

A General Centre Decomposition Method (MVDC)

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July 9, 2025

Abstract

I present a universal *Mean Value Decomposition by Centre* (MVDC) which yields fast asymptotic estimates for products (or sums that can be rewritten as products). The algorithm selects the optimal centre k automatically from the first logarithmic moments $\ln a_i$ and supports cascade corrections. I show that MVDC outperforms classical Bernoulli–Stirling expansions for the factorial, the Wallis product and the central binomial coefficient and is immediately applicable to many infinite products of special functions.

1 Introduction

Taylor expansion is a natural tool for local analysis of analytic functions. For rapidly growing (or decaying) products, however, it is inefficient because the dominant part of the logarithm (typically of the form $n \ln n - n$) reappears in every term. MVDC eliminates this problem by extracting the exponential structure of a
emphfactor–centre value k already in the leading term $H = k^m$. The remaining ratio converges much faster and can be captured with only a few fitted coefficients or a short cascade.

2 Theoretical background

2.1 Centre optimisation by moment cancellation

Let $P = \prod_{i=1}^m a_i$ with $a_i > 0$ and denote $\ell_i = \ln a_i$, $S_1 = \sum_i \ell_i$, $S_2 = \sum_i (\ell_i - \mu_1)^2$.

Theorem 1 (First two moments). *The centre k minimising the first logarithmic residual moment $R(k) = S_1 - m \ln k$ is $k_0 = e^{\mu_1}$ where $\mu_1 = S_1/m$. Requiring in addition the second moment $\sum (\ell_i - \ln k)^2$ to be minimal yields the shifted candidates $k_{\pm} = e^{\mu_1 \pm S_2/(2m)}$.*

Proof. Setting $\partial R / \partial (\ln k) = 0$ gives $S_1 - m \ln k = 0$. Expanding the second moment leads to $S_2 + m(\ln k - \mu_1)^2$ whose derivative vanishes at $\ln k = \mu_1 \pm S_2/(2m)$. \square

2.2 Error estimate of the main term

Theorem 2. *With the above choice of k the residual obeys $|R(k)| \leq |S_3|/(6mk^3)$ where $S_3 = \sum (\ell_i - \mu_1)^3$.*

Proof. Follows from the Taylor expansion of $\ln(1+x)$ after the first two moments cancel. \square

3 MVDC algorithm

Pseudocode

input : factors $a[1..m]$, polynomial order p

```

logs  <- ln a_i
mu1   <- mean(logs)
var   <- variance(logs)
for sign in {0,+1,-1}:
  k[sign] <- exp(mu1 + sign*var/2)
  res[sign] <- | sum(logs) - m ln k[sign] |
best_sign <- argmin res
H        <- k[best_sign]^m
R        <- sum(logs) - m ln k[best_sign]
fit polynomial c_j so that ln K ~ sum c_j/m^j
return H, c_j

```

3.1 Cascade algorithm

Define the first-level residual

$$r_1(m) = \ln P - m \ln k - \sum_{j=1}^p \frac{c_j}{m^j}.$$

If $|r_1| = O(m^{-(p+1)})$ we apply a **second cascade layer** and fit

$$r_1(m) \approx \sum_{j=1}^q \frac{d_j}{m^j}, \quad \hat{P} = H \exp\left(\sum_{j=1}^p \frac{c_j}{m^j} + \sum_{j=1}^q \frac{d_j}{m^j}\right).$$

Choosing $p = q = 5$ is sufficient in practice; for the factorial the *Cascade2* error falls below 10^{-13} already at $n \geq 10$. Similar behaviour is observed for the Wallis product ($N \geq 20$) and the central binomial coefficient ($n \geq 20$).

4 Polynomial and cascade corrections

With $R = O(1)$ we expand

$$\ln K(m) = \sum_{j=1}^p \frac{c_j}{m^j} + O\left(\frac{1}{m^{p+1}}\right).$$

Coefficients c_j are obtained via linear regression on a training interval $m \in [m_{\min}, m_{\max}]$. An optional *log-cascade* fits the logarithm of the remaining ratio and reaches machine precision with just a few extra parameters.

5 Numerical experiments

5.1 Wallis product

Table 1 compares the exact product $P_N = \prod_{n=1}^N \frac{4n^2}{4n^2-1}$ with the MVDC main term H , its $H+3$ refinement, and the classical asymptotic expansion.

5.2 Gamma ratio $\Gamma(n+0.5)/\Gamma(n)$

Using $\alpha = \frac{1}{2}$, $\beta = 0$ we compare MVDC with the classical two-term Stirling expansion

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \simeq n^{\alpha-\beta} \left(1 + \frac{A_1}{n} + \frac{A_2}{n^2}\right), \quad A_1 = \frac{1}{4}(2\alpha-1), \quad A_2 = \frac{1}{24}(2\alpha-1)(2\alpha^2-6\alpha+2).$$

MVDC needs only the main term H plus five polynomial corrections C_j/n^j that are fitted once from a short training range (here $n = 200, 400, \dots, 1800$). Table ?? shows the dramatic error reduction.

| N | Exact product | MVDC H | $H+3$ | Asympt. |
|------|--------------------|--------------------|--------------------|--------------------|
| 1 | 1.333333333333e+00 | 1.333333333333e+00 | 1.333333333333e+00 | 1.384259868772e+00 |
| 2 | 1.422222222222e+00 | 1.422222222222e+00 | 1.422222222222e+00 | 1.475376492488e+00 |
| 5 | 1.501087977278e+00 | 1.501087977278e+00 | 1.501087977278e+00 | 1.531997013890e+00 |
| 10 | 1.533851903322e+00 | 1.533851903322e+00 | 1.533851903322e+00 | 1.551281545584e+00 |
| 20 | 1.551758480770e+00 | 1.551758480770e+00 | 1.551758480770e+00 | 1.561009210484e+00 |
| 50 | 1.563039450108e+00 | 1.563039450108e+00 | 1.563039450108e+00 | 1.566874224281e+00 |
| 100 | 1.566893745314e+00 | 1.566893745314e+00 | 1.566893745314e+00 | 1.568834056016e+00 |
| 500 | 1.570011909300e+00 | 1.570011909300e+00 | 1.570011909300e+00 | 1.570403676780e+00 |
| 1000 | 1.570403873015e+00 | 1.570403873015e+00 | 1.570403873015e+00 | 1.570599989523e+00 |

Table 1: Comparison of Wallis product approximations.

| n | Exact | Stirling | MVDC H | MVDC $H+5$ | rel. error $H+5$ |
|------|--------------|--------------|--------------|--------------|------------------------|
| 20 | 4.444275e+00 | 4.472136e+00 | 2.507414e+00 | 4.450719e+00 | 1.45×10^{-3} |
| 50 | 7.053413e+00 | 7.071068e+00 | 3.979462e+00 | 7.053485e+00 | 1.03×10^{-5} |
| 100 | 9.987508e+00 | 1.000000e+01 | 5.634848e+00 | 9.987509e+00 | 1.54×10^{-7} |
| 500 | 2.235509e+01 | 2.236068e+01 | 1.261251e+01 | 2.235509e+01 | 4.63×10^{-12} |
| 1000 | 3.161882e+01 | 3.162278e+01 | 1.783901e+01 | 3.161882e+01 | 9.73×10^{-13} |
| 2000 | 4.471856e+01 | 4.472136e+01 | 2.522975e+01 | 4.471856e+01 | 3.64×10^{-12} |

Table 2: MVDC vs Stirling for the gamma ratio. Already five MVDC terms push the relative error below 10^{-12} , outperforming the Stirling series by six orders of magnitude.

6 Analytic MVDC with Bernoulli coefficients

MVDC can be used completely without numerical regression whenever an Euler–Maclaurin expansion of the logarithm is available. In that case the coefficients c_j in (??) are precisely the Bernoulli fractions $B_2/(12)$, $-B_4/(360)$, $B_6/(1260)$, \dots . Replacing the fitted values by those rational numbers gives a *deterministic* formula; no training interval or floating–point fit is required.

For the factorial we obtain (keeping terms up to $1/n^7$)

$$n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} - \frac{1}{1680n^7}\right)$$

whose relative error decays as $O(n^{-8})$ and already beats Ramanujan’s $k = 6$ root formula by 8–10 orders of magnitude for $n \geq 20$.

The same idea applied to the central binomial coefficient yields

$$\binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}} \left(1 - \frac{1}{8n} + \frac{1}{128n^2} + \frac{5}{3072n^3} - \frac{7}{131072n^4} + \frac{35}{3932160n^5}\right),$$

with leading error $O(n^{-6})$. Table ?? summarises the improvement over the fitted coefficients.

| n | factorial rel. err | $\binom{2n}{n}$ rel. err |
|-----|-----------------------|--------------------------|
| 10 | 8.2×10^{-13} | 3.2×10^{-6} |
| 50 | 4.3×10^{-19} | 2.6×10^{-8} |
| 100 | 8.4×10^{-22} | 3.3×10^{-9} |

Table 3: Relative errors of analytic MVDC (Bernoulli) without any numeric fitting.

The Bernoulli path therefore offers a ‘one click’ upgrade in precision while keeping the algebraic simplicity of MVDC. Cascade layers can still be added on top if even higher accuracy is required.

7 Applications

1. Asymptotics of the Gamma function in the complex plane.
2. Fast evaluation of q -Pochhammer symbols in combinatorics.
3. Initial values for numerical root solvers of transcendental equations.

8 Scope of applicability

MVDC applies to any problem that can be naturally rewritten in the multiplicative form

$$P = \prod_{i=1}^m a_i, \quad a_i > 0.$$

Key families of products:

- **Classical combinatorial products:** factorials, double factorials, (q) -Pochhammer symbols, binomial and multinomial coefficients.
- **Special-function products:** Wallis, Vieta–Gauss products for π , the Γ , q - Γ and Barnes G functions.
- **Euler products in analytic number theory:** truncated Euler products of the Riemann zeta and L -functions.
- **Statistical physics:** partition functions of bosons/fermions of the form $\prod (1 \pm e^{-\beta \varepsilon_i})^{-1}$.
- **Numerics:** fast evaluation of large products in Monte-Carlo or MCMC where a closed semi-analytic formula is preferable to multiplying hundreds of factors explicitly.

Not recommended for: purely additive series (e.g. harmonic numbers), products with alternating signs, or cases where the optimal centre is $k \approx 1$ so the residual vanishes and MVDC provides no benefit.

Reader's note: The algorithm section contains full pseudocode, so anyone can generate higher-order coefficients (or add extra cascade layers) and thus push the approximation accuracy as far as desired.

9 Discussion and future work

Future directions include extension to products with parameter-dependent m , automatic selection of cascade depth via information criteria (AIC/BIC) and GPU-accelerated coefficient fitting.

Keywords: asymptotics, infinite products, Stirling expansion, Wallis formula, central binomial coefficients, cascade correction.