Lesson 14. IP Formulations

1 Today

- LP review
- IP formulations

2 Solving Integer Programs can be *Really* Hard!

The following integer (linear) program (IP) seeks an objective-maximizing integer linear combination of a big number.

$$\begin{aligned} \text{maximize} \quad & 213x_1 - 1928x_2 - 11111x_3 - 2345x_4 + 9123x_5 \\ \text{subject to} \quad & 12223x_1 + 12224x_2 + 36674x_3 + 61119x_4 + 85569x_5 = 89643482 \\ & x_1,\ x_2,\ x_3,\ x_4,\ x_5 \geq 0,\ \text{integer} \end{aligned}$$

Problem 1. We will solve two versions of the problem above. In both cases, use the `tee=True` flag to see the solver output in Jupyter as follows:

```
solver_result = pyo.SolverFactory('glpk').solve(model, tee=True)
```

(a) First solve the **LP relaxation** of the IP, which means allowing the variables to be continuous rather than integer-valued: `domain=pyo.NonNegativeReals`

Does the solver arrive at an optimal solution? If so, list it here. If not, what happens?

(b) Now solve the IP as written, which means requiring the variables to take integer values: `domain=pyo.NonNegativeIntegers`

Does the solver arrive at an optimal solution? If so, list it here. If not, what happens?

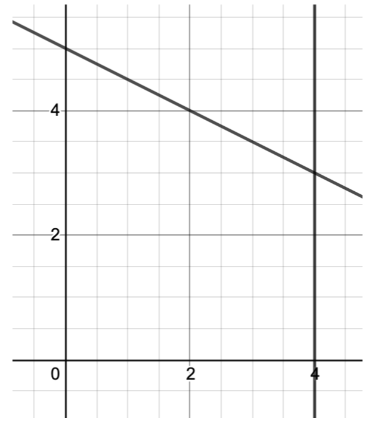
In general, ______ s are much harder to solve than ______ s.

- This week we will discuss why this is, and why the way we model IP problems (yes, there are choices!) can impact solver performance.
- Next week, we will learn about the **branch and bound algorithm** (B&B), which is used by most IP solvers. It is significantly more computationally-expensive than the LP simplex method.

3 LP Review

Problem 2. Solve the following LP graphically: Shade the feasible region, draw two objective contours: $x_1 + x_2 = 2$ and $x_1 + x_2 = 4$. Use arrows to indicate the direction of an increasing objective value. Label the optimal solution.

$$
\begin{aligned}
\max \quad & x_1 + x_2 \\
\text{s.t.} \quad & x_1 + 2x_2 \leq 10 \\
& x_1 \leq 4 \\
& x_1, x_2 \geq 0
\end{aligned}
$$



What is the optimal objective value?

Theorem: Every linear program (LP) has EXACTLY ONE of the following outcomes:

(1) Unique optimal solution
(2) Multiple optimal solutions
(3) Unbounded
(4) Infeasible

Problem 3. Sketch graphs that illustrate each of the LP outcomes.

Theorem: Integer programs (IPs) have the same four possible outcomes as LPs:

(1) Unique optimal solution
(2) Multiple optimal solutions
(3) Unbounded
(4) Infeasible

Theorem: If an LP has an optimal solution (the LP is not unbounded or infeasible), an optimal solution can always be found at a ________ ________ of the feasible region.

Question: Is the same true for IPs? In other words, if an optimal solution to an IP exists, can an optimal solution always be found at a corner point?

- Keep this question in mind. We will revisit it a little later.

4 IP Formulations

A **formulation** of an IP is a set of linear ________ that capture ALL of the ________ integer points, and NO OTHER integer points.

- We will see some examples of formulations of an IP in the next problem.

The **LP relaxation** of an IP is the LP that is formed by *relaxing* the integer requirement on the variables.

Problem 4. Below are two integer programs, along with the diagrams of their constraints. (Rader, examples 13.3, 13.4)

Problem A:

$$
\begin{aligned}
\text{maximize} \quad & 8x + 7y \\
\text{subject to} \quad & -18x + 38y \leq 133 \\
& 13x + 11y \leq 125 \\
& 10x - 8y \leq 55 \\
& x, y \in \mathbb{Z}^{\geq 0}
\end{aligned}
$$

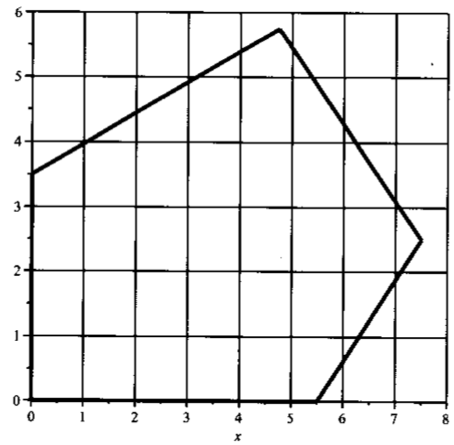


FIGURE 13.1 Feasible region for integer program (13.3).

Problem B:

$$
\begin{aligned}
\text{maximize} \quad & 8x + 7y \\
\text{subject to} \quad & -x + 2y \leq 6 \\
& x + y \leq 10 \\
& x - y \leq 5 \\
& x \leq 7 \\
& y \leq 5 \\
& x, y \in \mathbb{Z}^{\geq 0}
\end{aligned}
$$

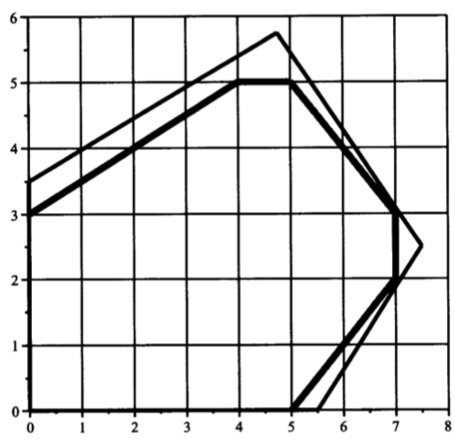


FIGURE 13.2 Feasible region for integer program (13.4).

(a) On the diagrams, identify all feasible solutions to both IPs.

(b) Are the feasible regions for problems A and B different or the same?

(c) What does this mean about the two problems A and B?

(d) Are the LP relaxations of A and B the same? If not, which has the higher optimal objective value?

Problem 5. Refer back to the previous problem to answer the following:

(a) Suppose an optimal solution to an IP exists.

i. Can an optimal solution *always* be found at a corner point? (The question from before.)

ii. Can an optimal solution *sometimes* be found at a corner point?

(b) Which of the two formulations of the IP in problem 3 is easier to solve? Why?

5 Comparing IP Formulations

- Often the decision of how to formulate an IP comes down to a trade-off between quality of formulation and number of constraints:
  - ◦ ______ constraints means a better (tighter) formulation, but too ______ constraints can cause memory pressure and slow down the solver. (More/Fewer, many/few)
  - ◦ ______ constraints means fewer memory problems, but results in a formulation that is not as good. (More/Fewer)

TAKEAWAY:

The way we choose to ______ the feasible region of an IP can have a big impact on solver performance.