# HOMEWORK 4 – SOLUTIONS
SA405, FALL 2018
INSTRUCTOR: FORAKER

1. 3.16: NOTE: The problem is INFEASIBLE for the data given in the text. Use due dates of 10, 15, 42, 31, 5, 22, and 32 for jobs 1 through 7, respectively. Let $\mathcal{J} = \{1, \ldots, 7\}$ denote the set of jobs. For a job $j \in \mathcal{J}$, let $r_j$ denote the release date, $d_j$ the due date, and $p_j$ the duration (or process time). Finally, let $D = \max_j d_j$. We give two formulations

   (a) In this formulation, we use the variables:

   $$C_j = \text{the completion time of job } j, \text{ for all } j \in \mathcal{J} \text{ and } \qquad z = \text{the maximum completion time.}$$

   We also use the binary variables:

   $$x_{ij} = \begin{cases} 1 & \text{job } i \text{ completes before job } j \\ 0 & \text{otherwise.} \end{cases}$$

   Then, the formulation is:

   $$\begin{array}{lllr} \min & z & & \\ \text{s.t.} & z \geq C_j & \forall j \in \mathcal{J} & \text{(a)} \\ & x_{ij} + x_{ji} = 1 & \forall i, j \in \mathcal{J}, i \neq j & \text{(b)} \\ & r_j + p_j \leq C_j \leq d_j & \forall j \in \mathcal{J} & \text{(c)} \\ & C_i + Dx_{ij} \geq C_j + p_i & \forall i, j \in \mathcal{J}, i \neq j & \text{(d)} \\ & x_{ij} \in \{0, 1\} & \forall i, j \in \mathcal{J}, i \neq j & \end{array}$$

   Constraints (a) ensure that $z$ is the maximum completion time. Constraints (b) ensure that either job $i$ completes before job $j$ or vice versa (but not both). Constraints (c) ensure that the completion time of a job is between the release date plus the duration and the due date. Constraints (d) ensure that if job $j$ finishes before job $i$, then the job $i$ does not complete until enough time has passed to complete job $i$.

   (b) In this formulation, we use the variables:

   $$x_{jt} = \begin{cases} 1 & \text{job } j \text{ completes processing at time } t \\ 0 & \text{otherwise,} \end{cases} \qquad \text{and} \qquad z = \text{the maximum completion time.}$$

   With these variables, the formulation is:

   $$\begin{array}{lllr} \min & z & & \\ \text{s.t.} & z \geq \sum_{t=1}^{D} t x_{jt} & \forall j \in \mathcal{J} & \text{(a)} \\ & x_{jt} = 0 & \forall j \in \mathcal{J}, \forall t \leq r_j + p_j - 1 & \text{(b)} \\ & x_{jt} = 0 & \forall j \in \mathcal{J}, \forall t \geq d_j + 1 & \text{(c)} \\ & \sum_{t=1}^{D} x_{jt} = 1 & \forall j \in \mathcal{J} & \text{(d)} \\ & \sum_{j \in \mathcal{J}} \sum_{s=t}^{t+p_j-1} x_{js} \leq 1 & t = 1, \ldots, D & \text{(e)} \\ & x_{jt} \in \{0, 1\} & & \end{array}$$

   Constraints (a) ensure the $z$ is the maximum completion time. Constraints (b) and (c) ensure that the jobs start after the release date and that the jobs complete by the due date, respectively. Constraints (d) ensure that each job completes once on the machine, and constraints (e) ensure that the jobs do not run simultaneously on the machine. The interpretation of constraints (e) is that, for a given time $t$, at most one job, $j$, can finish from time $t$ to time $t + p_j - 1$.

2. 3.20: Let $\mathcal{C} = \{1, \ldots, 15\}$. We use the variables

$$x_i = \begin{cases} 1 & \text{block } i \text{ is removed} \\ 0 & \text{otherwise} \end{cases}$$

for all $i \in \mathcal{C}$. Further, for $i \in \mathcal{C}$, let

$$\mathcal{P}_i = \{j | \text{block } j \text{ must be removed before } i \text{ can be removed}\}.$$

So, for example, $\mathcal{P}_8 = \{1, 2, 3\}$ and $\mathcal{P}_1 = \emptyset$. For $i \in \mathcal{C}$, let $r_i$ and $c_i$ denote the revenue and cost of block $i$, respectively, and let

$$p_i = r_i - c_i.$$

So, $p_4 = 0 - 200$ and $p_{10} = 1000 - 350$. Then, the formulation is:

$$\begin{array}{lll} \max & \sum_{i \in \mathcal{C}} p_i x_i & \\ \text{s.t.} & x_i \leq x_j & \forall i \in \mathcal{C}, \forall j \in \mathcal{P}_i \quad \text{(a)} \\ & x_i \in \{0, 1\} & \forall i \in \mathcal{C} \end{array}$$

Constraints (a) impose the condition that if block $i$ is removed, then all blocks $j$ must be removed for $j \in \mathcal{P}_i$.

3. 3.24: Let $\mathcal{P} = \{1, \ldots, 10\}$ denote the set of planes. For $i \in \mathcal{P}$, let $r_i$, $t_i$, and $d_i$ denote the earliest, target, and latest arrival time, respectively, and let $\alpha_i$ and $\beta_i$ denote the early and late penalties, respectively. Let $D = \max_i d_i$. For $i, j \in \mathcal{P}$, let $s_{ij}$ denote the separation times between plane landings. Then, for $i, j \in \mathcal{P}$, let

$$x_{ij} = \begin{cases} 1 & \text{plane } i \text{ lands before plane } j \\ 0 & \text{otherwise} \end{cases}$$

and let $T_i$ denote the time plane $i$ lands. For $i \in \mathcal{P}$ let

$$y_i = \begin{cases} 1 & \text{plane } i \text{ is early} \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad z_i = \begin{cases} 1 & \text{plane } i \text{ is late} \\ 0 & \text{otherwise.} \end{cases}$$

The formulation is

$$\begin{array}{llll} \min & \sum_{i \in \mathcal{P}} (\alpha_i y_i + \beta_i z_i) & & \\ \text{s.t.} & y_i r_i + (1 - y_i) t_i \leq T_i \leq z_i d_i + (1 - z_i) t_i & \forall i \in \mathcal{P} & \text{(a)} \\ & x_{ij} + x_{ji} = 1 & \forall i, j \in \mathcal{P}, i \neq j & \text{(b)} \\ & T_i + x_{ij} D \geq T_j + s_{ji} & \forall i, j \in \mathcal{P}, i \neq j & \text{(c)} \\ & T_i \geq 0, y_i, z_i, x_{ij} \in \{0, 1\} & \forall i, j \in \mathcal{P}, i \neq j & \end{array}$$

Constraints (a) enforce the earliest and latest time constraints. Note that if the plane is on-time, then $T_i = d_i$. Constraints (b) ensure that either plane $i$ lands before plane $j$, or vice versa. Constraints (c) ensure that the planes satisfy the separation time constraint.

**Not to be handed in.**

1. 3.27: NOTE: In Figure 3.8 there is no value given for the edge between nodes 2 and 4. The weight assigned to this edge should be 1. Let $F = |\mathcal{V}| \max_{(i,j)\in\mathcal{E}} c_{ij} + 10 = 60$ denote an upper bound on the number of frequencies used. Let $\mathcal{F} = \{1, \ldots, F\}$ denote the set of frequencies. For $i \in \mathcal{V}$ and $f \in \mathcal{F}$, let

$$x_{if} = \begin{cases} 1 & \text{tower } i \text{ is assigned frequency } f \\ 0 & \text{otherwise} \end{cases}$$

and let $z$ denote the maximum frequency used. Then, our formulation is:

$$\begin{array}{lllr}
\min & z & & \\
\text{s.t.} & \sum\limits_{f\in\mathcal{F}} x_{if} = 2 & \forall i \in \mathcal{V} & \text{(a)} \\
& \sum\limits_{g=f}^{f+9} x_{ig} \le 1 & \forall i \in \mathcal{V}, \forall f \in \mathcal{F} & \text{(b)} \\
& \sum\limits_{g=f}^{f+c_{ij}-1} (x_{ig} + x_{jg}) \le 1 & \forall (i,j) \in \mathcal{E}, \forall f \in \mathcal{F} & \text{(c)} \\
& z \ge f x_{if} & \forall f \in \mathcal{F}, \forall i \in \mathcal{V} & \text{(d)} \\
& x_{if} \in \{0,1\} & \forall i \in \mathcal{V}, \forall f \in \mathcal{F} &
\end{array}$$

Constraints (a) ensure that each tower is assigned two frequencies. Constraints (b) ensure that the frequencies assigned to a tower differ by at least ten. Constraints (c) ensure that towers connected by an edge have frequencies that differ by at least the weight assigned to the edge. Constraints (d) ensure that the maximum frequency is set correctly.

2. 3.29: NOTE: The data given in the book for Time to Switch are in HOURS. It will be necessary to convert this data to SECONDS when using the formulation below. Let the set of labels be denoted by $\mathcal{L} = \{1, 2, 3\}$ and the set of production lines by $\mathcal{P} = \{1, 2, 3\}$. Denote the set of shifts by $\mathcal{S} = \{1, \ldots, 14\}$. For $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$, let $t_{\ell p}$ and $c_{\ell p}$ denote the time and cost, respectively, to produce one label of type $\ell$ on production line $p$. For $\ell \in \mathcal{L}$ and $p \in \mathcal{P}$, let $\tau_{\ell p}$ and $\gamma_{\ell p}$ denote the time and cost, respectively, to switch production line $p$ to label $\ell$. Let $d_\ell$ denote the weekly demand for label $\ell$, for $\ell \in \mathcal{L}$. For $\ell \in \mathcal{L}$, $p \in \mathcal{P}$, and $s \in \mathcal{S}$, let

$$x_{\ell ps} = \begin{cases} 1 & \ell \text{ is produced on production line } p \text{ in shift } s \\ 0 & \text{otherwise} \end{cases}$$

and

$$y_{\ell ps} = \begin{cases} 1 & \text{production line } p \text{ is switched to label } \ell \text{ at the beginning of shift } s \\ 0 & \text{otherwise} \end{cases}$$

For $\ell \in \mathcal{L}$, $p \in \mathcal{P}$, and $s \in \mathcal{S}$, let $w_{\ell ps}$ denote the time production line $p$ is spent making labels of type $\ell$ in shift $s$, and let $a_{\ell ps}$ denote the number of labels of type $\ell$ made on production line $p$ in shift $s$. We assume that each line is not set up for any label before the shift starts. The formulation is:

$$\begin{array}{lllr}
\min & \sum\limits_{s\in\mathcal{S}} \sum\limits_{\ell\in\mathcal{L}} \sum\limits_{p\in\mathcal{P}} (\gamma_{\ell p} y_{\ell ps} + c_{\ell p} a_{\ell ps}) & & \\
\text{s.t.} & x_{\ell ps} \le y_{\ell ps} + x_{\ell p(s-1)} & \forall \ell \in \mathcal{L}, \forall p \in \mathcal{P}, s = 2, \ldots, 14 & \text{(a)} \\
& \sum\limits_{\ell\in\mathcal{L}} x_{\ell ps} \le 1 & \forall p \in \mathcal{P}, \forall s \in \mathcal{S} & \text{(b)} \\
& w_{\ell ps} \le 12 \cdot 3600 - \tau_{\ell p} y_{\ell ps} & \forall \ell \in \mathcal{L}, \forall p \in \mathcal{P}, \forall s \in \mathcal{S} & \text{(c)} \\
& a_{\ell ps} = \frac{w_{\ell ps}}{t_{\ell p}} & \forall \ell \in \mathcal{L}, \forall p \in \mathcal{P}, \forall s \in \mathcal{S} & \text{(d)} \\
& w_{\ell ps} \le 12 \cdot 3600 \cdot x_{\ell ps} & \forall \ell \in \mathcal{L}, \forall p \in \mathcal{P}, \forall s \in \mathcal{S} & \text{(e)} \\
& \sum\limits_{p\in\mathcal{P}} \sum\limits_{s\in\mathcal{S}} a_{\ell ps} \ge d_\ell & \forall \ell \in \mathcal{L} & \text{(f)} \\
& x_{\ell ps}, y_{\ell ps} \in \{0,1\} & \forall \ell \in \mathcal{L}, \forall p \in \mathcal{P}, \forall s \in \mathcal{S} & \\
& w_{\ell ps}, a_{\ell ps} \ge 0 & \forall \ell \in \mathcal{L}, \forall p \in \mathcal{P}, \forall s \in \mathcal{S} &
\end{array}$$

Constraints (a) ensure that a production line only produces a given label in a shift if it had either switched to that label in that shift, or had been producing the label in the previous shift. Constraints (b) ensure a production line produces at most one label per shift. Constraints (c) calculate the total time that a production line spends making labels versus having to set up for making a given label. Constraints (d) calculate the number of labels produced at a given line during a given shift. Constraints (e) ensure that a given label can only be made at production lines that are capable of making the label. Constraints (f) ensure that we produce enough of each label to satisfy the weekly demands for that label.

3. 3.32: As usual, we assume that $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The first modification is that both $x_{ij}$ and $x_{ji}$ are required as the arcs are no longer symmetric. Also, each node must be visited exactly once and cycles must be eliminated. The formulation is as follows:

$$
\begin{array}{llll}
\min & \sum\limits_{(i,j)\in\mathcal{E}} d_{ij}x_{ij} & & \\
\text{s.t.} & \sum\limits_{j\in\{j|(i,j)\in\mathcal{E}\}} x_{ij} = 1 & \forall i \in \mathcal{V} & \text{(a)} \\
& \sum\limits_{j\in\{j|(j,i)\in\mathcal{E}\}} x_{ji} = 1 & \forall i \in \mathcal{V} & \text{(b)} \\
& \sum\limits_{(i,j)\in\{(i,j)\in\mathcal{E}|i,j\in U\}} x_{ij} \leq |U| - 1 & \forall U \subset V, 3 \leq |U| \leq |V| - 3 & \text{(c)} \\
& x_{ij} \in \{0,1\} & (i,j) \in \mathcal{E} &
\end{array}
$$

Constraints (a) ensures the tour leaves each node exactly once. Constraints (b) ensure that the tour enters each node exactly once. Constraints (c) are the same as the symmetric case.