Analysis of the Collatz Conjecture: Excluding Infinite Growth

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Abstract

This paper (Part 2) follows up on the results about nontrivial cycles and the Diophantine bridge from the smooth model. By using the "One-Way Gate" (exclusion of returning to multiples of 3), a 3-adic analysis of the map T, and an exact Diophantine reduction with a telescoping identity, we show that infinite growth of a Collatz trajectory is impossible: every trajectory eventually reaches the target set $S = \{n \mid 3n+1 = 2^k\}$ and thus 1. Key points: (i) no nontrivial cycles; (ii) the fixed-point condition in integers $D \mid S(\sigma)$, $n = S(\sigma)/D$; (iii) exclusions of orders via inequalities and modular invariants.

1 Goal of the proof

The main idea is to prove the Collatz conjecture by showing that every sequence must necessarily reach a number n satisfying the condition:

$$3n + 1 = 2^k$$

for some natural number k. Once the sequence reaches such an n, the next term is 2^k, and then by repeated divisions by two it inevitably decreases to the number 1. This would prove the conjecture.

2 Step 1: Defining and Analyzing the Target Set S

Define the "target set" S as the set of all natural numbers n which, after one application of $3n+1$, become a power of two.

$$S = \{n \in \mathbb{N} \mid \exists k \in \mathbb{N} : 3n + 1 = 2^k\}$$

From this definition we can express n:

$$n = \frac{2^k - 1}{3}$$

For n to be an integer, the numerator $2^k - 1$ must be divisible by three. That is,

$$2^k - 1 \equiv 0 \pmod 3 \implies 2^k \equiv 1 \pmod 3$$

The condition $2^k \equiv 1 \pmod 3$ holds precisely when the exponent k is even. We can therefore write $k = 2m$ for $m \in \mathbb{N}$.

Thus we have clarified the form of the numbers in the set S:

$$n_m = \frac{2^{2m} - 1}{3} = \frac{4^m - 1}{3}$$

The first few elements of the set S are:

$$S = \{1, 5, 21, 85, 341, 1365, \dots\}$$

3 Step 2: Initial Modular Analysis

If there exists a sequence that avoids the goal (the number 1), it must avoid all numbers in the set S forever. We examine what conditions this imposes on these numbers.

Let us look at elements $n \in S$ modulo 4. For $m \geq 1$, $4^m \equiv 0 \pmod 4$. For $n \in S, n > 1$ we have:

$$3n = 4^m - 1 \implies 3n \equiv -1 \equiv 3 \pmod 4 \implies n \equiv 1 \pmod 4$$

Finding: All numbers in the target set S (except for the trivial case $n = 1$) give remainder 1 when divided by 4.

It follows directly that any odd number x that leaves remainder 3 upon division by 4 (i.e. $x \equiv 3 \pmod 4$) **cannot** be an element of the target set S.

4 Step 3: Analysis of the Transformation and Residues Modulo 3

Let $T(n) = \frac{3n+1}{2^p}$ be the transformation that maps an odd number n to the next odd number n' in the sequence, where $p = v_2(3n + 1)$ is the highest power of two dividing $3n + 1$.

Definition 1. *For a positive integer m we define the* 2-adic valuation $v_2(m)$ *as the largest integer $e \geq 0$ such that $2^e \mid m$ (and simultaneously $2^{e+1} \nmid m$). In other words, $v_2(m)$ counts how many times the factor 2 appears in the prime factorization of m.*

We examine the relationship between n and n' with respect to residues modulo 3.

Let n_i be an odd number in the sequence. Then $3n_i + 1$ is even. Let $p_i = v_2(3n_i + 1)$. The successor n_{i+1} is given by

$$n_{i+1} = \frac{3n_i + 1}{2^{p_i}}$$

Consider this equation in the context of congruences modulo 3:

$$n_{i+1} \equiv (3n_i + 1) \cdot (2^{p_i})^{-1} \pmod 3$$

Since $3n_i + 1 \equiv 1 \pmod 3$ and the inverse of 2 modulo 3 is 2 (because $2 \cdot 2 = 4 \equiv 1 \pmod 3$), it holds that $(2^{p_i})^{-1} \equiv (2^{-1})^{p_i} \equiv 2^{p_i} \pmod 3$. Substituting, we get

$$n_{i+1} \equiv 1 \cdot 2^{p_i} \pmod 3$$

$$n_{i+1} \equiv 2^{p_i} \pmod 3$$

4.1 Finding

The residue of the next odd number n_{i+1} modulo 3 is fully determined by the exponent $p_i = v_2(3n_i + 1)$ – the number of divisions by two.

- If p_i is **even**, then $n_{i+1} \equiv 2^{\text{even}} \equiv (2^2)^k \equiv 1^k \equiv 1 \pmod 3$.
- If p_i is **odd**, then $n_{i+1} \equiv 2^{\text{odd}} \equiv 2 \cdot (2^2)^k \equiv 2 \cdot 1^k \equiv 2 \pmod 3$.

4.2 Verification on the sequence for n=7

- $n_0 = 7$. $3 \cdot 7 + 1 = 22 = 2^1 \cdot 11$. $p_0 = 1$ (odd). $n_1 = 11$. $11 \equiv 2 \pmod 3$. OK.
- $n_1 = 11$. $3 \cdot 11 + 1 = 34 = 2^1 \cdot 17$. $p_1 = 1$ (odd). $n_2 = 17$. $17 \equiv 2 \pmod 3$. OK.
- $n_2 = 17$. $3 \cdot 17 + 1 = 52 = 2^2 \cdot 13$. $p_2 = 2$ (even). $n_3 = 13$. $13 \equiv 1 \pmod 3$. OK.
- $n_3 = 13$. $3 \cdot 13 + 1 = 40 = 2^3 \cdot 5$. $p_3 = 3$ (odd). $n_4 = 5$. $5 \equiv 2 \pmod 3$. OK.
- $n_4 = 5$. $3 \cdot 5 + 1 = 16 = 2^4 \cdot 1$. $p_4 = 4$ (even). $n_5 = 1$. $1 \equiv 1 \pmod 3$. OK.

4.3 Consequences for the structure of the sequence

We see that the residue modulo 3 in the sequence can change. The change is governed by the parity of the exponent p_i. A hypothetical "infinite" sequence would have to generate such a sequence of exponents p_i that the resulting numbers n_{i+1} always avoid the target set S.

The problem thus shifts to the analysis of the sequence of exponents $p_i = v_2(3n_i + 1)$. The Collatz conjecture is equivalent to the statement that there is no starting number n_0 that would generate an "infinite" sequence n_i such that $n_i \notin S$ for all $i \geq 0$.

5 Step 4: The One-Way Gate Theory

The previous analysis revealed a complex dependence of the residues $\bmod 3$ on the exponents of division by two. However, there is a much stronger and simpler rule that dramatically constrains the behavior of any Collatz sequence. We call this rule the "One-Way Gate".

Lemma 2 (One-Way Gate)**.** *If n_i is any odd number in a Collatz sequence, then the next odd number, n_{i+1},* ***can never be divisible by three****.*

Proof. The successor n_{i+1} is defined as $n_{i+1} = \frac{3n_i+1}{2^{p_i}}$. For n_{i+1} to be divisible by three, its numerator, $3n_i + 1$, would have to be divisible by three.

However, the term $3n_i$ is by definition divisible by three. It follows that the term $3n_i + 1$ always leaves remainder 1 modulo three.

$$3n_i + 1 \equiv 1 \pmod 3$$

Since the numerator $3n_i + 1$ is never divisible by three, the result n_{i+1} (after division by any power of two) can never be divisible by three. □

5.1 Consequences and Structure of Sequences

This lemma splits all Collatz sequences into two fundamentally different types:

1. **The "Genesis" path:** The sequence starts with a number n_0 that **is** a multiple of three (e.g. 3, 9, 21, ...). The very first successor, n_1, is no longer a multiple of three by the lemma. Hence neither $n_2, n_3, \ldots$ nor any later term. The property "being a multiple of three" appears only at the beginning and is immediately and irreversibly lost.
2. **The "Exile" path:** The sequence starts with a number n_0 that **is not** a multiple of three. By the lemma, none of the successors $(n_1, n_2, n_3, \ldots)$ will ever be a multiple of three. Such a sequence is forever "banished" from the set of multiples of three.

It follows that no multiple of three can appear in any Collatz sequence, except possibly for the first term n_0.

5.2 Closing: Why Can't the Sequence Avoid the Goal?

Our "One-Way Gate" lemma provides a key insight into the structure of Collatz sequences. It shows that once a sequence leaves (or never enters) the set of multiples of three, it is forever restricted to numbers that are not divisible by three. A hypothetical "infinite" sequence would then have to maneuver forever in this restricted space, avoiding all numbers from the target set S (specifically, those that are not multiples of three, such as $1, 5, 85, 341, \ldots$).

The transformation rules $(3n + 1)/2^p$ are fully deterministic. Although the behavior of sequences may appear chaotic, it is governed by these rules. The "One-Way Gate" of divisibility by three is one of the most important structural constraints of this motion, which removes the possibility for the sequence to arbitrarily return to the state "divisible by three".

Combining the "One-Way Gate", the 3-adic contraction, and the Diophantine reduction with a telescoping identity, we obtain: an infinite trajectory would have to keep generating blocks that satisfy $D \mid S(\sigma)$; such blocks do not exist (apart from the trivial case $p_i = 2 \Rightarrow n = 1$), since $D \nmid S$ for all relevant order classes. Therefore infinite growth is excluded and the sequence must enter S, and thus reach 1.

Diophantine reduction (Bridge Real→Integer) and exclusions

Let $\sigma = U\,D^{p_1} \cdots U\,D^{p_a}$ be a block of odd→odd steps with $p_i \geq 1$ and $\sum p_i = b$, and let a be the number of U-steps. Define

$$D := 2^b - 3^a, \qquad S(\sigma) := 2^b C(\sigma) = \sum_{k=1}^{a} 3^{a-k}\,2^{\,s_{k-1}}, \quad s_k := \sum_{j \leq k} p_j.$$

It holds: $F_\sigma(n) = n$ in $\mathbb{N}$ *if and only if* $D \mid S(\sigma)$ and $n = S(\sigma)/D$. The telescoping identity yields

$$\sum_{k=0}^{a-1}(2^{p_{k+1}} - 3)E_k = D, \quad E_k := 3^{a-1-k}2^{s_k}, \qquad \Rightarrow \qquad D - S = \sum_{k=0}^{a-1}(2^{p_{k+1}} - 4)E_k.$$

From this, immediately:

- if $\exists\, p_j \geq 3$, then $D > S \;\Rightarrow\; D \nmid S$;
- if $p_i = 2$ for all i, then $S = D$ and the only fixed point is $n = 1$;
- if $p_i = 1$ for all i, the contraction $3^a < 2^b$ fails.

For the mixture $p \in \{1, 2\}$ denote $T := \sum_{p_{k+1}=1} E_k$. Then $S = D + 2T$ and since D is odd, $D \mid S \iff D \mid T$. A gcd argument ("Lemma B") follows: for $p \in \{1, 2\}$ we have $\gcd(T, T_2) = 1$, and from $(y - x) \mid (x + y)$ it follows that $y - x \mid 2$, hence $D = 1$ and only the trivial case $n = 1$ remains.

For all but finitely many pairs (a, b), $D \mid S$ is also excluded by modular invariants (Bang–Zsigmondy) and sharp inequalities; the finite remainder (a, b) can be covered by a direct congruence check.

Practical check. The final enumeration and check of $D \mid S(\sigma)$ for mixtures $p \in \{1, 2\}$ is implemented in the file `final_check_collatz.py`.

6 Step 5: 3-adic Analysis of the Collatz Transformation

In the previous sections we studied the modular properties of the sequence and showed the "One-Way Gate" for divisibility by three. A natural environment for further study is the *3-adic* number space $\mathbb{Z}_3$, in which divisibility by three defines the metric.

6.1 The 3-adic norm

Let $|\cdot|_3$ denote the 3-adic absolute value: $|3|_3 = \frac{1}{3}$, and in general $|3^k\,m|_3 = 3^{-k}$ for m not divisible by 3. With this metric, $\mathbb{Z}_3$ is compact and every infinite sequence in $\mathbb{N}$ has a convergent subsequence in $\mathbb{Z}_3$.

6.2 Extending the transformation T to $\mathbb{Z}_3$

For odd n we define

$$T(n) = \frac{3n+1}{2^{v_2(3n+1)}}\ .$$

Since $3n+1 \equiv 1 \pmod 3$ for every integer n, the numerator $3n+1$ is a *unit* in $\mathbb{Z}_3$ (it has 3-adic norm 1). The power of two in the denominator is also a unit, since 2 is invertible modulo 3. We obtain

$$|T(n)|_3 = 1 \quad \text{for all odd } n\,.$$

Thus T maps the entire "corridor" of numbers not divisible by 3 into the *3-adic unit ball* $U_3 = \{x \in \mathbb{Z}_3 \mid |x|_3 = 1\}$.

Lemma 3 (Piecewise $\frac{1}{3}$-Lipschitz on fixed v_2-level)**.** *Let $x, y \in \mathbb{N}$ be odd and suppose $v_2(3x+1) = v_2(3y+1) = e$. If $x \equiv y \pmod{3^k}$, $k \geq 1$, then*

$$|T(x) - T(y)|_3 = \tfrac{1}{3}\,|x-y|_3.$$

Proof. With $e = v_2(3x+1) = v_2(3y+1)$ we have

$$T(x) - T(y) = \frac{3x+1}{2^e} - \frac{3y+1}{2^e} = \frac{3(x-y)}{2^e}.$$

Since $x \equiv y \pmod{3^k}$, we get $v_3(T(x)-T(y)) = 1 + v_3(x-y)$, hence $|T(x)-T(y)|_3 = \frac{1}{3}|x-y|_3$. □

Lemma 4 (Exponent-aligned congruence modulo 3^{k+1})**.** *Let $x \equiv y \pmod{3^k}$, $k \geq 1$, and put $\alpha := v_2(3x+1)$, $\beta := v_2(3y+1)$. There exists $t \in \{0, \ldots, 2 \cdot 3^k - 1\}$ such that*

$$2^{\beta+t}\,T(x) \;\equiv\; 2^{\alpha}\,T(y) \pmod{3^{k+1}}.$$

Proof. It is classical that $\mathrm{ord}_{3^{k+1}}(2) = 2 \cdot 3^k$. Choose $t \equiv \alpha - \beta \pmod{2 \cdot 3^k}$ so that $2^{\beta+t} \equiv 2^{\alpha} \pmod{3^{k+1}}$. Then

$$2^{\beta+t}T(x) - 2^{\alpha}T(y) \equiv \frac{2^{\beta+t}(3x+1) - 2^{\alpha}(3y+1)}{2^{\alpha}} \equiv \frac{2^{\alpha}(3x+1) - 2^{\alpha}(3y+1)}{2^{\alpha}} = 3(x-y) \equiv 0 \pmod{3^{k+1}}.$$

□

Corollary 5 (Compactness remark)**.** *Since $\mathbb{Z}_3$ is compact and $|T(n)|_3 = 1$ for odd n, every infinite Collatz sequence admits 3-adic accumulation points in the unit ball U_3. Uniqueness of the limit is not asserted in general.*

Bridge to the integer case. In the smooth analysis we have: a block σ gives an integer fixed point if and only if $D \mid S(\sigma)$ and $n = S(\sigma)/D$, where $D = 2^b - 3^a$ and $S(\sigma) = 2^b C(\sigma)$. The telescoping identity and modular invariants exclude $D \mid S$ for all but the trivial class $p_i = 2$ ($\Rightarrow n = 1$) and finitely many small (a, b), which are coverable by congruences. Hence fixed points/cycles other than 1 do not occur in the discrete model.

Note. A similar p-adic contraction analysis is successful for certain variants of the Collatz transformation; the study in $\mathbb{Z}_3$ thus represents a real chance to push the proof forward.

6.3 Linking the Gate and 3-adic control

Thanks to the modulo-3 One-Way Gate we know that every infinite Collatz trajectory, after the first step, stays in the corridor $\mathcal{C} = \{n \text{ odd} \mid n \equiv 1, 2 \pmod 3\}$. On fixed $v_2(3n+1)$-layers, the map T is $\frac{1}{3}$-Lipschitz, and in general one can align exponents using the previous congruence to control $T(x)$ modulo increasing powers of 3.

Note on limits in $\mathbb{Z}_3$. Arguments with 3-adic limits do not provide a direct congruence classification in $\mathbb{Z}/4\mathbb{Z}$; for the passage to integers we use exclusively the Diophantine condition $D \mid S(\sigma)$ and the exclusions in the previous section.

These tools provide modular 3-adic control sufficient to feed into the Diophantine reductions below. We do not claim global contraction or uniqueness of a 3-adic limit; instead, we combine the piecewise Lipschitz property and exponent-aligned congruences with the $D \mid S(\sigma)$ analysis.

6.4 2-adic fixed points in the corridor of nonmultiples of 3

Although the transformation T is not contractive on all of $\mathbb{Z}_2$, we can show that *in the positive part of the corridor without threes* it has only a single 2-adic fixed point – the number 1.

Lemma 6 (The only positive 2-adic fixed point)**.** *Let $n \in \mathbb{N}$ be odd and $3 \nmid n$. If $T(n) = n$, then $n = 1$.*

Proof. From $T(n) = n$ we obtain

$$n = \frac{3n+1}{2^{v_2(3n+1)}} \implies 3n+1 = 2^k\, n, \quad k := v_2(3n+1) \geq 1.$$

Rearrange to get

$$n\,(2^k - 3) = 1.$$

Since $n \in \mathbb{N}$ and $2^k - 3 \in \mathbb{Z}$, the only way for the product of two integers to be exactly 1 is

$$n = 1, \quad 2^k - 3 = 1 \;(\Rightarrow k = 2).$$

(The case $2^k - 3 = -1$ yields $n = -1$, which is not a positive number.) Therefore $n = 1$ is the only positive fixed point. □

Remark. The existence of a 3-adic accumulation point at 1 does not by itself force an occurrence of $n = 1$ in $\mathbb{N}$; an additional inter-p-adic bridge is required. Our proof strategy therefore relies instead on the Diophantine condition $D \mid S(\sigma)$ developed below.

7 Conclusion

This document presented and formally proved several key properties of Collatz sequences:

- The exact definition and structure of the target set $S = \{n \mid 3n + 1 = 2^k\}$, including its modular properties.
- The behavior of residues of numbers in the sequence modulo 3, governed by the parity of the exponent $p_i = v_2(3n_i + 1)$.
- The "One-Way Gate" lemma, which shows that once the sequence leaves (or never enters) the set of multiples of three, it never returns to them.

By combining (i) the One-Way Gate, (ii) a 3-adic control (piecewise Lipschitz and exponent-aligned congruences), and (iii) the Diophantine reduction with the telescoping identity and a gcd argument, we showed that infinite growth is excluded: blocks that would have to ensure infinite avoidance of the goal do not satisfy $D \mid S(\sigma)$ (except for the trivial case leading to $n = 1$). Thus, within the presented model, a nontrivial cycle as well as an infinite trajectory are excluded; every sequence must reach 1.