7 Markov Chains

For the Markov chain, the past and **the future are conditionally independent**. For the special case of random walk on an undirected network, the network structure is the key to determining the stationary distribution.

We can picture a Markov chain intuitively by imagining a system with *states* and someone randomly wandering around from state to state.

For many interesting Markov chains, the *stationary* distribution of the chain helps us understand how the chain will behave in the long run.

7.1 Markov property and transition matrix

A sequence of RVs $X_0, X_1, X_2, ...$ evolving over time. This is called a *stochastic process*.

Markov chains have a form of one-step dependence, allowing to do beyond IIDs bust still have very convenient structure.

Markov chains widely used for simulations of complex distributions, via algorithms known as *Markov chain Monte Carlo (MCMC)*.

Markov chains live in both space and time: the set of possible states X_n is called *state time*, and index n represents evolution of the process over *time*. The state space of can be discrete or continuous, and time can also be discrete or continuous. We will focus on *discrete-state*, *discrete-time* Markov Chains with a *finite* state space.

Markov Chain

A sequence of RVs $X_0, X_1, X_2, ...$ taking values in *state space* $\{1, 2, ..., M\}$ is called *Markov chain* $\forall\ n \geq 0$,

$$P(X_{n+1} = j | X_n = i, X_{n-1} = i - 1, ..., X_0 = i_0) = P(X_{n+1} = j | X_n = i)$$

$P(X_{n+1} = j | X_n = j)$ is called the *transition probability*. from state i to state j. This Markov chain is $time - homogeneous$, which means that

$P(X_{n+1} = j | X_n = j)$ is the same $\forall\ n$.

We can describe the probabilities of moving from state to state using a matrix called *translation matrix* whose i, j entry is probability of going from i-th to j-th state in a single step.

Translation matrix

Let $X_0, X_1, X_2, \ldots$ be a Markov chain $\{1, 2, \ldots, M\}$ and let $q_{ij} = P(X_{n+1} = j | X_n = i)$ be transition probability from state i to state j. The matrix $Q = (q_{ij})$ is the *transition matrix* of the chain. Q is nonnegative and each row sums to 1.

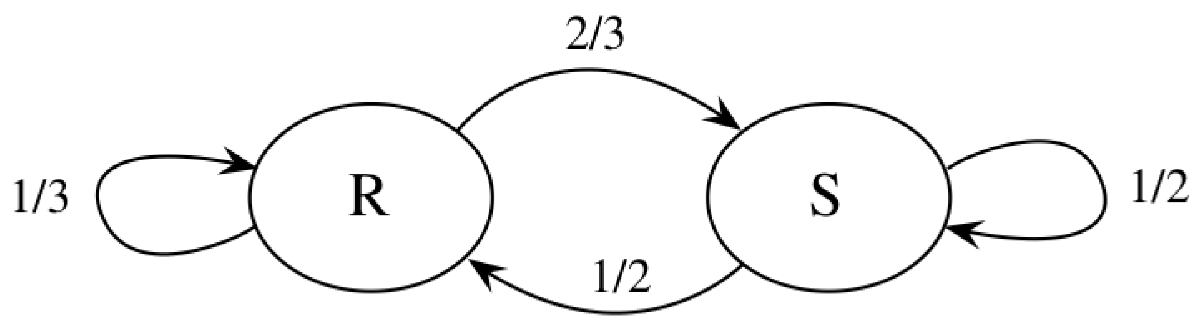
Example: Rainy-sunny Markov chain

If today is rainy, tomorrow it will be rainy with $P = 1/3$ and sunny with $P = 2/3$. If today is sunny, tomorrow it will be rainy with $P = 1/2$ and sunny with $P = 1/2$.

Let X_n be the weather on day n and $X_0, X_1, X_2, \ldots$ is a Markov chain on the state space $\{R, S\}$. Translation matrix of this chain is:

$$\begin{array}{c c} & \begin{array}{cc} R & S \end{array} \\ \begin{array}{c} R \\ S \end{array} & \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} \end{array}$$

Also we can represent this chain as graph:

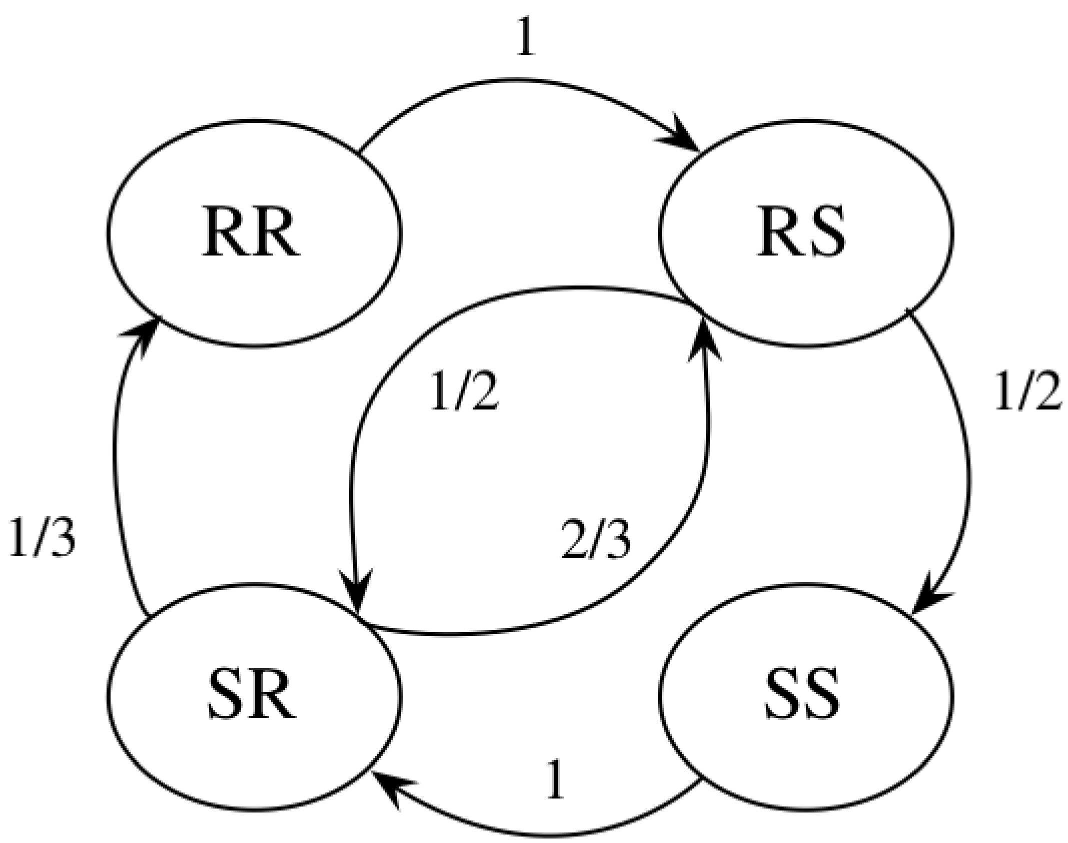


And what if tomorrow's weather depends on today's and yesterday's weather? To illustrate it, we can create a new Markov chain. Let $Y_n = (X_{n-1}, X_n)\ \forall n \geq 1$. Then $Y_1, Y_2, \ldots$ is a Markov chain on the state space $\{(R,R)(R,S),(S,R),(S,S)\}$.

Translation matrix of this chain is:

$$
\begin{array}{c c}
 & \begin{array}{cccc} (R,R) & (R,S) & (S,R) & (S,S) \end{array} \\
\begin{array}{c} (R,R) \\ (R,S) \\ (S,R) \\ (S,S) \end{array} &
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 1/2 & 1/2 \\
1/3 & 2/3 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

This Markov chain may be represented as the following graph:



Similarly, we can build a chain on n-order dependencies.

N-step transition probability

The n-step transition probability $q_{ij}^{(n)}$ from i to j is the probability of being at j exactly n steps after being at i.

$$q_{ij}^{(n)} = P(X_n = j | X_0 = i).$$

Of course,

$$q_{ij}^{(n)} = \sum_k q_{ik} q_{kj}.$$

The n-th power of the transition matrix gives the n-step transition probabilities $q_{ij}^{(n)}$ is the (i, j)-th entry of Q^n.

Marginal distribution of X_n

Let $\mathbf{t} = (t_1, t_2, \ldots)$ where $t_i = P(X_0 = i)$ and $\mathbf{t}$ is a row vector. Then *the marginal distribution* of X_n is given by the vector $\mathbf{t}Q^n$ is $P(X_n = j)$.

Proof:

By LOTA, conditioning on X_0, the probability that the chain is at j-th state after n steps is:

$$P(X_n = j) = \sum_{i=1}^{M} P(X_0 = i) P(X_n = j | X_0 = i) =$$

$$= \sum_{i=1}^{M} t_i q_{ij}^{(n)}$$

which is the jth component of $\mathbf{t}Q^n$.

7.2 Classification of states