2 Solving Integer Programs can be *Really* Hard!

The following integer (linear) program (IP) seeks an objective-maximizing integer linear combination of a big number.

$$\begin{aligned} &\text{maximize} && 213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5 \\ &\text{subject to} && 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 = 89643482 \\ &&& x_1, x_2, x_3, x_4, x_5 \geq 0, \text{ integer} \end{aligned}$$

Problem 1. We will solve two versions of the problem above. In both cases, use the `tee=True` flag to see the solver output in Jupyter as follows:

```
solver_result = pyo.SolverFactory('glpk').solve(model, tee=True)
```

- (a) First solve the **LP relaxation** of the IP, which means allowing the variables to be continuous rather than integer-valued: `domain=pyo.NonNegativeReals`

Does the solver arrive at an optimal solution? If so, list it here. If not, what happens?

- (b) Now solve the IP as written, which means requiring the variables to take integer values: `domain=pyo.NonNegativeIntegers`

Does the solver arrive at an optimal solution? If so, list it here. If not, what happens?

In general, s are much harder to solve than s.

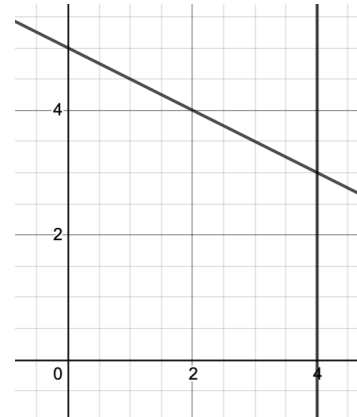
- This week we will discuss why this is, and why the way we model IP problems (yes, there are choices!) can impact solver performance.
- Next week, we will learn about the **branch and bound algorithm** (B&B), which is used by most IP solvers. It is significantly more computationally-expensive than the LP simplex method.

3 LP Review

Problem 2. Solve the following LP graphically: Shade the feasible region, draw two objective contours: $x_1 + x_2 = 2$ and $x_1 + x_2 = 4$. Use arrows to indicate the direction of an increasing objective value. Label the optimal solution.

$$\begin{aligned} \max \quad & x_1 + x_2 \\ \text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

What is the optimal objective value?



Theorem: Every linear program (LP) has EXACTLY ONE of the following outcomes:

- | | |
|--------------------------------|----------------|
| (1) Unique optimal solution | (3) Unbounded |
| (2) Multiple optimal solutions | (4) Infeasible |

Problem 3. Sketch graphs that illustrate each of the LP outcomes.

Theorem: Integer programs (IPs) have the same four possible outcomes as LPs:

- | | |
|--------------------------------|----------------|
| (1) Unique optimal solution | (3) Unbounded |
| (2) Multiple optimal solutions | (4) Infeasible |

Theorem: If an LP has an optimal solution (the LP is not unbounded or infeasible), an optimal solution can always be found at a of the feasible region.

Question: Is the same true for IPs? In other words, if an optimal solution to an IP exists, can an optimal solution always be found at a corner point?

- Keep this question in mind. We will revisit it a little later.

4 IP Formulations

A **formulation** of an IP is a set of linear that capture ALL of the integer points, and NO OTHER integer points.

- We will see some examples of formulations of an IP in the next problem.

The **LP relaxation** of an IP is the LP that is formed by *relaxing* the integer requirement on the variables.

Problem 4. Below are two integer programs, along with the diagrams of their constraints. (Rader, examples 13.3, 13.4)

Problem A:

$$\begin{aligned} &\text{maximize} && 8x + 7y \\ &\text{subject to} && -18x + 38y \leq 133 \\ &&& 13x + 11y \leq 125 \\ &&& 10x - 8y \leq 55 \\ &&& x, y \in \mathbb{Z}^{\geq 0} \end{aligned}$$

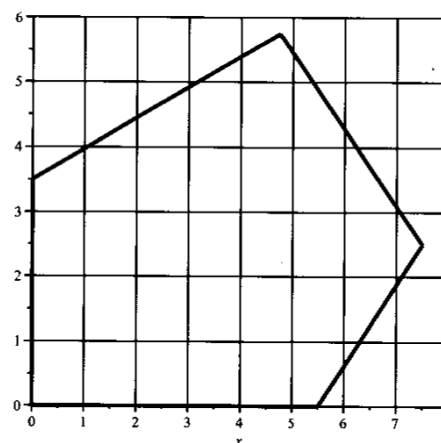


FIGURE 13.1 Feasible region for integer program (13.3).

Problem B:

$$\begin{aligned} &\text{maximize} && 8x + 7y \\ &\text{subject to} && -x + 2y \leq 6 \\ &&& x + y \leq 10 \\ &&& x - y \leq 5 \\ &&& x \leq 7 \\ &&& y \leq 5 \\ &&& x, y \in \mathbb{Z}^{\geq 0} \end{aligned}$$

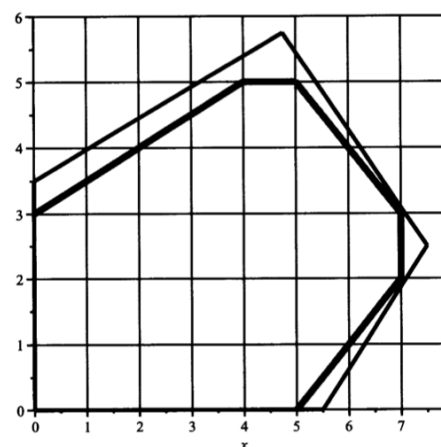


FIGURE 13.2 Feasible region for integer program (13.4).

(a) On the diagrams, identify all feasible solutions to both IPs.

(b) Are the feasible regions for problems A and B different or the same?

(c) What does this mean about the two problems A and B?

(d) Are the LP relaxations of A and B the same? If not, which has the higher optimal objective value?

Problem 5. Refer back to the previous problem to answer the following:

(a) Suppose an optimal solution to an IP exists.

i. Can an optimal solution *always* be found at a corner point? (The question from before.)

ii. Can an optimal solution *sometimes* be found at a corner point?

(b) Which of the two formulations of the IP in problem 3 is easier to solve? Why?

5 Comparing IP Formulations

- Often the decision of how to formulate an IP comes down to a trade-off between quality of formulation and number of constraints:

- constraints means a better (tighter) formulation, but too constraints can cause memory pressure and slow down the solver. (More/Fewer, many/few)
- constraints means fewer memory problems, but results in a formulation that is not as good. (More/Fewer)

TAKEAWAY:

The way we choose to the feasible region of an IP can have a big impact on solver performance.