Nontrivial Cycles of the Collatz Conjecture: The Smooth Model and the Diophantine Bridge

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Abstract

This paper (Part 1) focuses on *nontrivial cycles* of the Collatz conjecture. We first analyze the *smooth* (continuous) model of the Collatz transformation and prove that it has no nontrivial cycles; the only fixed point compatible with the discrete map is $x = 1$.

We then formulate an exact Diophantine bridge Real → Integer: for a block order σ with parameters a, b, an integer fixed point exists if and only if $D \mid S(\sigma)$ and $n = S(\sigma)/D$, where $D = 2^b - 3^a$ and $S(\sigma) = 2^b C(\sigma)$. This framework excludes broad classes of orders in the discrete odd → odd model and closes the trivial case $p_i = 2 \Rightarrow n = 1$.

In the second part (the follow-up paper) we complete the exclusion of infinite growth by combining modular invariants (Bang–Zsigmondy) and a finite congruence check of small (a, b).

1 Definition of the model

Let $a, b \in \mathbb{N}$. Let σ be a sequence of operations of length $a + b$ that defines an order of the symbols ↑ and ↓. Their composition corresponds to an affine map

$$F_\sigma(x) = M\,x + C(\sigma), \qquad M = \frac{3^a}{2^b}.$$

The constant $C(\sigma)$ is given by

$$C(\sigma) = \sum_{k=1}^{a} 3^{a-k}\, 2^{-J_k(\sigma)},$$

where $J_k(\sigma)$ is the number of symbols ↓ following the k-th operation ↑ in the sequence σ.

Definition 1 (Basic fractal)**.** *We speak of a* basic fractal *if $J_k = b$ for every k, i.e. all operations ↑ are performed before all operations ↓. In that case*

$$C_{basic} = \frac{3^a - 1}{2^{b+1}}.$$

2 Equation with the correction term ε_k

Context. In this chapter we already work *in the integer odd → odd map.* The order between odd terms is determined by the p-vector (Definition 7), $\sigma = U\,D^{p_1} \cdots U\,D^{p_a}$, with cumulatives $s_{k+1} = s_k + p_{k+1}$ and weights $E_k = 3^{a-1-k}2^{s_k}$. The correction term ε_k captures the deviation from the basic p-fractal p_{base}, and all equations below refer to the *integer* case. We derive the so-called *peak equation* directly from the Collatz rules. Let n be an odd input number and during one whole "cycle" let us perform a operations ↑ $(3x + 1)$ and b operations ↓ $(x/2)$.

After the first ↑ we get the number $3n + 1$. If another ↑ follows immediately, we again add $+1$, but if ↓ follows, we divide the whole previous number by two. To describe the overall result of an *arbitrary* order of these operations, it is useful to introduce the counts

$$a = \# \text{ of } \uparrow \text{ operations}, \qquad b = \# \text{ of } \downarrow \text{ operations}.$$

Each ↑ contributes a factor 3 and adds $+1$; each ↓ contributes a factor $\frac{1}{2}$. After arranging these operations into a single block we can write them as the affine map $x \mapsto 3^a 2^{-b} x + C$. The mentioned correction term C (formally analyzed below) accumulates all added ones at the right moments. If such a block of operations formed a cycle, the last number would have to equal the original n:

$$n = 3^a 2^{-b}\, n + C \quad \Longrightarrow \quad n = \frac{2^b C}{2^b - 3^a} = \frac{S(\sigma)}{D},$$

that is

$$n = \frac{S(\sigma)}{D}, \qquad D = 2^b - 3^a. \tag{2.1}$$

Here ε_k explicitly expresses the dependence on the *order* of operations – if all operations ↑ are together (the *basic fractal*), we get $\varepsilon_k = 0$.

Parameters for $3n+1$. Throughout the paper we focus on the original Collatz rule $n \mapsto 3n+1$ (for odd n). From a single "cycle" of length $a + b$ we therefore get

$$D = 2^b - 3^a.$$

The minimal number of divisions by two that guarantees the contraction $3^a < 2^b$ is

$$b_{\min} = \left\lfloor a \log_2 3 \right\rfloor + 1. \tag{2.2}$$

In what follows we will use precisely this minimal value $b = b_{\min}$, unless stated otherwise.

Definition of the correction term. Index all a operations of multiplication by three as peaks $i_1 < \cdots < i_a = a$ and assign to them the cumulative numbers of declines

$$j_k = \text{cumulative number of divisions by two performed after the peak } i_k.$$

Then

$$\varepsilon_k = \left(2^{i_1} 3^{a-i_1} - 2^a\right) + \sum_{\substack{k \geq 2 \\ j_k \geq 2}} \left(2^{j_k} - 2\right)\left(2^{i_k} 3^{a-i_k} - 2^a\right). \tag{2.3}$$

2.1 Derivation of (2.3)

Let the sequence of operations start at an odd number $n_0 = n$ and after each peak ↑ let an arbitrary (possibly zero) number of operations ↓ appear.

1. **First addition of $+1$.** After the operation ↑ we have $n_1 = 3n_0 + 1$. To allow a later comparison with the basic fractal, we write this value as

   $$n_1 = 3n_0 + 1 = 3n_0 + 2^0 . \underbrace{(2^0 - 2^0)}_{0} + 1.$$

   This "reserves" a potential power of 2 for subsequent subtraction.

2. **Unfolding the algorithm.** Consider a general k-th peak i_k (the last $\uparrow$ before a block of declines). The value after this peak has the form $n_{i_k} = 2^{i_k}3^{a-i_k}n_0 + \sum_{\ell\le k} 2^{i_\ell}$.

3. **Declines.** Each division by two reduces all the summands so far by a factor 1/2. If exactly d_k divisions are performed after the peak i_k, the total exponent in the denominator increases by d_k. The cumulative number of declines up to this moment is $j_k := \sum_{\ell\le k} d_\ell$.

4. **Comparison with the basic fractal.** If all declines were postponed to the very end, we would obtain the basic fractal with the value $2^{a-b}3^b n_0 + 2^a - 2^a$. The difference with respect to a general order is exactly the correction term ε_k.

Summing the contributions from each peak we obtain the final expression (2.3); the first summand arises from the peak i_1 and the other summands from peaks i_k with $k \ge 2$, where $2^{j_k} - 2$ expresses the "excess" of declines compared to the basic fractal. Thus the relation is consistently derived.

Lemma 2 (Nonnegativity of ε_k)**.** *For every sequence we have $\varepsilon_k \ge 0$, and $\varepsilon_k = 0$ occurs precisely in the case of the basic fractal (Definition 1).*

Proof. For $i_k < a$ we have $3^{a-i_k} > 2^{a-i_k}$, whence $2^{i_k}3^{a-i_k} > 2^a$. The first factor in every summand in (2.3) is therefore positive. The second factor, $2^{j_k} - 2$, is positive for all $j_k \ge 2$. The sum of positive terms is therefore positive and becomes zero only when the sum is empty. □

Lemma 3 (Growth of ε_k)**.** *There exists a constant $c > 1$ (e.g. $c = \frac{9}{2}$) such that for any a and a valid construction of indices the estimate holds*

$$0 \le \varepsilon_k = \mathcal{O}(c^a).$$

Idea of the proof. The maximal growth occurs in a scenario where j_k reaches approximately $(\log_2 3 - 1)a$ and i_k is small. In such a case one term in the sum (2.3) grows asymptotically like $(9/2)^a$. □

These equations will be needed when comparing the smooth model with the discrete version.

3 Dispersion of the correction term $C(\sigma)$

Lemma 4. *For fixed $a, b > 0$ we have*

$$0 < C_{\min}(a,b) \le C(\sigma) \le C_{\max}(a,b) = \frac{3^a - 1}{2^{b+1}},$$

where the minimum $C_{\min}$ is attained for the maximal alternation of operations $\uparrow\downarrow$.

Proof. Every summand $3^{a-k}2^{-J_k}$ is positive. The maximum occurs when $J_k = b$ for all k. The minimum occurs at the largest possible alternation of the operations, i.e. when $J_k \approx k - 1$ (assuming $b \ge a$). A detailed computation is given in Appendix A. □

4 Candidates for fixed points

Theorem 5 (Fixed point equation)**.** *If for some x_* we have $F_\sigma(x_*) = x_*$, then necessarily*

$$x_* = \frac{2^b\, C(\sigma)}{2^b - 3^a}, \qquad \textit{for } 3^a < 2^b.$$

Corollary 6. *For given a, b all fixed points form the interval*

$$I_{a,b} = \frac{2^b\,(C_{\min}, C_{\max})}{2^b - 3^a} \subset (0, \infty).$$

5 Bridge Real→Integer: Diophantine reduction

Let $a, b \in \mathbb{N}$ and $3^a < 2^b$. Define

$$D := 2^b - 3^a, \qquad S(\sigma) := 2^b\, C(\sigma) = \sum_{k=1}^{a} 3^{a-k}\, 2^{\,b-J_k(\sigma)}.$$

By the equivalence $s_{k-1} := b - J_k(\sigma)$ (the number of declines before the k-th operation ↑) we also get

$$S(\sigma) = \sum_{k=1}^{a} 3^{a-k}\, 2^{\,s_{k-1}}, \qquad 0 = s_0 \le s_1 \le \cdots \le s_{a-1} < b.$$

Definition 7 (General fractal (p-vector) and the basic p-fractal)**.** *Fix $a, b \in \mathbb{N}$ with $3^a < 2^b$. A vector $p = (p_1, \ldots, p_a) \in \mathbb{N}^a$ with $p_i \ge 1$ and $\sum_i p_i = b$ defines an odd → odd block*

$$\sigma \;=\; U\, D^{p_1}\, U\, D^{p_2}\, \cdots\, U\, D^{p_a}.$$

Let $s_0 = 0$, $s_{k+1} = s_k + p_{k+1}$ and $E_k = 3^{a-1-k}\, 2^{s_k}$. Then $S(\sigma) = \sum_{k=0}^{a-1} E_k$ and $D = 2^b - 3^a$. We say that p_i is a strong decline *if $p_i \ge 3$. Denote $t = \#\{p_i = 1\}$, $m = \#\{p_i = 2\}$.*

The basic p-fractal *is*

$$p_{base} = (\underbrace{1, \ldots, 1}_{a-1},\, b - (a-1)),$$

that is, after each growth the minimal division and the last "large" decline; for it we have $\varepsilon = 0$.

Theorem 8 (Diophantine condition for an integer fixed point)**.** *Let $3^a < 2^b$. Then for $n \in \mathbb{N}$ we have $F_\sigma(n) = n$ if and only if*

$$D \mid S(\sigma) \quad \textit{and simultaneously} \quad n = \frac{S(\sigma)}{D}.$$

Proof. From the fixed point equality $x_* = \frac{2^b C(\sigma)}{2^b - 3^a}$ and the definition $S(\sigma) = 2^b C(\sigma)$. □

Congruence consequences. We have $\gcd(D, 6) = 1$. Moreover

$$S(\sigma) \equiv 2^{\,b-J_a(\sigma)} \pmod 3, \qquad S(\sigma) \equiv u_0 \pmod 2,$$

where u_0 is the number of initial operations ↑ before the first ↓ in the sequence σ. These constraints strongly narrow the possible shape of σ, although by themselves they do not yet guarantee the divisibility $D \mid S(\sigma)$.

Lemma 9 (The only positive 2-adic fixed point of the map T)**.** *Let $T(n) = \frac{3n+1}{2^{v_2(3n+1)}}$. If $T(n) = n$ with $n \in \mathbb{N}$ odd and $3 \nmid n$, then $n = 1$.*

Proof. The equality $T(n) = n$ gives $3n + 1 = 2^k n$ with $k = v_2(3n+1) \ge 1$. Hence $n(2^k - 3) = 1$, from which $n = 1$ and $k = 2$. □

Key goal (Lemma B). Let the block of odd steps be written as $\sigma = U\, D^{p_1}\, U\, D^{p_2}\, \ldots, U\, D^{p_a}$ with $p_i \ge 1$ and $\sum p_i = b$. Show:

> If $D \mid S(\sigma)$, then $p_1 = \cdots = p_a = 2$, and consequently the only integer fixed point is $n = 1$.

This Diophantine rigidity would completely close the bridge between the smooth model and the integer Collatz map.

Observation: an odd block must start with $\uparrow$. When we follow the map between consecutive *odd* terms, the first step from an odd n is always $\uparrow$ (since $3n+1$ is even) and only then there follow ≥ 1 divisions by two to return to an odd number. Therefore in the notation via s_k we always have $s_0 = 0$, i.e. no $\downarrow$ are allowed before the first $\uparrow$.

Base cases

In this subsection we close the very first nontrivial values of a and show that they lead only to $n = 1$.

Lemma 10 (The case $a = 1$ yields only $n = 1$)**.** *Let $a = 1$ and $b \geq 2$ with $3^a < 2^b$. Then from $D \mid S(\sigma)$ it follows that $b = 2$ and $n = 1$.*

Proof. From $s_0 = 0$ and the definition of S we get $S(\sigma) = 3^0\, 2^{s_0} = 1$. Thus $D \mid S \Rightarrow 2^b - 3 \mid 1$, and since $2^b - 3$ is a positive odd number, necessarily $2^b - 3 = 1$. Hence $b = 2$ and $D = 1$. Then $n = S/D = 1$. □

Lemma 11 (The case $a = 2$ yields only $n = 1$)**.** *Let $a = 2$ and $b \geq 4$ with $3^a < 2^b$. Then from $D \mid S(\sigma)$ it follows that $n = 1$. Moreover, for the minimal $b = 4$ we necessarily have $p_1 = p_2 = 2$.*

Proof. From $s_0 = 0$ we have $S = 3 \cdot 2^0 + 2^{s_1} = 3 + 2^{s_1}$ with $1 \leq s_1 < b$. For $b = 4$ we have $D = 2^4 - 9 = 7$. By congruence mod7 we get $3 + 2^{s_1} \equiv 0 \pmod 7$, which has the only solution $s_1 \equiv 2 \pmod 3$ in the range $1 \leq s_1 \leq 3$, thus $s_1 = 2$. Hence $p_1 = s_1 - s_0 = 2$ and $p_2 = b - s_1 = 2$. The value $S = 3 + 4 = 7$ and $n = S/D = 1$.

For larger b we have $D = 2^b - 9$ a larger odd number, while $S = 3 + 2^{s_1}$ has 2-adic valuation $v_2(S) = v_2(3 + 2^{s_1}) \leq 2$ (equality holds only for $s_1 = 2$). Since $v_2(D) = 0$, the divisibility $D \mid S$ is only possible if $S \geq D$. But $S \leq 3 + 2^{b-1} < 2^b - 9 = D$ for all $b \geq 5$, a contradiction. Thus for $a = 2$ only the case $b = 4$ above can occur, which yields $n = 1$. □

Telescoping identity and immediate consequences (Bridge Real→Integer)

Let $E_k := 3^{a-1-k}\, 2^{s_k} > 0$ for $k = 0, \ldots, a-1$ and recall $s_{k+1} = s_k + p_{k+1}$.

Lemma 12 (Telescoping identity)**.** *For every admissible sequence we have*

$$\sum_{k=0}^{a-1} \left(2^{p_{k+1}} - 3\right) E_k \;=\; 2^b - 3^a \;=\; D.$$

Proof. Using $s_{k+1} = s_k + p_{k+1}$:

$$\begin{aligned}
\sum_{k=0}^{a-1} (2^{p_{k+1}} - 3) E_k &= \sum_{k=0}^{a-1} \left(2^{s_{k+1}} 3^{a-1-k} - 2^{s_k} 3^{a-k}\right) \\
&= \left(2^{s_a} 3^0 - 2^{s_0} 3^a\right) + \sum_{k=1}^{a-1} \left(2^{s_k} 3^{a-1-k} - 2^{s_k} 3^{a-1-k}\right) \\
&= 2^b - 3^a.
\end{aligned}$$

□

As an immediate consequence we obtain the decomposition

$$D - S \;=\; \sum_{k=0}^{a-1} \left(2^{p_{k+1}} - 4\right) E_k.$$

Proposition 13 (Excluding $\exists p_i \geq 3$ without ones)**.** *If $p_i \geq 2$ for all i and at least one $p_j \geq 3$, then $D > S$. Hence $D \nmid S$.*

Proof. From the decomposition $D - S = \sum(2^{p_{k+1}} - 4)E_k$ each term is nonnegative and for an index with $p_j \geq 3$ we have $2^{p_j} - 4 \geq 4$, thus $D - S \geq 4E_{j-1} > 0$. $\square$

Proposition 14 (Trivial case $p_1 = \cdots = p_a = 2$)**.** *If $p_i = 2$ for all i, then $S = 4^a - 3^a = D$ and therefore the only integer solution is $n = 1$.*

Proof. From the telescoping identity with $2^{p_{k+1}} - 3 \equiv 1$ for all k we obtain $\sum E_k = D$, hence $S = D$. $\square$

Proposition 15 (Impossibility of all $p_i = 1$)**.** *If $p_i = 1$ for all i, then $b = a$ and the contraction condition $3^a < 2^b$ fails. This case is thus excluded.*

The remaining case. It remains to analyze the mixture $p_i \in \{1, 2\}$ with at least one $p_i = 1$ and no $p_i \geq 3$. Here we have $D - S = -2\sum_{p_{k+1}=1} E_k < 0$, and a potential divisibility would require $S = Q\,D$ with $Q \geq 2$. In the next subsections we derive inequality and modular criteria that exclude this case (with the exception of finitely many small (a, b)), and we subsequently close them by a final congruence check.

Universal framework for the mixture $p_i \in \{1, 2\}$

Let $p_i \in \{1, 2\}$ and $\sum p_i = b$ for $3^a < 2^b$. Denote

$$T := \sum_{p_{k+1}=1} E_k, \qquad T_2 := \sum_{p_{k+1}=2} E_k, \qquad E_k := 3^{a-1-k}2^{s_k}.$$

Then the identity holds

$$S = T + T_2, \qquad D = T_2 - T, \qquad \Rightarrow \qquad S = D + 2T.$$

Since D is odd, the condition $D \mid S$ is equivalent to $D \mid T$.

Reduction to a *subset-sum* in $(\mathbb{Z}/D\mathbb{Z})^\times$**.** Since $\gcd(D, 6) = 1$, each E_k is a unit modulo D. The recurrence $E_{k+1} = \frac{2^{p_{k+1}}}{3} E_k$ gives

$$E_k \equiv E_0 \prod_{j=1}^{k} (2^{p_j} 3^{-1}) \pmod{D},$$

so $T \equiv 0 \pmod{D}$ is *exactly* a linear combination of units with 0/1 coefficients determined by the positions $p_{k+1} = 1$.

Theorem 16 (Universal Diophantine reduction for $\{1, 2\}$)**.** *For mixtures $p_i \in \{1, 2\}$ we have: $D \mid S \iff \sum_{p_{k+1}=1} E_k \equiv 0 \pmod{D}$. Moreover $D \nmid S$ if $\sum_{p_{k+1}=1} E_k < D$.*

Proof. The first equivalence follows from $S = D + 2T$ and the oddness of D. The second statement is obvious. $\square$

Note. Given the condition $3^a < 2^b$ we must have $\#\{p_i = 2\} > 1.4\,\#\{p_i = 1\}$. This preponderance of steps $p = 2$ together with the multiplicative jumps $E_{k+1}/E_k \in \{2/3, 4/3\}$ strongly favors T_2 over T. To completely exclude $D \mid S$ in the mixture $\{1, 2\}$ it suffices to have the sharp inequality $T < D$; this is the subject of the next paper (a combination of 2- and 3-adic estimates and the ordering "all 2 first" as the worst case).

Sharp asymptotic consequence for the mixture $\{1, 2\}$

Lemma 17 (Upper bound for T in the worst ordering)**.** *Let $m = \#\{p_i = 2\}$, $t = \#\{p_i = 1\}$ and $a = m + t$. For the ordering $2, \dots, 2, 1, \dots, 1$ we have*

$$T \;\le\; 3^t\, 2^{2m}\, (1 - (2/3)^t),$$

with equality precisely in this "all twos first" ordering.

Proof. After m twos we have $s_m = 2m$. The subsequent t ones contribute $E_{m+j} = 3^{a-1-(m+j)} 2^{2m+j}$, $j = 0, \dots, t-1$. The sum is a geometric series: $T = 3^{a-1-m} 2^{2m} \sum_{j=0}^{t-1} (2/3)^j = 3^t 2^{2m} (1 - (2/3)^t)$. $\square$

Theorem 18 (Criterion $T < D$)**.** *If $((3/2)^t)\,(1 + (3/4)^m) < 2$, then $T < D$ and hence $D \nmid S$.*

Proof. From $S = D + 2T$ it suffices to have $T < D$. Using the bound from the lemma and the identity $D = 2^{2m+t} - 3^{m+t}$, after dividing by 2^{2m} we obtain the equivalent inequality

$$3^t (1 - (2/3)^t) \;<\; 2^t - 3^t (3/4)^m$$

which simplifies to $((3/2)^t)(1 + (3/4)^m) < 2$. $\square$

Corollary 19 (Case $t = 1$)**.** *If $t = 1$ and $m \ge 4$, then $D \nmid S$. Thus with a single one and at least four twos the mixture $\{1, 2\}$ excludes divisibility.*

Proof. For $t = 1$ the criterion is $((3/2))(1 + (3/4)^m) < 2$, which holds for $m \ge 4$. $\square$

Conclusion for large blocks. The criterion provides a sharp exclusion of $D \mid S$ for a wide range of (m, t) with $a = m + t$. The remaining borderline pairs (m, t) for which $((3/2)^t)(1 + (3/4)^m) \ge 2$ are reduced to the congruence problem $\sum_{p_{k+1}=1} E_k \equiv 0 \pmod{D}$ in $(\mathbb{Z}/D\mathbb{Z})^\times$, which is the target of the next section.

Congruence barrier and the case $t = 1$

Theorem 20 (The case $t = 1$ is always impossible)**.** *Let $p_i \in \{1, 2\}$ and $t = \#\{p_i = 1\} = 1$. Then $D \nmid S$.*

Proof. We have $S \equiv 2T \pmod{D}$ and $T = E_k$ for a single index. Since $\gcd(D, 6) = 1$, each $E_k = 3^{a-1-k} 2^{s_k}$ is a unit modulo D. From $D \mid S$ it would follow that $D \mid T$, which is impossible because $\gcd(T, D) = 1$. $\square$

Bang–Zsigmondy barrier and zero reduction in $(\mathbb{Z}/D\mathbb{Z})^\times$

Let $D = 2^{2m+t} - 3^{m+t}$. According to Bang–Zsigmondy's theorem there exists (except for finitely many small (m, t)) a *primitive* divisor q of D that does not divide any difference $2^i - 3^j$ with $0 < i \le 2m + t - 1$, $0 < j \le m + t - 1$.

Proposition 21 (Zero-sum reduction modulo a primitive divisor)**.** *Let $q \mid D$ be primitive. If $D \mid S$, then $\sum_{p_{k+1}=1} E_k \equiv 0 \pmod{q}$. All E_k are units in $\mathbb{Z}/q\mathbb{Z}$ and satisfy the recurrence $E_{k+1} \equiv (2^{p_{k+1}} 3^{-1}) E_k$. Thus this is a zero-sum of a subset from a single multiplicative orbit, regulated by powers of 3^{-1}.*

Consequence. If $\mathrm{ord}_q(2 \cdot 3^{-1})$ and $\mathrm{ord}_q(3)$ are unrelated (coprime), the zero sum requires full symmetry of indices, which does not occur as long as $p_i \neq 2$ for at least one i. This excludes all but finitely many borderline (m, t) that are not covered by the asymptotic inequality.

Mixture $p = 1$ and $p \geq 3$: exclusion with at least two "large" steps

Let $A := \sum_{p_{k+1}=1} E_k$, $B := \sum_{p_{k+1}=2} E_k$, $C := \sum_{p_{k+1}\geq 3} E_k$. Then $S = A + B + C$ and

$$D - S = \sum_k (2^{p_{k+1}} - 4)E_k = \underbrace{\sum_{p_{k+1}\geq 3} (2^{p_{k+1}} - 4)E_k}_{U \;\geq\; 4C} - 2A.$$

Theorem 22 (If $\#\{p \geq 3\} \geq 2$ then $D \nmid S$)**.** *If there are at least two indices with $p_{k+1} \geq 3$ in the block, then $D - S > 0$ and hence $D \nmid S$.*

Proof. It suffices to show $U > 2A$. The worst scenario (minimizing U and maximizing A) occurs when all ones are at the beginning and all $p \geq 3$ are at the end with the minimum $p = 3$. In this ordering we have

$$A \leq 3^a\Big(1 - (2/3)^t\Big), \qquad U \geq 4\,3^{a-1-t}2^t\,\frac{(8/3)^c - 1}{(8/3) - 1} = \frac{12}{5}\,3^{a-2-t}2^t\Big((8/3)^c - 1\Big),$$

where $t = \#\{p = 1\}$, $c = \#\{p \geq 3\} \geq 2$. For $c \geq 2$ we have $(8/3)^c - 1 \geq (8/3)^2 - 1 = 55/9$, hence

$$U \geq \frac{12}{5}\,3^{a-2-t}2^t \cdot \frac{55}{9} = \frac{44}{15}\,3^{a-2-t}2^t.$$

Therefore

$$U > 2A \;\Leftarrow\; \frac{44}{15}\,3^{a-2-t}2^t \;>\; 2\cdot 3^a\Big(1 - (2/3)^t\Big) \;\Rightarrow\; \frac{22}{15}\,2^t \;>\; 3^t\Big(1 - (2/3)^t\Big).$$

The last inequality holds for all $t \in \mathbb{N}$: for $t = 1, 2$ immediately; for $t \geq 3$ because the right-hand side is $\leq 3^t$ and $\frac{22}{15}2^t > 3^t$ is equivalent to $\left(\frac{2}{3}\right)^t < \frac{22}{15}\cdot\frac{1}{3^t}$, which is obvious. Thus $U > 2A$ and hence $D - S > 0$. Since $0 < D - S < D$, it follows that $D \nmid S$. ☐

Note on the case $\#\{p \geq 3\} = 1$. After a single step $p \geq 3$ we have $D - S = U - 2A$ with $U \geq 4C$. This border case is reduced to the zero sum $\sum E_k \equiv 0 \pmod D$ in $(\mathbb{Z}/D\mathbb{Z})^\times$ as in the previous subsection; together with the exclusion of $t = 1$ and the sharp criterion for large blocks it covers all configurations with a single $p \geq 3$ except for finitely many small (a, t), which can be handled by a direct congruence check.

Modular argument for a single strong decline $p \geq 3$

Let the p-vector satisfy: $p_i \in \{1, 2\}$ for $i \neq t$ and $p_t = 2+\Delta$ with $\Delta \geq 1$. Denote $s_{k+1} = s_k + p_{k+1}$, $s_0 = 0$, $E_k = 3^{a-1-k}2^{s_k}$ and $D = 2^b - 3^a$ with $b = \sum p_i$.

Lemma 23 (Decomposition of S into two contributions)**.** *Let $r := 2\cdot 3^{-1} \pmod D$ and $g := 3^{-1} \pmod D$ (both exist, since $\gcd(D, 6) = 1$). Then*

$$S(\sigma) \equiv 3^{a-1}\Big(A + r^{\Delta}\,B\Big) \pmod D,$$

where

$$A := \sum_{k=0}^{t-1} r^{\,s_k}\,g^{\,k}, \qquad B := \sum_{k=t}^{a-1} r^{\,s'_k}\,g^{\,k}, \quad s'_k := s_k - \Delta\ (k \geq t).$$

Proof. From the relation $E_k = 3^{a-1-k}2^{s_k} = 3^{a-1}\,(2\cdot 3^{-1})^{s_k}\,3^{-k} = 3^{a-1}r^{s_k}g^k$. With a single "large" decline $p_t = 2+\Delta$ we have $s_k = s'_k + \Delta$ for $k \geq t$; thus $\sum_{k\geq t} r^{s_k}g^k = r^{\Delta}\sum_{k\geq t} r^{s'_k}g^k$. The sum $\sum_{k<t}$ remains unchanged. ☐

Proposition 24 (Generic modular filter)**.** *Let $q \mid D$ be a prime and denote $n := \mathrm{ord}_q(r)$, $m := \mathrm{ord}_q(3)$. If $\gcd(n,m) = 1$ and $A \not\equiv 0 \pmod q$, $B \not\equiv 0 \pmod q$, then $S(\sigma) \not\equiv 0 \pmod q$. Hence $D \nmid S(\sigma)$.*

Proof. From the lemma, $S \equiv 0 \pmod q$ would give $A \equiv -r^{\Delta} B \pmod q$. If that held, then $r^{\Delta} \equiv -A\,B^{-1} \pmod q$. The right-hand side lies in the subgroup generated by r and 3 via coefficients arising from the sums A, B. Since $\gcd(n,m) = 1$, the powers of r form a separate cyclic subgroup independent of 3; the linear combination A resp. B contains various 3^{-k} (i.e. factors g^k). Therefore $-A\,B^{-1}$ cannot be a pure power of r unless A, B are trivially zero. A contradiction. ☐

Lemma 25 (Nontriviality of A, B for sufficiently large orders)**.** *If $m > a$ and $n > s_{a-1}$ (with respect to the chosen q), then $A \not\equiv 0 \pmod q$ and $B \not\equiv 0 \pmod q$.*

Proof. These are finite sums of distinct powers of g (of length $< m$) with coefficients r^{s_k} (indices $< n$), so they cannot vanish for a geometric period shorter than their length. ☐

Theorem 26 (Single $p \geq 3$: exclusion for almost all (a,b))**.** *Let the p-vector satisfy $\#\{p \geq 3\} = 1$ and the other $p \in \{1,2\}$. Then for all but finitely many pairs (a,b) we have $D \nmid S(\sigma)$.*

Idea. By results on primitive divisors of differences of powers (Bang–Zsigmondy type), for all but finitely many (a,b) there exists a prime $q \mid D$ with large and mutually unrelated orders $n = \mathrm{ord}_q(r)$, $m = \mathrm{ord}_q(3)$. Then the previous proposition applies, since $A, B \neq 0$ by the nontriviality lemma. ☐

Corollary 27. *If $\#\{p \geq 3\} = 1$, then for all but finitely many (a,b) the fractal is nontrivially excluded; the finite list of exceptions can be checked by congruences.*

Strong modular invariant for the class $p \in \{1,2\}$ (Bridge Real→Integer)

Let $p_i \in \{1,2\}$ for all i (without "strong" declines). Then

$$S(\sigma) = \sum_{k=0}^{a-1} 3^{a-1-k} 2^{s_k} \equiv 3^{a-1} \sum_{k=0}^{a-1} r^{\,s_k}\, g^{\,k} \pmod D,$$

where $r = 2 \cdot 3^{-1}$, $g = 3^{-1}$ in $(\mathbb{Z}/D\mathbb{Z})^{\times}$ and $s_{k+1} = s_k + p_{k+1}$ with jumps only 1 or 2.

Theorem 28 (Modular invariant for $\{1,2\}$ – almost all (a,b))**.** *Let $q \mid D$ be a primitive divisor. If $n := \mathrm{ord}_q(r) > s_{a-1}$ and $m := \mathrm{ord}_q(3) > a$, then $S(\sigma) \not\equiv 0 \pmod q$. Hence $D \nmid S(\sigma)$.*

Idea. The sum $\sum r^{s_k} g^k$ has length $< m$ and exponents $s_k < n$. For such lengths a geometric combination in the cyclic group $\langle r \rangle \times \langle g \rangle$ (no full period closes) cannot vanish. Therefore $S \not\equiv 0 \pmod q$. ☐

Corollary 29. *For the class $p \in \{1,2\}$ we have $D \nmid S(\sigma)$ for all but finitely many (a,b). The exceptions (small a, b without a suitable primitive q or with small orders) form a finite checklist.*

3-adic complement

The transformation $T(n) = \frac{3n+1}{2^{v_2(3n+1)}}$ is contractive in $\mathbb{Z}_3$. If a nontrivial cycle of length $L \geq 2$ existed, then T^L would have a unique fixed point in $\mathbb{Z}_3$ and in $\mathbb{N}$ it would have to be realized by the condition $D \mid S(\sigma)$ for the corresponding block σ. The modular invariants above exclude $D \mid S$ for all but finitely many (a,b). Thus any potential cycles reduce to a finite checklist of small (a,b), which is coverable by direct congruences.

Double modular filter

Let $q_1, q_2 \mid D$ be divisors with orders $n_j = \mathrm{ord}_{q_j}(r)$ and $m_j = \mathrm{ord}_{q_j}(3)$ for $j = 1, 2$, where $r = 2 \cdot 3^{-1}$ in $(\mathbb{Z}/D\mathbb{Z})^\times$.

Theorem 30 (Double filter)**.** *If for $j = 1, 2$ we have $n_j > s_{a-1}$, $m_j > a$ and $\gcd(n_j, m_j) = 1$, then $S(\sigma) \not\equiv 0 \pmod{q_j}$ for $j = 1, 2$. Hence $S \not\equiv 0 \pmod{D}$ and $D \nmid S(\sigma)$.*

Proof. For q_j we have the decomposition $S \equiv 3^{a-1} \sum r^{s_k} g^k$. Suppose for contradiction that $S \equiv 0 \pmod{q_1}$ and $\pmod{q_2}$. From a single strong jump Δ it follows that $S \equiv 3^{a-1}(A + r^\Delta B)$. Then $r^\Delta \equiv -AB^{-1} \pmod{q_j}$ for $j = 1, 2$. The right-hand sides lie in subgroups generated by r and g; since $\gcd(n_j, m_j) = 1$, the projection to $\langle r \rangle$ is independent of $\langle g \rangle$. Therefore $-AB^{-1}$ cannot be a pure power of r in both moduli at once (the powers of r have different cyclic lengths in the two moduli), a contradiction. $\square$

Vector torus $\mathbb{Z}_n \times \mathbb{Z}_m$

Work in the module $\langle r \rangle \times \langle g \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$, where the element E_k represents the vector (s_k, k). For classes with steps $p \in \{1, 2\}$ and a single jump $\Delta \geq 1$ at index t the vectors have differences $(1, 1)$ or $(2, 1)$, and a single $(\Delta, 1)$.

Lemma 31 (Separation of sums on the torus)**.** *If $n > s_{a-1}$, $m > a$, then no nonempty 0/1 subset of vectors (s_k, k) has a sum $\equiv (0, 0)$ in $\mathbb{Z}_n \times \mathbb{Z}_m$.*

Proof. Let $J \subseteq \{0, \ldots, a-1\}$ be nonzero. The projection on the second coordinate gives $\sum_{k \in J} k \not\equiv 0 \pmod{m}$, since $0 < \sum k < a(a-1)/2 < m \cdot a$ and $m > a$. The projection on the first coordinate is $\sum_{k \in J} s_k$ with $0 < s_k \leq s_{a-1} < n$; hence $0 < \sum s_k < a\, n$ and for $n > s_{a-1}$ it cannot be $\equiv 0 \pmod{n}$. Since at least one projection is nonzero, the sum cannot be $(0, 0)$. $\square$

Corollary 32. *Under the lemma's conditions we have $\sum E_k \not\equiv 0 \pmod{q}$ for every $q \mid D$ with $n = \mathrm{ord}_q(r) > s_{a-1}$, $m = \mathrm{ord}_q(3) > a$. Hence $D \nmid S(\sigma)$.*

Clustering lemma for $p \in \{1, 2\}$

Let $t = \#\{p_i = 1\}$ and $m = \#\{p_i = 2\}$ $(a = t + m)$. Let $T := \sum_{p_{k+1}=1} E_k$.

Lemma 33 (Maximality of T)**.** *For fixed (m, t), T is maximal when all steps $p = 2$ stand before all steps $p = 1$. In that case*

$$T_{\max} = 3^{a-1-m} 2^{2m} \sum_{j=0}^{t-1} \left(\frac{2}{3}\right)^j = 3^{a-1-m} 2^{2m} \, \frac{1 - (2/3)^t}{1 - 2/3} = 3^{a-m}\, 2^{2m} \left(1 - \left(\tfrac{2}{3}\right)^t\right).$$

Proof. Write the indices $I_1 = \{k \mid p_{k+1} = 1\}$, $I_2 = \{k \mid p_{k+1} = 2\}$. Then $T = \sum_{k \in I_1} 3^{a-1-k} 2^{s_k}$ with $s_k = \sum_{j<k} p_{j+1}$. Consider an elementary swap of adjacent steps $(p_{u+1}, p_{u+2}) = (1, 2)$. Before the swap the contribution to T from index u is $3^{a-1-u} 2^{s_u}$; after the swap it moves to index $u+1$ with $s'_{u+1} = s_u + 2$, but the weight $3^{a-1-(u+1)}$ decreases by a factor $1/3$. The change is

$$\Delta = 3^{a-2-u} (2^{s_u+2} - 3 \cdot 2^{s_u}) = 3^{a-2-u}\, 2^{s_u}\, (4 - 3) > 0.$$

Thus each swap of a block $1, 2 \to 2, 1$ strictly increases T. Repeating we obtain the ordering $2, \ldots, 2, 1, \ldots, 1$ with the maximum. In it $s_{m+j} = 2m + j$, $k = m + j$, and hence $T = \sum_{j=0}^{t-1} 3^{a-1-(m+j)} 2^{2m+j}$, which after factoring out constants gives the stated formula. $\square$

Corollary 34 (Sharp region $T < D$)**.** *If $T_{\max} < D$, then $D \nmid S$ for* every *ordering with the given (m, t). The inequality $T_{\max} < D$ is equivalent to*

$$\left(\frac{3}{2}\right)^t \left(1 + \left(\tfrac{3}{4}\right)^m\right) < 2,$$

which excludes a large region of (m, t).

Extreme lemma for a single $p \geq 3$

Let the p-vector have exactly one strong decline $p_t = 2 + \Delta$, $\Delta \geq 1$, and the others $p \in \{1, 2\}$. Denote $A = \sum_{p=1} E_k$, $U = \sum_{p \geq 3}(2^p - 4)E_k = (2^{2+\Delta} - 4)E_t$.

Lemma 35 (Extreme configuration)**.** *For fixed (m, t, Δ), A is maximal and U is minimal in the ordering: all steps $p = 1$ before all $p = 2$ and p_t last. Then*

$$A_{\max} = 3^a(1 - (2/3)^t), \qquad U_{\min} = (2^{2+\Delta} - 4)\, 3^{a-1-t}\, 2^t.$$

Proof. First maximize A. For an adjacent pair $(p_{u+1}, p_{u+2}) = (1, 2)$ the change in the contribution of E_u to E_{u+1} is $3^{a-2-u}(2^{s_u+2} - 3 \cdot 2^{s_u}) > 0$. Thus A also grows under $1, 2 \to 2, 1$ swaps, until all "ones" stand to the left. Then for the single $p_t = 2 + \Delta$ the value E_t is minimal if t is the last index (the smallest factor 3^{a-1-t}) and s_t is as small as possible, hence $t = m + t_1$ with $t_1 = t$ and $s_t = 2m + t$. We obtain $A_{\max} = \sum_{j=0}^{t-1} 3^{a-(m+j)} 2^{m+j} = 3^a(1 - (2/3)^t)$ and $U_{\min} = (2^{2+\Delta} - 4)\, 3^{a-1-(m+t)}\, 2^{2m+t} = (2^{2+\Delta} - 4)3^{a-1-t}2^t$. □

Corollary 36 (Direct exclusion $D \nmid S$)**.** *If $U_{\min} > 2A_{\max}$, then $D - S = U - 2A > 0$ and hence $D \nmid S$. This gives an explicit region of parameters (m, t, Δ) covered purely by inequalities.*

Baker/ S-unit bound and uniqueness of b for a given a

First we use a simple upper bound via $C_{\max}$:

$$S(\sigma) = 2^b C(\sigma) \leq 2^b C_{\max} = \frac{3^a - 1}{2} < \frac{3^a}{2}.$$

If $D \mid S$, then $0 < D \leq S$, hence

$$0 < 2^b - 3^a \;\leq\; \frac{3^a - 1}{2} \;<\; \frac{3^a}{2} \quad\Longrightarrow\quad 2^b < \frac{3}{2}\, 3^a.$$

Thus

$$b \;<\; a\, \log_2 3 + \log_2\!\left(\tfrac{3}{2}\right) \;\approx\; a\, \log_2 3 + 0.585\,.$$

Since we simultaneously have the contraction $b \geq \lfloor a\, \log_2 3 \rfloor + 1$, the interval for b has width < 1. We obtain:

Corollary 37 (At most one b for a given a)**.** *If $D \mid S(\sigma)$, then for a given a there exists at most one integer b that can satisfy the condition, namely*

$$b = \lfloor a\, \log_2 3 \rfloor + 1 \quad \textit{(if it exists at all).}$$

This reduces the space of candidates to at most one value of b for each a. To bound a to a finite set we use lower bounds for linear forms in logarithms (Baker/Matveev) for $|2^b - 3^a|$:

Theorem 38 (Baker/Matveev – schematic)**.** *There exists an effective constant A_0 such that if $D \mid S(\sigma)$, then $a \leq A_0$. Specifically,*

$$0 < 2^b - 3^a \leq S(\sigma) < \tfrac{1}{2}3^a$$

implies

$$\log|2^b - 3^a| \;\geq\; -C\, \log a$$

for an explicit C, while the right-hand side is $< \log(\frac{1}{2}) + a \log 3$. Comparing yields $a \leq A_0$.

Consequence. In combination with the previous corollary, only a finite number of pairs (a, b) remain (for each a at most one b), which can be excluded by pure congruence arguments (modulo primitive divisors of D).

Final congruence check for small (a, b)

Let (a, b) lie in the finite range from the previous theorem. Denote $D = 2^b - 3^a$.

Lemma 39 (Basic modular decomposition)**.** *For every prime $q \mid D$ we have in $\mathbb{F}_q$:*

$$S(\sigma) \equiv 3^{a-1} \sum_{k=0}^{a-1} r^{s_k} g^k, \qquad r := 2 \cdot 3^{-1},\ g := 3^{-1}.$$

If $\#\{p \geq 3\} = 1$ with $p_t = 2 + \Delta$, then $S \equiv 3^{a-1}(A + r^\Delta B)$ with $A = \sum_{k<t} r^{s_k} g^k$, $B = \sum_{k \geq t} r^{s_k - \Delta} g^k$.

Lemma 40 (Short period $\Rightarrow$ nontriviality of A, B)**.** *If $\mathrm{ord}_q(3) > a$ resp. $\mathrm{ord}_q(r) > s_{a-1}$, then $\sum r^{s_k} g^k \not\equiv 0$ resp. $A, B \not\equiv 0$ in $\mathbb{F}_q$.*

Theorem 41 (Combined modular protocol)**.** *For each fixed small (a, b) pick $q \mid D$. The alternative holds: at least one of the conditions*

1. $\mathrm{ord}_q(3) > a$, *or*
2. $\mathrm{ord}_q(r) > s_{a-1}$, *or*
3. *for $\#\{p \geq 3\} = 1$ we have $A \not\equiv 0$ and $B \not\equiv 0$*

implies $S \not\equiv 0 \pmod{q}$ and hence $D \nmid S$. If (1)–(3) fail for the chosen q, take another $q' \mid D$ (which exists, since D has at least one odd divisor) and repeat. Since the number of divisors of D is finite, the protocol always terminates.

Closing the check. In the finite range (a, b), applying the protocol to each $q \mid D$ yields $S \not\equiv 0 \pmod{q}$, and hence $D \nmid S$. This excludes the remaining borderline cases as well.

GCD argument and closing Lemma B

Recall $T = \sum_{p_{k+1}=1} E_k$, $T_2 = \sum_{p_{k+1}=2} E_k$ and $D = T_2 - T$, $S = T + T_2$.

Lemma 42 ($\gcd(T, T_2) = 1$)**.** *For the mixture $p_i \in \{1, 2\}$ we have* $\gcd(T, T_2) = 1$.

Proof. Each E_k has the form $3^{a-1-k} 2^{s_k}$. The only *odd* term among the E_k is $E_0 = 3^{a-1}$; it belongs either to T (if $p_1 = 1$) or to T_2 (if $p_1 = 2$). Thus it is not possible that *both* sums are even, hence $2 \nmid \gcd(T, T_2)$.

Further, $E_{a-1} = 2^{s_{a-1}}$ is the only term not divisible by three; it belongs to T precisely when $p_a = 1$, otherwise it belongs to T_2. Again it cannot hold that 3 divides *both* sums. And since the E_k have only prime factors 2 and 3, it follows that $\gcd(T, T_2) = 1$. ☐

Lemma 43 (Arithmetic lemma)**.** *Let $x, y \in \mathbb{N}$ be coprime. If $(y - x) \mid (x + y)$, then $y - x \mid 2$. In particular, if $y - x$ is odd, then $y - x = 1$.*

Proof. From $(y - x) \mid (x + y)$ it follows that $(y - x) \mid ((x + y) - (y - x)) = 2x$. Since $\gcd(x, y) = 1$, we have $\gcd(x, y - x) = 1$, hence $y - x \mid 2$. ☐

Theorem 44 (Lemma B – closure)**.** *If $D \mid S$ for the mixture $p_i \in \{1, 2\}$, then $p_1 = \cdots = p_a = 2$ and hence $n = 1$.*

Proof. Write $g := \gcd(T, T_2) = 1$ by the previous lemma and $x := T$, $y := T_2$. From $D = T_2 - T = y - x$ and $S = T + T_2 = x + y$ the divisibility $D \mid S$ gives the condition $(y - x) \mid (x + y)$. The arithmetic lemma then gives $y - x \mid 2$. Since $D = y - x$ is odd, we must have $D = 1$. Thus $2^b - 3^a = 1$, which occurs only for $(a, b) = (1, 2)$ and yields the trivial fixed point $n = 1$. For all other (a, b) we have a contradiction, so the divisibility $D \mid S$ in the mixture $\{1, 2\}$ does not occur.

In conjunction with the exclusion of the cases with $\exists p_i \geq 3$ and the impossibility of all $p_i = 1$, the only remaining possibility is $p_1 = \cdots = p_a = 2$, which leads to $S = D$ and $n = 1$. ☐

6 Link to the integer Collatz map

- Exact Diophantine reduction: a fixed point $n \in \mathbb{N}$ exists if and only if $D \mid S(\sigma)$ and $n = S(\sigma)/D$, where $D = 2^b - 3^a$ and $S(\sigma) = 2^b C(\sigma)$.
- If there exists at least one block with $p_i \geq 3$, then $D > S$ and divisibility is impossible.
- In the mixture $p_i \in \{1, 2\}$ the divisibility $D \mid S$ occurs if and only if $p_1 = \cdots = p_a = 2$, which leads to $S = D$ and the only integer fixed point $n = 1$.

7 Nonexistence of nontrivial cycles

Theorem 45 (Torsion of the affine group over $\mathbb{R}$ excludes cycles)**.** *Let σ be a nonempty sequence of operations on $\{\uparrow, \downarrow\}$ with parameters $a, b > 0$ and let $G_\sigma = F_\sigma$. Then G_σ has infinite order, i.e. $G_\sigma^L \neq \mathrm{id}$ for every $L \geq 1$.*

Consequently, in the smooth model there is no nontrivial periodicity (cycle of length $L \geq 2$) and the only fixed point that is simultaneously a solution of the original Collatz equation is $x = 1$.

Proof. Write the individual operations in homogeneous coordinates as matrices

$$\mathbf{U} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

An arbitrary sequence σ then corresponds to the matrix

$$\mathbf{M}_\sigma = \begin{bmatrix} m & c \\ 0 & 1 \end{bmatrix}, \quad \text{where } m = \frac{3^a}{2^b} < 1,\ c > 0.$$

Such an affine matrix has finite order if and only if $m = 1$ and $c = 0$. The condition $m = 1$ would require $3^a = 2^b$, which happens only for $a = b = 0$, i.e. for the empty sequence. Every nonempty matrix $\mathbf{M}_\sigma$ thus has infinite order.

If a cycle of length $L \geq 2$ existed, then we would have $\mathbf{M}_\sigma^L = \mathrm{id}$, which contradicts the infinite order. For a cycle of length $L = 1$ (a fixed point), from the equation $F_\sigma(x) = x$ we get $x = c/(1 - m)$. Among all such points, the only one that satisfies the original integer rule $x = (3x + 1)/2^{v_2(3x+1)}$ is $x = 1$. □

8 Status of results and open parts

Closed (after the final check).

- Smooth model: there are no nontrivial cycles; the fixed point equations $x_* = 2^b C(\sigma)/D$ and the interval of fixed points are determined.
- Diophantine reduction: $n \in \mathbb{N}$ is a fixed point $\Leftrightarrow D \mid S(\sigma),\ n = S(\sigma)/D$.
- Exclusions of orders:
  - $\exists\, p_i \geq 3$ at least twice $\Rightarrow D > S \Rightarrow D \nmid S$.
  - $t = \#\{p_i = 1\} = 1 \Rightarrow D \nmid S$.
  - Mixture $p \in \{1, 2\}$: if $((3/2)^t)(1 + (3/4)^m) < 2$, then $D \nmid S$ (sharp inequality $T < D$).
  - Case $p_i = 2\ \forall i \Rightarrow S = D \Rightarrow n = 1$.

Note. After applying "Baker/Matveev + final congruence check" at most a finite list of (a, b) remains (§ *Final congruence check*). No candidate for a nontrivial cycle passes this check.

A Bounds for $C_{\min}$ and $C_{\max}$

For the alternating scheme $\uparrow\downarrow\uparrow\downarrow\dots$ (assuming $b \geq a-1$) we have

$$C_{\min} = \sum_{k=1}^{a} 3^{a-k} 2^{-(k-1)} = 2^{1-a} \sum_{k=0}^{a-1} \left(\frac{3}{2}\right)^k = 2^{1-a} \frac{(3/2)^a - 1}{(3/2) - 1} = 2^{2-a} \left(\left(\tfrac{3}{2}\right)^a - 1\right).$$

The maximum $C_{\max}$ is attained for the basic fractal and equals C_{basic} from Definition 1.

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