5 Averages, Law of Large Numbers, and Central Limit Theorem

Expectation, standard deviation and correlation.

Expectation is the most widely used notion of average in statistics, because of its intuitive interpretations and convenient properties.

Linearity is the most important property of expectation.

The law of large numbers (LLN) and central limit theorem (CLT) are powerful results about the sample mean of a large number of IID RVs.

The LLN says that the sample mean is likely to be close to the theoretical expectation. The CLT says that the sample mean will be approximately Normal.

5.1 Expectation

Arithmetic mean of $x_1, x_2, \ldots, x_n$:

$$\bar{x} = \frac{1}{n} \sum_{j=1}^{n} x_j.$$

More generally, weighted mean of $x_1, x_2, \ldots, x_n$:

$$weighted\_mean(x) = \sum_{j=1}^{n} x_j p_j.$$

Where the weights $p_1, p_2, \ldots, p_n$ are pre-specified nonnegative numbers with $\sum_{j=1}^{n} p_i = 1$ (so for unweighted mean $p_j = 1/n \; \forall \; i$).

The definition of *expectation of DRV* is inspired by weighted mean, weights are probabilities:

The **expected value** (or **expected mean**) of DRV X whose possible values are $x_1, x_2, \ldots$ is:

$$E(X) = \sum_{j=1}^{\infty} x_j P(X = x_j).$$

If support is finite,

$$E(X) = \sum_{x} x P(X = x)$$

Where x is value and $P(X = x)$ is PMF at x.

The expectation is undefined if $\sum_{j=1}^{\infty} |x_j| P(X = x_j)$ diverges, since then the series for $E(X)$ diverges or depends on the order in which the x_j are listed.

Warning!

For any DRV X, $E(X)$ is a number (if exists). Common mistake: to replace an RV by its expectation which is wrong because X is a function, $E(X)$ is a constant, and ignores the variability of X.

Notation:

$E(X)$ is abbreviating to EX, similarly EX^2 is $E(X^2)$ not $(E(X))^2$!

The order of operations here is very important!

5.2 Linearity of expectation

The most important property of expectation is *linearity*: expected sum of RVs is the sum of expectations: $\forall X, Y$ $\forall$ constant c:

$$E(X + Y) = E(X) + E(Y),$$
$$E(cX) = cE(X).$$

Averages can be calculated in two ways, *ungrouped* or *grouped*, is all that is needed to prove linearity!

It allows us to work with the distribution X directly without returning to the sample space.

But we can't use the same super-pebbles for another RV Y on the same sample space.

We can take a weighted average of the values of individual pebbles. If $X(s)$ is the value assigns to pebble s:

$$E(X) = \sum_s X(s)P(\{s\}),$$

where $P(s)$ is the weight of pebble s. This corresponds to the ungrouped way of taking averages! It breaks down the sample space into the smallest possible units, so we are now using the *same* weights $P(\{s\})$ for every random variable:

$$E(Y) = \sum_s Y(s)P(\{s\}).$$

Now we can combine $E(X)$ and $E(Y)$:

$$E(X) + E(Y) = \sum_s X(s)P(\{s\}) + \sum_s Y(s)P(\{s\}) = \\ \sum_s (X+Y)(s)P(\{s\}) = E(X+Y)$$

Using this property, we can calculate expectations for *Binomial* and *Hypergeometric* distributions!

Binomial expectation: for $X \sim Bin(n,p)$:

$$E(X) = \sum_{k=0}^{n} kP(X=k) = \sum_{k=0}^{n} k\binom{n}{k}p^k q^{n-k}.$$

Linearity of expectation provides a much shorter path to the same result: RV X is the sum of n independent $Bern(p)$ RVs:

$$X = I_1 + \ldots + I_n,$$

each I_j has $E(I_j) = 1p + 0q = p$. By linearity,

$$E(X) = E(I_1) + \ldots E(I_n) = np.$$

Hypergeometric expectation: for $X \sim HGeom(w, b, n)$:

We can write X as a sum of Bernoulli RVs,

$$X = I_1 + \ldots + I_n,$$

Where I_j equals 1 if jth ball is white and 0 otherwise. By symmetry, $I_j \sim Bern(p)$, where $p = w/(w+b)$.

These I_j aren't independent, since balls aren't replacing. However, linearity still holds for dependent RVs. Thus,

$$E(X) = nw/(w+b).$$

5.3 Geometric and Negative Binomial

Geometric distribution:

A sequence of independent Bernoulli trials, each with the same success probability $p \in (0,1)$, with trials performed until a success occurs. RV X is the number of failures before the first successful trial. X has the Geometric distribution with parameter p: $X \sim Geom(p)$.

Geometric PDF: If $X \sim Geom(p)$, then PMF of X is:

$$P(X = k) = q^k p$$

for $k = 0, 1, 2, \ldots$ where $q = 1 - p$.

This is a valid PMF because

$$\sum_{k=0}^{\infty} q^k p = p \sum_{k=0}^{\infty} q^k = p \frac{1}{1-q} = 1.$$

Warning: In our convention, the *Geometric* distribution **excludes** the success, and the *First Success* distribution **includes** the success $Y \sim FS(p)$.

If $Y \sim FS(p)$, then $Y - 1 \sim Geom(p)$

Geometric expectation:

By definition,

$$E(X) = \sum_{k=0}^{\infty} kq^k p,$$

where $q = 1 - p$. Each term looks similar to kq^{k-1}.

$$\sum_{k=0}^{\infty} q^k = \frac{1}{1-q}.$$

This series converges because $0 < q < 1$. Differentiating both sides,

$$\sum_{k=0}^{\infty} kq^{k-1} = \frac{1}{(1-q)^2}.$$

Multiply both sides by pq:

$$E(X) = \sum_{k=0}^{\infty} kq^k p = pq \sum_{k=0}^{\infty} kq^{k-1} = pq \frac{1}{(1-q)^2} = \frac{q}{p}$$

First Success expectation:

$$E(Y) = E(X+1) = \frac{q}{p} + 1 = \frac{1}{p}.$$

Negative Binomial distribution

Sequence of independent Bernoulli trials which success probability p, X is the number of failures before the rth success, $X \sim NBin$.

Binomial counts the number of *successes* in a fixed number of *trials*, while the *Negative Binomial* counts the number of *failures* until a fixed number of *successes*.

Negative Binomial PDF: If $X \sim NBin(r, p)$, then PMF of X is:

$$P(X = n) = \binom{n+r-1}{r-1} p^r q^n$$

for $n = 0, 1, 2, \dots$ where $q = 1 - p$.

Theorem:

If $X \sim NBin(r, p)$, We can write $X = X_1 + \dots X_r$ were X_i are IIDs $\sim Geom(p)$.

Negative Binomial expectation:

By previous theorem and linearity,

$$E(X) = E(X_1) + \dots + E(X_r) = \frac{rq}{p}.$$

5.4 Indicator RVs and the fundamental bridge

Indicator RVs (look at 3.3) are extremely useful tool for calculating expected values.

Indicator RV properties: A and B are events.

1. $(I_A)^k = I_A \; \forall \; k > 0$.
2. $I_{A^c} = 1 - I_A$.
3. $I_{A \cap B} = I_A I_B$.
4. $I_{A \cup B} = I_A + I_B - I_A I_B$.

Proofs:

Property 3: $I_A I_B$ if both I_A and I_B are 1, 0 otherwise. Property 4: $I_{A \cup B} = 1 - I_{A^c \cap B^c} = 1 - I_{A^c} I_{B^c} = 1 - (1 - I_A)(1 - I_B) = I_A + I_B - I_A I_B$

Fundamental bridge:

The probability of an event A is the expected value of its indicator RV I_A:

$$P(A) = E(I_A).$$

Fundamental bridge allows to express *any* probability as an expectation. We can express a DRV as a sum of a sum of indicator RVs. Using fundamental bridge, we can find the expectation of indicators and find the expectation of original RV!

5.5 Law of the unconscious statistician (LOTUS)

$E(g(X))$ doesn't equal $g(E(X))$ if g isn't linear. How to correctly calculate $E(g(X))$?

LOTUS:

If X is a DRV and g is a function $\mathbb{R} \to \mathbb{R}$, then:

$$E(g(X)) = \sum_x g(x) P(X = x),$$

the sum takes over all possible values of X.

So we need only to know PMF of X, not PMF of $g(X)$.

Let X have support $0, 1, 2...$ with probabilities $p_0, p_1, p_2, \ldots$, so the PMF is $P(X = n) = p_n$, then X^3 as support $0^3, 1^3, 2^3, \ldots$ so:

$$E(X) = \sum_{n=0}^{\infty} n p_n,$$

$$E(X^3) = \sum_{n=0}^{\infty} n^3 p_n.$$

5.6 Variance

Important application of LOTUS is finding the *variance* of an RV. It tells how spread out the distribution is.

Variance of an RV X:

$$Var(X) = E(X - EX)^2.$$

The square root of the variance is called the **standard deviation**:

$$SD(X) = \sqrt{Var(X)}$$

Important: when we write $E(X - EX)^2$ it is the expectation of an RV $(X - EX)^2$, **not** $(E(X - EX))^2$ which is 0 by linearity.

Why squaring? because $E(X - EX) = 0$ because positive and negative deviations cancel each other. However, variation is in squared units so we need SD to receive number in correct units.

Equivalent expression for variance is $Var(X) = E(X^2) - (EX)^2$. This expression is easier to work with sometimes.

Variance properties:

1. $Var(X + c) = Var(X) \ \forall \ const \ c$
2. $Var(cX) = c^2 Var(X) \ \forall \ const \ c$
3. If RVs X and Y are independent, $Var(X + Y) = Var(X) + Var(Y)$. Not true in general if general if RVs are dependent.
4. $Var(X) \geq 0$ and 0 if and only if $P(X = c) = 1$.

Geometric and Negative Binomial variance:

If $X \sim Geom(p)$, we already know that $E(X) = q/p$.

$$Var(X) = E(X^2) - (EX)^2 = \frac{q}{p^2}$$

This is also the variance of First Success distribution. If an RV $X \sim NBin(r, p)$, it can be represened as a sum of r IIDs $Geom(p)$ so its variance is:

$$Var(X) = r\frac{q}{p^2}$$

Binomial variance:

If $X \sim Bin(n,p)$, it can be represened as sum of indicator RVs:
$X = I_1 + I_2 + \ldots I_n$ where I_j is indicator for $j-th$ trial. Variance of each I_j:

$$Var(I_j) = E(I_j^2) - (E(I_j))^2 = p - p^2 = p(1-p).$$

Since the I_j are independent,

$$Var(X) = Var(I_1) + \ldots + Var(I_n) = np(1-p).$$

5.7 Poisson

Poisson distribution is an extremely popular distribution for modeling discrete data.

Poisson distribution:

An RV X has the *Poisson distribution* with parameter λ where $\lambda > 0$ if its PMF is:

$$P(X=k) = \frac{e^{-\lambda}\lambda^k}{k!}, k = 0, 1, 2, \ldots$$

$X \sim Pois(\lambda)$.

Poisson expectation and variance:

Let $X \sim Pois(\lambda)$. Then the mean and variance are both equal to λ.

$$E(X) = e^{-\lambda}\sum_{k=0}^{\infty} k\frac{\lambda^k}{k!} = e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^k}{k!} =$$
$$= \lambda e^{-\lambda}\sum_{k=1}^{\infty} k\frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda}e^{\lambda} = \lambda$$

We dropped $k=0$ because it was 0, then took out λ out of sum, so inside remained a Taylor series for e^{λ}.

To get the variance, let's find $E(X^2)$. By LOTUS,

$$E(X^2) = \sum_{k=0}^{\infty} k^2 P(X = k) = e^{-\lambda} \sum_{k=0}^{\infty} k^2 \frac{\lambda^k}{k!}.$$

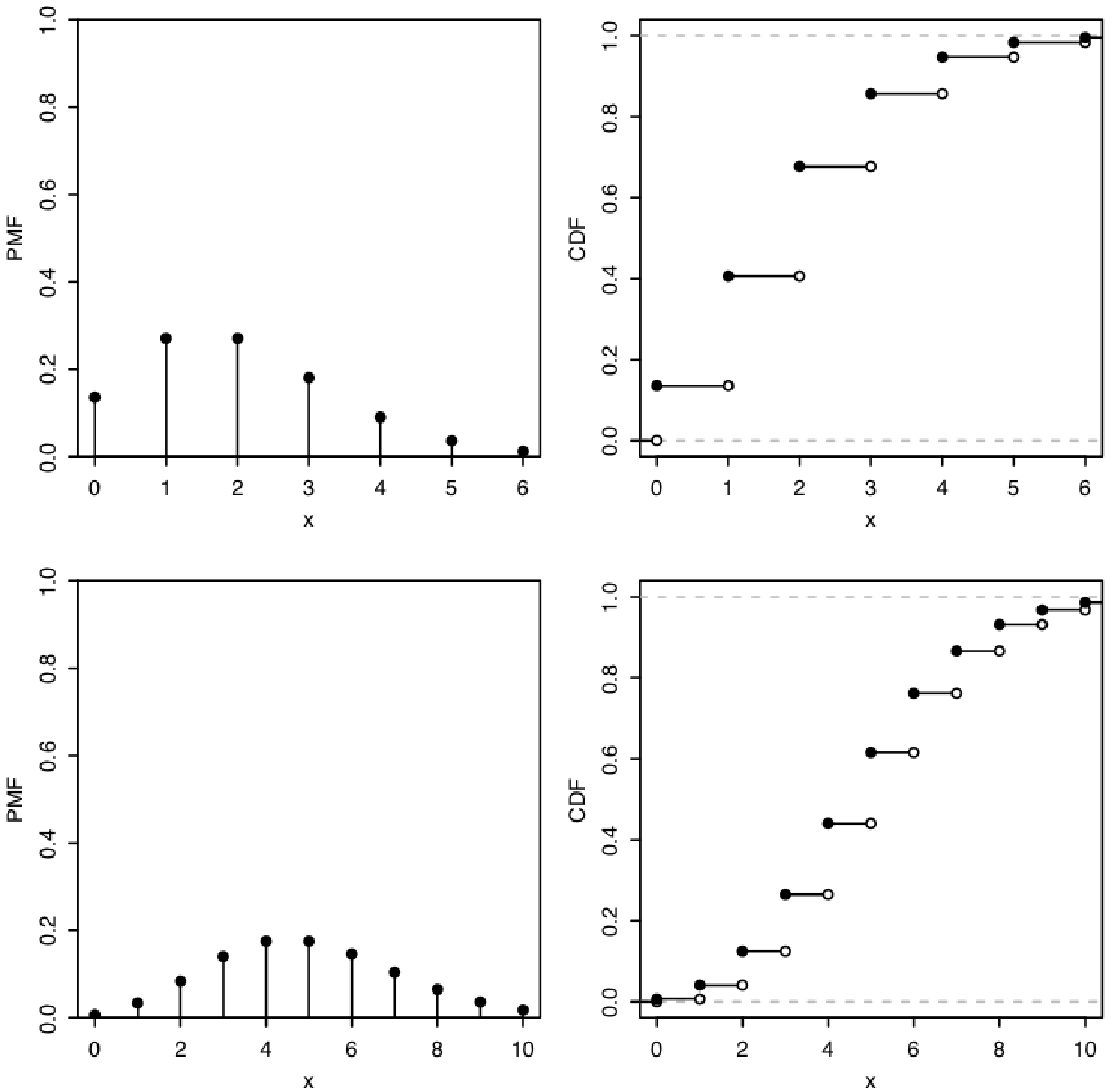
Using the same method we used for $Geom$,

$$E(X^2) = \lambda(1 + \lambda).$$

So the variance is:

$$Var(X) = E(X^2) - (EX)^2 = \lambda(1 + \lambda) - \lambda^2 = \lambda.$$

Look at these PMF and CDF for $Pois(2)$ and $Pois(5)$! PMF of $Pois(2)$ is highly skewed, and with big λs PMF becomes well-shapred.



Parameter λ is interpreted as the rate of occurrence of these rare events!

Poisson approximation:

Let $A_1, A_2, \ldots, A_n$ be events $p_j = P(A_j)$ where n is large, the p_j are small, and A_j are independent. Let

$$X = \sum_{j=1}^{n} I(A_j)$$

count how many of the A_j occur. Then X is approximately $Pois(\lambda)$ with $\lambda = \sum_{j=1}^{n} p_j$.

The Poisson paradigm is also called the *law of rare events*. (It means that p_j are small).

The Poisson distribution often gives *good approximations*. The Poisson is a popular model, or at least a starting point, for data whose values are nonnegative integers (called *count data* in statistics).

Sum of independent Poissons:

If $X \sim Pois(\lambda_1), Y \sim Pois(\lambda_2)$, and X is independent of Y, then $X + Y \sim Pois(\lambda_1 + \lambda_2)$.

5.8 Expectation of a continuous RV

Expectation of a continuous RV:

The *expected value* or *expectation* or *mean* of a continuous RV X with PDF f is:

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx.$$

As in the discrete case, this expectation may not exist.

Linearity of expectation and LOTUS also holds for continuous RV:

If X is a CRV with PDF f and g is a function $\mathbb{R} \to \mathbb{R}$,

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx.$$

Mean and Var of Uniform RVs:

Let $X \sim Unif(a, b)$ So the expectation is pretty simple to find:

$$E(U) = \int_a^b x\frac{1}{b-a}dx = \frac{1}{b-a}(\frac{b^2}{2} - \frac{a^2}{2}) = \frac{a+b}{2}$$

For the variance:

$$E(U^2) = \int_a^b x^2\frac{1}{b-a}dx = \frac{1}{3}\frac{b^3-a^3}{b-a}$$

So the variance is:

$$Var(U) = E(U^2) - (EU)^2 = \frac{1}{3}\frac{b^3-a^3}{b-a} - (\frac{a+b}{2})^2 = \frac{(b-a)^2}{12}$$

Mean and Var of Normal RVs:

Let $X \sim \mathcal{N}(\mu, \sigma^2)$. This RV has $E(X) = \mu$ and $Var(X) = \sigma^2$.

Mean and variance of Exponential RVs:

Let's start with $X \sim Expo(1)$:

$$E(X) = \int_0^{\infty} xe^{-x}dx = 1.$$

and by LOTUS,

$$E(X^2) = \int_0^{\infty} x^2xe^{-x}dx = 2.$$

So the variance is:

$$Var(X) = E(X^2) - (EX)^2 = 1.$$

Now $Y = X/\lambda \sim Expo(\lambda)$. Then,

$$E(Y) = \frac{1}{\lambda}E(X) = \frac{1}{\lambda},$$
$$Var(Y) = \frac{1}{\lambda^2}E(X) = \frac{1}{\lambda^2}.$$

5.9 Law of Large Numbers

Let $X_1, X_2, \ldots, X_n$ are IIDs with finite mean μ and var σ^2. Let

$$\overline{X}_n = \frac{X_1, \ldots, X_n}{n}$$

be sample mean of X_1 through X_n. The sample mean is also an RV with mean μ and var σ^2/n:

$$E(\overline{X}_n) = \frac{1}{n}E(X_1 + \ldots + X_n) = \frac{1}{n}(E(X_1) + \ldots + E(X_n)) = \mu$$
$$Var(\overline{X}_n) = \frac{1}{n^2}Var(X_1 + \ldots + X_n) = \frac{1}{n^2}(Var(X_1) + \ldots + Var(X_n)) = \frac{\sigma^2}{n}$$

Law of large numbers (LLN) says that as n grows, sample mean $\overline{X}_n \to \mu$. LLN has two versions:

Strong LLN:

The sample mean $\overline{X}_n \to \mu$ pointwise as $n \to \infty$ with probability 1. (The event $\overline{X}_n \to \mu$ has probability 1).

Weak LLN:

$\forall\, \epsilon > 0$, $P(|\overline{X}_n - \mu| > \epsilon) \to 0$ as $n \to \infty$ (This is *convergence in probability*)

LLN is essential!

Every time we use the proportion of times that something happened as an approximation to its probability, we are *implicitly appealing to LLN*. Every time we use the average value in the replications of some quantity to approximate its theoretical average, we are *implicitly appealing to LLN*.

5.10 Central Limit Theorem

LLN says that $n \to \infty$, $\overline{X}_n \to \mu$. What is its distribution along this way?

The Central Limit stays that for large n, the distribution $\overline{X}_n$ approaches to Normal distribution.

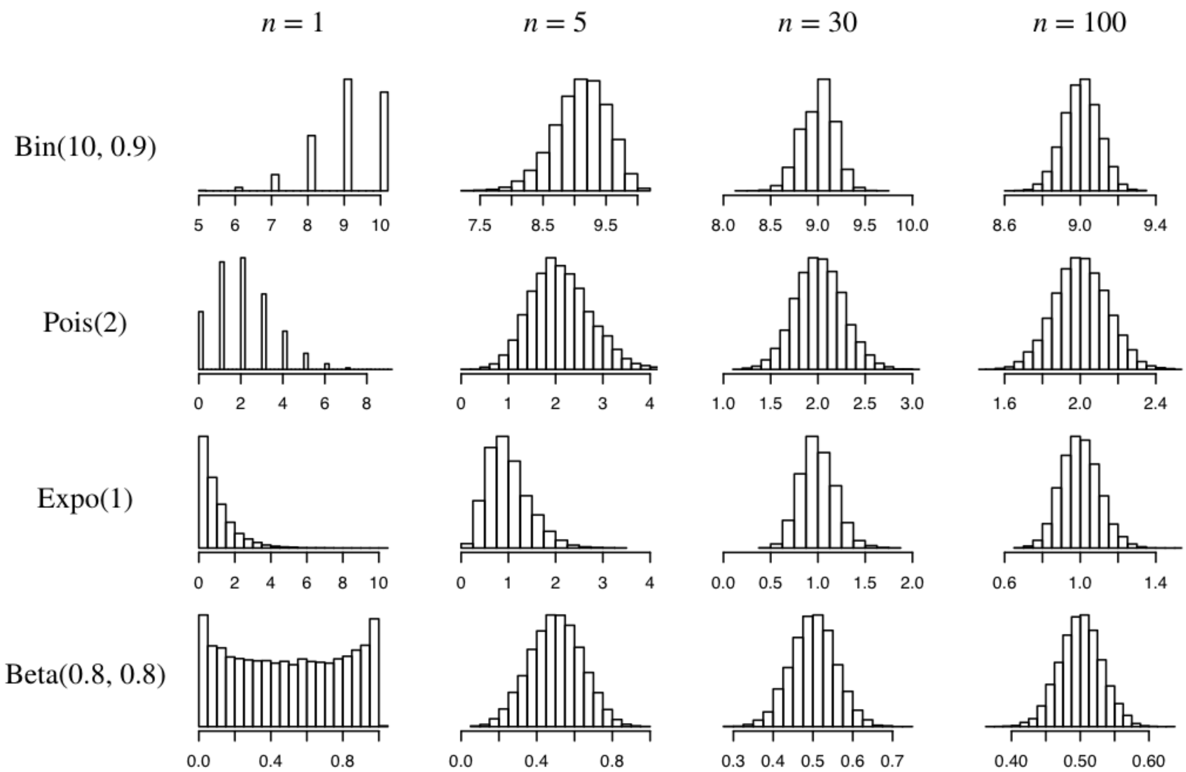
CLT: As $n \to \infty$,

$$\sqrt{n}\frac{\overline{X}_n - \mu}{\sigma} \to \mathcal{N}(0,1)$$

in distribution. Or CDF approaches Φ, the CDF of $\mathcal{N}(0,1)$. It has asymptotic result, but also suggests an approximation for the distribution of $\overline{X}_n$ when n is a large finite number.

For large n, the distribution of $\overline{X}_n$ is approximately $\mathcal{N}(\mu, \sigma^2/n)$.

This result is very general: the distribution of X_j can be *anything in the world* as long as it has finite variance and mean. Example how CLT works for different distributions:



As you may notice, it doesn't mean that the distribution of X_j is irrelevant: if X_j has highly skewed or multimodel distribution, we may need very large n.

The sum $W_n = X_1 + \ldots + X_n = n\overline{X}_n$. So the CLT also implies that W_n is also Normal. The CLT then states that for large n,

$$W_n \sim \mathcal{N}(n\mu, n\sigma^2).$$

This also can be useful!

Poisson convergence to Normal:

Let $Y \sim Pois(n)$. We can consider Y to be sum of n IID $Pois(1)$ RVs. Therefore, for large n:

$$Y \sim \mathcal{N}(n, n).$$

Binomial convergence to Normal:

Let $Y \sim Bin(n, p)$, we consider Y to be a sum of n IID $Bern(p)$ RVs. Therefore, for large n:

$$Y \sim \mathcal{N}(np, np(1-p)).$$