ASE 381P Problem Set #1

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You need not hand in anything. Instead, be prepared to answer any of these problems on an upcoming take-home exam. You may discuss your solutions with classmates up until the time that the exam becomes available, but do not swap work.

1. Prove that for any square matrix S, there exists a symmetric matrix P such that $\boldsymbol{x}^T S \boldsymbol{x} = \boldsymbol{x}^T P \boldsymbol{x}$ for all $\boldsymbol{x}$ of appropriate size. Derive a formula for P in terms of S. This formula may be useful later in the course for converting a non-standard quadratic form (i.e., $\boldsymbol{x}^T S \boldsymbol{x}$, where $S \neq S^T$) to a standard quadratic form (i.e., $\boldsymbol{x}^T P \boldsymbol{x}$, where $P = P^T$).

2. Prove that the l_2 norm (the Euclidean norm) satisfies the 3 properties of a norm. Do not use Schwartz's inequality to prove the triangle inequality unless you also prove Schwartz's inequality. Hint: To prove the triangle inequality, it might be helpful to decompose one of the two vectors into two components, one that is parallel to the other vector and one that is perpendicular to the other vector.

3. Suppose that A is an $m \times n$ matrix with $\rho(A) = l < \min(m, n)$ (where $\rho(A)$ denotes the rank of A), and suppose that the first l columns of A are linearly independent. Show that the QR factorization of A has the form

$$A = QR = Q \begin{bmatrix} R_{l\times l} & X_{l\times(n-l)} \\ 0_{(m-l)\times l} & 0_{(m-l)\times(n-l)} \end{bmatrix}$$

where Q is orthonormal, $R_{l\times l}$ is upper-triangular and nonsingular, $X_{l\times(n-l)}$ is some $l\times(n-l)$ matrix, and all other entries in R are zero. Hint: left-multiplication by a nonsingular matrix does not change the linear dependence/independence of the columns of a matrix.

4. Prove that a Householder transformation is symmetric and orthonormal.

5. Find two Householder transformations H_1 and H_2 such that

$$H_1 H_2 \begin{bmatrix} 8 & 7 \\ 3 & -14 \\ -13 & 16 \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \\ 0 & 0 \end{bmatrix}$$

6. Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ if A is an $m \times n$ matrix and B is an $n \times m$ matrix.

7. Prove that the Gaussian distribution of a scalar random variable x

$$p_x(\xi) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[\frac{-(\xi-\mu)^2}{2\sigma^2}\right]$$

has mean $\bar{x} = \mu$ and variance $\sigma_x^2 = \sigma^2$. Hints: Use transformation of variables, integration by parts, and the fact that the integral of $p_x(\xi)$ over all ξ is equal to 1.

8. (Matlab exercise) Plot the histogram produced by 100,000 samples of a uniform distribution (use `rand()` in Matlab). Then plot the histogram of 100,000 random variables that are each the average of 100 uniform random variables. See if you can plot in probability density units instead of histogram units. This will require you to get the position of the bin centers from the `hist()` function.

   Discuss how these results reflect on the central limit theorem. Do they help convince you of its correctness?

9. Prove that the Gaussian distribution of an n-dimensional vector $\boldsymbol{x}$:

$$p(\boldsymbol{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}[\det(P)]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\boldsymbol{x}-\bar{\boldsymbol{x}})^T P^{-1}(\boldsymbol{x}-\bar{\boldsymbol{x}})\right]$$

   has a mean of $\bar{\boldsymbol{x}}$ and a covariance matrix of P. Also, prove that the distribution is normalized; i.e., that its integral over the whole sample space is equal to 1. Assume that P is symmetric.

   Hints: Any symmetric matrix P can be factorized as $P = V\Lambda V^T$ where V is an orthonormal matrix and Λ is a diagonal matrix. (Just for your information, the columns of V are the eigenvectors of P and the diagonal elements of Λ are the eigenvalues of P, but it's not necessary to know this to do the proof.) Assume this factorization and make the transformation $\boldsymbol{z} = V^T\boldsymbol{x}$. The elements of $\boldsymbol{z}$ will be statistically independent. This will allow you to factorize the probability distribution for the z_i components into n scalar Gaussian distributions. You can then use the known properties of scalar distributions to prove the results you're after. It may be helpful to know that $[\det(V)]^2 = 1$ for V orthonormal. Also recall that the determinant of a product is the product of the determinants if all the matrices in question are square.

10. Given the joint probability distribution for $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$:

$$p(\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{1}{(2\pi)^{\frac{m}{2}}[\det(P_x)]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}(\boldsymbol{x}_1^T H_{11}\boldsymbol{x}_1 + 2\boldsymbol{x}_1^T H_{12}\boldsymbol{x}_2 + \boldsymbol{x}_2^T H_{22}\boldsymbol{x}_2)\right]$$

   where $\boldsymbol{x}_1$ is an n-dimensional vector and $\boldsymbol{x}_2$ is an $(m-n)$-dimensional vector, and where

$$\begin{bmatrix} H_{11} & H_{12} \\ H_{12}^T & H_{22} \end{bmatrix} = P_x^{-1}$$

   Prove that the integral of $p(\boldsymbol{x}_1,\boldsymbol{x}_2)d\boldsymbol{x}_1$ over the whole $\boldsymbol{x}_1$ space is equal to

$$\frac{(2\pi)^{\frac{n}{2}}}{(2\pi)^{\frac{m}{2}}[\det(P_x)]^{\frac{1}{2}}[\det(H_{11})]^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\boldsymbol{x}_2^T(H_{22} - H_{12}^T H_{11}^{-1} H_{12})\boldsymbol{x}_2\right]$$

   Hints: In the expression for $p(\boldsymbol{x}_1,\boldsymbol{x}_2)$, express the argument of the exponential function in the form

$$-\frac{1}{2}(\boldsymbol{x}_1-\bar{\boldsymbol{x}}_1)^T H_{11}(\boldsymbol{x}_1-\bar{\boldsymbol{x}}_1) - \frac{1}{2}\boldsymbol{x}_2^T G\boldsymbol{x}_2$$

for an appropriately chosen vector $\bar{\boldsymbol{x}}_1$ and an appropriately chosen matrix G. Then use the fact that the integral over the whole $\boldsymbol{x}_1$ space of $\exp\left[-\frac{1}{2}\boldsymbol{x}_1^T H_{11}\boldsymbol{x}_1\right]$ must equal $(2\pi)^{n/2}[\det(H_{11}^{-1})]^{1/2}$ in order for the corresponding multivariate Gaussian function to be normalized.

11. Suppose you roll a single die twice. What is the probability that the first roll is a 5 conditioned on knowledge that the sum of the two rolls is 9?

Also do the following problems from Bar Shalom:
1-1,1-6,1-7,1-15