

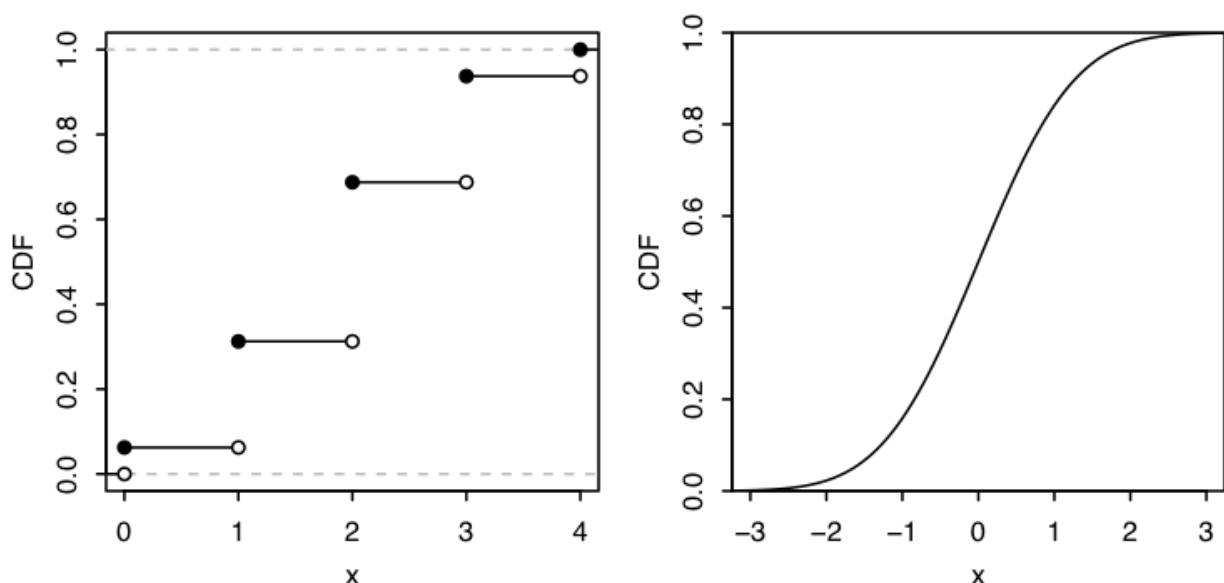
4 Continuous Random Variables

Together, discrete and continuous approaches form a powerful framework for modeling the world.

4.1 Probability density function

Continuous RVs!

An RV has a *continuous distribution* if its CDF is differentiable. Endpoints of CDF may be continuous but not differentiable. A continuous RV is a RV with a continuous distribution.



For a continuous RV X with CDF F , the PDF of X is derivative f of the CDF: $f(x) = F'(x)$

The support of X : all x where $f(x) > 0$.

The PDF is kinda similar to PMF, but for PDF quantity of $f(x)$ is **not a probability**. To obtain the probability, we need to **integrate** PDF.

We can be carefree about including or excluding endpoints as above for continuous RVs, but we must not be careless about this for discrete RVs.

Valid PDF of a continuous RV:

1. Nonnegative: $f(x) \geq 0$
2. Integrates to 1: $\int_{-\infty}^{\infty} f(x)dx = 1$

Example: logistic distribution.

$X \sim \text{Logistic}$.

CDF:

$$F(x) = \frac{e^x}{1 + e^x}, x \in \mathbb{R}$$

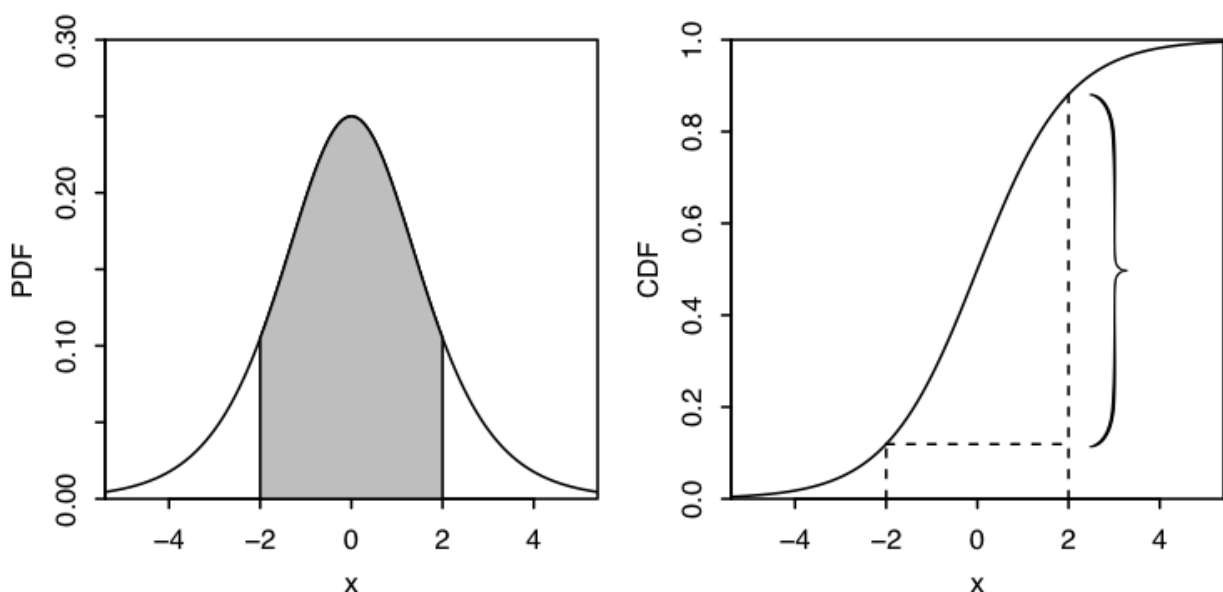
PDF:

$$f(x) = \frac{e^x}{(1 + e^x)^2}, x \in \mathbb{R}$$

To find $P(-2 < X < 2)$, we need to integrate PDF from -2 to 2 :

$$P(-2 < X < 2) = \int_{-2}^2 \frac{e^x}{(1 + e^x)^2} = F(2) - F(-2) \approx 0.76$$

Or $P(-2 < X < 2)$ is indicated by the shaded area under the PDF and the height of the curly brace on the CDF.



4.2 Uniform distribution

A continuous RV U has the *Uniform distribution* $X \sim \text{Unif}(a, b)$ on the interval (a, b) if its PDF is:

$$f(x) = \frac{1}{b-a} \quad \forall a < x < b,$$
$$f(x) = 0 \text{ otherwise}$$

The CDF is the accumulated area under the PDF:

$$F(x) = 0 \quad \forall x \leq a,$$
$$F(x) = \frac{x-a}{b-a} \quad a < x < b,$$
$$F(x) = 1 \quad \forall x \geq b.$$

$\text{Unif}(0, 1)$ is the standard Uniform.

For Uniform distributions, *probability is proportional to length*.

Location-scale transformation.

The RV Y has been obtained as a *location-scale transformation* of X if $Y = \sigma X + \mu$. μ controls the location and σ controls the scale.

Warning: if Y is a linear function of X , the Uniformity is preserved, but if Y is defined as a *nonlinear* transformation of X , Y will not be Uniform.

Warning: When using location-scale transformations, the shifting and scaling should be applied to the *random variables* themselves, not to their PDFs.

4.3 Universality of the Uniform distribution

Given a $\text{Unif}(0, 1)$ RV, we can construct an RV with *any continuous distribution we want*.

Other names of the universality of Uniform:

- probability integral transform,

- inverse transform sampling,
- the quantile transformation,
- the fundamental theorem of simulation.

Theorem:

F is a CDF which is continuous function and strictly increasing on the support of distribution. This ensures that the inverse function F^{-1} exists as function $(0, 1) \rightarrow \mathbb{R}$. Results:

1. Let $U \sim \text{Unif}(0, 1)$ and $X = F^{-1}(U)$. Then X is an RV with CDF F .
2. Let X be an RV with CDF F . Then $F(X) \sim \text{Unif}(0, 1)$.

What this theorem is saying about?

First part: Since F^{-1} is a function (**quantile function**), U is a RV, and a function of RV is RV, $F^{-1}(U)$ is a RV; universality of the Uniform says its CDF is F .

Second part: reverse direction! Starting from RV X whose CDF is F and then creating RV $\text{Unif}(0, 1)$. Universality of the Uniform says that the distribution of $F(X)$ is Uniform on $(0, 1)$.

Warning: potential notational collusion!

$F(x) = P(X \leq x)$ by definition, but $F(X) = P(X \leq X) = 1$ is incorrect by definition. Rather, we should first find an expression for the CDF as a function of x , then replace x with X to obtain a random variable. For example, if the CDF of X is $F(x) = 1 - e^{-x}$ for $x > 0$, then $F(X) = 1 - e^{-X}$.

Example: percentiles

Exam, grades 0-100, RV X is the score of random student. We approximate the discrete distribution of scores using continuous distribution. So X is continuous RV, CDF is strictly increasing on $(0, 100)$. Suppose median score is 60. So $F(60) = 1/2$ or $F^{-1}(1/2) = 60$

If student scores 72 on the exam, then his **percentile** is the fraction of students who's score is below 72. This is $F(72)$ which is number $(0.5, 1)$. Other way, if we have percentile 0.95, the score is $F^{-1}(0.95)$. Percentile is also called a *quantile*, F^{-1} is *quantile function*.

The universality property says that $F(X) \sim Unif(0, 1)$.

So! **50** of students have a percentile of at least **0.5**. **10** have a percentile between $(0, 0.1)$, and between $(0.1, 0.2)$, ...

4.4 Normal distribution

A famous continuous distribution with a bell-shaped PDF!

A continuous RV Z has the *standard Normal distribution* if its PDF φ is:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty.$$

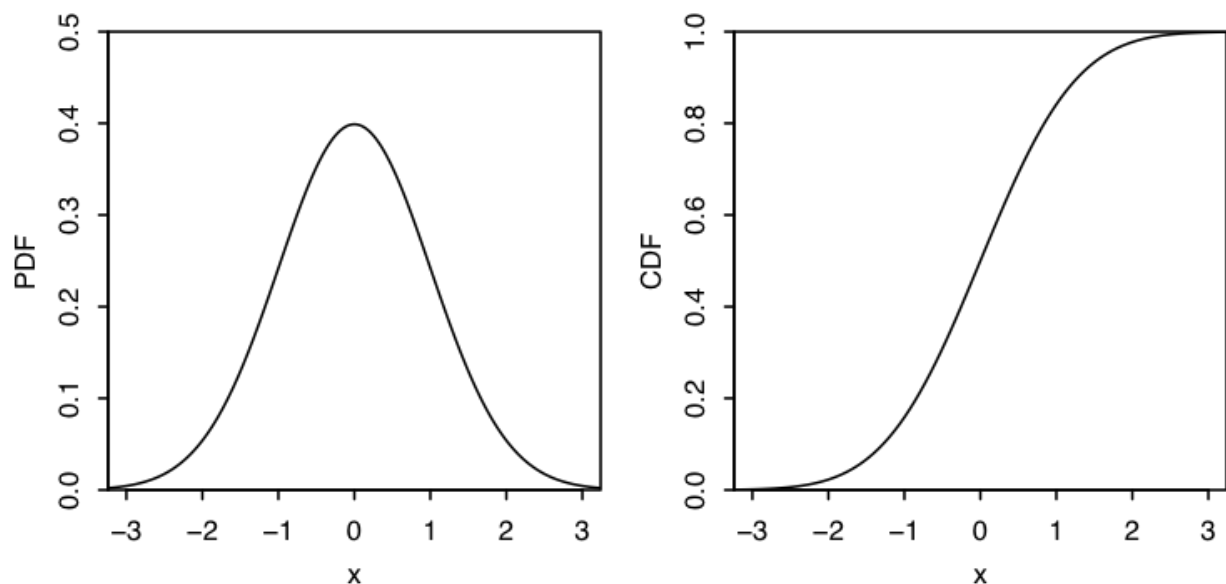
We write this as $Z \sim \mathcal{N}(0, 1)$.

It is widely used in statistics because of the *central limit theorem*, which says that under very weak assumptions, the sum of a large number of IID (independent and identically distributed) RVs has an approximately Normal distribution, regardless of the distribution of the individual RVs.

Why it has $1/\sqrt{2\pi}$ in PDF? We need this constant to integrate PDF to **1**. Such constants are called *normalizing constants*.

The standard Normal Φ CDF is the accumulated area under the PDF:

$$\Phi(z) = \int_{-\infty}^z \varphi(t) dt = \int_{-\infty}^z \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt$$



So φ is a standard Normal PDF, Φ is a standard Normal CDF, Z is standard Normal RV.

Normal PDF and CDF are looking similar to Logistic ones, but Normal PDF decays to 0 more quickly: almost all the area under φ is between $-3, 3$ while for Logistic PDF it is between $-5, 5$.

Properties of Normal PDF and CDF:

1. *Symmetry of PDF*: $\varphi(z) = \varphi(-z)$, φ is an even function.
2. *Symmetry of tail areas*: The area under PDF curve is left to -2 equals to area to the right of 2 ,

$$\Phi(z) = 1 - \Phi(-z) \quad \forall z$$

Proof:

$$\Phi(z) = \int_{-\infty}^{-z} \varphi(t) dt = \int_z^{\infty} \varphi(u) du = 1 - \int_{-\infty}^z \varphi(u) du = 1 - \Phi(z)$$

3. *Symmetry of Z and $-Z$* : If $Z \sim \mathcal{N}(0, 1)$, then $-Z \sim \mathcal{N}(0, 1)$ as well.

Proof:

$$P(-Z \leq z) = P(Z \leq -z) = 1 - \Phi(-z) = \Phi(z)$$

Normal distribution:

If $Z \sim \mathcal{N}(0, 1)$, then:

$$X = \mu + \sigma Z$$

Is normal distribution with mean parameter μ and variance parameter σ^2 , \forall real μ, σ^2 and $\sigma > 0$. We denote this by: $X \sim \mathcal{N}(\mu, \sigma^2)$.

If $X \sim \mathcal{N}(\mu, \sigma^2)$,

$$\frac{X - \mu}{\sigma} \sim \mathcal{N}(0, 1).$$

It is called *standardization*. We can use it to find PDF and CDF of X :

$$F(x) = \Phi\left(\frac{x - \mu}{\sigma}\right),$$
$$f(x) = \varphi\left(\frac{x - \mu}{\sigma}\right) \frac{1}{\sigma}.$$

Important numbers if $X \sim \mathcal{N}(\mu, \sigma^2)$, then

$$P(|X - \mu| < \sigma) \approx 0.68$$
$$P(|X - \mu| < 2\sigma) \approx 0.95$$
$$P(|X - \mu| < 3\sigma) \approx 0.997$$

4.5 Exponential distribution