Parameter Synthesis for Continuous Time Markov Chains

Multi-Agent Systems Project

Tanja Derin [SM3800013],
Roberta Lamberti [SM3600007],

Data Science and Artificial Intelligence

Supervised by Prof. Tatjana Petrov

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1 Introduction

Continuous-Time Markov Chains (CTMCs) are a widely used concept for modeling stochastic systems that evolve in continuous time. They provide a natural framework to describe systems in which transitions between states occur randomly with exponentially distributed waiting times. In this project, we consider a CTMC with three states A, B, C and transition rates parameterized by (r, s). The objective is to determine parameter values such that the probability of reaching state C within time $t = 5$ exceeds a threshold of 0.8. To this end, we derive the transient distribution both symbolically and numerically, formulate the synthesis goal as a reachability constraint, and then explore the parameter space to identify feasible regions. Finally, we analyze the sensitivity of the reachability probability with respect to r and s, discussing the uniqueness or multiplicity of solutions.

2 Symbolic Derivation

We consider the CTMC with generator

$$Q = \begin{bmatrix} -r & r & 0 \\ s & -(r+s) & r \\ 0 & s & -s \end{bmatrix}, \qquad \pi(0) = (1, 0, 0).$$

To compute the reachability of C, we make C absorbing:

$$Q^* = \begin{bmatrix} -r & r & 0 \\ s & -(r+s) & r \\ 0 & 0 & 0 \end{bmatrix}.$$

With Kolmogorov forward equations from Q^* we obtain

$$\frac{dp_A}{dt} = -rp_A(t) + sp_B(t),$$
$$\frac{dp_B}{dt} = rp_A(t) - (r+s)p_B(t),$$
$$\frac{dp_C}{dt} = rp_B(t),$$

Since probabilities sum to 1, we can compute

$$p_C(t) = 1 - p_A(t) - p_B(t).$$

Reduction and solving the 2×2 system

We focus on (p_A, p_B) and finding the eigenvalues

$$\frac{d}{dt}\begin{bmatrix} p_A \\ p_B \end{bmatrix} = \begin{bmatrix} -r & s \\ r & -(r+s) \end{bmatrix} \begin{bmatrix} p_A \\ p_B \end{bmatrix}.$$

The eigenvalues λ of the coefficient matrix are found from the characteristic equation

$$\lambda^2 + (2r + s)\lambda + r^2 = 0$$

Hence the two (real and negative) eigenvalues are

$$\lambda_{1,2} = \frac{-(2r+s) \pm \Delta}{2}; \qquad \Delta = \sqrt{s^2 + 4rs}.$$

For eigenvalue λ, the eigenvector satisfies $y = \frac{\lambda + r}{s} x$. Thus the ratios between B- and A-components are

$$k_1 = \tfrac{\Delta - s}{2s}, \qquad k_2 = -\tfrac{\Delta + s}{2s}.$$

Therefore

$$p_A(t) = \alpha_1 e^{\lambda_1 t} + \alpha_2 e^{\lambda_2 t}, \qquad p_B(t) = k_1\alpha_1 e^{\lambda_1 t} + k_2\alpha_2 e^{\lambda_2 t}.$$

Applying the initial conditions at $t = 0$ we require $p_A(0) = 1$, $p_B(0) = 0$. This gives

$$\alpha_1 + \alpha_2 = 1, \qquad k_1\alpha_1 + k_2\alpha_2 = 0,$$

with solution

$$\alpha_1 = \frac{\Delta + s}{2\Delta}, \qquad \alpha_2 = \frac{\Delta - s}{2\Delta}.$$

Moreover,

$$\beta_1 = k_1\alpha_1 = \frac{\Delta^2 - s^2}{4s\Delta} \qquad \beta_2 = k_2\alpha_2 = \frac{s^2 - \Delta^2}{4s\Delta}$$

The final closed form is

$$\boxed{p_A(t) = \frac{\Delta + s}{2\Delta}\, e^{\lambda_1 t} + \frac{\Delta - s}{2\Delta}\, e^{\lambda_2 t}},$$

$$\boxed{p_B(t) = \frac{\Delta^2 - s^2}{4s\Delta}\, e^{\lambda_1 t} + \frac{s^2 - \Delta^2}{4s\Delta}\, e^{\lambda_2 t}}.$$

Using $p_C(t) = 1 - p_A(t) - p_B(t)$, one obtains (after simplification)

$$\boxed{p_C(t) = 1 - \frac{(\Delta + s)^2 e^{\lambda_1 t} - (\Delta - s)^2 e^{\lambda_2 t}}{4s\,\Delta}}.$$

Synthesis constraint

At time $t = 5$, the requirement is

$$p_C(5) \;\geq\; 0.8.$$

While $p_C(t)$ itself admits a symbolic expression, this inequality is transcendental in the parameters (r, s) and cannot be solved algebraically. Therefore, we evaluate $p_C(5)$ numerically on a grid of parameter values and visualize the feasible region using heatmaps and contour plots.

3 Results

The implementation of this project is available at this repository.

3.1 Checks

Before performing a full grid exploration, we tested the probability functions on a few parameter pairs (r, s). For each case, we compared the occupancy of states B and C at time $t = 5$ with the reachability probability of C by $t = 5$.

These checks confirm that:

- The reachability of C is consistently higher than the occupancy of C, as expected.
- The occupancy of B remains low (never exceeding ≈ 0.35), showing that the original formulation of the synthesis goal for state B would not be feasible.
- The functions behave consistently with intuition about the CTMC dynamics, validating our implementation before moving to the synthesis stage.

3.2 Symbolic vs. numerical solution

To verify the correctness of the symbolic derivation, we compared the closed-form expression for $p_C(t)$ with the numerical computation using the matrix exponential $\pi(t) = \pi(0)e^{Qt}$ implemented in `scipy.linalg.expm`. Table 1 reports the results for several parameter pairs.

(r, s)	Numerical	Symbolic	Difference
(1.0, 1.0)	0.826595	0.826595	$1.11 \cdot 10^{-16}$
(0.7, 0.2)	0.806731	0.806731	$2.22 \cdot 10^{-16}$
(1.2, 0.1)	0.975039	0.975039	$1.11 \cdot 10^{-16}$

Table 1: Comparison of numerical and symbolic results for $p_C(5)$.

The results confirm that the two methods are in complete agreement, with differences only at the level of machine precision (10^{-16}).

3.3 Feasible Regions

We performed a grid search over $r, s \in [0.1, 3.0]$ and computed the reachability of C by $t = 5$.

The results show that the reachability of C yields a broad feasible region, with many (r, s) pairs satisfying $\Pr(\tau_C \leq 5) \geq 0.8$.

The choice of the grid is motivated by the dynamics of exponential holding times. For example, with $r = 3$ the mean holding time in A is $1/3 \approx 0.33$. In a time horizon of $t = 5$, this corresponds to about $5/0.33 \approx 15$ expected opportunities to move along the path $A \to B \to C$ which is sufficient to almost guarantee reaching C before $t = 5$.

Table 2 lists example parameter pairs (r, s) where the reachability probability exceeds the threshold.

r	s	$\Pr(\tau_C \leq 5)$
0.7	0.1	0.834
0.7	0.2	0.807
0.8	0.1	0.885
0.8	0.2	0.862
0.8	0.3	0.840
0.9	0.2	0.902
0.9	0.3	0.885

Table 2: Examples of feasible parameter pairs (r, s).

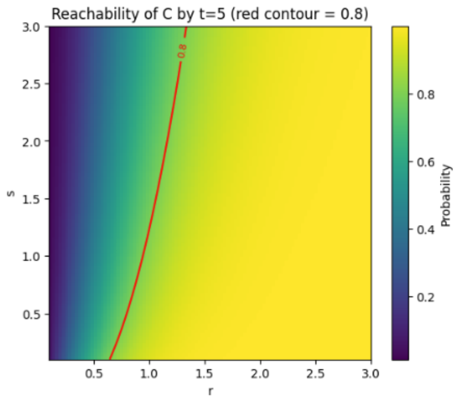

Figure 1: Feasible region for parameters (r, s)

3.4 Sensitivity Analysis

Figure 2 shows the sensitivity of $\Pr(\tau_C \leq 5)$ with respect to the parameters. For fixed $s = 0.2$, the probability increases with r, crossing the 0.8 threshold near $r \approx 0.75$ and then saturating close to 1 (diminishing returns). For fixed $r = 0.8$, the probability decreases with s; feasibility remains valid up to about $s \approx 0.5$. These trends confirm the intuition: higher r accelerates progress along $A \to B \to C$, while larger s increases the chance of diversion. Thus the feasible region is not a single point but a band of (r, s) values.

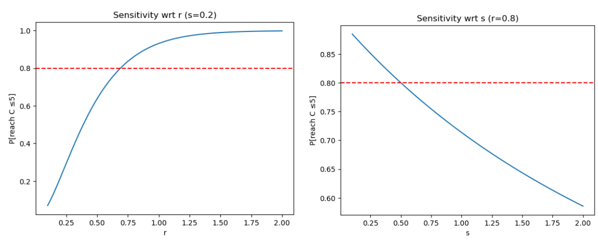

Figure 2: Sensitivity of reachability

Conclusion

The reachability of C by time $t = 5$ admits a broad feasible region with many parameter pairs (r, s) exceeding the threshold. The symbolic derivation and the numerical computations agree up to machine precision, which validates the correctness of the approach. Overall, the exercise shows how parameter synthesis can be applied to a simple CTMC to identify non-unique feasible solutions.