Lesson 7. Set Covering, Packing, and Partitioning

1 Today...

- We consider three very common modeling situations and their corresponding constraints.

2 Covering Students

USNA would like for all students to hear a presentation on updated COVID procedures. They decide to send a representative into classes to present the information. The presenter, Hannah, who is an Operations Research major, needs to ensure that every student sees the presentation, but would like to visit as few classes as possible. She develops the following mini-version of the problem in order to help write a model that will solve the large-scale optimization problem.

- Let S be the set of students:

  $S :=$ { Kyle, Aaron, Ryan, Jordan, Monika, Brandon, Samnang, Adam, Natalie, Joshua }

- Let $\mathcal{C}$ be the set of classes:

  $\mathcal{C} :=$ { Naval history, Fencing, Sailing, Boxing, Wrestling, AMP }

- Each element C of $\mathcal{C}$ is itself a set, and $C \subseteq S$, for all $C \in \mathcal{C}$:

Naval history	:= { Kyle, Ryan, Monika, Brandon }
Fencing	:= { Kyle, Jordan, Samnang, Natalie }
Sailing	:= { Aaron, Monika, Adam }
Boxing	:= { Aaron, Ryan, Jordan, Samnang }
Wrestling	:= { Brandon, Ryan, Joshua }
AMP	:= { Adam, Natalie, Joshua }

- Hannah defines the following set of binary **decision variables**:

$$z_C := \begin{cases} 1 & \text{if she should visit class } C \\ 0 & \text{if she should not visit class } C \end{cases}, \text{ for } C \in \mathcal{C}$$

3 Set Covering

Problem 1. Consider the problem described on the previous page.

a. Write a concrete constraints that ensures that Ryan will see the presentation, and one that ensures that Brandon will see the presentation.

$x_N + x_B + x_W \geq 1$ $\quad$ $x_N + x_W \geq 1$

b. Find a feasible solution and illustrate it below:

$$z_C := \begin{cases} 1 & \text{if } C \in \{N, F, S, W\} \\ 0 & \text{otherwise} \end{cases}$$

Kyle	Aaron	Ryan	Jordan	Monika
Brandon	Samnang	Adam	Natalie	Joshua

c. Why are these called **set-covering constraints**? (~~Think of the set of students.~~)

We need to determine which sets should be chosen in order to "cover" every student.

d. Write an abstract model to find a set of classes that covers all students while requiring the fewest possible presentations.

Minimize $\sum_{i \in C} z_i$

s.t. $\sum_{i \in C : s \in C^i} z \geq 1 \quad \forall s \in S$

- How many set covering constraints are required?

One for each student.

- What set should we iterate the constraints over?

$S > \forall s \in S$

- How do we know which classes (z-variables) to include in each constraint?

We have to make sure that the student is in the class.

e. How would we write the set of abstract covering constraints in Python?

4 Set Packing

Problem 2. Eventually Hannah realizes that no student can stand to hear the presentation multiple times, but that she really wants lots of practice with public speaking. She wants to give the presentation as many times as possible without any student seeing it more than once.

a. Write a concrete constraint that ensures that Ryan will see the presentation *at most once*.

$$x_N + x_B + x_W \leq 1$$

b. Find a feasible solution and illustrate it below:

$$z_C := \begin{cases} 1 & \text{if } C \in \{\} \\ 0 & \text{otherwise} \end{cases}$$

→ Don't visit any classes.

∴ ☺ Don't overthink it.

Kyle	Aaron	Ryan	Jordan	Monika
Brandon	Samnang	Adam	Natalie	Joshua

c. Why are these called **set-packing constraints**? (Think of the set of classes.)

We are trying to include as many sets (pack) as we can without any overlap of students.

d. Write an abstract model to find a collection of classes that maximizes the number of classes Hannah visits, while not seeing any student more than once.

$$\text{Max} \sum_{i \in C} z_i$$

$$\text{s.t.} \sum_{i \in C : s \in C^i} z_i \leq 1 \quad \forall s \in S$$

$$z_i \in \{0, 1\} \quad \forall i \in C$$

5 Set Partitioning

Problem 3. Hannah receives a message of encouragement from the Chief of Staff and is told to be sure to show the presentation to *every single student*. But she still knows that no student can possibly sit through it twice, so she must revise her model again.

a. Write a concrete constraint that ensures that Samnang will see the presentation *exactly once*.

$$z_F + z_B = 1$$

b. Find a feasible solution and illustrate it below:

$$z_C := \begin{cases} 1 & \text{if } C \in \boxed{\text{DNE}} \\ 0 & \text{otherwise} \end{cases}$$

Kyle	Aaron	Ryan	Jordan	Monika
Brandon	Samnang	Adam	Natalie	Joshua

c. Why are these called **set-partitioning constraints**? (Think of the set of students.)

We are trying to divide out (partition) all the students into only a single set w/ no overlap.

d. Write an abstract model to find a collection of classes that minimizes the number of classes Hannah visits, while seeing every student exactly once.

$$\text{Minimize} \sum_{i \in C} z_i$$

$$\text{s.t.} \sum_{i \in C : s \in C^i} z \geq 1 \quad \forall s \in S$$