Lesson 13. Facility Location

1 Today...

– Facility Location Introduction

– Two Variations of the Facility Location Problem

○ Set Covering Location Problem

○ Maximal Covering Location Problem

2 Facility Location Introduction

In these problems, the **input data** is...

– a network of *customers* (demand nodes),

– a set of **possible** *facilities* (supply nodes),

– a set of edges between customers and facilities that **could** serve them

– distances on the edges (which could represent distance, time, cost, or some combination of these factors)

The **goal** is to choose a set of supply facilities to serve the customers' demand based on some metric(s).

– For example: minimize the number of supply facilities opened while requiring that all customers are served.

– Real world problems of this type include locating

○ military installations,

○ fire/police stations,

○ cell phone towers,

○ retail distribution centers and stores,

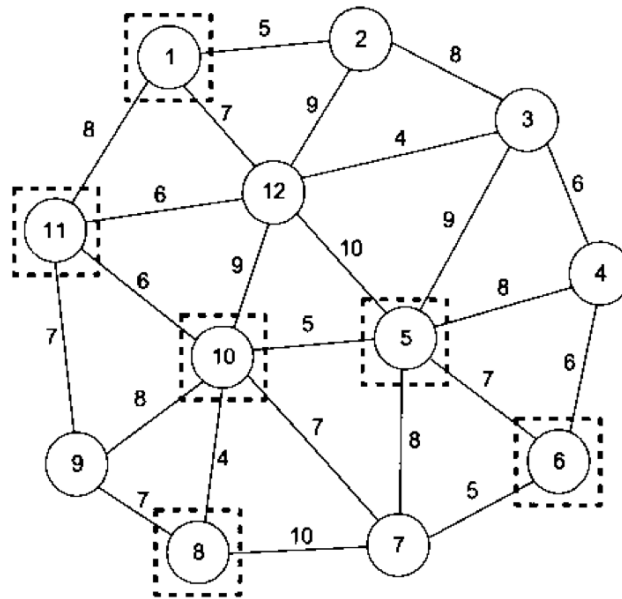○ schools.

3 *General* Facility Location Problems

FIGURE 4.1 Example problem used for facility location models.

General Facility Location Problem goal:

Choose a set of supply facilities to meet customer demand according to some metric.

Notation:

Sets:

$C =$ set of customer nodes

$S =$ set of possible supply nodes

$E =$ edges (c, s) connecting a customer c with a supply facility s that *could* serve the customer

Parameters:

$d_{cs} =$ distance (or cost or time) between customer c and supply location s, for $(c, s) \in E$

$h_c =$ demand of customer c, for $c \in C$

Decision Variables:

$$x_s = \begin{cases} 1 \text{ if } \boxed{\text{facility } s \text{ is opened}} \\ 0 \text{ otherwise} \end{cases} \text{, for all } s \in S$$

Problem 1. We will use the network and data on the following page for all of our example problems.

(a) *All* vertices represent customers. *Boxed* vertices represent possible supply locations. Use set notation to list the elements of the sets C and S.

$$C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\} \qquad S = \{1, 5, 6, 8, 10, 11\}$$

(b) The distance between a customer c and a supplier s is the length of the shortest path between them. Find $d_{1,1}$, $d_{4,1}$, and $d_{8,5}$. (Note that the edges in the model do not correspond directly to the edges in the graph.)

(Just eyeball it for now)

$$d_{11} = 0 \qquad d_{41} = 17 \qquad d_{85} = 9$$

(c) How should the columns and rows in the distance matrix be labeled? Do your answers in part (a) agree with the corresponding values in the distance matrix?

See Below! ☺

DATA for FACILITY LOCATION EXAMPLES:

FIGURE 4.1 Example problem used for facility location models.

$$
\mathbf{d} =
\begin{array}{c|cccccc}
 & 1 & 5 & 6 & 8 & 10 & 11 \\
\hline
1 & 0 & 17 & 23 & 18 & 14 & 8 \\
2 & 5 & 17 & 20 & 22 & 18 & 13 \\
3 & 11 & 9 & 12 & 17 & 13 & 10 \\
4 & 17 & 8 & 6 & 17 & 13 & 16 \\
5 & 17 & 0 & 7 & 9 & 5 & 11 \\
6 & 23 & 7 & 0 & 15 & 12 & 18 \\
7 & 21 & 8 & 5 & 10 & 7 & 13 \\
8 & 18 & 9 & 15 & 0 & 4 & 10 \\
9 & 15 & 13 & 20 & 7 & 8 & 7 \\
10 & 14 & 5 & 12 & 4 & 0 & 6 \\
11 & 8 & 11 & 18 & 10 & 6 & 0 \\
12 & 7 & 10 & 16 & 13 & 9 & 6 \\
\end{array}.
$$

DEMANDS

$\mathbf{h} = (100,\ 90,\ 110,\ 120,\ 80,\ 100,\ 95,\ 75,\ 110,\ 90,\ 120,\ 85).$

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4   *Set Covering* Facility Location Problem

Goal of the **set covering facility location problem**:

> Open the fewest number of facilities so that every customer is "covered" by some facility.

Problem 2. Complete the set covering facility location formulation below by adding the objective function description (1) and the missing constraint (2).

Set covering facility location model

***New* Sets:**

N_c = the facilities in S that *could* serve customer $\forall c \in C$

($N_c \subseteq S$, for all $c \in C$. N_c is called the "neighborhood" of c.)

Objective and constraint descriptions:

(1) Minimize the total # of opened facilities

(2) Ensure that customer c is covered by an open facility, for all $c \in C$.

$$\text{minimize} \quad \sum_{s \in S} x_s \tag{1}$$

$$\text{subject to} \quad \boxed{\sum_{s \in N_c} x_s \geq 1} \text{, for all } \boxed{c \in C} \tag{2}$$

$$x_s \in \{0, 1\}, \text{ for } s \in S$$

4.1 Example: Set Covering Facility Location (Neighborhoods determined by distance.)

Problem 3. Find the minimum number of facilities required to serve all customers. A facility must be within $D = 9$ miles of a customer in order to serve the customer.

(a) For each customer c, the neighborhood of N_c is the set of facilities that can cover customer c:

$$N_c = \left\{ s \in S : \boxed{d_{c,s} \leq D} \right\}, \quad \text{for all } c \in C.$$

$$\hookrightarrow D = 9$$

(b) Complete the missing neighborhoods.
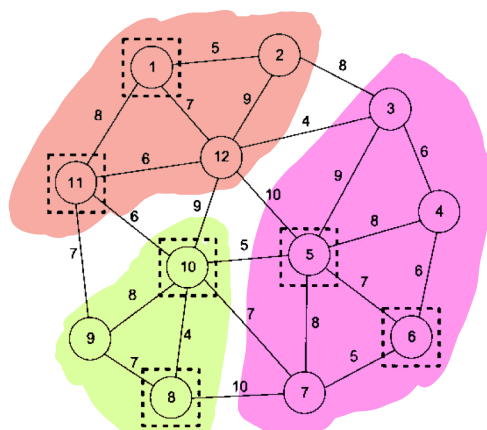
FIGURE 4.1 Example problem used for facility location models.

$$N_1 = \{1, 11\} \qquad\qquad N_7 = \{5, 6, 10\}$$

$$N_2 = \{1\} \qquad\qquad N_8 = \boxed{\{5, 8, 10\}}$$

$$N_3 = \boxed{\{5\}} \qquad\qquad N_9 = \{8, 10, 11\}$$

$$N_4 = \{5, 6\} \qquad\qquad N_{10} = \boxed{\{5, 8, 10, 11\}}$$

$$N_5 = \{5, 6, 8, 10\} \qquad\qquad N_{11} = \{1, 10, 11\}$$

$$N_6 = \{5, 6\} \qquad\qquad N_{12} = \boxed{\{1, 10, 11\}}$$

Facility 1 covers: 1, 2, 11, 12
Facility 5 covers: 3, 3, 4, 6, 7, 10   Facility 8 covers 8, 9, 10

(c) How many facilities are required? List the values of the decision variables that correspond to an optimal feasible solution.

Three facilities. Must open ①, ⑤ and one of ⑧, ⑩, or ⑪.

(d) Write an abbreviated version of the concrete model.

$$\text{Min } X_1 + X_5 + X_8 + X_{10} + X_{11}$$

$$\text{s.t.} \quad X_1 + X_{11} \geq 1 \quad (\text{Customer } \#1)$$
$$X_1 \geq 1 \quad (\text{Customer } \#2)$$
$$X_5 \geq 1 \quad (\text{Customer } \#3)$$
$$\vdots$$
$$X_1 + X_{10} + X_{11} \geq 1 \quad (\text{Customer } \#12)$$

$$\sum_{s \in N_{12}} X_s \geq 1$$

$$X_s \in \{0, 1\} \ \forall s \in S$$

5   *Maximal Covering* Location Problem

The goal of the **maximal covering location problem**:

> Select the p facilities to open in order to maximize the customer demand that is covered. (A customer, c, can only be covered by a supply facility, s, in its neighborhood: $s \in N_c$.)

Problem 4. Complete the maximal covering facility location formulation below by adding the objective function (3), the missing constraint (4), and the description for constraint (5).

Maximal covering facility location model

New Parameters:

h_c = the demand at customer c, for all $c \in C$

p = the number of facilities to open

New Decision Variables:

$$y_c = \begin{cases} 1 \text{ if a facility } s \text{ in the neighborhood of } c \text{ has been selected} \\ 0 \text{ otherwise} \end{cases}, \text{ for all } c \in C$$

Objective and constraint descriptions:

(3) Maximize the total customer demand that is covered.

(4) Ensure that exactly p facilities are opened.

(5) *If no supply facility in the neighborhood of customer c is selected, then customer c is not covered $(y_c = 0)$, $\forall c \in C$*

$$\text{maximize} \quad \boxed{\sum_{c \in C} h_c y_c} \tag{3}$$

$$\text{subject to} \quad \boxed{\sum_{s \in S} x_s = p} \tag{4}$$

$$\sum_{s \in N_c} x_s \geq y_c, \text{ for } c \in C \tag{5}$$

Counts the # of facilities selected in the neighborhood of c. If the count is 0, then the constraint is: $0 \geq y_c$.

$$x_s \in \{0, 1\}, \text{ for } s \in S$$

$$y_c \in \{0, 1\}, \text{ for } c \in C$$

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5.1 Example: Maximal Covering Facility Location Problem

Assume the neighborhoods are defined as before...

Problem 5. We saw in the last example that every customer's demand can be met by three facilities. Suppose that we can only afford to build and maintain $p = 2$ facilities, and the demand values for the customers (in order) are

$$\mathbf{h} = (100, 90, 110, 120, 80, 100, 95, 75, 110, 90, 120, 85).$$

Opening which two facilities will allow us to cover the most total customer demand?

(a) The optimal solution is to choose facilities 5 and 11. List the values of the decision variables x_s and y_c in the optimal solution. Illustrate the solution on the graph of the network.

$$X_5 = X_{11} = 1, \quad X_1 = X_6 = X_8 = X_{10} = 0$$

$$Y_1 = Y_3 = Y_4 = Y_5 = Y_6 = Y_7 = Y_8 = Y_9 = Y_{10} = Y_{11} = Y_{12} = 1$$
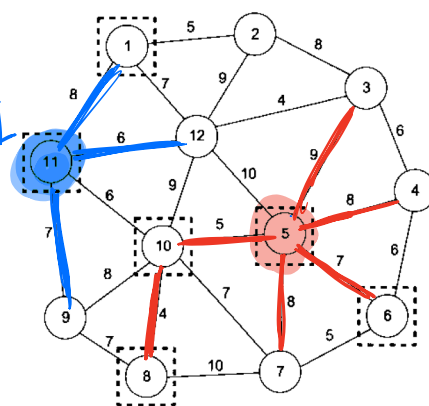
$$Y_2 = 0$$

FIGURE 4.1 Example problem used for facility location models.

(b) Find the optimal objective function value. What does it mean?

> The total amount of demand met $=$

(c) Write concrete versions of the following:

  i objective function:

> $$\text{Max} \quad 100 y_1 + 90 y_2 + \ldots + 85 y_{12}$$

  ii constraint (4):

> $$X_1 + X_5 + X_6 + X_8 + X_{10} + X_{12} = 2$$

  iii constraints (5) for customers 4 and 8:

> $$Y_4 \leq X_5 + X_6$$
> $$Y_8 \leq X_5 + X_8 + X_{10}$$

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