Lesson 4. Shortest Path

1 Today...

- Model a **shortest path** network flow problem, "in disguise".

2 Print Shop – Copier Purchase Plan

(Similar to Example 2.13, p. 64) In Scranton, PA, Dunder Mifflin prints high volumes of photocopying to meet their high demand. The office manager, Michael Scott, is interested in determining when to purchase a new high-speed copier over the next 4 years. During the years that a copier is not purchased, maintenance must be performed. The maintenance cost depends on the age of the copier. The table below provides estimated maintenance cost per age of machine.

Age at Beginning of Year	Maintenance Cost for the Coming Year
0	\$2000
1	\$3500
2	\$6000
3	\$9500

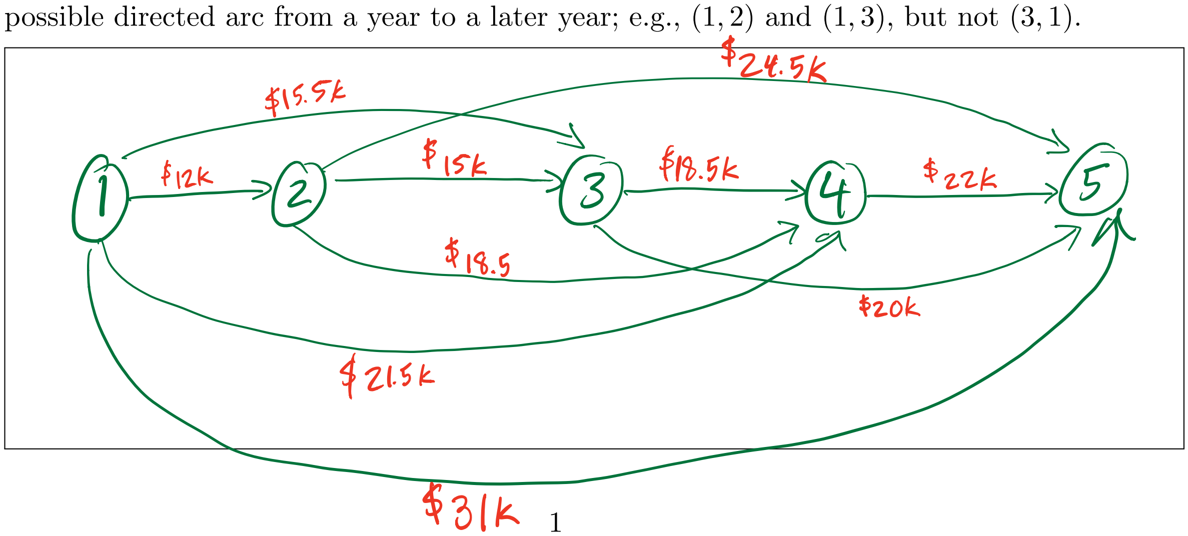
The cost (in today's dollars) of purchasing copiers at the beginning of each year is given below.

Year	Purchase Cost
1	\$10,000
2	\$13,000
3	\$16,500
4	\$20,000

Determine the years in which a new copier should be purchased in order to minimize the cost (purchase + maintenance) of having a machine for 4 years.

3 How is this a network flow problem?

- Draw a node for each year, 1 through 5, from left to right. Since we want to account for four full years, we need 5 nodes to bring us to the end of year 4 / beginning of year 5. Draw every possible directed arc from a year to a later year; e.g., $(1, 2)$ and $(1, 3)$, but not $(3, 1)$.



- Arc (i,j) represents purchasing a copier at the beginning of year i and maintaining it until the beginning of year j. For example, the cost incurred by selecting arc $(1,4)$ (in thousands) is $\$10 + \$2 + \$3.5 + \$6 = \$21.5$: which is the cost of purchasing a new copier in year 1, then maintaining it through years 1, 2, and 3. Add arc costs to the network diagram.

4 How is the printer problem a shortest path problem?

A **path** is an ordered sequence of connected arcs such that any node is "visited" at most once.

- In this problem, the minimum cost strategy corresponds to the minimum cost *path* from node $\boxed{1}$ to node $\boxed{5}$.
- Therefore, this is a $\boxed{\text{SHORTEST}}$ $\boxed{\text{PATH}}$ **network flow** problem.
- This kind of problem requires *supplies* and *demands*, like the bakery problem.
  - What should the *supply/demand* be at node 1? $\boxed{1}$
  - What should the *supply/demand* be at node 5? $\boxed{-1}$
  - What are the relay nodes in this network? $\boxed{2, 3, 4}$

Shortest path is a special case of the $\boxed{\text{Minimum}}$ $\boxed{\text{cost}}$ network flow problem.

- What applications of network flow problems can you imagine? How about specific Naval applications? Write at least two ideas. *(Think about all the types of network flows we have seen: transportation, minimum cost, maximum flow, and shortest path.)*

Wireless Sensor Networks gathering and transmitting data.
What's the cheapest way to move USMC through maritime settings.

5 Concrete and Abstract models.

Often problems are too large for it to be reasonable to write out an entire concrete model, but it is still good to write out (at least) an **abbreviated concrete model**, in order to fully understand the logic of the model before writing the abstract version.

In an abbreviated (shortened) concrete model, it is common to use an ellipsis, "...", to represent repetitive elements of the model that are left out, like terms in a long summation, or constraints in a large class of constraints of the same type. *Standard practice is to write the first two terms (or constraints), then (...), then the last term (or constraint).* This way, the patterns in the model are evident.

- Write an abbreviated concrete model for the copier shortest path problem.

Min $12x_{12} + 15x_{13} + \ldots + 22x_{45}$

s.t.

$$
\begin{aligned}
x_{12} + x_{13} + x_{14} + x_{15} &= 1 \\
x_{23} + x_{24} + x_{25} - x_{12} &= 0 \\
x_{34} + x_{35} - x_{13} - x_{23} &= 0 \\
&\vdots \\
-x_{15} - x_{25} - x_{35} - x_{45} &= -1 \\
x_{12}, x_{13}, \ldots, x_{45} &\geq 0 \text{ (BINARY?)}
\end{aligned}
$$

- Write an abstract model for the copier problem. Use the general form for the balance of flow constraints for the nodes of the network.

Minimize $\sum_{(i,j)\in A} c_{ij} x_{ij}$

s.t. $\sum_{j:(i,j)\in A} x_{ij} - \sum_{h:(h,i)\in A} x_{hi} = b_i \;\; \forall i \in V$

$x_{ij} \in \{0,1\} \;\; \forall (i,j) \in A$

Sets

$V :=$ Set of Nodes (Years)

$V := \{1,2,3,4,5\}$

$A :=$ Set of Arcs

$A := \{(1,2),(1,3),\ldots,(4,5)\}$

Parameters

$c_{ij} =$ the cost of buying a copier in year i and maintaining it until year j

$b_i :=$ the balance at each node

$b_1 = 1 \quad b_2 = b_3 = b_4 = 0 \quad b_5 = -1$

Variables

$x_{ij} \in \{0,1\} = 1$ if a copier is purchased in year i and maintained until year j.