Lesson 3. Network Flows

1 Today...

- We model two **network flow problems**:
  - a **transportation problem**, and
  - a **minimum cost network flow problem**.

2 Transportation Model

(Example 2.11, p.57) A local baked goods company has two bakeries where they bake their goods, which they then ship to three different area stores to sell. Each bakery can produce up to 50 truckloads of baked goods per week, and each bakery can supply any of the stores. The weekly demands (in truckloads) anticipated at each store along with the transportation costs (per truckload) are provided in the tables below. Note that partial truckloads cost just as much as full truckloads. How many truckloads should be sent from each bakery to each store in order to minimize total shipping cost?

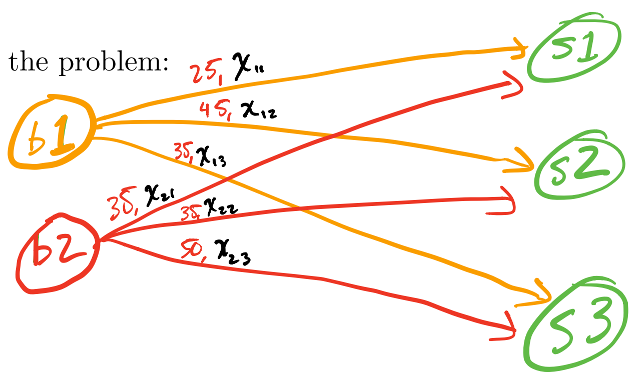
	Demand
Store 1	30
Store 2	20
Store 3	40

	Store 1	Store 2	Store 3
Bakery 1	\$20	\$45	\$35
Bakery 2	\$35	\$35	\$50

2.1 Graphical representation of the network.

- Bakeries and stores are represented by *nodes* (or *vertices*). The bakeries are called **supply nodes**, and the stores are called **demand nodes**. Label each node ($b1$, $b2$, $s1$, $s2$, $s3$). Write the supply or demand amounts next to each node.
- **Directed arcs** (or **directed edges**) represent flow of goods. Use an arrow on the demand end of each arc to indicate the direction of the flow of goods. Label each arc with the cost per truckload.

Draw a **directed graph** to represent the problem:



Find a feasible (not necessarily optimal) solution, just to get a feel for the problem.

bakery 1 30 → store #1
20 → store #2

bakery 2 40 → store #3

2.2 Concrete model: transportation problem

Formulate a concrete model that finds a feasible transportation strategy that minimizes cost:

- Define decision variables.

x_{11} = # of truckloads from bakery #1 to store #1
x_{12} = " " " " " " " " " #2
x_{13} = " " " " " " " " " #3

Same for
x_{21}
x_{22}
x_{23}

- Add the decision variables to the network diagram on page 1.
- Describe objective and constraints.

Minimize the total cost to ship truckloads from bakeries to stores.

Constraints
→ Non-negativity and integer
→ Must meet demand at each store

- Write a concrete model for this problem using the decision variables defined above.

Minimize $20x_{11} + 45x_{12} + 35x_{13} + 35x_{21} + 35x_{22} + 50x_{23}$

s.t.

$$x_{11} + x_{21} \geq 30$$

$$x_{12} + x_{22} \geq 20$$

$$x_{13} + x_{23} \geq 40$$

$$x_{11}, x_{21}, x_{12}, x_{22}, x_{13}, x_{23} \geq 0 \text{ INTEGER}$$

2.3 Abstract model: tranportation problem

Formulate an abstract model for the bakery problem. Clearly define sets, parameters, and variables.

SETS

$B :=$ Set of Bakeries

$B := \{b1, b2\}$

$S :=$ Set of Stores

$S := \{s1, s2, s3\}$

PARAMETERS

$c_{ij} \geq 0 \quad \forall i \in B, j \in S$

↳ cost to transmit a single truckload from bakery i to store j.

$d_j \geq 0 \quad \forall j \in S$

↳ the demand in truckloads at each store $j \in S$.

VARIABLES

$x_{ij} \geq 0$ Integer $\forall i \in B, j \in S$

= the # of truckloads shipped from bakery i to store j.

$$\text{Minimize} \sum_{i \in B, j \in S} c_{ij} x_{ij}$$

$$\text{s.t.} \quad \sum_{i \in B} x_{ij} \geq d_j \quad \forall j \in S$$

$$x_{ij} \geq 0 \text{ Integer } \forall i \in B, j \in S.$$

2.4 Balancing supply and demand

- When *total supply* > *total demand*, we can model and solve the problem, as we saw in the bakery problem. When *total supply* < *total demand*, the problem is $\boxed{\text{INFEASIBLE}}$, but we still might want to solve the problem to meet as much demand as possible at a minimum cost.
- When *total supply* = *total demand*, we can use = in the place of all inequalities in our constraints and we can write all of our constraints using the same format (we will see this soon), which is really nice.

We can always write a balanced problem with *total supply* = *total demand* by adding a **dummy node** and **dummy arcs** with well-chosen parameter values (supply/demand and arc costs).

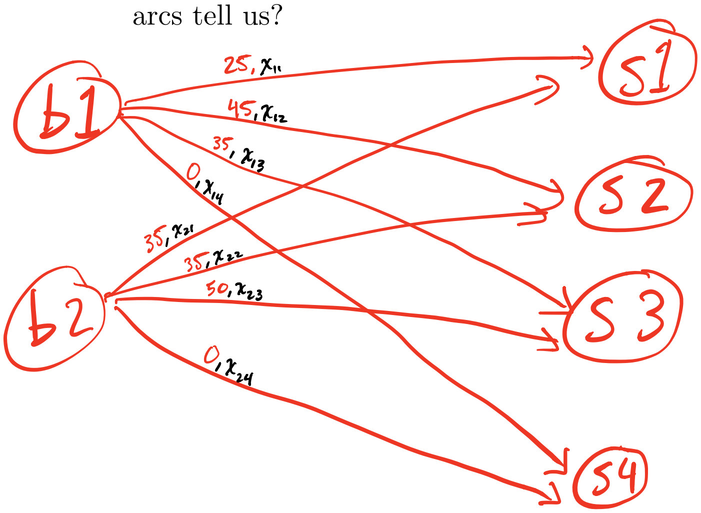
Problem 1.

(a) In the original bakery problem, we had $100 = \textit{total supply} > \textit{total demand} = 90$. We wish to add a dummy node and arcs so that supply and demand are balanced and that the original arc variables still provide the optimal solution to the original problem.

(i) Add a dummy node, along with its supply or demand. Also add any dummy arcs entering or leaving the node, as well as their shipping costs and associated variables.

(ii) In an optimal solution, what do the values of the variables associated with the dummy arcs tell us?

They tell us how much of the stock at each bakery will be leftover---not sent to a store.

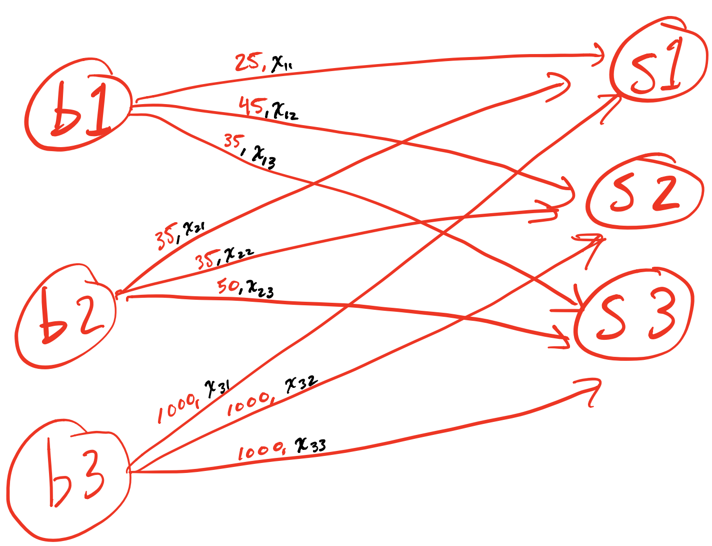


(b) Now suppose we have the original bakery problem, but now the demand at each store increases by 10, so $\textit{total supply} = 100$, and $\textit{total demand} = 120$. We still want to know the minimum cost way to meet as much demand as possible.

(i) Add a dummy node, along with its supply or demand to accurately capture the logic of the problem. Also add any dummy arcs entering or leaving the node, as well as their shipping costs and associated variables.

(ii) In an optimal solution, what do the values of the variables associated with the dummy arcs tell us?

These variables refer to the amount of demand that cannot be met at one of the stores.



3 Minimum cost network flow model

(This is similar to Example 2.12, p 60) Consider the bakery again, only this time there are two warehouses that baked goods must be delivered to before they are taken to their final destinations. The warehouses ($w1$ and $w2$) have no supply nor demand. These are the transportation costs:

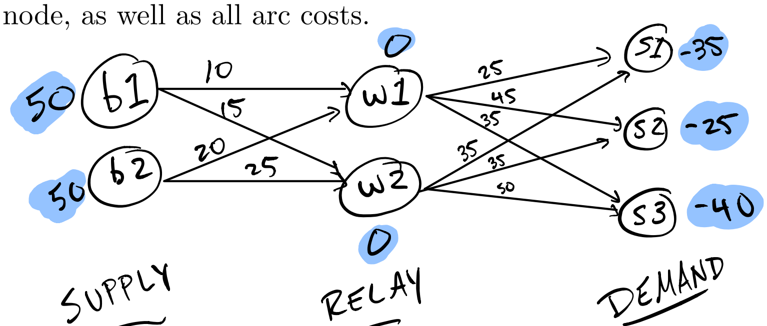
	Warehouse 1	Warehouse 2
Bakery 1	\$10	\$15
Bakery 2	\$20	\$25

	Store 1	Store 2	Store 3
Warehouse 1	\$20	\$45	\$35
Warehouse 2	\$35	\$35	\$50

Also, assume that the store demands are $d_{s1} = 35$, $d_{s2} = 25$, and $d_{s3} = 40$, so that *total supply* = *total demand* = 100.

- The warehouses are called **transshipment nodes**. For the sake of simplicity, I'll refer to these as *relay* nodes.
- This is called a **transshipment problem** or a **minimum cost network flow problem**.

Draw the new network diagram. It is standard practice to combine supply and demand at each node into a single parameter, $b = supply - demand$. Add labels for $b = supply - demand$ for each node, as well as all arc costs.



3.1 Concrete model: minimum cost network flow

Write a concrete model for the bakery problem with warehouses.

Min $10x_{b1,w1} + 15x_{b1,w2} + 20x_{b2,w1} + 25x_{b2,w2} + 25x_{w1,s1} + 45x_{w1,s2} + 35x_{w2,s1} + 35x_{w2,s2} + 50x_{w2,s3}$

s.t. $x_{b1,w1} + x_{b1,w2} = 50$

$x_{b1,w2} + x_{b2,w2} = 50$

$x_{w1,s1} + x_{w1,s2} + x_{w1,s3} - x_{b1,w1} - x_{b2,w1} = 0$

$x_{w1,s1} + x_{w1,s2} + x_{w1,s3} - x_{b1,w1} - x_{b2,w1} = 0$

3.2 Balance of flow constraints

Before writing the abstract model, it will be helpful to rearrange the constraints so that they all have the same form. First, verify that you have **one constraint per node**, and that every constraints as one of the following forms:

$$\textit{supply} \;=\; \textit{flow out} \qquad \text{or} \qquad \textit{flow in} \;=\; \textit{demand} \qquad \text{or} \qquad \textit{flow in} \;=\; \textit{flow out}.$$

We can write all of our constraints using the same form by adding the first two forms above:

$$\textit{supply} + \textit{flow in} \;=\; \textit{demand} + \textit{flow out}.^{*}$$

These are called **flow-balance** constraints.

*Note that the validity of these constraints require that *total supply* = *total demand* in the problem.

- Why does it make sense to combine *supply* with *flow out* and *demand* with *flow in* in the balance of flow constraints?

- Write a general form for the **flow-balance** using $b = \textit{supply} - \textit{demand}$:

Balance of flow:

$$= \; b$$

Must be included for *every node* in the network.

3.3 Abstract model: minimum cost network flow

- Write the abstract model for this minimum cost network flow problem.