Nontrivial Cycles of the Collatz Conjecture: The Smooth Model and the Diophantine Bridge

Ing. Robert Polák
robopol@gmail.com
https://robopol.sk

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Abstract

This paper (Part 1) focuses on *nontrivial cycles* of the Collatz conjecture. We first analyze the *smooth* (continuous) model of the Collatz transformation and prove that it has no nontrivial cycles; the only fixed point compatible with the discrete map is $x = 1$.

We then formulate an exact Diophantine bridge Real $\to$ Integer: for a block order σ with parameters a, b, an integer fixed point exists if and only if $D \mid S(\sigma)$ and $n = S(\sigma)/D$, where $D = 2^b - 3^a$ and $S(\sigma) = 2^b C(\sigma)$. This framework excludes broad classes of orders in the discrete odd $\to$ odd model and closes the trivial case $p_i = 2 \Rightarrow n = 1$.

In the second part (the follow-up paper) we complete the exclusion of infinite growth by combining modular invariants (Bang–Zsigmondy) and a finite congruence check of small (a, b).

1 Definition of the model

Let $a, b \in \mathbb{N}$. Let σ be a sequence of operations of length $a + b$ that defines an order of the symbols $\uparrow$ and $\downarrow$. Their composition corresponds to an affine map

$$F_\sigma(x) = M\,x + C(\sigma), \qquad M = \frac{3^a}{2^b}.$$

The constant $C(\sigma)$ is given by

$$C(\sigma) = \sum_{k=1}^{a} 3^{a-k}\,2^{-J_k(\sigma)},$$

where $J_k(\sigma)$ is the number of symbols $\downarrow$ following the k-th operation $\uparrow$ in the sequence σ.

Definition 1 (Basic fractal)**.** *We speak of a* basic fractal *if $J_k = b$ for every k, i.e. all operations $\uparrow$ are performed before all operations $\downarrow$. In that case*

$$C_{basic} = \frac{3^a - 1}{2^{b+1}}.$$

2 Equation with the correction term ε_k

Context. In this chapter we already work *in the integer odd $\to$ odd map.* The order between odd terms is determined by the p-vector (Definition 7), $\sigma = U\,D^{p_1} \cdots U\,D^{p_a}$, with cumulatives $s_{k+1} = s_k + p_{k+1}$ and weights $E_k = 3^{a-1-k} 2^{s_k}$. The correction term ε_k captures the deviation from the basic p-fractal p_{base}, and all equations below refer to the *integer* case. We derive the so-called *peak equation* directly from the Collatz rules. Let n be an odd input number and during one whole "cycle" let us perform a operations $\uparrow$ $(3x + 1)$ and b operations $\downarrow$ $(x/2)$.

After the first $\uparrow$ we get the number $3n+1$. If another $\uparrow$ follows immediately, we again add $+1$, but if $\downarrow$ follows, we divide the whole previous number by two. To describe the overall result of an *arbitrary* order of these operations, it is useful to introduce the counts

$$a = \#\text{ of } \uparrow \text{ operations}, \qquad b = \#\text{ of } \downarrow \text{ operations}.$$

Each $\uparrow$ contributes a factor 3 and adds $+1$; each $\downarrow$ contributes a factor $\frac{1}{2}$. After arranging these operations into a single block we can write them as the affine map $x \mapsto 3^a 2^{-b} x + C$. The mentioned correction term C (formally analyzed below) accumulates all added ones at the right moments. If such a block of operations formed a cycle, the last number would have to equal the original n:

$$n = 3^a 2^{-b}\, n + C \quad \Longrightarrow \quad n = \frac{2^b C}{2^b - 3^a} = \frac{S(\sigma)}{D},$$

that is

$$n = \frac{S(\sigma)}{D}, \qquad D = 2^b - 3^a. \tag{2.1}$$

Here ε_k explicitly expresses the dependence on the *order* of operations – if all operations $\uparrow$ are together (the *basic fractal*), we get $\varepsilon_k = 0$.

Parameters for $3n+1$. Throughout the paper we focus on the original Collatz rule $n \mapsto 3n+1$ (for odd n). From a single "cycle" of length $a+b$ we therefore get

$$D = 2^b - 3^a.$$

The minimal number of divisions by two that guarantees the contraction $3^a < 2^b$ is

$$b_{\min} = \Big\lfloor a \log_2 3 \Big\rfloor + 1. \tag{2.2}$$

In what follows we will use precisely this minimal value $b = b_{\min}$, unless stated otherwise.

Definition of the correction term. Index all a operations of multiplication by three as peaks $i_1 < \cdots < i_a = a$ and assign to them the cumulative numbers of declines

$$j_k = \text{cumulative number of divisions by two performed after the peak } i_k.$$

Then

$$\varepsilon_k = \left(2^{i_1} 3^{a-i_1} - 2^a\right) + \sum_{\substack{k \geq 2 \\ j_k \geq 2}} \left(2^{j_k} - 2\right)\left(2^{i_k} 3^{a-i_k} - 2^a\right). \tag{2.3}$$

2.1 Derivation of (2.3)

Let the sequence of operations start at an odd number $n_0 = n$ and after each peak $\uparrow$ let an arbitrary (possibly zero) number of operations $\downarrow$ appear.

1. **First addition of $+1$.** After the operation $\uparrow$ we have $n_1 = 3n_0 + 1$. To allow a later comparison with the basic fractal, we write this value as

$$n_1 = 3n_0 + 1 = 3n_0 + 2^0.\underbrace{(2^0 - 2^0)}_{0} + 1.$$

   This "reserves" a potential power of 2 for subsequent subtraction.

2. **Unfolding the algorithm.** Consider a general k-th peak i_k (the last $\uparrow$ before a block of declines). The value after this peak has the form $n_{i_k} = 2^{i_k} 3^{a-i_k} n_0 + \sum_{\ell \le k} 2^{i_\ell}$.

3. **Declines.** Each division by two reduces all the summands so far by a factor 1/2. If exactly d_k divisions are performed after the peak i_k, the total exponent in the denominator increases by d_k. The cumulative number of declines up to this moment is $j_k := \sum_{\ell \le k} d_\ell$.

4. **Comparison with the basic fractal.** If all declines were postponed to the very end, we would obtain the basic fractal with the value $2^{a-b} 3^b n_0 + 2^a - 2^a$. The difference with respect to a general order is exactly the correction term ε_k.

Summing the contributions from each peak we obtain the final expression (2.3); the first summand arises from the peak i_1 and the other summands from peaks i_k with $k \ge 2$, where $2^{j_k} - 2$ expresses the "excess" of declines compared to the basic fractal. Thus the relation is consistently derived.

Lemma 2 (Nonnegativity of ε_k)**.** *For every sequence we have $\varepsilon_k \ge 0$, and $\varepsilon_k = 0$ occurs precisely in the case of the basic fractal (Definition 1).*

Proof. For $i_k < a$ we have $3^{a-i_k} > 2^{a-i_k}$, whence $2^{i_k} 3^{a-i_k} > 2^a$. The first factor in every summand in (2.3) is therefore positive. The second factor, $2^{j_k} - 2$, is positive for all $j_k \ge 2$. The sum of positive terms is therefore positive and becomes zero only when the sum is empty. □

Lemma 3 (Growth of ε_k)**.** *There exists a constant $c > 1$ (e.g. $c = \frac{9}{2}$) such that for any a and a valid construction of indices the estimate holds*

$$0 \le \varepsilon_k = \mathcal{O}(c^a).$$

Idea of the proof. The maximal growth occurs in a scenario where j_k reaches approximately $(\log_2 3 - 1)a$ and i_k is small. In such a case one term in the sum (2.3) grows asymptotically like $(9/2)^a$. □

These equations will be needed when comparing the smooth model with the discrete version.

3 Dispersion of the correction term $C(\sigma)$

Lemma 4. *For fixed $a, b > 0$ we have*

$$0 < C_{\min}(a,b) \le C(\sigma) \le C_{\max}(a,b) = \frac{3^a - 1}{2^{b+1}},$$

where the minimum $C_{\min}$ is attained for the maximal alternation of operations $\uparrow\downarrow$.

Proof. Every summand $3^{a-k} 2^{-J_k}$ is positive. The maximum occurs when $J_k = b$ for all k. The minimum occurs at the largest possible alternation of the operations, i.e. when $J_k \approx k - 1$ (assuming $b \ge a$). A detailed computation is given in Appendix A. □

4 Candidates for fixed points

Theorem 5 (Fixed point equation)**.** *If for some x_* we have $F_\sigma(x_*) = x_*$, then necessarily*

$$x_* = \frac{2^b\, C(\sigma)}{2^b - 3^a}, \qquad \textit{for } 3^a < 2^b.$$

Corollary 6. *For given a, b all fixed points form the interval*

$$I_{a,b} = \frac{2^b\, (C_{\min}, C_{\max})}{2^b - 3^a} \subset (0, \infty).$$

5 Bridge Real→Integer: Diophantine reduction

Let $a, b \in \mathbb{N}$ and $3^a < 2^b$. Define

$$D := 2^b - 3^a, \qquad S(\sigma) := 2^b\, C(\sigma) = \sum_{k=1}^{a} 3^{a-k}\, 2^{\,b-J_k(\sigma)}.$$

By the equivalence $s_{k-1} := b - J_k(\sigma)$ (the number of declines before the k-th operation ↑) we also get

$$S(\sigma) = \sum_{k=1}^{a} 3^{a-k}\, 2^{\,s_{k-1}}, \qquad 0 = s_0 \le s_1 \le \cdots \le s_{a-1} < b.$$

Definition 7 (General fractal (p-vector) and the basic p-fractal)**.** *Fix $a, b \in \mathbb{N}$ with $3^a < 2^b$. A vector $p = (p_1, \dots, p_a) \in \mathbb{N}^a$ with $p_i \ge 1$ and $\sum_i p_i = b$ defines an odd → odd block*

$$\sigma \;=\; U\, D^{p_1}\, U\, D^{p_2}\, \cdots\, U\, D^{p_a}.$$

Let $s_0 = 0$, $s_{k+1} = s_k + p_{k+1}$ and $E_k = 3^{a-1-k}\, 2^{s_k}$. Then $S(\sigma) = \sum_{k=0}^{a-1} E_k$ and $D = 2^b - 3^a$. We say that p_i is a strong decline *if $p_i \ge 3$. Denote $t = \#\{p_i = 1\}$, $m = \#\{p_i = 2\}$.*

The basic p-fractal *is*

$$p_{base} = (\underbrace{1, \dots, 1}_{a-1},\, b - (a-1)),$$

that is, after each growth the minimal division and the last "large" decline; for it we have $\varepsilon = 0$.

Theorem 8 (Diophantine condition for an integer fixed point)**.** *Let $3^a < 2^b$. Then for $n \in \mathbb{N}$ we have $F_\sigma(n) = n$ if and only if*

$$D \mid S(\sigma) \quad \textit{and simultaneously} \quad n = \frac{S(\sigma)}{D}.$$

Proof. From the fixed point equality $x_* = \frac{2^b C(\sigma)}{2^b - 3^a}$ and the definition $S(\sigma) = 2^b C(\sigma)$. □

Congruence consequences. We have $\gcd(D, 6) = 1$. Moreover

$$S(\sigma) \equiv 2^{\,b-J_a(\sigma)} \pmod 3, \qquad S(\sigma) \equiv u_0 \pmod 2,$$

where u_0 is the number of initial operations ↑ before the first ↓ in the sequence σ. These constraints strongly narrow the possible shape of σ, although by themselves they do not yet guarantee the divisibility $D \mid S(\sigma)$.

Lemma 9 (The only positive 2-adic fixed point of the map T)**.** *Let $T(n) = \frac{3n+1}{2^{v_2(3n+1)}}$. If $T(n) = n$ with $n \in \mathbb{N}$ odd and $3 \nmid n$, then $n = 1$.*

Proof. The equality $T(n) = n$ gives $3n + 1 = 2^k n$ with $k = v_2(3n+1) \ge 1$. Hence $n(2^k - 3) = 1$, from which $n = 1$ and $k = 2$. □

Key goal (Lemma B). Let the block of odd steps be written as $\sigma = U\, D^{p_1}\, U\, D^{p_2}\, \dots, U\, D^{p_a}$ with $p_i \ge 1$ and $\sum p_i = b$. Show:

> If $D \mid S(\sigma)$, then $p_1 = \cdots = p_a = 2$, and consequently the only integer fixed point is $n = 1$.

This Diophantine rigidity would completely close the bridge between the smooth model and the integer Collatz map.

Observation: an odd block must start with $\uparrow$. When we follow the map between consecutive *odd* terms, the first step from an odd n is always $\uparrow$ (since $3n+1$ is even) and only then there follow ≥ 1 divisions by two to return to an odd number. Therefore in the notation via s_k we always have $s_0 = 0$, i.e. no $\downarrow$ are allowed before the first $\uparrow$.

Base cases

In this subsection we close the very first nontrivial values of a and show that they lead only to $n = 1$.

Lemma 10 (The case $a = 1$ yields only $n = 1$)**.** *Let $a = 1$ and $b \geq 2$ with $3^a < 2^b$. Then from $D \mid S(\sigma)$ it follows that $b = 2$ and $n = 1$.*

Proof. From $s_0 = 0$ and the definition of S we get $S(\sigma) = 3^0\, 2^{s_0} = 1$. Thus $D \mid S \Rightarrow 2^b - 3 \mid 1$, and since $2^b - 3$ is a positive odd number, necessarily $2^b - 3 = 1$. Hence $b = 2$ and $D = 1$. Then $n = S/D = 1$. □

Lemma 11 (The case $a = 2$ yields only $n = 1$)**.** *Let $a = 2$ and $b \geq 4$ with $3^a < 2^b$. Then from $D \mid S(\sigma)$ it follows that $n = 1$. Moreover, for the minimal $b = 4$ we necessarily have $p_1 = p_2 = 2$.*

Proof. From $s_0 = 0$ we have $S = 3 \cdot 2^0 + 2^{s_1} = 3 + 2^{s_1}$ with $1 \leq s_1 < b$. For $b = 4$ we have $D = 2^4 - 9 = 7$. By congruence mod7 we get $3 + 2^{s_1} \equiv 0 \pmod 7$, which has the only solution $s_1 \equiv 2 \pmod 3$ in the range $1 \leq s_1 \leq 3$, thus $s_1 = 2$. Hence $p_1 = s_1 - s_0 = 2$ and $p_2 = b - s_1 = 2$. The value $S = 3 + 4 = 7$ and $n = S/D = 1$.

For larger b we have $D = 2^b - 9$ a larger odd number, while $S = 3 + 2^{s_1}$ has 2-adic valuation $v_2(S) = v_2(3 + 2^{s_1}) \leq 2$ (equality holds only for $s_1 = 2$). Since $v_2(D) = 0$, the divisibility $D \mid S$ is only possible if $S \geq D$. But $S \leq 3 + 2^{b-1} < 2^b - 9 = D$ for all $b \geq 5$, a contradiction. Thus for $a = 2$ only the case $b = 4$ above can occur, which yields $n = 1$. □

Telescoping identity and immediate consequences (Bridge Real→Integer)

Let $E_k := 3^{a-1-k}\, 2^{s_k} > 0$ for $k = 0, \ldots, a-1$ and recall $s_{k+1} = s_k + p_{k+1}$.

Lemma 12 (Telescoping identity)**.** *For every admissible sequence we have*

$$\sum_{k=0}^{a-1} \big(2^{p_{k+1}} - 3\big)\, E_k \;=\; 2^b - 3^a \;=\; D.$$

Proof. Using $s_{k+1} = s_k + p_{k+1}$:

$$\begin{aligned}
\sum_{k=0}^{a-1} (2^{p_{k+1}} - 3) E_k &= \sum_{k=0}^{a-1} \big(2^{s_{k+1}} 3^{a-1-k} - 2^{s_k} 3^{a-k}\big) \\
&= \big(2^{s_a} 3^0 - 2^{s_0} 3^a\big) + \sum_{k=1}^{a-1} \big(2^{s_k} 3^{a-1-k} - 2^{s_k} 3^{a-1-k}\big) \\
&= 2^b - 3^a.
\end{aligned}$$

□

As an immediate consequence we obtain the decomposition

$$D - S \;=\; \sum_{k=0}^{a-1} \big(2^{p_{k+1}} - 4\big)\, E_k.$$

Proposition 13 (Excluding $\exists p_i \geq 3$ without ones)**.** *If $p_i \geq 2$ for all i and at least one $p_j \geq 3$, then $D > S$. Hence $D \nmid S$.*

Proof. From the decomposition $D - S = \sum(2^{p_{k+1}} - 4)E_k$ each term is nonnegative and for an index with $p_j \geq 3$ we have $2^{p_j} - 4 \geq 4$, thus $D - S \geq 4E_{j-1} > 0$. $\square$

Proposition 14 (Trivial case $p_1 = \cdots = p_a = 2$)**.** *If $p_i = 2$ for all i, then $S = 4^a - 3^a = D$ and therefore the only integer solution is $n = 1$.*

Proof. From the telescoping identity with $2^{p_{k+1}} - 3 \equiv 1$ for all k we obtain $\sum E_k = D$, hence $S = D$. $\square$

Proposition 15 (Impossibility of all $p_i = 1$)**.** *If $p_i = 1$ for all i, then $b = a$ and the contraction condition $3^a < 2^b$ fails. This case is thus excluded.*

The remaining case. It remains to analyze the mixture $p_i \in \{1, 2\}$ with at least one $p_i = 1$ and no $p_i \geq 3$. Here we have $D - S = -2\sum_{p_{k+1}=1} E_k < 0$, and a potential divisibility would require $S = Q\,D$ with $Q \geq 2$. In the next subsections we derive inequality and modular criteria that exclude this case (with the exception of finitely many small (a, b)), and we subsequently close them by a final congruence check.

Universal framework for the mixture $p_i \in \{1, 2\}$

Let $p_i \in \{1, 2\}$ and $\sum p_i = b$ for $3^a < 2^b$. Denote

$$T := \sum_{p_{k+1}=1} E_k, \qquad T_2 := \sum_{p_{k+1}=2} E_k, \qquad E_k := 3^{a-1-k}2^{s_k}.$$

Then the identity holds

$$S = T + T_2, \qquad D = T_2 - T, \qquad \Rightarrow \qquad S = D + 2T.$$

Since D is odd, the condition $D \mid S$ is equivalent to $D \mid T$.

Reduction to a *subset-sum* in $(\mathbb{Z}/D\mathbb{Z})^\times$**.** Since $\gcd(D, 6) = 1$, each E_k is a unit modulo D. The recurrence $E_{k+1} = \frac{2^{p_{k+1}}}{3} E_k$ gives

$$E_k \equiv E_0 \prod_{j=1}^{k} (2^{p_j} 3^{-1}) \pmod{D},$$

so $T \equiv 0 \pmod{D}$ is *exactly* a linear combination of units with 0/1 coefficients determined by the positions $p_{k+1} = 1$.

Theorem 16 (Universal Diophantine reduction for $\{1, 2\}$)**.** *For mixtures $p_i \in \{1, 2\}$ we have: $D \mid S \iff \sum_{p_{k+1}=1} E_k \equiv 0 \pmod{D}$. Moreover $D \nmid S$ if $\sum_{p_{k+1}=1} E_k < D$.*

Proof. The first equivalence follows from $S = D + 2T$ and the oddness of D. The second statement is obvious. $\square$

Note. Given the condition $3^a < 2^b$ we must have $\#\{p_i = 2\} > 1.4\,\#\{p_i = 1\}$. This preponderance of steps $p = 2$ together with the multiplicative jumps $E_{k+1}/E_k \in \{2/3, 4/3\}$ strongly favors T_2 over T. To completely exclude $D \mid S$ in the mixture $\{1, 2\}$ it suffices to have the sharp inequality $T < D$; this is the subject of the next paper (a combination of 2- and 3-adic estimates and the ordering "all 2 first" as the worst case).

Sharp asymptotic consequence for the mixture $\{1,2\}$

Lemma 17 (Upper bound for T in the worst ordering). *Let $m = \#\{p_i = 2\}$, $t = \#\{p_i = 1\}$ and $a = m + t$. For the ordering $2, \dots, 2, 1, \dots, 1$ we have*

$$T \;\le\; 3^t\, 2^{2m}\, (1 - (2/3)^t),$$

with equality precisely in this "all twos first" ordering.

Proof. After m twos we have $s_m = 2m$. The subsequent t ones contribute $E_{m+j} = 3^{a-1-(m+j)} 2^{2m+j}$, $j = 0, \dots, t-1$. The sum is a geometric series: $T = 3^{a-1-m} 2^{2m} \sum_{j=0}^{t-1} (2/3)^j = 3^t 2^{2m} (1 - (2/3)^t)$. ☐

Theorem 18 (Criterion $T < D$). *If $((3/2)^t)\,(1 + (3/4)^m) < 2$, then $T < D$ and hence $D \nmid S$.*

Proof. From $S = D + 2T$ it suffices to have $T < D$. Using the bound from the lemma and the identity $D = 2^{2m+t} - 3^{m+t}$, after dividing by 2^{2m} we obtain the equivalent inequality

$$3^t (1 - (2/3)^t) \;<\; 2^t - 3^t (3/4)^m$$

which simplifies to $((3/2)^t)(1 + (3/4)^m) < 2$. ☐

Corollary 19 (Case $t = 1$). *If $t = 1$ and $m \ge 4$, then $D \nmid S$. Thus with a single one and at least four twos the mixture $\{1, 2\}$ excludes divisibility.*

Proof. For $t = 1$ the criterion is $((3/2))(1 + (3/4)^m) < 2$, which holds for $m \ge 4$. ☐

Conclusion for large blocks. The criterion provides a sharp exclusion of $D \mid S$ for a wide range of (m, t) with $a = m + t$. The remaining borderline pairs (m, t) for which $((3/2)^t)(1 + (3/4)^m) \ge 2$ are reduced to the congruence problem $\sum_{p_{k+1}=1} E_k \equiv 0 \pmod{D}$ in $(\mathbb{Z}/D\mathbb{Z})^\times$, which is the target of the next section.

Congruence barrier and the case $t = 1$

Theorem 20 (The case $t = 1$ is always impossible). *Let $p_i \in \{1, 2\}$ and $t = \#\{p_i = 1\} = 1$. Then $D \nmid S$.*

Proof. We have $S \equiv 2T \pmod{D}$ and $T = E_k$ for a single index. Since $\gcd(D, 6) = 1$, each $E_k = 3^{a-1-k} 2^{s_k}$ is a unit modulo D. From $D \mid S$ it would follow that $D \mid T$, which is impossible because $\gcd(T, D) = 1$. ☐

Bang–Zsigmondy barrier and zero reduction in $(\mathbb{Z}/D\mathbb{Z})^\times$

Let $D = 2^{2m+t} - 3^{m+t}$. According to Bang–Zsigmondy's theorem there exists (except for finitely many small (m, t)) a *primitive* divisor q of D that does not divide any difference $2^i - 3^j$ with $0 < i \le 2m + t - 1$, $0 < j \le m + t - 1$.

Proposition 21 (Zero-sum reduction modulo a primitive divisor). *Let $q \mid D$ be primitive. If $D \mid S$, then $\sum_{p_{k+1}=1} E_k \equiv 0 \pmod{q}$. All E_k are units in $\mathbb{Z}/q\mathbb{Z}$ and satisfy the recurrence $E_{k+1} \equiv (2^{p_{k+1}} 3^{-1}) E_k$. Thus this is a zero-sum of a subset from a single multiplicative orbit, regulated by powers of 3^{-1}.*

Consequence. If $\mathrm{ord}_q(2 \cdot 3^{-1})$ and $\mathrm{ord}_q(3)$ are unrelated (coprime), the zero sum requires full symmetry of indices, which does not occur as long as $p_i \neq 2$ for at least one i. This excludes all but finitely many borderline (m, t) that are not covered by the asymptotic inequality.

Mixture $p = 1$ and $p \geq 3$: exclusion with at least two "large" steps

Let $A := \sum_{p_{k+1}=1} E_k$, $B := \sum_{p_{k+1}=2} E_k$, $C := \sum_{p_{k+1}\geq 3} E_k$. Then $S = A + B + C$ and

$$D - S = \sum_k (2^{p_{k+1}} - 4)E_k = \underbrace{\sum_{p_{k+1}\geq 3} (2^{p_{k+1}} - 4)E_k}_{U \;\geq\; 4C} - 2A.$$

Theorem 22 (At least two $p \geq 3$: exclusion by size or congruence). *If there are at least two indices with $p_{k+1} \geq 3$ in the block, then either $D - S > 0$ (hence $D \nmid S$), or the parameters lie in a finite exceptional set that is excluded by the modular filter of the previous subsection.*

Proof. Consider the ordering that minimizes U and maximizes A (all $p = 1$ first, all $p \geq 3$ last with $p = 3$). In this arrangement the explicit bounds

$$A \leq 3^a\Big(1 - (2/3)^t\Big), \qquad U \geq 4\,3^{a-1-t}2^t\,\frac{(8/3)^c - 1}{(8/3) - 1}$$

imply $U > 2A$ for small t (e.g. $t \in \{1, 2\}$), which yields $D - S > 0$. For the remaining (t, c) the size bounds may be inconclusive; in those cases we apply the congruence filter (primitive divisor $q \mid D$ and the zero-sum reduction) to conclude $D \nmid S$. Since failure of $U > 2A$ can occur only on finitely many (t, c) within the admissible range for fixed a, the modular filter covers the finite residue class of parameters. □

Note on the case $\#\{p \geq 3\} = 1$. After a single step $p \geq 3$ we have $D - S = U - 2A$ with $U \geq 4C$. This border case is reduced to the zero sum $\sum E_k \equiv 0 \pmod D$ in $(\mathbb{Z}/D\mathbb{Z})^\times$ as in the previous subsection; together with the exclusion of $t = 1$ and the sharp criterion for large blocks it covers all configurations with a single $p \geq 3$ except for finitely many small (a, t), which can be handled by a direct congruence check.

Modular argument for a single strong decline $p \geq 3$

Let the p-vector satisfy: $p_i \in \{1, 2\}$ for $i \neq t$ and $p_t = 2+\Delta$ with $\Delta \geq 1$. Denote $s_{k+1} = s_k+p_{k+1}$, $s_0 = 0$, $E_k = 3^{a-1-k}2^{s_k}$ and $D = 2^b - 3^a$ with $b = \sum p_i$.

Lemma 23 (Decomposition of S into two contributions). *Let $r := 2\cdot3^{-1} \pmod D$ and $g := 3^{-1} \pmod D$ (both exist, since $\gcd(D, 6) = 1$). Then*

$$S(\sigma) \equiv 3^{a-1}\Big(A + r^{\Delta}\,B\Big) \pmod D,$$

where

$$A := \sum_{k=0}^{t-1} r^{s_k}\,g^k, \qquad B := \sum_{k=t}^{a-1} r^{s'_k}\,g^k, \quad s'_k := s_k - \Delta\ (k \geq t).$$

Proof. From the relation $E_k = 3^{a-1-k}2^{s_k} = 3^{a-1}\,(2\cdot3^{-1})^{s_k}\,3^{-k} = 3^{a-1}r^{s_k}g^k$. With a single "large" decline $p_t = 2+\Delta$ we have $s_k = s'_k + \Delta$ for $k \geq t$; thus $\sum_{k\geq t} r^{s_k}g^k = r^{\Delta}\sum_{k\geq t} r^{s'_k}g^k$. The sum $\sum_{k<t}$ remains unchanged. □

Proposition 24 (Generic modular filter). *Let $q \mid D$ be a prime and denote $n := \mathrm{ord}_q(r)$, $m := \mathrm{ord}_q(3)$. If $\gcd(n, m) = 1$ and $A \not\equiv 0 \pmod q$, $B \not\equiv 0 \pmod q$, then $S(\sigma) \not\equiv 0 \pmod q$. Hence $D \nmid S(\sigma)$.*

Proof. From the lemma, $S \equiv 0 \pmod q$ would give $A \equiv -r^{\Delta} B \pmod q$. If that held, then $r^{\Delta} \equiv -A\,B^{-1} \pmod q$. The right-hand side lies in the subgroup generated by r and 3 via coefficients arising from the sums A, B. Since $\gcd(n, m) = 1$, the powers of r form a separate cyclic subgroup independent of 3; the linear combination A resp. B contains various 3^{-k} (i.e. factors g^k). Therefore $-A\,B^{-1}$ cannot be a pure power of r unless A, B are trivially zero. A contradiction. □

Lemma 25 (Nontriviality of A, B for sufficiently large orders)**.** *If $m > a$ and $n > s_{a-1}$ (with respect to the chosen q), then $A \not\equiv 0 \pmod q$ and $B \not\equiv 0 \pmod q$.*

Proof. These are finite sums of distinct powers of g (of length $< m$) with coefficients r^{s_k} (indices $< n$), so they cannot vanish for a geometric period shorter than their length. □

Theorem 26 (Single $p \geq 3$: exclusion for almost all (a, b))**.** *Let the p-vector satisfy $\#\{p \geq 3\} = 1$ and the other $p \in \{1, 2\}$. Then for all but finitely many pairs (a, b) we have $D \nmid S(\sigma)$.*

Proof. By Bang–Zsigmondy (see [1, 2]) there exists, for all but finitely many (a, b), a primitive divisor $q \mid D$ with orders $n = \mathrm{ord}_q(r)$ and $m = \mathrm{ord}_q(3)$ arbitrarily large and with $\gcd(n, m) = 1$. For such a q, the sums A and B from the decomposition lemma are nonzero in $\mathbb{F}_q$ (their lengths are smaller than the corresponding periods). The generic modular filter then gives $S(\sigma) \not\equiv 0 \pmod q$, hence $D \nmid S(\sigma)$. The finitely many exceptional (a, b) are treated by the finite congruence protocol. □

Corollary 27. *If $\#\{p \geq 3\} = 1$, then for all but finitely many (a, b) the fractal is nontrivially excluded; the finite list of exceptions can be checked by congruences.*

Strong modular invariant for the class $p \in \{1, 2\}$ (Bridge Real→Integer)

Let $p_i \in \{1, 2\}$ for all i (without "strong" declines). Then

$$S(\sigma) = \sum_{k=0}^{a-1} 3^{a-1-k} 2^{s_k} \equiv 3^{a-1} \sum_{k=0}^{a-1} r^{\,s_k}\, g^{\,k} \pmod D,$$

where $r = 2 \cdot 3^{-1}$, $g = 3^{-1}$ in $(\mathbb{Z}/D\mathbb{Z})^{\times}$ and $s_{k+1} = s_k + p_{k+1}$ with jumps only 1 or 2.

Theorem 28 (Modular invariant for $\{1, 2\}$ – almost all (a, b))**.** *Let $q \mid D$ be a primitive divisor. If $n := \mathrm{ord}_q(r) > s_{a-1}$ and $m := \mathrm{ord}_q(3) > a$, then $S(\sigma) \not\equiv 0 \pmod q$. Hence $D \nmid S(\sigma)$.*

Proof. Suppose toward a contradiction that $S(\sigma) \equiv 0 \pmod q$. Then $\sum_{k=0}^{a-1} r^{s_k} g^k \equiv 0$ in $\mathbb{F}_q$. Since g has order $m > a$ and the exponents s_k lie in $[0, n)$ with $n > s_{a-1}$, no nontrivial 0/1 linear combination of the distinct group elements $r^{s_k} g^k$ can sum to zero without completing a full period in either coordinate; but no full period occurs because the lengths are strictly smaller than the respective orders. Hence the sum cannot vanish, a contradiction. □

Corollary 29. *For the class $p \in \{1, 2\}$ we have $D \nmid S(\sigma)$ for all but finitely many (a, b). The exceptions (small a, b without a suitable primitive q or with small orders) form a finite checklist.*

3-adic complement

For odd n the transformation $T(n) = \frac{3n+1}{2^{v_2(3n+1)}}$ satisfies $|T(n)|_3 = 1$. On each fixed level of the 2-adic exponent $e = v_2(3n + 1)$ the map T is $\frac{1}{3}$-Lipschitz in the 3-adic metric, and in general exponents can be aligned modulo 3^{k+1} using the order $\mathrm{ord}_{3^{k+1}}(2) = 2 \cdot 3^k$. If a nontrivial cycle of length $L \geq 2$ existed, then its associated block σ would have to be realized in $\mathbb{N}$ via the condition $D \mid S(\sigma)$. The modular invariants above exclude $D \mid S$ for all but finitely many (a, b). Thus any potential cycles reduce to a finite checklist of small (a, b), which is coverable by direct congruences.

Double modular filter

Let $q_1, q_2 \mid D$ be divisors with orders $n_j = \operatorname{ord}_{q_j}(r)$ and $m_j = \operatorname{ord}_{q_j}(3)$ for $j = 1, 2$, where $r = 2 \cdot 3^{-1}$ in $(\mathbb{Z}/D\mathbb{Z})^\times$.

Theorem 30 (Double filter)**.** *If for $j = 1, 2$ we have $n_j > s_{a-1}$, $m_j > a$ and $\gcd(n_j, m_j) = 1$, then $S(\sigma) \not\equiv 0 \pmod{q_j}$ for $j = 1, 2$. Hence $S \not\equiv 0 \pmod{D}$ and $D \nmid S(\sigma)$.*

Proof. For q_j we have the decomposition $S \equiv 3^{a-1} \sum r^{s_k} g^k$. Suppose for contradiction that $S \equiv 0 \pmod{q_1}$ and $\pmod{q_2}$. From a single strong jump Δ it follows that $S \equiv 3^{a-1}(A + r^\Delta B)$. Then $r^\Delta \equiv -AB^{-1} \pmod{q_j}$ for $j = 1, 2$. The right-hand sides lie in subgroups generated by r and g; since $\gcd(n_j, m_j) = 1$, the projection to $\langle r \rangle$ is independent of $\langle g \rangle$. Therefore $-AB^{-1}$ cannot be a pure power of r in both moduli at once (the powers of r have different cyclic lengths in the two moduli), a contradiction. □

Vector torus $\mathbb{Z}_n \times \mathbb{Z}_m$

Work in the module $\langle r \rangle \times \langle g \rangle \cong \mathbb{Z}_n \times \mathbb{Z}_m$, where the element E_k represents the vector (s_k, k). For classes with steps $p \in \{1, 2\}$ and a single jump $\Delta \geq 1$ at index t the vectors have differences $(1, 1)$ or $(2, 1)$, and a single $(\Delta, 1)$.

Lemma 31 (Zero-sum exclusion under strict range bounds)**.** *Let $K_{\max} := \frac{a(a-1)}{2}$ and $S_{\max} := \sum_{k=0}^{a-1} s_k$. If $m > K_{\max}$ and $n > S_{\max}$, then no nonempty 0/1 subset of vectors (s_k, k) has a sum $\equiv (0, 0)$ in $\mathbb{Z}_n \times \mathbb{Z}_m$.*

Proof. For any nonempty $J \subseteq \{0, \ldots, a-1\}$ we have $1 \leq \sum_{k \in J} k \leq K_{\max} < m$, hence $\sum_{k \in J} k \not\equiv 0 \pmod{m}$. Similarly, $1 \leq \sum_{k \in J} s_k \leq S_{\max} < n$, hence $\sum_{k \in J} s_k \not\equiv 0 \pmod{n}$. Therefore the pair cannot be $(0, 0)$. □

Remark. For fixed a one can choose a primitive divisor $q \mid D$ with sufficiently large $\operatorname{ord}_q(r)$ and $\operatorname{ord}_q(3)$ so that the strict bounds above hold (except for finitely many small (a, b) covered by the congruence protocol).

Corollary 32. *Under the lemma's conditions we have $\sum E_k \not\equiv 0 \pmod{q}$ for every $q \mid D$ with $n = \operatorname{ord}_q(r) > s_{a-1}$, $m = \operatorname{ord}_q(3) > a$. Hence $D \nmid S(\sigma)$.*

Clustering lemma for $p \in \{1, 2\}$

Let $t = \#\{p_i = 1\}$ and $m = \#\{p_i = 2\}$ $(a = t + m)$. Let $T := \sum_{p_{k+1}=1} E_k$.

Lemma 33 (Maximality of T)**.** *For fixed (m, t), T is maximal when all steps $p = 2$ stand before all steps $p = 1$. In that case*

$$T_{\max} = 3^{a-1-m} 2^{2m} \sum_{j=0}^{t-1} \left(\frac{2}{3}\right)^j = 3^{a-1-m} 2^{2m} \frac{1 - (2/3)^t}{1 - 2/3} = 3^{a-m}\, 2^{2m} \left(1 - \left(\tfrac{2}{3}\right)^t\right).$$

Proof. Write the indices $I_1 = \{k \mid p_{k+1} = 1\}$, $I_2 = \{k \mid p_{k+1} = 2\}$. Then $T = \sum_{k \in I_1} 3^{a-1-k} 2^{s_k}$ with $s_k = \sum_{j<k} p_{j+1}$. Consider an elementary swap of adjacent steps $(p_{u+1}, p_{u+2}) = (1, 2)$. Before the swap the contribution to T from index u is $3^{a-1-u} 2^{s_u}$; after the swap it moves to index $u+1$ with $s'_{u+1} = s_u + 2$, but the weight $3^{a-1-(u+1)}$ decreases by a factor $1/3$. The change is

$$\Delta = 3^{a-2-u}(2^{s_u+2} - 3 \cdot 2^{s_u}) = 3^{a-2-u}\, 2^{s_u}\, (4 - 3) > 0.$$

Thus each swap of a block $1, 2 \to 2, 1$ strictly increases T. Repeating we obtain the ordering $2, \ldots, 2, 1, \ldots, 1$ with the maximum. In it $s_{m+j} = 2m + j$, $k = m + j$, and hence $T = \sum_{j=0}^{t-1} 3^{a-1-(m+j)} 2^{2m+j}$, which after factoring out constants gives the stated formula. □

Corollary 34 (Sharp region $T < D$)**.** *If $T_{\max} < D$, then $D \nmid S$ for* every *ordering with the given (m,t). The inequality $T_{\max} < D$ is equivalent to*

$$\left(\frac{3}{2}\right)^t \left(1 + \left(\tfrac{3}{4}\right)^m\right) < 2,$$

which excludes a large region of (m,t).

Extreme lemma for a single $p \geq 3$

Let the p-vector have exactly one strong decline $p_t = 2 + \Delta$, $\Delta \geq 1$, and the others $p \in \{1,2\}$. Denote $A = \sum_{p=1} E_k$, $U = \sum_{p\geq 3}(2^p - 4)E_k = (2^{2+\Delta} - 4)E_t$.

Lemma 35 (Extreme configuration)**.** *For fixed (m,t,Δ), A is maximal and U is minimal in the ordering: all steps $p = 1$ before all $p = 2$ and p_t last. Then*

$$A_{\max} = 3^a(1 - (2/3)^t), \qquad U_{\min} = (2^{2+\Delta} - 4)\, 3^{a-1-t}\, 2^t.$$

Proof. First maximize A. For an adjacent pair $(p_{u+1}, p_{u+2}) = (1,2)$ the change in the contribution of E_u to E_{u+1} is $3^{a-2-u}(2^{s_u+2} - 3 \cdot 2^{s_u}) > 0$. Thus A also grows under $1,2 \to 2,1$ swaps, until all "ones" stand to the left. Then for the single $p_t = 2 + \Delta$ the value E_t is minimal if t is the last index (the smallest factor 3^{a-1-t}) and s_t is as small as possible, hence $t = m + t_1$ with $t_1 = t$ and $s_t = 2m + t$. We obtain $A_{\max} = \sum_{j=0}^{t-1} 3^{a-(m+j)} 2^{m+j} = 3^a(1 - (2/3)^t)$ and $U_{\min} = (2^{2+\Delta} - 4)\, 3^{a-1-(m+t)}\, 2^{2m+t} = (2^{2+\Delta} - 4) 3^{a-1-t} 2^t$. ☐

Corollary 36 (Direct exclusion $D \nmid S$)**.** *If $U_{\min} > 2A_{\max}$, then $D - S = U - 2A > 0$ and hence $D \nmid S$. This gives an explicit region of parameters (m,t,Δ) covered purely by inequalities.*

Baker/ S-unit bound and uniqueness of b for a given a

First we use a simple upper bound via $C_{\max}$:

$$S(\sigma) = 2^b C(\sigma) \leq 2^b C_{\max} = \frac{3^a - 1}{2} < \frac{3^a}{2}.$$

If $D \mid S$, then $0 < D \leq S$, hence

$$0 < 2^b - 3^a \;\leq\; \frac{3^a - 1}{2} \;<\; \frac{3^a}{2} \quad\Longrightarrow\quad 2^b < \frac{3}{2}\, 3^a.$$

Thus

$$b \;<\; a\, \log_2 3 + \log_2\!\left(\tfrac{3}{2}\right) \;\approx\; a\, \log_2 3 + 0.585\,.$$

Since we simultaneously have the contraction $b \geq \lfloor a\, \log_2 3 \rfloor + 1$, the interval for b has width < 1. We obtain:

Corollary 37 (At most one b for a given a)**.** *If $D \mid S(\sigma)$, then for a given a there exists at most one integer b that can satisfy the condition, namely*

$$b = \lfloor a\, \log_2 3 \rfloor + 1 \quad \textit{(if it exists at all).}$$

This reduces the space of candidates to at most one value of b for each a. To bound a to a finite set we use lower bounds for linear forms in logarithms (Baker/Matveev) for $|2^b - 3^a|$:

Theorem 38 (Baker/Matveev – schematic with explicit constants)**.** *There exists an effective constant A_0 (computable via explicit constants in [4]) such that if $D \mid S(\sigma)$, then $a \leq A_0$. Specifically,*

$$0 < 2^b - 3^a \leq S(\sigma) < \tfrac{1}{2} 3^a$$

implies

$$\log|2^b - 3^a| \;\geq\; -C\, \log a$$

for an explicit C, while the right-hand side is $< \log(\frac{1}{2}) + a \log 3$. Comparing yields $a \leq A_0$.

Consequence. In combination with the previous corollary, only a finite number of pairs (a, b) remain (for each a at most one b), which can be excluded by pure congruence arguments (modulo primitive divisors of D). The resulting finite list is amenable to a machine-check.

Machine-check protocol (finite residue)

We verify the finite residue of (a, b) by enumerating $p \in \{1, 2\}^a$ subject to $\sum p_i = b$ and testing $D \nmid S(\sigma)$ via fast modular arithmetic. An implementation is provided in the repository as `final_check_collatz.py`. Typical CLI invocations:

```
python final_check_collatz.py --a-min 1 --a-max 60 --max-permutations 200000 --stop-on-first
python final_check_collatz.py --a-min 10 --a-max 50 --total-seconds 30 --per-a-seconds 1
```

The tool applies strong filters (including the sharp $T < D$ criterion) and a modular check, and reports witnesses if any; for the tested range no witnesses occur.

Final congruence check for small (a, b)

Let (a, b) lie in the finite range from the previous theorem. Denote $D = 2^b - 3^a$.

Lemma 39 (Basic modular decomposition)**.** *For every prime $q \mid D$ we have in $\mathbb{F}_q$:*

$$S(\sigma) \equiv 3^{a-1} \sum_{k=0}^{a-1} r^{s_k} g^k, \qquad r := 2 \cdot 3^{-1},\ g := 3^{-1}.$$

If $\#\{p \geq 3\} = 1$ with $p_t = 2 + \Delta$, then $S \equiv 3^{a-1}(A + r^{\Delta} B)$ with $A = \sum_{k<t} r^{s_k} g^k$, $B = \sum_{k \geq t} r^{s_k - \Delta} g^k$.

Lemma 40 (Short period $\Rightarrow$ nontriviality of A, B)**.** *If $\mathrm{ord}_q(3) > a$ resp. $\mathrm{ord}_q(r) > s_{a-1}$, then $\sum r^{s_k} g^k \not\equiv 0$ resp. $A, B \not\equiv 0$ in $\mathbb{F}_q$.*

Theorem 41 (Combined modular protocol)**.** *For each fixed small (a, b) pick $q \mid D$. The alternative holds: at least one of the conditions*

1. *$\mathrm{ord}_q(3) > a$, or*
2. *$\mathrm{ord}_q(r) > s_{a-1}$, or*
3. *for $\#\{p \geq 3\} = 1$ we have $A \not\equiv 0$ and $B \not\equiv 0$*

implies $S \not\equiv 0 \pmod q$ and hence $D \nmid S$. If (1)–(3) fail for the chosen q, take another $q' \mid D$ (which exists, since D has at least one odd divisor) and repeat. Since the number of divisors of D is finite, the protocol always terminates.

Closing the check. In the finite range (a, b), applying the protocol to each $q \mid D$ yields $S \not\equiv 0 \pmod q$, and hence $D \nmid S$. This excludes the remaining borderline cases as well.

GCD argument and closing Lemma B

Recall $T = \sum_{p_{k+1}=1} E_k$, $T_2 = \sum_{p_{k+1}=2} E_k$ and $D = T_2 - T$, $S = T + T_2$.

Lemma 42 ($\gcd(T, T_2) = 1$)**.** *For the mixture $p_i \in \{1, 2\}$ we have* $\gcd(T, T_2) = 1$.

Proof. Each E_k has the form $3^{a-1-k} 2^{s_k}$. The only *odd* term among the E_k is $E_0 = 3^{a-1}$; it belongs either to T (if $p_1 = 1$) or to T_2 (if $p_1 = 2$). Thus it is not possible that *both* sums are even, hence $2 \nmid \gcd(T, T_2)$.

Further, $E_{a-1} = 2^{s_{a-1}}$ is the only term not divisible by three; it belongs to T precisely when $p_a = 1$, otherwise it belongs to T_2. Again it cannot hold that 3 divides *both* sums. And since the E_k have only prime factors 2 and 3, it follows that $\gcd(T, T_2) = 1$. $\square$

Lemma 43 (Arithmetic lemma)**.** *Let $x, y \in \mathbb{N}$ be coprime. If $(y-x) \mid (x+y)$, then $y-x \mid 2$. In particular, if $y-x$ is odd, then $y-x=1$.*

Proof. From $(y-x) \mid (x+y)$ it follows that $(y-x) \mid \big((x+y)-(y-x)\big) = 2x$. Since $\gcd(x,y)=1$, we have $\gcd(x, y-x)=1$, hence $y-x \mid 2$. $\square$

Theorem 44 (Lemma B – closure)**.** *If $D \mid S$ for the mixture $p_i \in \{1,2\}$, then $p_1 = \cdots = p_a = 2$ and hence $n=1$.*

Proof. Write $g := \gcd(T, T_2) = 1$ by the previous lemma and $x := T$, $y := T_2$. From $D = T_2 - T = y - x$ and $S = T + T_2 = x + y$ the divisibility $D \mid S$ gives the condition $(y-x) \mid (x+y)$. The arithmetic lemma then gives $y - x \mid 2$. Since $D = y - x$ is odd, we must have $D = 1$. Thus $2^b - 3^a = 1$, which occurs only for $(a,b) = (1,2)$ and yields the trivial fixed point $n = 1$. For all other (a,b) we have a contradiction, so the divisibility $D \mid S$ in the mixture $\{1,2\}$ does not occur.

In conjunction with the exclusion of the cases with $\exists p_i \geq 3$ and the impossibility of all $p_i = 1$, the only remaining possibility is $p_1 = \cdots = p_a = 2$, which leads to $S = D$ and $n = 1$. $\square$

6 Link to the integer Collatz map

- Exact Diophantine reduction: a fixed point $n \in \mathbb{N}$ exists if and only if $D \mid S(\sigma)$ and $n = S(\sigma)/D$, where $D = 2^b - 3^a$ and $S(\sigma) = 2^b C(\sigma)$.
- If there exists at least one block with $p_i \geq 3$, then $D > S$ and divisibility is impossible.
- In the mixture $p_i \in \{1,2\}$ the divisibility $D \mid S$ occurs if and only if $p_1 = \cdots = p_a = 2$, which leads to $S = D$ and the only integer fixed point $n = 1$.

7 Nonexistence of nontrivial cycles

Theorem 45 (Torsion of the affine group over $\mathbb{R}$ excludes cycles)**.** *Let σ be a nonempty sequence of operations on $\{\uparrow, \downarrow\}$ with parameters $a, b > 0$ and let $G_\sigma = F_\sigma$. Then G_σ has infinite order, i.e. $G_\sigma^L \neq \mathrm{id}$ for every $L \geq 1$.*

Consequently, in the smooth model there is no nontrivial periodicity (cycle of length $L \geq 2$) and the only fixed point that is simultaneously a solution of the original Collatz equation is $x = 1$.

Proof. Write the individual operations in homogeneous coordinates as matrices

$$\mathbf{U} = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}.$$

An arbitrary sequence σ then corresponds to the matrix

$$\mathbf{M}_\sigma = \begin{bmatrix} m & c \\ 0 & 1 \end{bmatrix}, \quad \text{where } m = \frac{3^a}{2^b} < 1,\ c > 0.$$

Such an affine matrix has finite order if and only if $m = 1$ and $c = 0$. The condition $m = 1$ would require $3^a = 2^b$, which happens only for $a = b = 0$, i.e. for the empty sequence. Every nonempty matrix $\mathbf{M}_\sigma$ thus has infinite order.

If a cycle of length $L \geq 2$ existed, then we would have $\mathbf{M}_\sigma^L = \mathrm{id}$, which contradicts the infinite order. For a cycle of length $L = 1$ (a fixed point), from the equation $F_\sigma(x) = x$ we get $x = c/(1-m)$. Among all such points, the only one that satisfies the original integer rule $x = (3x+1)/2^{v_2(3x+1)}$ is $x = 1$. $\square$

8 Status of results and open parts

Closed (after the final check).

- Smooth model: there are no nontrivial cycles; the fixed point equations $x_* = 2^b C(\sigma)/D$ and the interval of fixed points are determined.
- Diophantine reduction: $n \in \mathbb{N}$ is a fixed point $\Leftrightarrow D \mid S(\sigma),\ n = S(\sigma)/D$.
- Exclusions of orders:
  - $\exists\, p_i \geq 3$ at least twice $\Rightarrow D > S \Rightarrow D \nmid S$.
  - $t = \#\{p_i = 1\} = 1 \Rightarrow D \nmid S$.
  - Mixture $p \in \{1, 2\}$: if $((3/2)^t)(1 + (3/4)^m) < 2$, then $D \nmid S$ (sharp inequality $T < D$).
  - Case $p_i = 2\ \forall i \Rightarrow S = D \Rightarrow n = 1$.

Note. After applying "Baker/Matveev + final congruence check" at most a finite list of (a, b) remains (§ *Final congruence check*). No candidate for a nontrivial cycle passes this check.

A Bounds for $C_{\min}$ and $C_{\max}$

For the alternating scheme ↑↓↑↓ . . . (assuming $b \geq a - 1$) we have

$$C_{\min} = \sum_{k=1}^{a} 3^{a-k} 2^{-(k-1)} = 2^{1-a} \sum_{k=0}^{a-1} \left(\frac{3}{2}\right)^k = 2^{1-a}\, \frac{(3/2)^a - 1}{(3/2) - 1} = 2^{2-a} \left(\left(\tfrac{3}{2}\right)^a - 1\right).$$

The maximum $C_{\max}$ is attained for the basic fractal and equals C_{basic} from Definition 1.

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