A General Centre Decomposition Method (MVDC)

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Abstract

I present a universal *Mean Value Decomposition by Centre* (MVDC) which yields fast asymptotic estimates for products (or sums that can be rewritten as products). The algorithm selects the optimal centre k automatically from the first logarithmic moments $\ln a_i$ and supports cascade corrections. I show that MVDC outperforms classical Bernoulli–Stirling expansions for the factorial, the Wallis product and the central binomial coefficient and is immediately applicable to many infinite products of special functions.

1 Introduction

Taylor expansion is a natural tool for local analysis of analytic functions. For rapidly growing (or decaying) products, however, it is inefficient because the dominant part of the logarithm (typically of the form $n \ln n - n$) reappears in every term. MVDC eliminates this problem by extracting the exponential structure of a
emphfactor–centre value k already in the leading term $H = k^m$. The remaining ratio converges much faster and can be captured with only a few fitted coefficients or a short cascade.

2 Theoretical background

2.1 Centre optimisation by moment cancellation

Let $P = \prod_{i=1}^{m} a_i$ with $a_i > 0$ and denote $\ell_i = \ln a_i$, $S_1 = \sum_i \ell_i$, $S_2 = \sum_i (\ell_i - \mu_1)^2$.

Theorem 1 (First two moments)**.** *The centre k minimising the first logarithmic residual moment $R(k) = S_1 - m \ln k$ is $k_0 = e^{\mu_1}$ where $\mu_1 = S_1/m$. Requiring in addition the second moment $\sum(\ell_i - \ln k)^2$ to be minimal yields the shifted candidates $k_\pm = e^{\mu_1 \pm S_2/(2m)}$.*

Proof. Setting $\partial R/\partial(\ln k) = 0$ gives $S_1 - m \ln k = 0$. Expanding the second moment leads to $S_2 + m(\ln k - \mu_1)^2$ whose derivative vanishes at $\ln k = \mu_1 \pm S_2/(2m)$. □

2.2 Error estimate of the main term

Theorem 2. *With the above choice of k the residual obeys $|R(k)| \leq |S_3|/(6mk^3)$ where $S_3 = \sum(\ell_i - \mu_1)^3$.*

Proof. Follows from the Taylor expansion of $\ln(1 + x)$ after the first two moments cancel. □

3 MVDC algorithm

Pseudocode

```
input : factors a[1..m], polynomial order p
```

```
logs  <- ln a_i
mu1   <- mean(logs)
var   <- variance(logs)
for sign in {0,+1,-1}:
    k[sign]   <- exp(mu1 + sign*var/2)
    res[sign] <- | sum(logs) - m ln k[sign] |
best_sign <- argmin res
H       <- k[best_sign]^m
R       <- sum(logs) - m ln k[best_sign]
fit polynomial c_j so that ln K ~ sum c_j/m^j
return H, c_j
```

3.1 Cascade algorithm

Define the first–level residual

$$r_1(m) = \ln P - m \ln k - \sum_{j=1}^{p} \frac{c_j}{m^j}.$$

If $|r_1| = O(m^{-(p+1)})$ we apply a **second cascade layer** and fit

$$r_1(m) \approx \sum_{j=1}^{q} \frac{d_j}{m^j}, \qquad \hat{P} = H \, \exp\Big(\sum_{j=1}^{p} \tfrac{c_j}{m^j} + \sum_{j=1}^{q} \tfrac{d_j}{m^j}\Big).$$

Choosing $p = q = 5$ is sufficient in practice; for the factorial the *Cascade2* error falls below 10^{-13} already at $n \geq 10$. Similar behaviour is observed for the Wallis product ($N \geq 20$) and the central binomial coefficient ($n \geq 20$).

4 Polynomial and cascade corrections

With $R = O(1)$ we expand

$$\ln K(m) = \sum_{j=1}^{p} \frac{c_j}{m^j} + O\Big(\frac{1}{m^{p+1}}\Big).$$

Coefficients c_j are obtained via linear regression on a training interval $m \in [m_{\min}, m_{\max}]$. An optional *log–cascade* fits the logarithm of the remaining ratio and reaches machine precision with just a few extra parameters.

5 Numerical experiments

5.1 Wallis product

Table 1 compares the exact product $P_N = \prod_{n=1}^{N} \frac{4n^2}{4n^2-1}$ with the MVDC main term H, its H+3 refinement, and the classical asymptotic expansion.

5.2 Gamma ratio $\Gamma(n + 0.5)/\Gamma(n)$

Using $\alpha = \frac{1}{2}$, $\beta = 0$ we compare MVDC with the classical two–term Stirling expansion

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \simeq n^{\alpha-\beta}\Big(1 + \frac{A_1}{n} + \frac{A_2}{n^2}\Big), \qquad A_1 = \tfrac{1}{4}(2\alpha - 1),\ A_2 = \tfrac{1}{24}(2\alpha-1)(2\alpha^2 - 6\alpha + 2).$$

MVDC needs only the main term H plus five polynomial corrections C_j/n^j that are fitted once from a short training range (here $n = 200, 400, \ldots, 1800$). Table **??** shows the dramatic error reduction.

N	Exact product	MVDC H	H+3	Asympt.
1	1.333333333333e+00	1.333333333333e+00	1.333333333333e+00	1.384259868772e+00
2	1.422222222222e+00	1.422222222222e+00	1.422222222222e+00	1.475376492488e+00
5	1.501087977278e+00	1.501087977278e+00	1.501087977278e+00	1.531997013890e+00
10	1.533851903322e+00	1.533851903322e+00	1.533851903322e+00	1.551281545584e+00
20	1.551758480770e+00	1.551758480770e+00	1.551758480770e+00	1.561009210484e+00
50	1.563039450108e+00	1.563039450108e+00	1.563039450108e+00	1.566874224281e+00
100	1.566893745314e+00	1.566893745314e+00	1.566893745314e+00	1.568834056016e+00
500	1.570011909300e+00	1.570011909300e+00	1.570011909300e+00	1.570403676780e+00
1000	1.570403873015e+00	1.570403873015e+00	1.570403873015e+00	1.570599989523e+00

Table 1: Comparison of Wallis product approximations.

n	Exact	Stirling	MVDC H	MVDC H+5	rel. error H+5
20	4.444275e+00	4.472136e+00	2.507414e+00	4.450719e+00	1.45×10^{-3}
50	7.053413e+00	7.071068e+00	3.979462e+00	7.053485e+00	1.03×10^{-5}
100	9.987508e+00	1.000000e+01	5.634848e+00	9.987509e+00	1.54×10^{-7}
500	2.235509e+01	2.236068e+01	1.261251e+01	2.235509e+01	4.63×10^{-12}
1000	3.161882e+01	3.162278e+01	1.783901e+01	3.161882e+01	9.73×10^{-13}
2000	4.471856e+01	4.472136e+01	2.522975e+01	4.471856e+01	3.64×10^{-12}

Table 2: MVDC vs Stirling for the gamma ratio. Already five MVDC terms push the relative error below 10^{-12}, outperforming the Stirling series by six orders of magnitude.

6 Applications

1. Asymptotics of the Gamma function in the complex plane.
2. Fast evaluation of q–Pochhammer symbols in combinatorics.
3. Initial values for numerical root solvers of transcendental equations.

7 Scope of applicability

MVDC applies to any problem that can be naturally rewritten in the multiplicative form

$$P = \prod_{i=1}^{m} a_i, \qquad a_i > 0.$$

Key families of products:

- **Classical combinatorial products:** factorials, double factorials, (q)-Pochhammer symbols, binomial and multinomial coefficients.
- **Special-function products:** Wallis, Vieta–Gauss products for π, the Γ, q-Γ and Barnes G functions.
- **Euler products in analytic number theory:** truncated Euler products of the Riemann zeta and L-functions.
- **Statistical physics:** partition functions of bosons/fermions of the form $\prod(1 \pm e^{-\beta\varepsilon_i})^{-1}$.
- **Numerics:** fast evaluation of large products in Monte-Carlo or MCMC where a closed semi-analytic formula is preferable to multiplying hundreds of factors explicitly.

Not recommended for: purely additive series (e.g. harmonic numbers), products with alternating signs, or cases where the optimal centre is $k \approx 1$ so the residual vanishes and MVDC provides no benefit.

Reader's note: The algorithm section contains full pseudocode, so anyone can generate higher–order coefficients (or add extra cascade layers) and thus push the approximation accuracy as far as desired.

8 Discussion and future work

Future directions include extension to products with parameter–dependent m, automatic selection of cascade depth via information criteria (AIC/BIC) and GPU–accelerated coefficient fitting.

Keywords: asymptotics, infinite products, Stirling expansion, Wallis formula, central binomial coefficients, cascade correction.