Lesson 1: Mathematical Modeling Review

1 Goals

- Write a concrete Linear Programming (LP) model.
- Introduce an integrality requirement on variables.
- Convert the linear program to abstract form.

2 Concrete Model

Chelsea is heading out on a camping trip, and she wants to carry only one pack that has 5.3 ft^3 of volumetric space. To keep from hurting her back, she needs to make sure that the contents of her backpack weigh no more than 12.5 lbs. You can assume the backpack weight is negligible. See the list of items that she is able to bring:

ID	Item	Volume (ft^3)	Usefulness Factor	Weight (lbs.)
1	Rope	2	1	3
2	Matches	0.01	5	0.1
3	Tent	3	7	10
4	Sleeping bag	2	6	4
5	Hammock	0.4	4.5	4
6	Granola bars	0.67	8	2

Problem 1. Write a concrete linear program whose solution maximizes the usefulness of the contents of Chelsea's bag given volume and weight requirements.

a) Define decision variables and the describe (in words) the objective function and the role of each constraint.

x_1 = # of Ropes she brings
x_2 = # of Matches she brings
x_3 = # of Tents she brings
x_4 = # of Sleeping bags she brings
x_5 = # of Hammocks she brings
x_6 = # of Granola Bars she brings

⇒ Maximize the total usefulness of all the items Chelsea packs.

→ Cannot exceed the total weight of 12.5 lbs.

→ Cannot exceed the volume of 5.3 cubic feet.

b) Write the concrete model.

Maximize $x_1 + 5x_2 + 7x_3 + 6x_4 + 4.5x_5 + 8x_6$

s.t.

$$3x_1 + 0.1x_2 + 10x_3 + 4x_4 + 4x_5 + 5x_6 \le 12.5$$

$$2x_1 + 0.01x_2 + 3x_3 + 2x_4 + 0.4x_5 + 0.67x_6 \le 5.3$$

$$x_1, x_2, x_3, x_4, x_5, x_6 \ge 0$$

3 Understanding Integrality Restrictions

- **Continuous Linear Program (LP):** Suppose that Chelsea is allowed to bring fractional amounts of each item, so that *variables can take on any nonnegative values.* Let z be the optimal objective function value to this problem.
- **Integer Linear Program (IP):** Suppose that Chelsea can either bring the entire item or not, so that *variable values are restricted to 0 or 1.* Let $\bar{z}$ be the optimal objective function value to this problem.

Problem 2. How does z compare to $\bar{z}$? Provide justification for your response.

$$z \ge \bar{z}$$

We know that any solution to the IP is also feasible to the LP. But, the opposite is not necessarily true. Some, but not all LP solutions are feasible to the IP. Therefore, if we allow fractional solutions, then our feasible region gets larger, and we can only improve our objective (Increase our objective function value.)

4 Convert to Abstract Notation

Problem 3. Assuming integrality restrictions, convert your model to abstract notation. Clearly define all sets, parameters, and decision variables.

Sets

$I :=$ Set of items Chelsea can bring

Parameters

$u_i \ \forall i \in I :=$ the usefulness of item $i \in I$.

$w_i \ \forall i \in I :=$ the weight of item $i \in I$.

$v_i \ \forall i \in I :=$ the volume of item $i \in I$.

$W :=$ Max weight allowed

$V :=$ Max volume allowed

Variables

$x_i \in \mathbb{Z}^+ \ \forall i \in I$

↳ The number of item $i \in I$ that Chelsea brings.

$$\text{Max} \sum_{i \in I} u_i x_i$$

s.t.

$$\sum_{i \in I} w_i x_i \leq W$$

$$\sum_{i \in I} v_i x_i \leq V$$

$$x_i \in \mathbb{Z}^+$$

How could I further make this abstract?

5 Next time...

In the next lesson we will implement this abstract model in a `Jupyter` notebook using `Pyomo`, and solve it using `GLPK`. Before the next class, make sure that you have `Python`, `Pyomo`, `GLPK`, and `Jupyter` on your computer. You used this set-up for SA305 and/or SM286D. If you need to install or reinstall any of these, see the instructions provided with Lesson 1. (*Be careful not to install a second version of* `Anaconda`*, because this can create problems.* If you need to reinstall, uninstall everything first.)