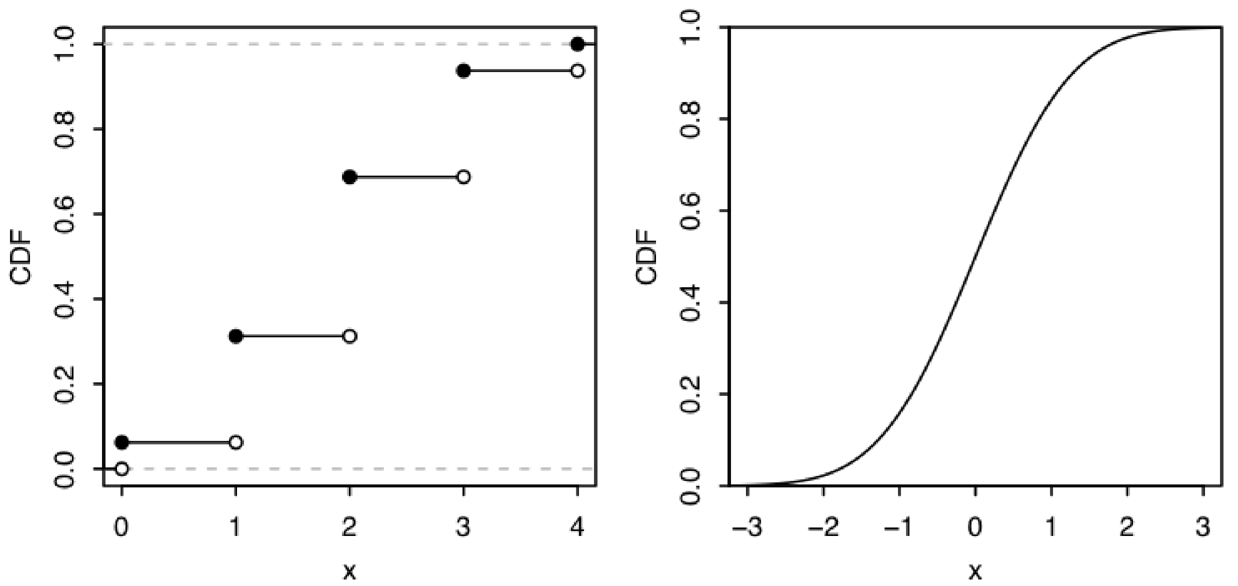
4 Continuous Random Variables

Together, discrete and continuous approaches form a powerful framework for modeling the world.

4.1 Probability density function

Continuous RVs!

An RV has a *continuous distribution* if its CDF is differentiable. Endpoints of CDF may be continuous but not differentiable. A continuous RV is a RV with a continuous distribution.



For a continuous RV X with CDF F, the PDF of X is derivative f of the CDF: $f(x) = F'(x)$

The support of X: all x where $f(x) > 0$.

The PDF is kinda similar to PMF, but for PDF quantity of $f(x)$ **is not a probability**. To obtaion the probability, we need to **integrate** PDF.

We can be carefree about including or excluding endpoints as above for continuous RVs, but we must not be careless about this for discrete RVs.

Valid PDF of a continuous RV:

1. Nonnegative: $f(x) \geq 0$
2. Integrates to 1: $\int_{-\infty}^{\infty} f(x)dx = 1$

Example: logistic distribution.

$X \sim$ Logistic.

CDF:

$$F(x) = \frac{e^x}{1+e^x}, x \in \mathbb{R}$$

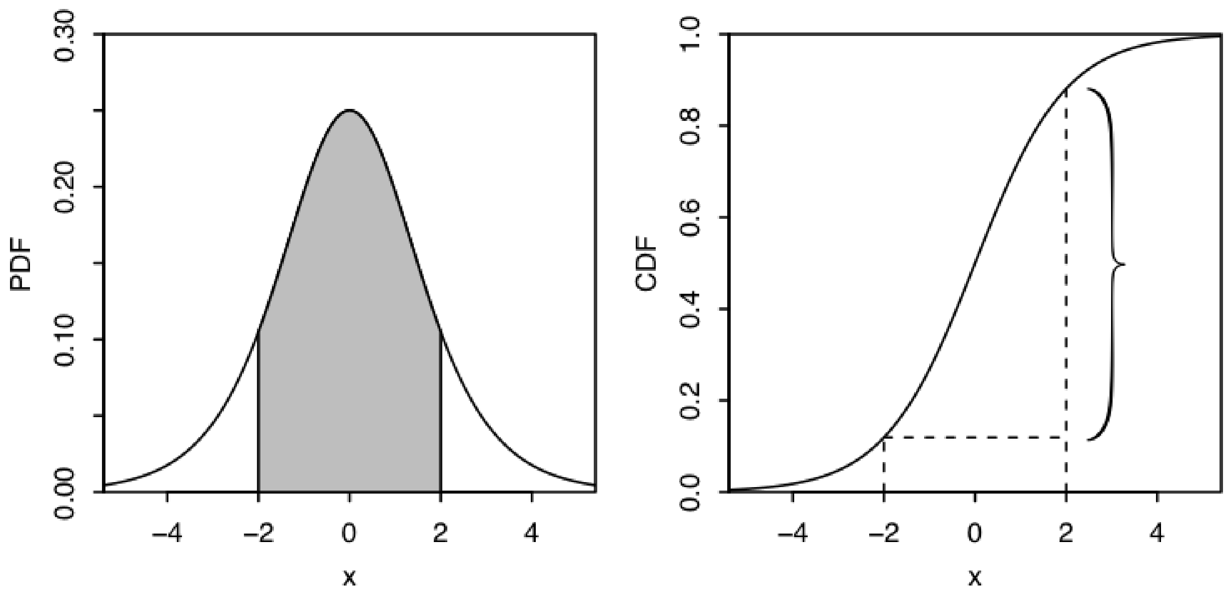
PDF:

$$F(x) = \frac{e^x}{(1+e^x)^2}, x \in \mathbb{R}$$

To find $P(-2 < X < 2)$, we need to integrate PDF from -2 to 2:

$$P(-2 < X < 2) = \int_{-2}^{2} \frac{e^x}{(1+e^x)^2} = F(2) - F(-2) \approx 0.76$$

Or $P(-2 < X < 2)$ is indicated by the shaded area under the PDF and the height of the curly brace on the CDF.



4.2 Uniform distribution

A continuous RV U has the *Uniform distribution* $X \sim Unif(a, b)$ on the interval (a, b) if its PDF is:

$$f(x) = \frac{1}{b-a} \, \forall a < x < b,$$
$$f(x) = 0 \; otherwise$$

The CDF is the accumulated area under the PDF:

$$F(x) = 0 \, \forall x \leq a,$$
$$F(x) = \frac{x-a}{b-a} \; a < x < b,$$
$$F(x) = 1 \, \forall x \geq b.$$

$Unif(0, 1)$ is the standard Uniform.

For Uniform distributions, *probability is proportional to length.*

Location-scale transformation.

The RV Y has been obtained as a *location-scale transformation* of X if $Y = \sigma X + \mu$. μ controls the location and σ controls the scale.

Warning: if Y is a linear function of X, the Uniformity is preserved, but if Y is defined as a *nonlinear* transformation of X, Y will not be Uniform.

Warning: When using location-scale transformations, the shifting and scaling should be applied to the *random variables* themselves, not to their PDFs.

4.3 Universality of the Uniform

Given a $Unif(0, 1)$ RV, we can construct an RV with *any continuous distribution we want.*