Lesson 15. IP Formulations, Part 1

1 Today

- LP review
- IP formulations

2 Solving Integer Programs can be *Really* Hard!

The following integer (linear) program (IP) seeks an objective-maximizing integer linear combination of a big number.

$$\begin{aligned} \text{maximize} \quad & 213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5 \\ \text{subject to} \quad & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 = 89643482 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0,\ \text{integer} \end{aligned}$$

Problem 1. We will solve two versions of the problem above. In both cases, use the `tee=True` flag to see the solver output in Jupyter as follows:

```
solver_result = pyo.SolverFactory('glpk').solve(model, tee=True)
```

(a) First solve the **LP relaxation** of the IP, which means allowing the variables to be continuous rather than integer-valued: `domain=pyo.NonNegativeReals`

Does the solver arrive at an optimal solution? If so, list it here. If not, what happens?

Yes! (0, 0, 0, 0, 1047.62)

(b) Now solve the IP as written, which means requiring the variables to take integer values: `domain=pyo.NonNegativeIntegers`

Does the solver arrive at an optimal solution? If so, list it here. If not, what happens?

Took too long!

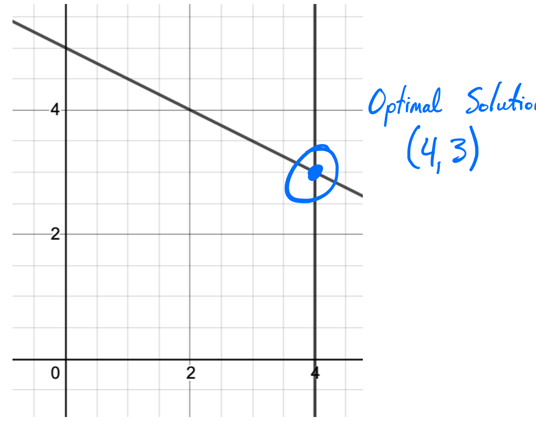
In general, $\boxed{\text{IP}}$ s are much harder to solve than $\boxed{\text{LP}}$ s.

- This week we will discuss why this is, and why the way we model IP problems (yes, there are choices!) can impact solver performance.
- Next week, we will learn about the **branch and bound algorithm** (B&B), which is used by most IP solvers. It is significantly more computationally-expensive than the LP simplex method.

3 LP Review

Problem 2. Solve the following LP graphically: Shade the feasible region, draw two objective contours: $x_1 + x_2 = 2$ and $x_1 + x_2 = 4$. Use arrows to indicate the direction of an increasing objective value. Label the optimal solution.

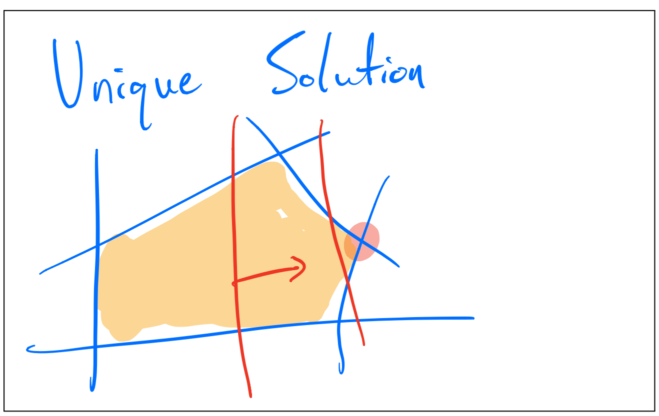
$$
\begin{array}{ll}
\max & x_1 + x_2 \\
\text{s.t.} & x_1 + 2x_2 \le 10 \\
 & x_1 \le 4 \\
 & x_1, x_2 \ge 0
\end{array}
$$

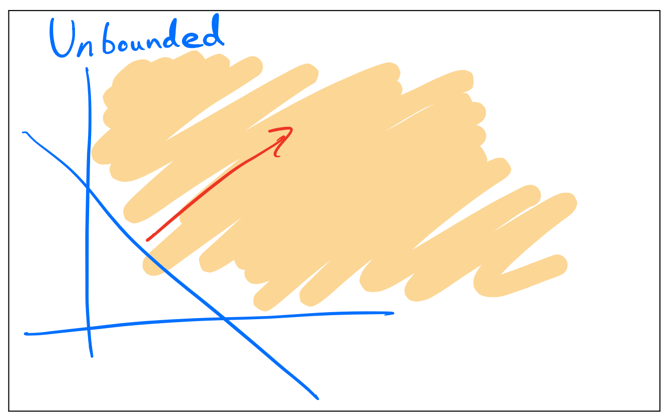


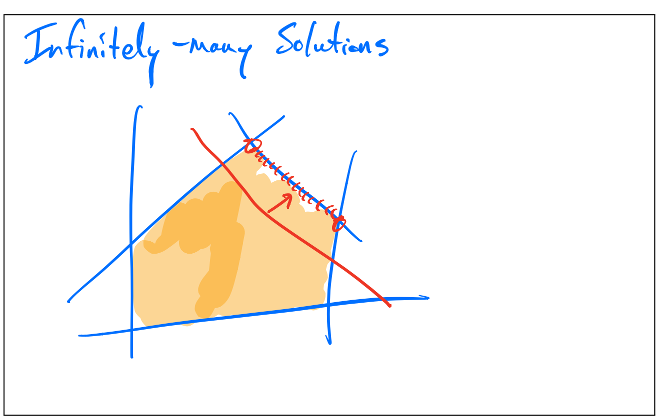
What is the optimal objective value? $\boxed{7}$

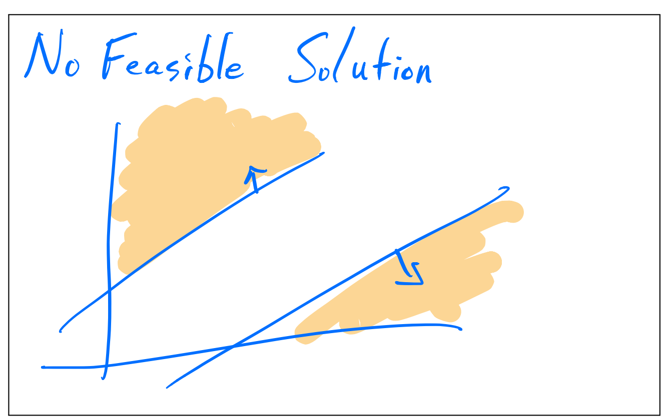
> **Theorem:** Every linear program (LP) has EXACTLY ONE of the following outcomes:
>
> (1) Unique optimal solution
> (2) Multiple optimal solutions
> (3) Unbounded
> (4) Infeasible

Problem 3. Sketch graphs that illustrate each of the LP outcomes.









Theorem: Integer programs (IPs) have the same four possible outcomes as LPs:

(1) Unique optimal solution
(2) Multiple optimal solutions
(3) Unbounded
(4) Infeasible

Theorem: If an LP has an optimal solution (the LP is not unbounded or infeasible), an optimal solution can always be found at a CORNER POINT of the feasible region.

Question: Is the same true for IPs? In other words, if an optimal solution to an IP exists, can an optimal solution always be found at a corner point?

- Keep this question in mind. We will revisit it a little later.

How easy the problem is to solve?

Linear Programs LP: Easy as pie (Almost always)

Integer Programs IP: Eh... Alright (Lots are easy, though.)

Non-linear Programs NLP: Depends on the non-linearity. Some are alright.

Mixed-integer non-linear Programs MINLP: BAD!

4 IP Formulations

A **formulation** of an IP is a set of linear constraints that capture ALL of the feasible integer points, and NO OTHER integer points.

- We will see some examples of formulations of an IP in the next problem.

The **LP relaxation** of an IP is the LP that is formed by *relaxing* the integer requirement on the variables.

Problem 4. Below are two integer programs, along with the diagrams of their constraints. (Rader, examples 13.3, 13.4)

Problem A:

$$
\begin{aligned}
\text{maximize} \quad & 8x + 7y \\
\text{subject to} \quad & -18x + 38y \leq 133 \\
& 13x + 11y \leq 125 \\
& 10x - 8y \leq 55 \\
& x, y \in \mathbb{Z}^{\geq 0}
\end{aligned}
$$

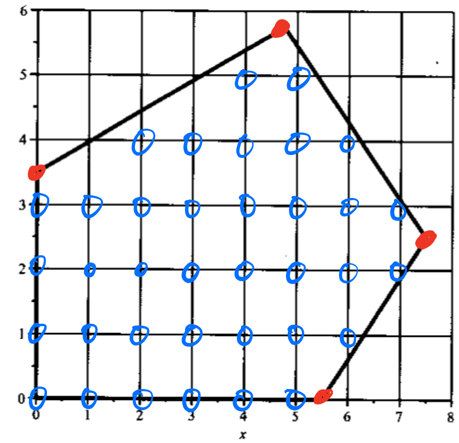


FIGURE 13.1 Feasible region for integer program (13.3).

Problem B:

$$
\begin{aligned}
\text{maximize} \quad & 8x + 7y \\
\text{subject to} \quad & -x + 2y \leq 6 \\
& x + y \leq 10 \\
& x - y \leq 5 \\
& x \leq 7 \\
& y \leq 5 \\
& x, y \in \mathbb{Z}^{\geq 0}
\end{aligned}
$$

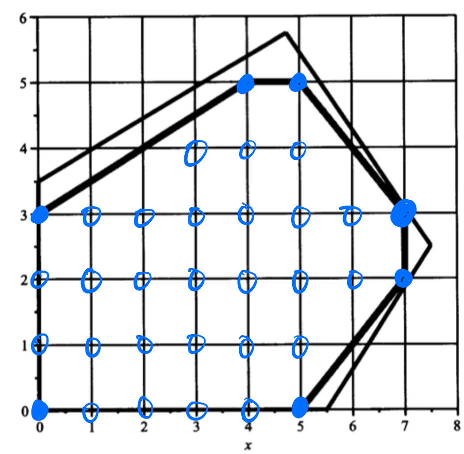


FIGURE 13.2 Feasible region for integer program (13.4).

(a) On the diagrams, identify all feasible solutions to both IPs.

(b) Are the feasible regions for problems A and B different or the same? Nope!

(c) What does this mean about the two problems A and B?

A+B have the same objective function and feasible region.
They can both be used to model and solve the same problem, but they are different formulations

(d) Are the LP relaxations of A and B the same? If not, which has the higher optimal objective value?

Nope! They are different because they have different feasible regions
Problem A has the higher optimal objective value.

Problem 5. Refer back to the previous problem to answer the following:

(a) Suppose an optimal solution to an IP exists.

i. ~~Can~~ Will an optimal solution *always* be found at a corner point? (The question from before.)

Nope!

ii. Can an optimal solution *sometimes* be found at a corner point?

Yes! But not generally

(b) Which of the two formulations of the IP in problem 3 is easier to solve? Why?

Formulation B, because every corner point is an integer solution.

5 Comparing IP Formulations

- Often the decision of how to formulate an IP comes down to a trade-off between quality of formulation and number of constraints:
  - $\boxed{\text{More}}$ constraints means a better (tighter) formulation, but too $\boxed{\text{many}}$ constraints can cause memory pressure and slow down the solver. (More/Fewer, many/few)
  - $\boxed{\text{Fewer}}$ constraints means fewer memory problems, but results in a formulation that is not as good. (More/Fewer)

TAKEAWAY:

The way we choose to $\boxed{\text{formulate}}$ the feasible region of an IP can have a big impact on solver performance.