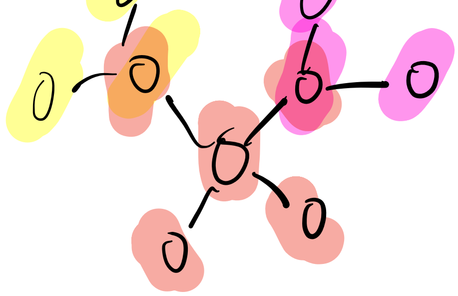

Lesson 7: Set Covering, Packing, and Partitioning

1 Covering Students

The USNA would like for all students to hear a presentation on an update to yard-wide COVID procedures. They decide to send a representative into classes to present the information. The presenter, Hannah, who was an Operations Research major, needs to ensure that every student sees the presentation, but would like to visit as few classes as possible. She develops the following mini-version of the problem in order to help write a model that will solve the large-scale optimization problem.

Let S be the set of students:

$S :=$ { Kyle, Aaron, Ryan, Jordan, Monika, Brandon, Samnang, Adam, Natalie, Joshua }

Let $\mathscr{C}$ be the set of classes:

$\mathscr{C} :=$ { Naval history, Fencing, Sailing, Boxing, Wrestling, AMP }

Each element C of $\mathscr{C}$ is itself a set, a subset of S ($C \subseteq S$, for all $C \in \mathscr{C}$):

Naval history	:=	{ Kyle, Ryan, Monika, Brandon }
Fencing	:=	{ Kyle, Jordan, Samnang, Natalie }
Sailing	:=	{ Aaron, Monika, Adam }
Boxing	:=	{ Aaron, Ryan, Jordan, Samnang }
Wrestling	:=	{ Jordan, Brandon, Joshua }
AMP	:=	{ Adam, Natalie, Joshua }

Hannah defines the following set of binary variables:

$$z_C := \begin{cases} 1 & \text{if she should visit class } i \\ 0 & \text{if she should not visit class } i \end{cases}, \text{ for } C \in \mathscr{C}$$

6 classes
6 variables

z_N
z_F
z_S
$\vdots$
z_A

If $z_N = 1$ then everyone in the Naval History class sees the presentation and are covered → Kyle, Ryan, Monika, Brandon are covered.

2 Set Covering

1. Write two concrete constraints: one that ensures that Jordan will see the presentation, and one that ensures that Brandon will see the presentation.

Jordan: → wresting, → Fencing, → Boxing → For Jordan to see the presentation she'd have to visit at least one of these classes

$$\underbrace{z_W + z_F + z_B}_{\text{each class hes in}} \geq 1$$

visited at least once

Brandon → N, → W $\quad z_N + z_W \geq 1$

2. Why are these called **set covering constraints**? (Think of the set of students.)

Ensures each student sees presentation and "covers" them at least once

3. How many set covering constraints are needed?

One constraint for each student so one constraint for each $s \in S$

4. Using the same sets as above and the variable z_c, how would we write a general parameterized set covering constraint for the students?

For each student $s \in S$ visit at least 1 of class $c \in C$ s.t. $s \subseteq c$

Variable z_c for $c \in C$

$$\sum_{\substack{c \in C: \\ s \in c}} z_c \geq 1 \quad \text{for all } s \in S$$

Sum across all $c \in C$, such that s is in c.

The parameterized constraint above works but is a bit messy. There's another way to parameterize it using what's called an **adjacency matrix**. The adjacency matrix is a matrix where the rows correspond to the classes and the columns correspond to the students.

5. Let the adjacency matrix be $a_{c,s}$ for all $c \in \mathscr{C}$ and all $s \in \mathscr{S}$ Illustrate this matrix.

a_{cs} for
$c \in C$
$s \in S$

$$A = \begin{array}{c} \\ N \\ F \\ S \\ B \\ W \\ A \end{array} \begin{array}{c} \begin{array}{ccccc} K & A & R & \cdots & J \end{array} \\ \left[\begin{array}{ccccc} 1 & 0 & 1 & & 0 \\ 1 & 0 & 0 & & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 1 & 1 & & 0 \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & & 1 \end{array}\right] \end{array}$$

$a_{cs} = 1$ if student s is in class c. Parameter

6. Write the parameterized set covering constraints using the adjacency matrix.

$$\sum_{c \in C} a_{cs} z_c \geq 1 \text{ for } s \in S$$

$$s = K: \quad 1 \cdot z_N + 1 \cdot z_F + 0 \cdot z_S + 0 \cdot z_B + 0 \cdot z_W + 0 \cdot z_A \geq 1$$

Either approach works, it's really up to you when it comes to modeling.

7. Write a condensed ~~abstract~~ parameterized model to find a set of classes that covers all students while requiring the fewest possible presentations using the sets, variables, and parameters defined above.

Objective

min number of presentations: $\sum_{c \in C} z_c$

Constraints

$$\sum_{c \in C} a_{cs} z_c \geq 1 \text{ for } s \in S \quad \text{(All sets covered)}$$

$$z_c \in \{0, 1\} \ \forall c \in C$$

3 Set Packing

Eventually Hannah realizes that no student can stand to hear the presentation multiple times, but that she really wants lots of practice with public speaking. She wants to give the presentation as many times as possible without any student seeing it more than once.

1. Write two concrete constraints: one that ensures that Ryan will see the presentation *at most once*, and one that ensures that Brandon will see the presentation *at most once*.

Ryan: N B

$$\underbrace{Z_N + Z_B}_{\text{All the classes he's in}} \underbrace{\leq 1}_{\text{at most once}}$$

Brandon: N W

$$Z_N + Z_W \leq 1$$

2. Why are these called **set packing constraints**? (Think of the set of classes.)

Goal is to select as many sets as possible without any set being selected more than once.

3. Write a condensed ~~abstract~~ model to find a collection of classes that maximizes the number of classes Hannah visits, while not seeing any student more than once.

Objective

Maximize number of visited classes:

$$\sum_{c \in C} z_c$$

Constraints

$$\sum_{c \in C} a_{cs} z_c \leq 1 \quad \forall\ s \in S$$

$$z_c \in \{0,1\} \quad \forall\ c \in C$$

Hardest version to solve → Not gauranteed to have a solution

4 Set Partitioning

Hannah receives a message of encouragement from the Chief of Staff and is told to be sure to show the presentation to *every single student*. But she still knows that no student can possibly sit through it twice, so she must revise her model again.

1. Write two concrete constraints: one that ensures that Aaron will see the presentation *exactly once*, and one that ensures that Samnang will see the presentation *exactly once*.

A: S, B $\quad z_S + z_B = 1$ ← He sees it exactly once

S: F, B $\quad z_F + z_B = 1$

2. Why are these called **set partitioning constraints**? (Think of the set of students.)

The main set (students) is being partitioned into subsets with no overlap.

3. Write an abstract model to find a collection of classes that minimizes the number of classes Hannah visits, while seeing every student exactly once.

Objective

Min # of classes : $\sum_{c \in C} z_c$

Constraints

$$\sum_{c \in C} a_{cs} z_c = 1 \quad \forall s \in S$$

$$z_c \in \{0,1\} \quad \forall c \in C$$