6 Joint Distributions and Conditional Expectation

With data coming from several groups, we should consider both *within* group variation and *between* group variation.

Using *conditional expectation*, we can predict the value of one random variable, given the information we have about other random variables.

6.1 Joint, marginal, and conditional distributions

We introduce multivariate analogs of the CDF, PMF, and PDF.

Key concepts:

- **Distribution** of RV X provides complete information about the probability of X into any subset of real line.
- **Joint distribution** of two RVs X and Y and provides complete information about the probability of the vector (X, Y).
- **Marginal distribution** of X is the individual distribution of X ignoring the value of Y.
- **Conditional distribution** of X given $Y = y$ is the updated distribution of X after observing $Y = y$.

Discrete joint CDF, PMF:

The **joint CDF** of RVs X and Y is the function F_{XY} given by:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

analogously for joint CDF of n RVs.

The **joint PMF** of discrete RVs X and Y is the function p_{XY}, given by:

$$p_{XY}(x, y) = P(X = x, Y = y)$$

analogously for joint PMF of n RVs.

We require valid joint PMF to be nonnegative and sum to 1:

$$\sum_x \sum_y P(X = x, Y = y) = 1.$$

Marginal PMF: For discrete RVs X and Y, the *marginal PMF* of X is:

$$P(X = x) = \sum_y P(X = x, Y = y)$$

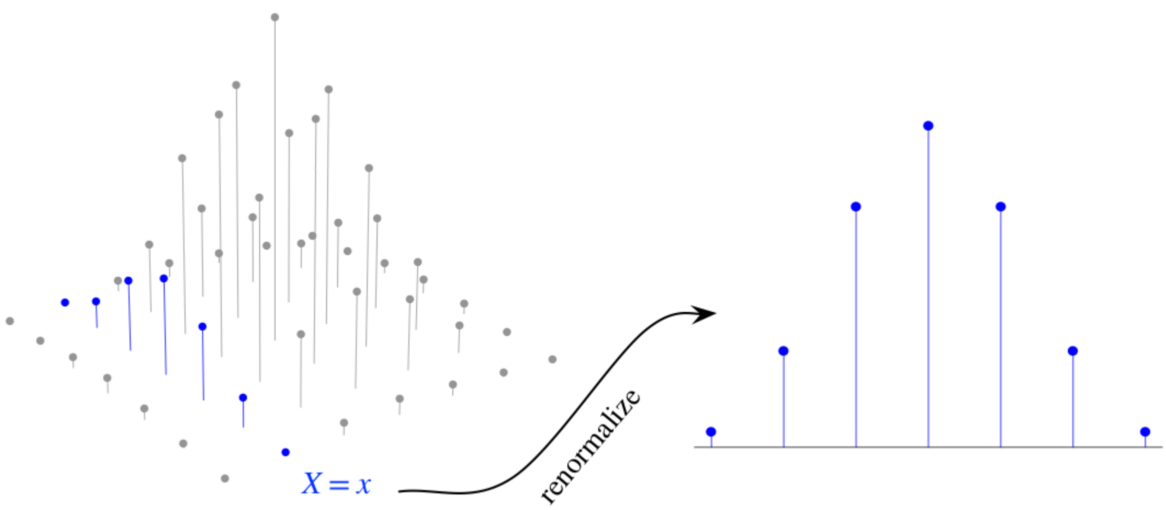
The operation of summering over the possible values of Y in order to convert the joint PMF to marginal PMF is *marginalizing* out of Y.

Now: we observe the value of X and want to update the distribution of Y using this information. Using the marginal PMF isn't good idea because it doesn't take into account any info about X. Instead,

Conditional PMF:

$$P(Y = y|X = x) = \frac{P(X = x, Y = y)}{P(X = x)}.$$

Where x is the observed value of X. The conditional PMF $P(Y = y|X = x)$ is obtained by renormalizing the column of the joint PMF that is compatible with the event $X = x$.



We can also obtain the conditional distribution using Bayes' rule:

$$P(Y=y|X=x)=\frac{P(X=x|Y=y)P(Y=y)}{P(X=x)}.$$

Using LOTP, we have another way to get the marginal PMF:

$$P(X=x)=\sum_y P(X=x|Y=y)P(Y=y)$$

Now we can revisit the definition of **independence**:

RVs X and Y are *independent* if $\forall x, y$,

$$F_{X,Y}(x,y)=F_X(x)F_Y(y).$$

If X and Y are discrete, it is equivalent to:

$$P(X=x,Y=y)=P(X=x)P(Y=y)$$

$\forall x, y$ and it is also equivalent to:

$$P(Y=y|X=x)=P(Y=y)$$

$\forall y$ and $\forall x$ such that $P(X=x)>0$.