7 Markov Chains

For the Markov chain, the past and **the future are conditionally independent**. For the special case of random walk on an undirected network, the network structure is the key to determining the stationary distribution.

We can picture a Markov chain intuitively by imagining a system with *states* and someone randomly wandering around from state to state.

For many interesting Markov chains, the *stationary* distribution of the chain helps us understand how the chain will behave in the long run.

7.1 Markov property and transition matrix

A sequence of RVs $X_0, X_1, X_2, \ldots$ evolving over time. This is called a *stochastic process*.

Markov chains have a form of one-step dependence, allowing to do beyond IIDs bust still have very convenient structure.

Markov chains widely used for simulations of complex distributions, via algorithms known as *Markov chain Monte Carlo (MCMC)*.

Markov chains live in both space and time: the set of possible states X_n is called *state time*, and index n represents evolution of the process over *time*. The state space of can be discrete or continuous, and time can also be discrete or continuous. We will focus on *discrete-state*, *discrete-time* Markov Chains with a *finite* state space.

Markov Chain

A sequence of RVs $X_0, X_1, X_2, \ldots$ taking values in *state space* $\{1, 2, \ldots, M\}$ is called *Markov chain* $\forall\ n \geq 0$,

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i-1, \ldots, X_0 = i_0) = P(X_{n+1} = j | X_n = i)$$

$P(X_{n+1} = j | X_n = j)$ is called the *transition probability*. from state i to state j. This Markov chain is $time - homogeneous$, which means that

$P(X_{n+1} = j | X_n = j)$ is the same $\forall\ n$.

We can describe the probabilities of moving from state to state using a matrix called *translation matrix* whose i, j entry is probability of going from i-th to j-th state in a single step.

Translation matrix

Let $X_0, X_1, X_2, \ldots$ be a Markov chain $\{1, 2, \ldots, M\}$ and let $q_{ij} = P(X_{n+1} = j | X_n = i)$ be transition probability from state i to state j. The matrix $Q = (q_{ij})$ is the *transition matrix* of the chain. Q is nonnegative and each row sums to 1.

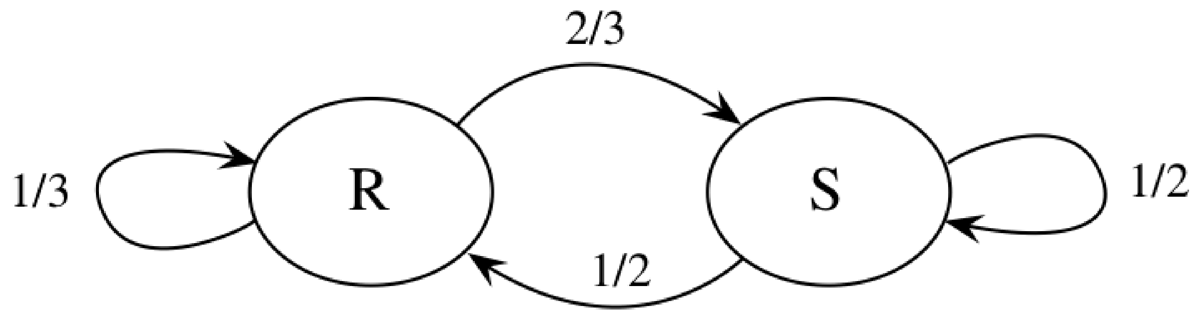
Example: Rainy-sunny Markov chain

If today is rainy, tomorrow it will be rainy with $P = 1/3$ and sunny with $P = 2/3$. If today is sunny, tomorrow it will be rainy with $P = 1/2$ and sunny with $P = 1/2$.

Let X_n be the weather on day n and $X_0, X_1, X_2, \ldots$ is a Markov chain on the state space $\{R, S\}$. Translation matrix of this chain is:

$$\begin{array}{cc} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

Also we can represent this chain as graph:

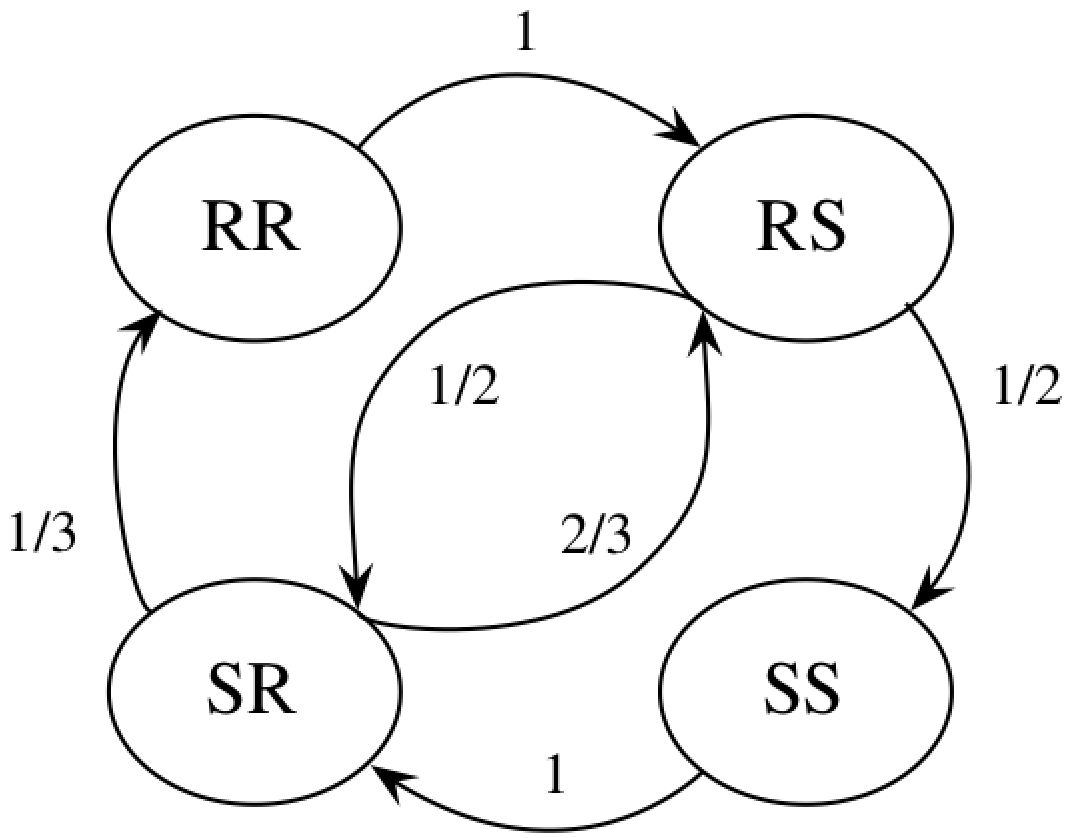


And what if tomorrow's weather depends on today's and yesterday's weather? To illustrate it, we can create a new Markov chain. Let $Y_n = (X_{n-1}, X_n)\ \forall n \geq 1$. Then $Y_1, Y_2, \ldots$ is a Markov chain on the state space $\{(R,R)(R,S),(S,R),(S,S)\}$.

Translation matrix of this chain is:

$$
\begin{array}{c c}
 & \begin{array}{cccc} (R,R) & (R,S) & (S,R) & (S,S) \end{array} \\
\begin{array}{c} (R,R) \\ (R,S) \\ (S,R) \\ (S,S) \end{array} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{array}
$$

This Markov chain may be represented as the following graph:



Similarly, we can build a chain on n-order dependencies.

N-step transition probability

The n-step transition probability $q_{ij}^{(n)}$ from i to j is the probability of being at j exactly n steps after being at i.

$$q_{ij}^{(n)} = P(X_n = j | X_0 = i).$$

Of course,

$$q_{ij}^{(n)} = \sum_k q_{ik} q_{kj}.$$

The n-th power of the transition matrix gives the n-step transition probabilities $q_{ij}^{(n)}$ is the (i, j)-th entry of Q^n.

Marginal distribution of X_n

Let $\mathbf{t} = (t_1, t_2, \dots)$ where $t_i = P(X_0 = i)$ and $\mathbf{t}$ is a row vector. Then *the marginal distribution* of X_n is given by the vector $\mathbf{t}Q^n$ is $P(X_n = j)$.

Proof:

By LOTA, conditioning on X_0, the probability that the chain is at j-th state after n steps is:

$$P(X_n = j) = \sum_{i=1}^{M} P(X_0 = i) P(X_n = j | X_0 = i) =$$
$$= \sum_{i=1}^{M} t_i q_{ij}^{(n)}$$

which is the jth component of $\mathbf{t}Q^n$.

7.2 Classification of states

States may be *recurrent* or *transient*. Recurrent ones will be visited over and over again in the long one while transient ones will be constantly abandoned.

Also states may be classified using their *period* which is a possible integer summarizing the amount of time that can be elapsed between visits to this state.

Recurrent and transient states:

State i of a Markov chain is recurrent if starting from i, the $P = 1$ that the chain will return to i. The state i is transient if the chain starts from i there is $P > 0$ that the chain will never return to i.

As long as there is a positive probability of leaving i forever, the chain eventually will leave i forever!

If i is a transient state of a Markov chain, and the probability of never returning to i starting from i is a positive number $p > 0$. Then the number of returns to i before leaving it forever is $\sim Geom(p)$.

Irreducible and reducible chain:

A Markov chain with transition matrix Q is *irreducible* if for $\forall\ i, j$ it is possible to go from i to j in a finite number of steps with $P > 0$. So $\forall\ i, j$ there is integer $n > 0$ that (i, j)-th entry of Q^n is positive.

Not *irreducible* Markov chain is *reducible*.

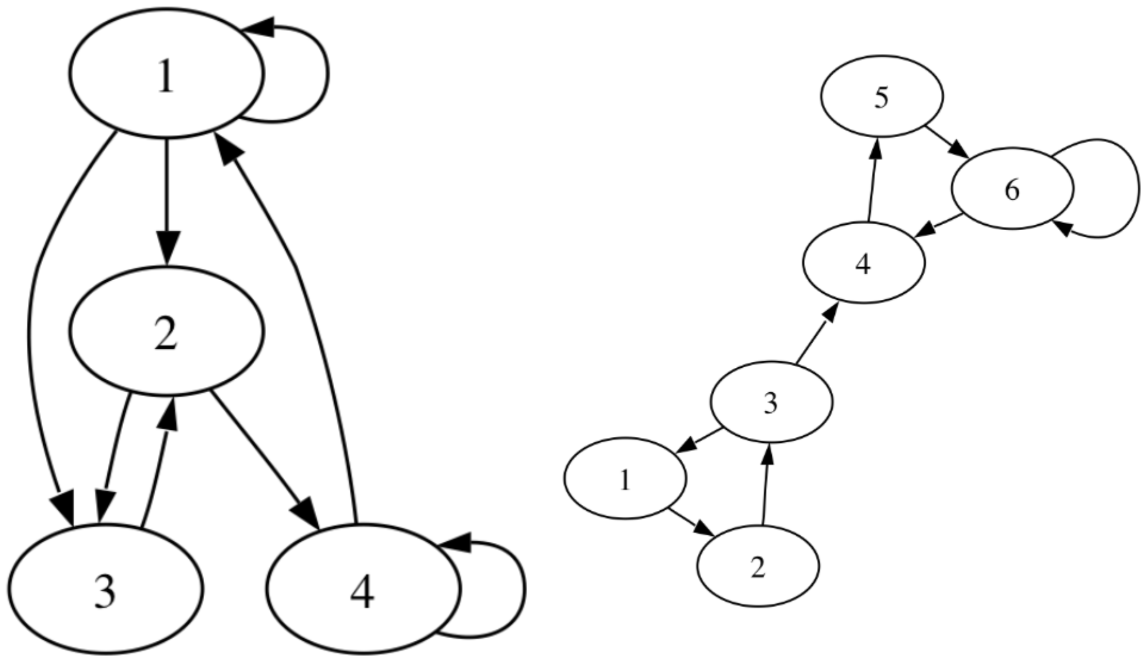
In an irreducible Markov chain with a finite state space, all states are recurrent.

Period of a state, periodic and aperiodic chain:

The *period* of a state i in a Markov chain is the greatest common divisor (gcd) of the possible numbers of steps it can take to return to i when starting at i. The period of i is the greatest common divisor of numbers n such that (i, j)-th entry of Q^n is positive.

A state is called *aperiodic* if period $= 1$ and *periodic* otherwise. The chain is *aperiodic* of all the states are *aperiodic*, and *periodic* otherwise.

Examples:



Left: aperiodic Markov chain

Right: periodic Markov chain with states $1, 2, 3$ with period 3.

7.3 Stationary distribution

What fraction of time will it spend in each recurrent of states? This question is answered by *stationary distribution* a.k.a. *steady-state distribution*.

Stationary distribution:

A row vector $\mathbf{s} = (s_1, \ldots, s_M)$ such that $s_i \geq 0$ and $\sum_i s_i = 1$ is a *stationary distribution* for a Markov chain with transition matrix Q if:

$$\sum_i s_i q_{ij} = s_j$$

$\forall\ j$, or equivalently,

$$\mathbf{s}Q = \mathbf{s}.$$

If $\mathbf{s}$ is the distribution of X_0, then $\mathbf{s}Q$ is the marginal distribution of X_1. But the equation $\mathbf{s}Q = \mathbf{s}$ means that X_1 has the distribution $\mathbf{s}$. Same for X_2, X_3, etc.

Properties:

Existence and uniqueness:

Any irreducible Markov chain has a unique stationary distribution. In this distribution, every state has a positive probability.

Convergence:

Let $X_0, X_1, \dots$ be a Markov chain with a stationary distribution $\mathbf{s}$ and transition matrix Q, such that some power Q^m is positive in all entries. Then $P(X_n = i)$ converges to s_i as $n \to \infty$. In terms of transition matrix, Q^n converges to $\mathbf{s}$ in each row.

Expected time to return:

Let $X_0, X_1, \dots$ be a Markov chain with a stationary distribution $\mathbf{s}$. Let r_i be the expected time it takes the chain to return to i, given that it starts at i. Then, $s_i = 1/r_i$.

Example of usage: Google PageRank. Founders of Google modeled web-surfing as Markov chain and then used its stationary distribution to rank the relevance of webpages.

7.4 Reversibility

Stationary distribution is useful for understanding long-run behavior, but it may be computationally difficult to find the stationary distribution when state space is large.

Reversibility:

Let $Q = (q_{ij})$ is the transition matrix of a Markov chain. Suppose $\mathbf{s} = (s_1, \dots, s_M)$ where $s_i \geq 0, \sum_i s_i = 1$ such that

$$s_i q_{ij} = s_j q_{ji}$$

$\forall\ i, j$ This equation is *reversibility* or *detailed balance* condition, and the chain is *reversible* with respect to $\mathbf{s}$ if it holds.

So with a transition matrix, we can find a nonnegative vector $\mathbf{s}$ which sums to 1 then it is a stationary distribution!

Reversible implies stationary

If $Q = (q_{ij})$ is a transition matrix of a Markov chain that is reversible with respect to a nonnegative $\mathbf{s} = (s_1, \ldots, s_M)$ and $\sum_i s_i = 1$, $\mathbf{s}$ is a stationary distribution of the chain.

Why?

$$\sum_i s_i q_{ij} = \sum_i s_j q_{ji} = s_j \sum_i q_{ji} = s_j$$

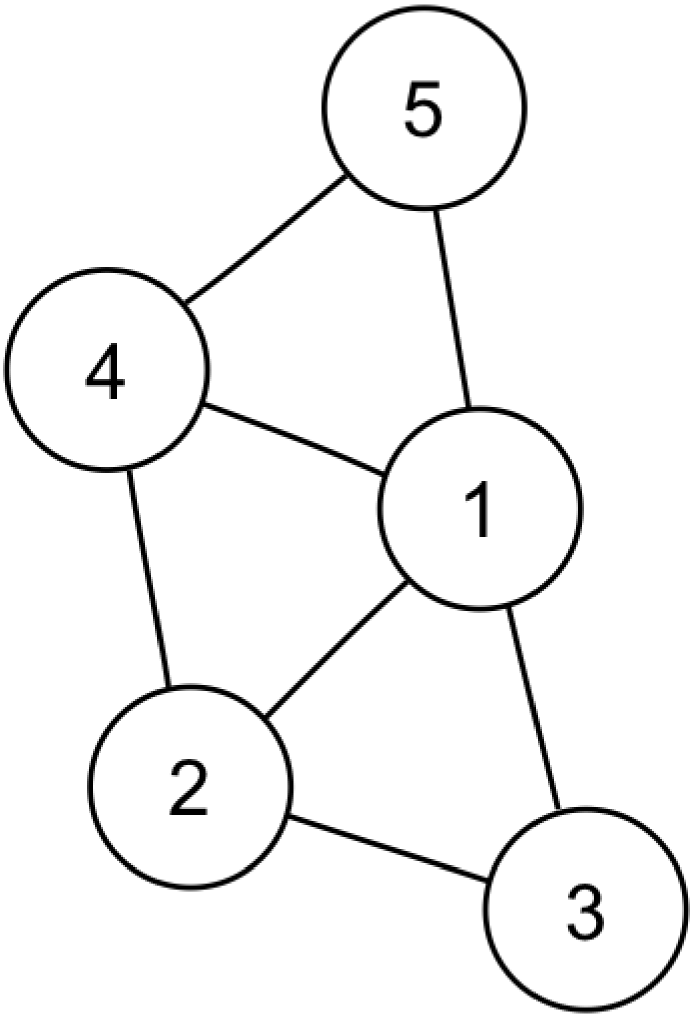
Using this result, we can easily verify the reversibility condition which may be simpler than solving the system of equations $\mathbf{s}Q = \mathbf{s}$. We will look at 3 types of Markov chains where is is possible to find an $\mathbf{s}$ that satisfies the reversibility. These Markov chains are called *reversible*.

If each column of Q sums to 1, then the uniform distribution over all states, $(1/M, 1/M, \ldots, 1/M)$ is a stationary distribution. A nonnegative matrix whose columns are all equal to 1 is *doubly stochastic matrix*.

If the Markov chain is a *random walk on an undirected network*, then there is a simple formula for the stationary distribution.

Network is a collection of *nodes* joined by *edges*; it is undirected if you can travel through edges in both directions.

Example:



The *degree* of a node is the number of attached edges. The *degree sequence* is vector $(d_1, \ldots, d_n)$ where d_j is a degree of j-th node. If the edge is from node to itself, it is called a *self-loop* and counts as 1 in the degree of that node.

For network above, it has a degree sequence $\mathbf{d} = (4, 3, 2, 3, 2)$. Also

$$d_i q_{ij} = d_j q_{ji} \; \forall i, j$$

because q_{ij} is $1/d_i$ if there's an edge, else 0. Therefore, the stationary distribution is proportional to the degree sequence. For the network above, $\mathbf{s} = (\frac{4}{14}, \frac{3}{14}, \frac{2}{14}, \frac{3}{14}, \frac{2}{14})$.

FIN.