1. (15 points) Consider the following **MINIMIZING**, canonical form linear program, labeled (P):

$$\begin{array}{lll} \min & 3x_1 \quad - x_3 & \\ \text{s.t.} & x_1 + x_2 + x_3 + x_4 & = 4 \\ & -2x_1 + x_2 - x_3 & = 1 \\ & \quad 3x_2 + x_3 + x_4 & = 9 \\ & x_1, x_2, x_3, x_4 \geq 0 & \end{array} \tag{P}$$

(a) (*3 points*) Because (P) is in canonical form, it is in the form minimize $\mathbf{c}^T\mathbf{x}$ such that $A\mathbf{x} = \mathbf{b}$. Find the matrix A and the vectors $\mathbf{b}$, and $\mathbf{c}$.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & -1 & 0 \\ 0 & 3 & 1 & 1 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} 4 \\ 1 \\ 9 \end{bmatrix}, \mathbf{c}^T = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

(b) (*2 points*) A Basic Feasible Solution to (P) will have __3__ basic variables and __1__ nonbasic variables.

(c) (*4 points*) Given the feasible direction $d^{x_3} = [-2/3, -1/3, 1, 0]^T$, determine whether or not it is an improving direction. Based on your answer about whether the direction is improving, state whether the simplex algorithm would continue or terminate at this point. **Briefly** explain your answer for full credit. You may find it helpful to recall that $\bar{c}_k = c^T d^k = c_k + \sum_{i \in \mathcal{B}} c_i d_i^k$.

We calculate the reduced cost for the nonbasic variable x_3 as follows: $\bar{c}_{x_3} = [3, 0, -1, 0] \begin{bmatrix} \frac{-2}{3} \\ \frac{-1}{3} \\ 1 \\ 0 \end{bmatrix} = -2 - 1 = -3$. Because we have a minimization problem and the reduced cost is negative, this is an improving simplex direction. This means our current solution is not optimal, and the simplex algorithm will continue.

(d) (*3 points*) If the initial solution is $x^0 = [1,3,0,0]^T$, and the simplex direction is $d^{x_3} = [-2/3, -1/3, 1, 0]^T$, use the ratio test to find the maximum step size, λ. Based on your answer about λ, which variable will leave the basis and become nonbasic? You may find it helpful to recall that

$$\lambda_{max} = \min\left\{\frac{x_j}{-d_j^k} : d_j^k < 0\right\}$$

The improving simplex direction has two negative components, so we apply the rule given above to find $\lambda_{max} = \min\left\{\frac{1}{\frac{2}{3}}, \frac{3}{\frac{1}{3}}\right\} = \min\left\{\frac{3}{2}, 9\right\} = \frac{3}{2} = 1.5$. Because x_1 defines the maximum step size, it will leave the basis and become nonbasic.

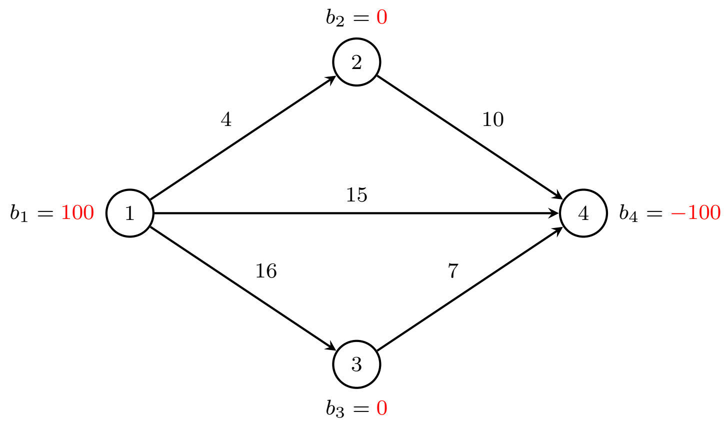
(e) (*3 points*) If the initial solution is $x^0 = [1,3,0,0]^T$, use your answer from part d above to compute the new solution generated by this iteration of the Simplex Method. If you were not able to answer part d, use $\lambda_{max} = 3$.

$$x^1 = x^0 + \lambda_{max} d = [1,3,0,0]^T + \frac{3}{2}\left[\frac{-2}{3}, \frac{-1}{3}, 1, 0\right]^T = [0, 2.5, 1.5, 0]^T$$

If you did not successfully answer parts a and c, you should have

$$x^1 = x^0 + \lambda_{max} d = [1,3,0,0]^T + 3\left[\frac{-2}{3}, \frac{-1}{3}, 1, 0\right]^T = [-1, 2, 3, 0]^T$$

2. (25 points) The directed network shown below illustrates the supply chain for a small beverage company, NEHI. Node 1 represents a factory, nodes 2 and 3 are warehouses, and node 4 is a retail store. NEHI can ship crates of RC cola along any arc. The number c_{ij} on the arc (i, j) represents the cost per crate to ship from node i to node j. Below, we show the start of a model whose purpose is to minimize shipping costs while meeting demand constraints.



<u>Indices</u>

$i \in N$	nodes (alias j), $N = \{1, 2, 3, 4\}$
$(i, j) \in A$	arc directed from node i to node j

<u>Data</u>

c_{ij}	shipping cost in dollars/case from node i to node j
b_i	net supply in crates at node i
$capacity_2 = 60$	upper bound on warehouse 2 in cases
$capacity_3 = 40$	upper bound on warehouse 3 in cases

<u>Decision Variables</u>

X_{ij}	Number of cases shipped on arc (i, j)

(a) (*4 points*) If the factory at node 1 has a supply of 100 cases, nodes 2 and 3 are warehouses with no supply or demand, and node 4 is a store with a demand of 100 cases, fill in the appropriate values for the net supplies b_i's on the figure above.

(b) (*6 points*) Write the balance of flow constraint for node 4.

$$-X_{24} - X_{14} - X_{34} = -100 \qquad \text{(BOF, Node 4)}$$

(c) (*6 points*) The warehouses at nodes 2 and 3 can only be used if a fixed charge of \$200 and \$300, respectively, is paid to open the warehouse. Write an objective function that minimizes the total cost of shipping and the fixed charges associated with opening one (or both) of the warehouses. Be sure to define any items you use in your objective function that are not already given above. This problem can be done using set notation or not using set notation, as you choose.

<u>Binary Decision Variables</u>

$OPEN_2$ 1 if warehouse 2 is open, 0 otherwise

$OPEN_3$ 1 if warehouse 3 is open, 0 otherwise

$$\min_{X,OPEN} \sum_{(i,j)\in A} c_{ij}X_{ij} \quad +200OPEN_2 + 300OPEN_3$$

(d) (*6 points*) Write constraints that ensure cases are only shipped to warehouses that are open. Be sure to define any items you use in your constraints that are not already given above.

$$\sum_{i\in N|(i,2)\in A} X_{i2} \quad \leq capacity_2 OPEN_2$$

$$\sum_{i\in N|(i,3)\in A} X_{i3} \quad \leq capacity_3 OPEN_3$$

or, more simply

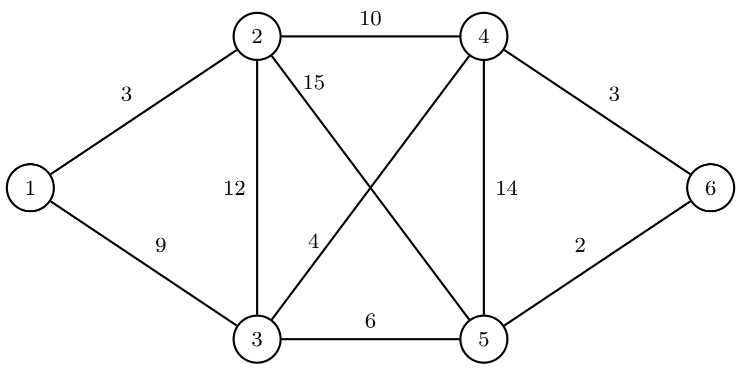
$$X_{12} \quad \leq capacity_2 OPEN_2$$

$$X_{13} \quad \leq capacity_3 OPEN_3$$

(e) (*3 points*) The brothers that run NEHI have just learned of a new, unusual, shipping requirement. If they send more than 20 cases from the factory to the warehouse at node 2 then they must also send at least 30 cases from the factory to the warehouse at node 3. Introduce constraints that impose this condition. Be sure to define any items you use in your constraints that are not already given above.

We first add a binary decision variable Z to the formulation, where $Z = 1$ if $X_{12} \geq 21$ and is 0 otherwise. We then add the constraint $X_{ij} \geq 0$ for all $(i,j) \in A$. Next we add the term $+100Z$ to the objective function. Finally, we add the following constraints: $100Z \geq X_{12} - 20$, and $X_{13} \geq 30Z$.

3. (20 points) As a newly commissioned 2nd Lt you have been tasked with connecting 6 forward operating bases (located at nodes 1 through 6) at minimal cost. Connecting a base to another base has a cost, and each base must have a "path" of wiring to every other base. However, not every base can be wired directly to every other base. Your boss, Capt May B. Wright has determined that the locations and possible connections form the following network, with costs (c_{ij} in thousands of dollars) listed next to the arcs (you may assume he is correct in his drawing). He has denoted the network $\mathcal{G} = (\mathcal{V}, \mathcal{E})$:



Thus, you are trying to determine a minimum spanning tree on the given network. We denote the edge between two nodes i and j by the pair (i, j) with $i < j$ and define the binary variables X_{ij} to be one when the edge (i, j) is in the minimum spanning tree and zero otherwise. Capt Wright started formulating the integer program below, labeled (MST):

$$
\begin{aligned}
\min \quad & \sum_{(i,j)\in\mathcal{E}} c_{ij} X_{ij} & & \\
\text{s.t.} \quad & \sum_{\{j|(i,j)\in\mathcal{E}\}} X_{ij} + \sum_{\{j|(j,i)\in\mathcal{E}\}} X_{ji} \geq ?, \quad \forall i \in \mathcal{V} & & \text{(a)} \\
& \sum_{(i,j)\in\mathcal{E}} X_{ij} = |\mathcal{V}| - 1 & & \text{(b)} \\
& X_{ij} \in \{0,1\} \qquad \forall (i,j) \in \mathcal{E}. & &
\end{aligned}
\qquad \text{(MST)}
$$

(a) (*4 points*) Capt Wright wasn't sure what should go on the right hand side of the inequality for constraint (a) in (MST). Specify what should replace the ? on the right hand side of the inequality. For full credit you MUST also explain in words what this constraint is enforcing.

The ? should be replaced with the number 1. The purpose of this constraint is to ensure that every vertex i is adjacent to at least one edge in the spanning tree.

(b) (*6 points*) Write down the associated constraint of type (b) from (MST). For full credit you MUST explain in words what this constraint is enforcing.

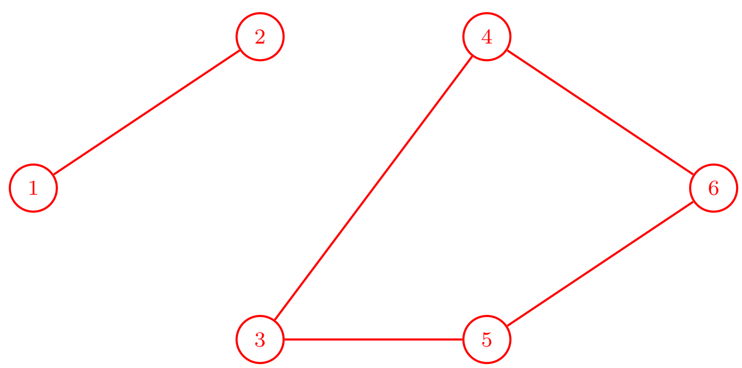
$$\sum_{(i,j)\in\mathcal{E}} X_{ij} = 5$$

or

$$X_{12} + X_{13} + X_{23} + X_{24} + X_{25} + X_{34} + X_{35} + X_{45} + X_{46} + X_{56} = 5$$

This constraint ensures there are exactly 5 arcs in the spanning tree.

(c) (*6 points*) Capt Wright solves (MST) and obtains the solution $X_{12} = X_{35} = X_{46} = X_{56} = X_{34} = 1$, all other $X_{ij} = 0$. Graphically illustrate (using nodes and edges) what this solution represents.



(d) (*4 points*) Does the solution in part c above represent a minimum cost spanning tree? If your answer is **YES**, then give the value of the objective function for this solution. If your answer is **NO**, use the variables X_{ij} to write a constraint that you would add to (MST) before you attempt to solve the model again.

YES Objective function value is 18.

NO, which is the correct answer based on the illustration in part c above.

$$X_{34} + X_{46} + X_{56} + X_{35} \leq 3$$

This constraint eliminates the cycle 3-4-6-5.

SA405, Test 1 September 26, 2017

INSTRUCTOR: CAPT FORAKER

Name (please print): ______________________________

Instructions:

- Do **not** write your name on each page, only write your name above.
- No books or notes are allowed.
- You may use your calculator on this test.
- Show all work clearly. (Little or no credit will be given for a numerical answer without the correct accompanying work. Partial credit is given where appropriate.)
- If you need more space than is provided, use the back of the previous page.
- Please read the question carefully. If you are not sure what a question is asking, ask for clarification.
- If you start over on a problem, please CLEARLY indicate what your final answer is, along with its accompanying work.
- All formulations must have descriptions of any indices, parameters, and decision variables used. All constraints must be described.

Problem	Points	Score
1	15	
2	25	
3	20	
Total	60	