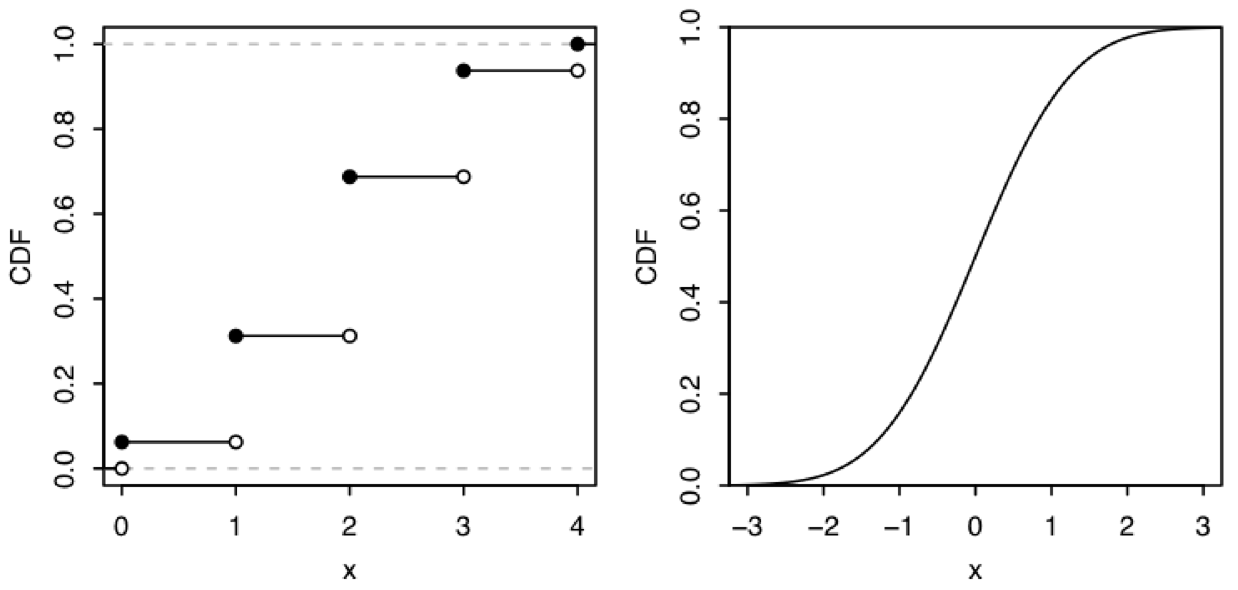
4 Continuous Random Variables

Together, discrete and continuous approaches form a powerful framework for modeling the world.

4.1 Probability density function

Continuous RVs!

An RV has a *continuous distribution* if its CDF is differentiable. Endpoints of CDF may be continuous but not differentiable. A continuous RV is a RV with a continuous distribution.



For a continuous RV X with CDF F, the PDF of X is derivative f of the CDF: $f(x) = F'(x)$

The support of X: all x where $f(x) > 0$.

The PDF is kinda similar to PMF, but for PDF quantity of $f(x)$ **is not a probability**. To obtaion the probability, we need to **integrate** PDF.

We can be carefree about including or excluding endpoints as above for continuous RVs, but we must not be careless about this for discrete RVs.

Valid PDF of a continuous RV:

1. Nonnegative: $f(x) \geq 0$
2. Integrates to 1: $\int_{-\infty}^{\infty} f(x)dx = 1$

Example: logistic distribution.

$X \sim$ Logistic.

CDF:

$$F(x) = \frac{e^x}{1+e^x}, x \in \mathbb{R}$$

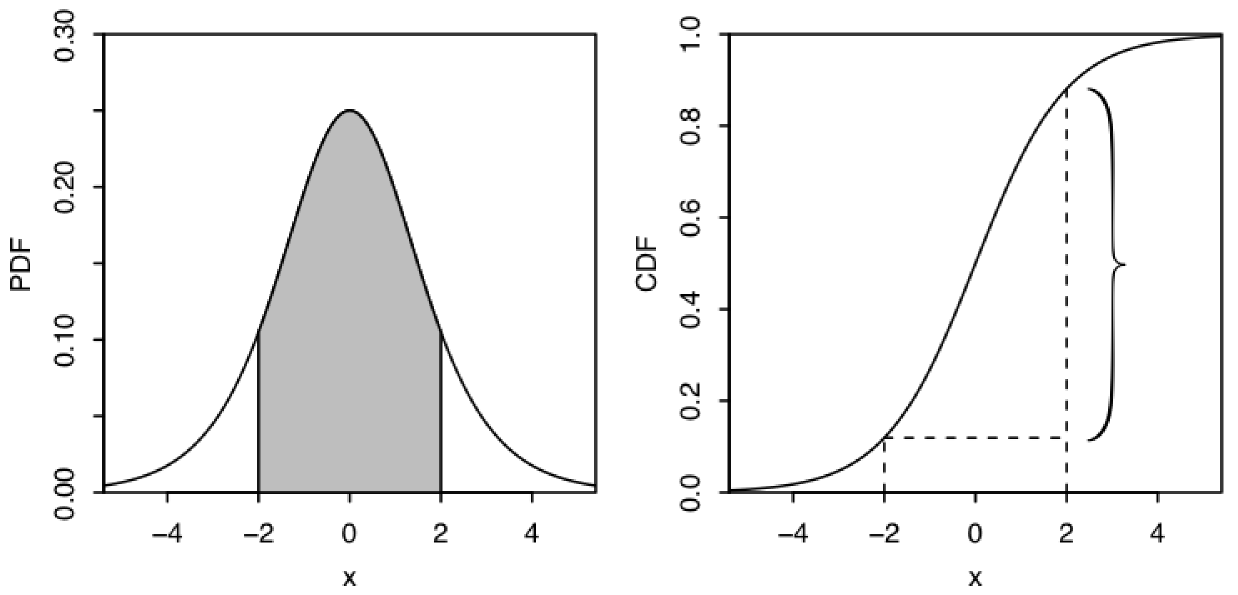
PDF:

$$F(x) = \frac{e^x}{(1+e^x)^2}, x \in \mathbb{R}$$

To find $P(-2 < X < 2)$, we need to integrate PDF from -2 to 2:

$$P(-2 < X < 2) = \int_{-2}^{2} \frac{e^x}{(1+e^x)^2} = F(2) - F(-2) \approx 0.76$$

Or $P(-2 < X < 2)$ is indicated by the shaded area under the PDF and the height of the curly brace on the CDF.



4.2 Uniform distribution

A continuous RV U has the *Uniform distribution* $X \sim Unif(a, b)$ on the interval (a, b) if its PDF is:

$$f(x) = \frac{1}{b-a} \, \forall a < x < b,$$
$$f(x) = 0 \; otherwise$$

The CDF is the accumulated area under the PDF:

$$F(x) = 0 \, \forall x \leq a,$$
$$F(x) = \frac{x-a}{b-a} \; a < x < b,$$
$$F(x) = 1 \, \forall x \geq b.$$

$Unif(0, 1)$ is the standard Uniform.

For Uniform distributions, *probability is proportional to length.*

Location-scale transformation.

The RV Y has been obtained as a *location-scale transformation* of X if $Y = \sigma X + \mu$. μ controls the location and σ controls the scale.

Warning: if Y is a linear function of X, the Uniformity is preserved, but if Y is defined as a *nonlinear* transformation of X, Y will not be Uniform.

Warning: When using location-scale transformations, the shifting and scaling should be applied to the *random variables* themselves, not to their PDFs.

4.3 Universality of the Uniform distribution

Given a $Unif(0, 1)$ RV, we can construct an RV with *any continuous distribution we want*.

Other names of the universality of Uniform:

- probability integral transform,

- inverse transform sampling,
- the quantile transformation,
- the fundamental theorem of simulation.

Theorem:

F is a CDF which is continuous function and strictly increasing on the support of distribution. This ensures that the inverse function F^{-1} exists as function $(0, 1) \to \mathbb{R}$. Results:

1. Let $U \sim Unif(0, 1)$ and $X = F^{-1}(U)$. Then X is an RV with CDF F.
2. Let X be an RV with CDF F. Then $F(X) \sim Unif(0, 1)$.

What this theorem is saying about?

Fist part: Since F^{-1} is a function (**quantile function**), U is a RV, and a function of RV is RV, $F^{-1}(U)$ is a RV; universality of the Uniform says its CDF is F.

Second part: reverse direction! Starting from RV X whose CDF is F and then creating RV $Unif(0, 1)$. Universality of the Uniform says that the distribution of $F(X)$ is Uniform on $(0, 1)$.

Warning: potential notational collusion!

$F(x) = P(X \leq x)$ by definition, but $F(X) = P(X \leq X) = 1$ is incorrect by definition. Rather, we should first find an expression for the CDF as a function of x, then replace x with X to obtain a random variable. For example, if the CDF of X is $F(x) = 1 - e^{-1}$ for $x > 0$, then $F(X) = 1 - e^{-X}$.

Example: percentiles

Exam, grades 0-100, RV X is the score of random student. We approximate the discrete distribution of scores using continuous distribution. So X is continuous RV, CDF is strictly increasing on $(0, 100)$. Suppose median score is 60. So $F(60) = 1/2$ or $F^{-1}(1/2) = 60$

If student scores 72 on the exam, then his **percentile** is the fraction of students who's score is below 72. This is $F(72)$ which is number $(0.5, 1)$. Other way, if we have percentile 0.95, the score is $F^{-1}(0.95)$. Percentile is also called a *quantile*, F^{-1} is *quantile function*.

The universality property says that $F(X) \sim Unif(0, 1)$.

So! 50 of students have a percentile of at least 0.5. 10 have a percentile between $(0, 0.1)$, and between $(0.1, 0.2)$, …

4.4 Normal distribution

A famous continuous distribution with a bell-shaped PDF!

A continuous RV Z has the *standard Normal distribution* if its PDF φ is:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

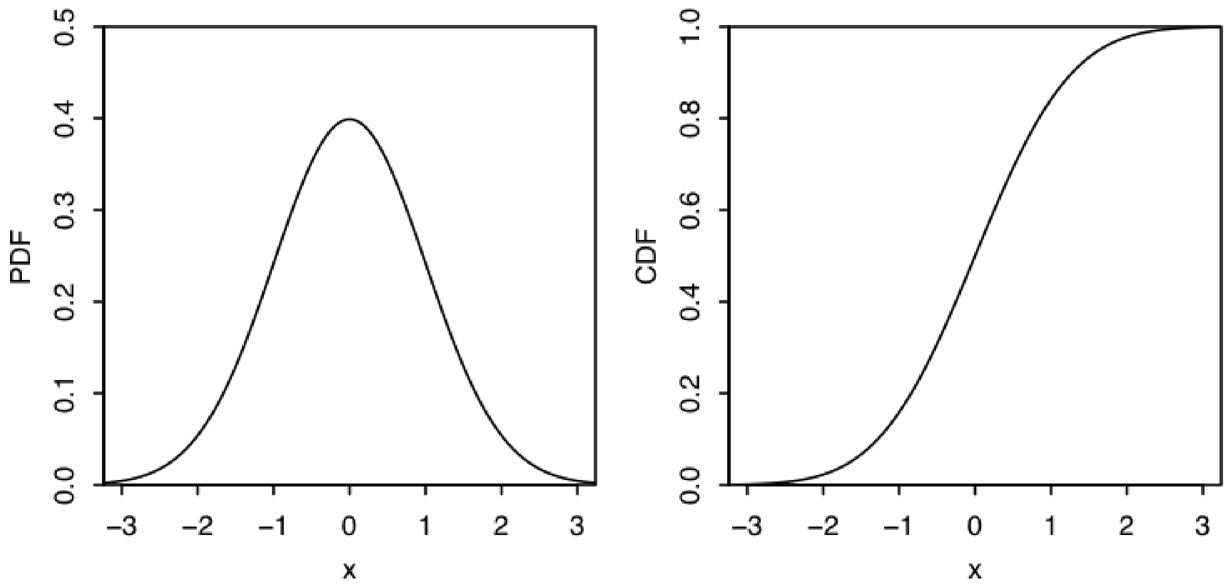
We write this as $Z \sim \mathcal{N}(0, 1)$.

It is widely used in statistics because of the *central limit theorem*, which says that under very weak assumptions, the sum of a large number of IID (independent and identically distributed) RVs has an approximately Normal distribution, regardless of the distribution of the individual RVs.

Why it has $1/\sqrt{2\pi}$ in PDF? We need this constant to integrate PDF to 1. Such constants are called *normalizing constants*.

The standard Normal Φ CDF is the accumulated area under the PDF:

$$\Phi(z) = \int_{-\infty}^{z} \varphi(z) dt = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$



So φ is a standard Normal PDF, Φ is a standard Normal CDF, Z is standard Normal RV.

Normal PDF and CDF are looking similar to Logistic ones, but Normal PDF decays to 0 more quickly: almost all the area under φ is between $-3, 3$ while for Logistic PDF it is between $-5, 5$.

Properties of Normal PDF and CDF:

1. *Symmetry of PDF*: $\varphi(z) = \varphi(-z)$, φ is an even function.
2. *Symmetry of tail areas*: The area under PDF curve is left to -2 equals to area to the right of 2,

$$\Phi(z) = 1 - \Phi(-z) \ \forall z$$

Proof:

$$\Phi(z) = \int_{-\infty}^{-z} \varphi(t)dt = \int_{z}^{\infty} \varphi(u)du = 1 - \int_{-\infty}^{z} \varphi(u)du = 1 - \Phi(z)$$

3. *Symmetry of Z and $-Z$*: If $Z \sim \mathcal{N}(0, 1)$, then $-Z \sim \mathcal{N}(0, 1)$ as whell.

Proof:

$$P(-Z \leq z) = P(Z \leq -z) = 1 - \Phi(-z) = \Phi(z)$$

Normal distribution:

If $Z \sim \mathcal{N}(0,1)$, then:

$$X = \mu + \sigma Z$$

Is normal distribution with mean parameter μ and variance parameter σ^2, $\forall$ real μ, σ^2 and $\sigma > 0$. We denote this by: $X \sim \mathcal{N}(\mu, \sigma^2)$.

If $X \sim \mathcal{N}(\mu, \sigma^2)$,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0,1).$$

It is called *standardization*. We can use it to find PDF and CDF of X:

$$F(x) = \Phi(\frac{x-\mu}{\sigma}),$$
$$f(x) = \varphi(\frac{x-\mu}{\sigma})\frac{1}{\sigma}.$$

Important numbers if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$P(|X-\mu| < \sigma) \approx 0.68$$
$$P(|X-\mu| < 2\sigma) \approx 0.95$$
$$P(|X-\mu| < 3\sigma) \approx 0.997$$

4.5 Exponential distribution

It is widely used as a simple model for the waiting time for a certain kind of event to occur.

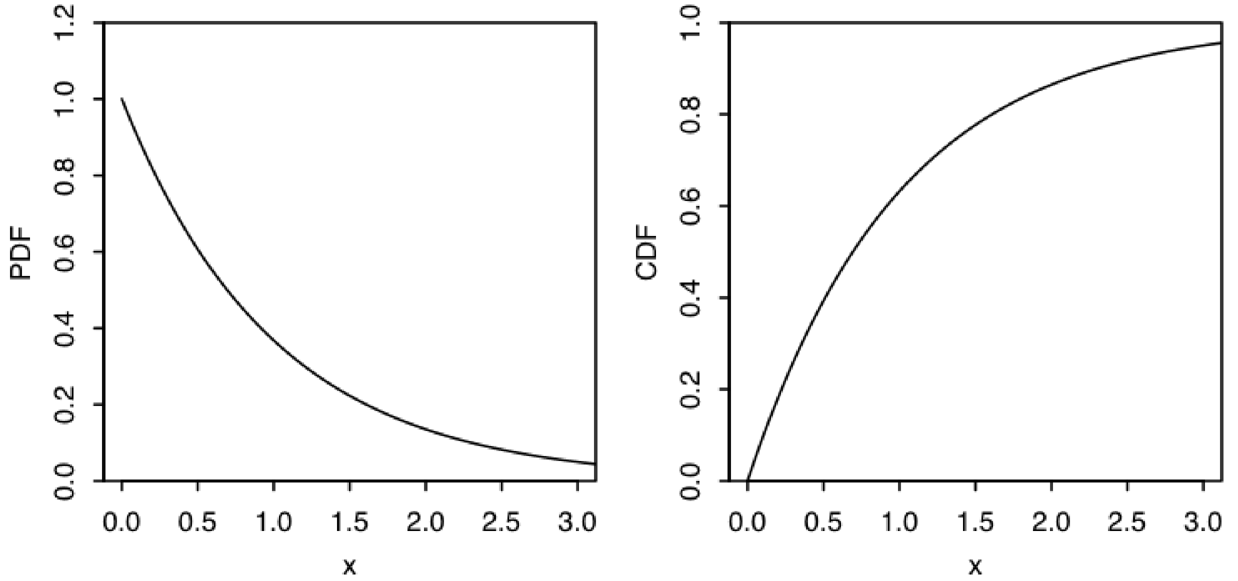
A continuous RV has an **Exponential distribution** with the parameter λ, where $\lambda > 0$ if its PDF is:

$$f(x) = \lambda e^{-\lambda x},\ x > 0.$$

We denote this by $X \sim Expo(\lambda)$. CDF:

$$F(x) = 1 - e^{-\lambda x},\ x > 0.$$

Plotted $Expo(1)$ PDF and CDF:



Scale transformations for exponential distributions: If $X \sim Expo(1)$, then:

$$Y = \frac{X}{Y} \sim Expo(\lambda)$$

because

$$P(Y \leq y) = P(X/\lambda \leq y) = P(X \leq \lambda y) = 1 - e^{\lambda y},\ y > 0.$$

If $Y \sim Expo(\lambda)$, then $\lambda Y \sim Expo(1)$.

Memoryless property:

If the waiting time for a certain event to occur is Exponential, your additional waiting time is still Exponential!

To have a *memoryless property*, an RV X should satisfy:

$$P(X \geq s + t | X \geq t) = P(X \geq t),\ s, t > 0$$

s represents time already spent on waiting, t is additional time. Another way to state the memoryless property: conditional on $X \geq s$, the additional waiting time $X - s$ is still $\sim Expo(\lambda)$.

Proof:

$$P(X \geq s+t|X \geq s) = \frac{P(X \geq s+t)}{P(X \geq s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda s} = P(X \geq t)$$

Why then do we care about the Exponential distribution?

1. Some physical phenomena, such as radioactive decay, truly do exhibit the memoryless property.
2. The Exponential distribution is well-connected to other named distributions.
3. The Exponential serves as a building block for more flexible distributions.

4.6 Poisson processes

Closely connected to exponential distribution! Exponential and Poisson are linked by a common story, the *Poisson process*:

A process is called **Poisson process** with rate λ if:

1. The number of arrivals that occur in an interval of length t is a RV $\sim Pois(\lambda t)$.

2. The number of arrivals that occur in disjoint intervals are independent.

Example:

The arrivals are emails landing according to a Poisson process with rate λ. How many emails will arrive in one hour? Number of emails/hour is $\sim Pois(\lambda)$

How long does it take until the fist email arrives? It will be a distribution on $(0, \infty)$. Let T_1 to be the time until the 1st e-mail arrives. Saying that the 1st email arrives = there's no emails arrived between $0, T_1$

$$T_1 > t \ same \ event \ as \ N_t = 0$$

This is a *count-time duality* because it connects a discrete RV N_t (counts) with continuous RV T_1 (time).

So these events have same probability:

$$P(T_1 > t) = P(N_t = 0) = \frac{e^{-\lambda t}(\lambda t)^0}{0!} = e^{-\lambda t}$$

Therefore $P(T_1 \leq t) = 1 - e^{-\lambda t}$, so $T_1 \sim Expo(\lambda)$!

To summarize: in a Possion process of rate λ,

- the number of arrivals in an interval of length 1 is $Pois(\lambda)$.
- the times between arrivals are IID $Expo(\lambda)$.