1. (15 points) Consider the following **MINIMIZING**, canonical form linear program, labeled (P):

$$
\begin{array}{lllr}
\min & 2x_1 \quad - 3x_3 + 18x_4 & & \\
\text{s.t.} & x_1 - x_2 + 2x_3 + x_4 & = 4 & \\
& x_1 + x_2 \quad + 3x_4 & = 2 & \qquad (P) \\
& x_1, x_2, x_3, x_4 \geq 0 & &
\end{array}
$$

(a) (*6 points*) Assume that x_1 and x_3 are basic. Solve for the current basic feasible solution.

Because x_2 and x_4 are nonbasic we have $x_2 = 0$ and $x_4 = 0$. Then, by inspection of the second constraint, $x_1 = 2$. It then follows from the first constraint that $x_3 = 1$.

(b) (*3 points*) Given the feasible direction $d^{x_2} = [-1, 1, 1, 0]^T$ associated with the basis $\mathcal{B} = \{x_1, x_3\}$, determine whether or not it is an improving direction. Based on your answer about whether the direction is improving, state whether the simplex algorithm would continue or terminate at this point. **Briefly** explain your answer for full credit. You may find it helpful to recall that $\bar{c}_k = c^T d^k = c_k + \sum_{i \in \mathcal{B}} c_i d_i^k$.

We calculate the reduced cost for the nonbasic variable x_2 as follows: $\bar{c}_{x_2} = [2, 0, -3, 18] \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = -2 - 3 = -5$. Because we have a minimization problem and the reduced cost is negative, this is an improving simplex direction. This means our current solution is not optimal, and the simplex algorithm will continue.

(c) (*3 points*) For the simplex direction $d^{x_2} = [-1, 1, 1, 0]^T$ associated with the basis $\mathcal{B} = \{x_1, x_3\}$, use the ratio test to find the maximum step size, λ. Based on your answer about λ, which variable will leave the basis and become nonbasic? You may find it helpful to recall that

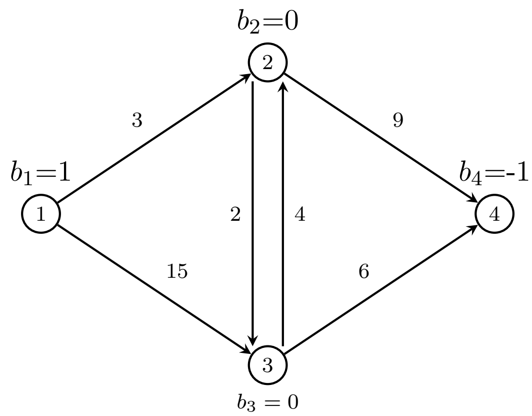
$$\lambda_{max} = \min \left\{ \frac{x_j}{-d_j^k} : d_j^k < 0 \right\}$$

The improving simplex direction has one negative component, so we apply the rule given above to find $\lambda_{max} = \frac{2}{1} = 2$. Because x_1 defines the maximum step size, it will leave the basis and become nonbasic.

(d) (*3 points*) Use your answers from parts a and c above to compute the new solution generated by this iteration of the Simplex Method.

$$x_{\text{new}} = x_{\text{old}} + \lambda_{max} d = [2, 0, 1, 0]^T + 2[-1, 1, 1, 0]^T = [0, 2, 3, 0]^T$$

2. (20 points) Consider the directed network shown below, where the numbers on the arcs represent cost, c_{ij}, to send one unit of flow along the arc. Use the start of a formulation given below to answer the following questions.



Indices

$i \in N$	nodes, $N = \{1,2,3,4\}$
$(i,j) \in A$	arcs

Data

c_{ij}	cost to flow one unit (defined on figure above)

Decision Variables

X_{ij}	Amount of flow on arc (i,j)

(a) (*6 points*) We place one unit of supply at node 1, 1 unit of demand at node 4 and zero units of supply at all other nodes. Formulate an objective function to compute the minimum cost network flow.

$$\min_X \quad 3X_{12} + 15X_{13} + 2X_{23} + 4X_{32} + 9X_{24} + 6X_{34}$$

or

$$\min_X \quad \sum_{(i,j)\in A} c_{ij}X_{ij}$$

(b) (*4 points*) Label all four nodes in the network diagram above with their "net supply" or b_i value.

See the labels on the figure above. Note that there is one unit of supply at node 1, one unit of demand at node 4, and nodes 2 and 3 are transshipment nodes.

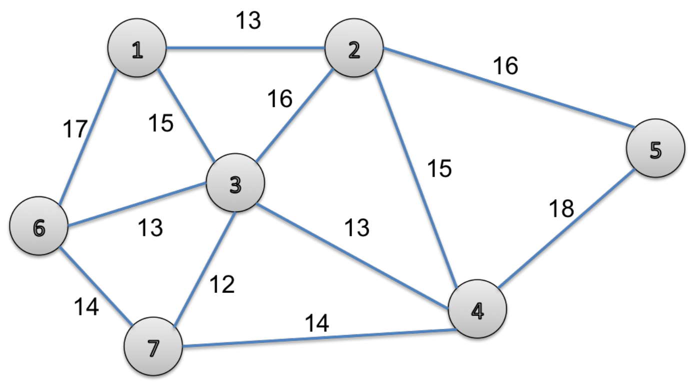
(c) (*6 points*) Write the balance of flow constraint for node 3.

$$X_{32} + X_{34} - X_{13} - X_{23} = 0 \qquad \text{(BOF, Node 3)}$$

(d) (*4 points*) Find the optimal objective value by inspection, and write the corresponding values of the decision variables for this solution.

An optimal solution to the model shown is $X_{12} = X_{23} = X_{34} = 1,\ X_{13} = X_{32} = X_{24} = 0$, which gives an optimal objective value of 11.

3. (20 points) US NorthWest plans to install fiber in a metro area network that is expected to experience increased demand due to the opening of a large manufacturing facility. The Central Offices (COs) in the network are represented by vertices in the graph below. The edges in the graph represent the possible fiber paths linking the COs. The number on edge (i, j) represents the cost c_{ij} (in \$1000) of installing fiber between COs i and j.



Network planners developed the following integer program to solve for the collection of edges that will minimize the cost of connecting all the central offices in the network via a fiber spanning tree. (Let V represent the set of vertices in the graph. Let E represent the set of edges in the graph.)

$$
\begin{array}{lllr}
\min & \sum\limits_{(i,j)\in E} c_{ij}X_{ij} & & \\
\text{s.t.} & \sum\limits_{j:(i,j)\in E} X_{ij} + \sum\limits_{j:(j,i)\in E} X_{ji} \geq 1, & \forall i \in V & \text{(a)} \\
& \sum\limits_{(i,j)\in E} X_{ij} = \ ? & & \text{(b)} \\
& \sum\limits_{\substack{i,j\in V' \\ (i,j)\in E}} X_{ij} \leq |V'| - 1 & \forall V' \subseteq V & \text{(c)} \\
& X_{ij} \in \{0,1\} & \forall (i,j) \in E. &
\end{array}
\qquad \text{(MST)}
$$

(a) (*4 points*) Write down the constraint of type (a) from (MST) for vertex 4.

$$\sum_{j:(4,j)\in E} X_{4j} + \sum_{j:(j,4)\in E} X_{j4} \geq 1$$

or

$$X_{45} + X_{47} + X_{24} + X_{34} \geq 1$$

This constraint ensures that vertex 4 is connected to at least one other node.

(b) (*2 points*) What GUSEK code would implement the objective function? Assume that c, X and E have already been defined for you.

```
minimize obj:sum{(i,j) in E} c[i,j] * X[i,j];
```

(c) (*6 points*) Architect Ima Klutz spilled cappuccino on the formulation rendering the right hand side of constraint (b) illegible. What is the purpose of constraint (b), and what should the missing number be?

$$\sum_{(i,j)\in E} X_{ij} = n - 1 = 6$$

The purpose of this constraint is to ensure we have $n-1$ edges in our tree that contains n nodes.

(d) (*4 points*) The modeling team learns that city ordinances require that if link $(3,7)$ is built, then neither $(1,2)$ nor $(3,4)$ may be built. Write a constraint to model this new requirement.

$$2 - 2X_{37} \geq X_{12} + X_{34}$$

We can derive this constraint from the rule that says that when we want to model the constraint "(binary variable) $Y = 1$ implies $f \leq 0$" we use $f \leq M(1-Y)$ where M is a constant chosen so that $f \leq M$ always holds. Here the variable X_{37} plays the role of Y and requiring that $X_{12} = 0$ and $X_{34} = 0$ is the same as requiring that $X_{12} + X_{34} \leq 0$, so $X_{12} + X_{34}$ plays the role of f. The quantity f is clearly always less than 2 so we can take $M = 2$. This gives the answer above.

(e) (*4 points*) The Central Office represented by vertex 3 is centrally located, but the equipment there is outdated. If vertex 3 is used as a hub, meaning three or more fiber paths meet at vertex 3, the Central Office there will require a $25,000 upgrade. The modeling team adds a new binary variable Y that indicates whether or not vertex 3 is used as a hub in the network design. They modify the objective function by adding the term $25 * Y$. Write a constraint that forces Y to be 1 if three or more selected edges connect to vertex 3.

$$X_{13} + X_{23} + X_{34} + X_{36} + X_{37} \leq 3Y + 2$$

We obtain this solution as follows. First we see that we want to impose the condition "$X_{13} + X_{23} + X_{34} + X_{36} + X_{37} \geq 3$ implies $Y = 1$". This is logically equivalent to the condition that $Y = 0$ implies that $X_{13}+X_{23}+X_{34}+X_{36}+X_{37} \leq 2$. We'd like to use the rule mentioned in part (d) but our assumption is that $Y = 0$, not $Y = 1$. To fix this we introduce the variable $Z = 1 - Y$. Now $Y = 0$ precisely when $Z = 1$, so we are trying to model "$Z = 1$ implies $X_{13}+X_{23}+X_{34}+X_{36}+X_{37}-2 \leq 0$". This can be done using the rule with $f = X_{13} + X_{23} + X_{34} + X_{36} + X_{37} - 2$ and $M = 3$. We get

$$X_{13} + X_{23} + X_{34} + X_{36} + X_{37} - 2 \leq 3(1 - Z),$$

and substituting $Z = 1 - Y$ we get

$$X_{13} + X_{23} + X_{34} + X_{36} + X_{37} - 2 \leq 3Y,$$

which is equivalent to the condition that we gave as the answer above.

SA405, Practice Test 1 September 21, 2017

Name (please print): SOLUTIONS

Instructions:

- Do **not** write your name on each page, only write your name above.
- No books or notes are allowed.
- You may use your calculator on this test.
- Show all work clearly. (Little or no credit will be given for a numerical answer without the correct accompanying work. Partial credit is given where appropriate.)
- If you need more space than is provided, use the back of the previous page.
- Please read the question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.
- All formulations must have descriptions of any indices, parameters, and decision variables used. All constraints must be described.

Problem	Points	Score
1	15	
2	20	
3	20	
Total	55	