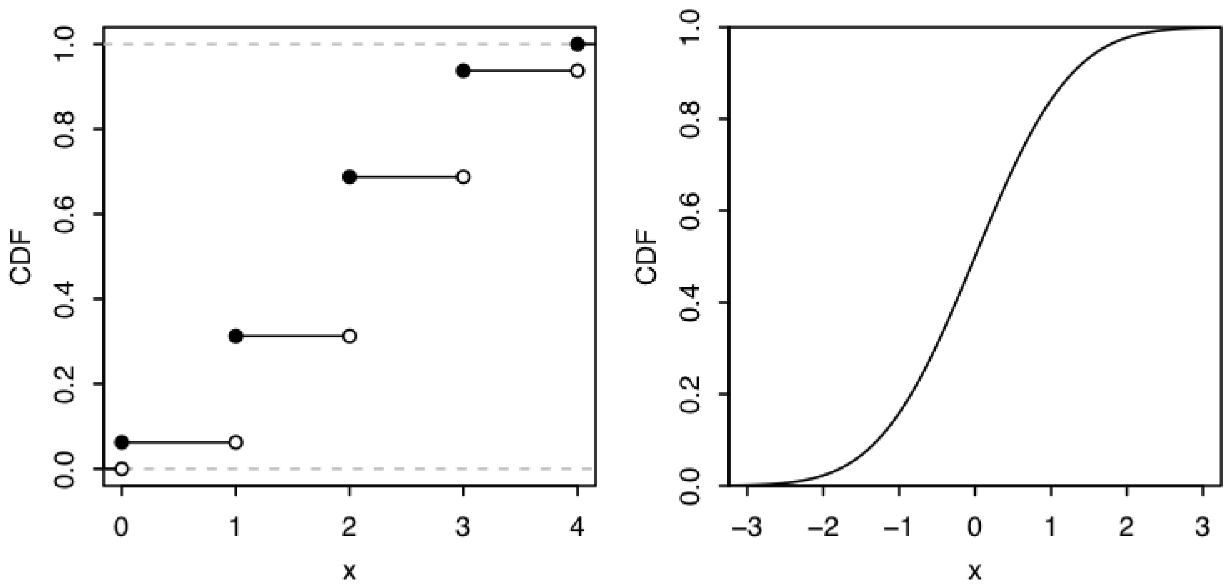
4 Continuous Random Variables

Together, discrete and continuous approaches form a powerful framework for modeling the world.

4.1 Probability density function

Continuous RVs!

An RV has a *continuous distribution* if its CDF is differentiable. Endpoints of CDF may be continuous but not differentiable. A continuous RV is a RV with a continuous distribution.



For a continuous RV X with CDF F, the PDF of X is derivative f of the CDF: $f(x) = F'(x)$

The support of X: all x where $f(x) > 0$.

The PDF is kinda similar to PMF, but for PDF quantity of $f(x)$ **is not a probability**. To obtaion the probability, we need to **integrate** PDF.

We can be carefree about including or excluding endpoints as above for continuous RVs, but we must not be careless about this for discrete RVs.

Valid PDF of a continuous RV:

1. Nonnegative: $f(x) \geq 0$
2. Integrates to 1: $\int_{-\infty}^{\infty} f(x)dx = 1$

Example: logistic distribution.

$X \sim$ Logistic.

CDF:

$$F(x) = \frac{e^x}{1+e^x}, x \in \mathbb{R}$$

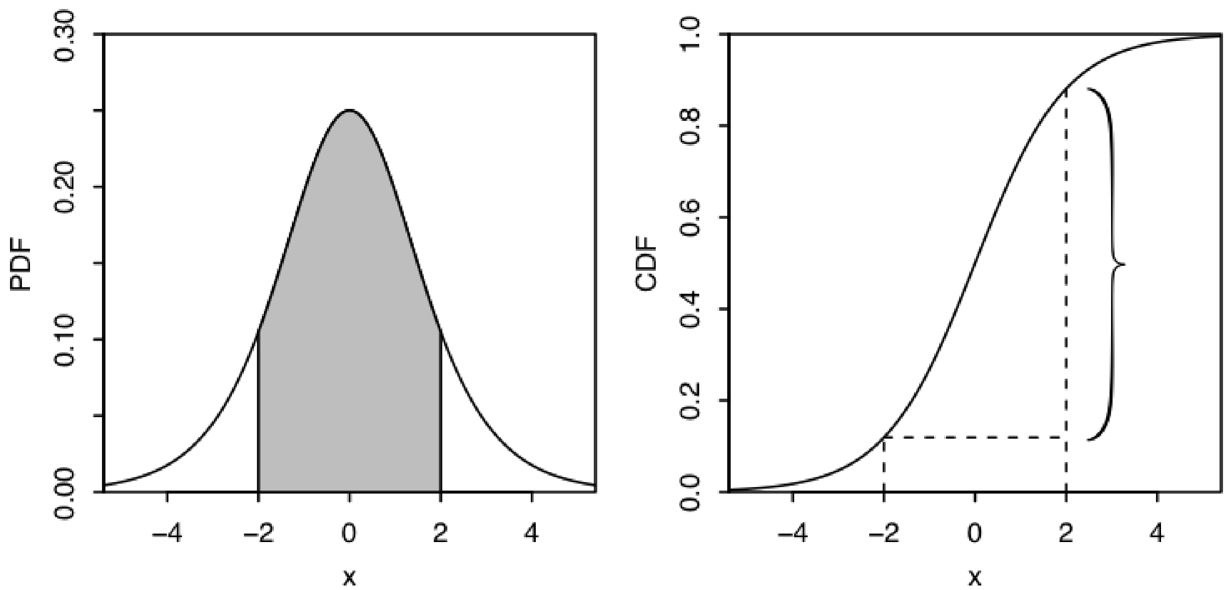
PDF:

$$F(x) = \frac{e^x}{(1+e^x)^2}, x \in \mathbb{R}$$

To find $P(-2 < X < 2)$, we need to integrate PDF from -2 to 2:

$$P(-2 < X < 2) = \int_{-2}^{2} \frac{e^x}{(1+e^x)^2} = F(2) - F(-2) \approx 0.76$$

Or $P(-2 < X < 2)$ is indicated by the shaded area under the PDF and the height of the curly brace on the CDF.



4.2 Uniform distribution

A continuous RV U has the *Uniform distribution* $X \sim Unif(a, b)$ on the interval (a, b) if its PDF is:

$$f(x) = \frac{1}{b-a} \, \forall a < x < b,$$
$$f(x) = 0 \; otherwise$$

The CDF is the accumulated area under the PDF:

$$F(x) = 0 \, \forall x \leq a,$$
$$F(x) = \frac{x-a}{b-a} \; a < x < b,$$
$$F(x) = 1 \, \forall x \geq b.$$

$Unif(0, 1)$ is the standard Uniform.

For Uniform distributions, *probability is proportional to length.*

Location-scale transformation.

The RV Y has been obtained as a *location-scale transformation* of X if $Y = \sigma X + \mu$. μ controls the location and σ controls the scale.

Warning: if Y is a linear function of X, the Uniformity is preserved, but if Y is defined as a *nonlinear* transformation of X, Y will not be Uniform.

Warning: When using location-scale transformations, the shifting and scaling should be applied to the *random variables* themselves, not to their PDFs.

4.3 Universality of the Uniform distribution

Given a $Unif(0, 1)$ RV, we can construct an RV with *any continuous distribution we want*.

Other names of the universality of Uniform:

- probability integral transform,

- inverse transform sampling,
- the quantile transformation,
- the fundamental theorem of simulation.

Theorem:

F is a CDF which is continuous function and strictly increasing on the support of distribution. This ensures that the inverse function F^{-1} exists as function $(0,1) \to \mathbb{R}$. Results:

1. Let $U \sim Unif(0,1)$ and $X = F^{-1}(U)$. Then X is an RV with CDF F.
2. Let X be an RV with CDF F. Then $F(X) \sim Unif(0,1)$.

What this theorem is saying about?

Fist part: Since F^{-1} is a function (**quantile function**), U is a RV, and a function of RV is RV, $F^{-1}(U)$ is a RV; universality of the Uniform says its CDF is F.

Second part: reverse direction! Starting from RV X whose CDF is F and then creating RV $Unif(0,1)$. Universality of the Uniform says that the distribution of $F(X)$ is Uniform on $(0,1)$.

Warning: potential notational collusion!

$F(x) = P(X \leq x)$ by definition, but $F(X) = P(X \leq X) = 1$ is incorrect by definition. Rather, we should first find an expression for the CDF as a function of x, then replace x with X to obtain a random variable. For example, if the CDF of X is $F(x) = 1 - e^{-1}$ for $x > 0$, then $F(X) = 1 - e^{-X}$.

Example: percentiles

Exam, grades 0-100, RV X is the score of random student. We approximate the discrete distribution of scores using continuous distribution. So X is continuous RV, CDF is strictly increasing on $(0,100)$. Suppose median score is 60. So $F(60) = 1/2$ or $F^{-1}(1/2) = 60$

If student scores 72 on the exam, then his **percentile** is the fraction of students who's score is below 72. This is $F(72)$ which is number $(0.5,1)$. Other way, if we have percentile 0.95, the score is $F^{-1}(0.95)$. Percentile is also called a *quantile*, F^{-1} is *quantile function*.

The universality property says that $F(X) \sim Unif(0, 1)$.

So! 50 of students have a percentile of at least 0.5. 10 have a percentile between $(0, 0.1)$, and between $(0.1, 0.2)$, …

4.4 Normal distribution