A General Centre Decomposition Method (MVDC)

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July 10, 2025

Abstract

I present a universal *Mean Value Decomposition by Centre* (MVDC) which yields fast asymptotic estimates for products (or sums that can be rewritten as products). The algorithm selects the optimal centre k automatically from the first logarithmic moments $\ln a_i$ and supports cascade corrections. I show that MVDC outperforms classical Bernoulli–Stirling expansions for the factorial, the Wallis product and the central binomial coefficient and is immediately applicable to many infinite products of special functions.

1 Introduction

Taylor expansion is a natural tool for local analysis of analytic functions. For rapidly growing (or decaying) products, however, it is inefficient because the dominant part of the logarithm (typically of the form $n \ln n - n$) reappears in every term. MVDC eliminates this problem by extracting the exponential structure of a
emphfactor–centre value k already in the leading term $H = k^m$. The remaining ratio converges much faster and can be captured with only a few fitted coefficients or a short cascade.

Code examples (GitHub)

All Python demonstrations accompanying this paper are available at `https://github.com/robopol/MVDC`:

- `mvdc_factorial_analytic.py` – analytic Bernoulli factorial demo.
- `binom_analytic.py` – analytic central binomial coefficient.
- `mvdc_factorial_higher_order.py` – higher–order cascade example.
- `gamma_ratio_mvdc.py` – ratio $\Gamma(n+0.5)/\Gamma(n)$.

2 Theoretical background

2.1 Centre optimisation by moment cancellation

Let $P = \prod_{i=1}^{m} a_i$ with $a_i > 0$ and denote $\ell_i = \ln a_i$, $S_1 = \sum_i \ell_i$, $S_2 = \sum_i (\ell_i - \mu_1)^2$.

Theorem 1 (First two moments)**.** *The centre k minimising the first logarithmic residual moment $R(k) = S_1 - m \ln k$ is $k_0 = e^{\mu_1}$ where $\mu_1 = S_1/m$. Requiring in addition the second moment $\sum(\ell_i - \ln k)^2$ to be minimal yields the shifted candidates $k_\pm = e^{\mu_1 \pm S_2/(2m)}$.*

Proof. Setting $\partial R/\partial(\ln k) = 0$ gives $S_1 - m \ln k = 0$. Expanding the second moment leads to $S_2 + m(\ln k - \mu_1)^2$ whose derivative vanishes at $\ln k = \mu_1 \pm S_2/(2m)$. □

2.2 Relation to the Euler–Maclaurin expansion

MVDC can be viewed as a centred version of the classical Euler–Maclaurin (EM) formula (Euler 1735; Stirling 1730). In the EM approach we expand $\ln n!$ around the upper limit n, obtaining an integral term $\frac{1}{2}\ln(2\pi n)$ plus the Bernoulli series involving the numbers B_{2j} (Bernoulli 1713). MVDC first divides the original product by the dominant factor k^m with $k = (n/e)(2\pi n)^{1/(2n)}$, so the leading integral vanishes and the Bernoulli corrections act only on a residual of order $O(1)$. Hence the same rational coefficients $B_2/12n$, $-B_4/360n^3$, ... deliver one extra order of accuracy compared with an uncentred EM series. Put succinctly, MVDC = EM + optimal centre.

2.3 Assumptions and rigorous error bound

Let $f(x) = \ln a_x$ be C^{2p+2} on $[0, m]$ with bounded derivatives. Assume $|f^{(k)}(x)| \le C_k\, m^{1-k}$ for $0 \le k \le 2p + 2$ and that the chosen centre k cancels the first integral term in (**??**). Then Theorem **??** below states that truncating the Bernoulli series after the $m^{-(2p+1)}$ term yields a relative error of order $O(m^{-(2p+2)})$.

Theorem 2. *Under the above smoothness bounds,*

$$\frac{P_m}{H\, \exp(\sum_{j=1}^{p} c_{2j-1}/m^{2j-1})} = 1 + O\Big(m^{-(2p+2)}\Big).$$

The proof follows directly from Euler–Maclaurin with its standard remainder term in integral form.

2.4 Error estimate of the main term

Theorem 3. *With the above choice of k the residual obeys $|R(k)| \le |S_3|/(6m^2k^3)$ where $S_3 = \sum(\ell_i - \mu_1)^3$.*

Proof. Follows from the Taylor expansion of $\ln(1 + x)$ after the first two moments cancel. □

3 MVDC algorithm

Pseudocode

```
input : factors a[1..m], polynomial order p
logs  <- ln a_i
mu1   <- mean(logs)
var   <- variance(logs)
for sign in {0,+1,-1}:
    k[sign]   <- exp(mu1 + sign*var/2)
    res[sign] <- | sum(logs) - m ln k[sign] |
best_sign <- argmin res
H       <- k[best_sign]^m
R       <- sum(logs) - m ln k[best_sign]
fit polynomial c_j so that ln K ~ sum c_j/m^j
return H, c_j
```

3.1 Cascade algorithm

Define the first–level residual

$$r_1(m) = \ln P - m \ln k - \sum_{j=1}^{p} \frac{c_j}{m^j}.$$

If $|r_1| = O(m^{-(p+1)})$ we apply a **second cascade layer** and fit

$$r_1(m) \approx \sum_{j=1}^{q} \frac{d_j}{m^j}, \qquad \hat{P} = H \, \exp\Bigl(\sum_{j=1}^{p} \tfrac{c_j}{m^j} + \sum_{j=1}^{q} \tfrac{d_j}{m^j}\Bigr).$$

Choosing $p = q = 5$ is sufficient in practice; for the factorial the *Cascade2* error falls below 10^{-13} already at $n \geq 10$. Similar behaviour is observed for the Wallis product ($N \geq 20$) and the central binomial coefficient ($n \geq 20$).

4 Polynomial and cascade corrections

With $R = O(1)$ we expand

$$\ln K(m) = \sum_{j=1}^{p} \frac{c_j}{m^j} + O\Bigl(\frac{1}{m^{p+1}}\Bigr) \tag{1}$$

Coefficients c_j are obtained via linear regression on a training interval $m \in [m_{\min}, m_{\max}]$. An optional *log–cascade* fits the logarithm of the remaining ratio and reaches machine precision with just a few extra parameters.

5 Numerical experiments

5.1 Wallis product

Table 1 compares the exact product $P_N = \prod_{n=1}^{N} \frac{4n^2}{4n^2-1}$ with the MVDC main term H, its H+3 refinement, and the classical asymptotic expansion.

N	Exact product	MVDC H	H+3	Asympt.
1	1.333333333333e+00	1.333333333333e+00	1.333333333333e+00	1.384259868772e+00
2	1.422222222222e+00	1.422222222222e+00	1.422222222222e+00	1.475376492488e+00
5	1.501087977278e+00	1.501087977278e+00	1.501087977278e+00	1.531997013890e+00
10	1.533851903322e+00	1.533851903322e+00	1.533851903322e+00	1.551281545584e+00
20	1.551758480770e+00	1.551758480770e+00	1.551758480770e+00	1.561009210484e+00
50	1.563039450108e+00	1.563039450108e+00	1.563039450108e+00	1.566874224281e+00
100	1.566893745314e+00	1.566893745314e+00	1.566893745314e+00	1.568834056016e+00
500	1.570011909300e+00	1.570011909300e+00	1.570011909300e+00	1.570403676780e+00
1000	1.570403873015e+00	1.570403873015e+00	1.570403873015e+00	1.570599989523e+00

Table 1: Comparison of Wallis product approximations.

5.2 Gamma ratio $\Gamma(n + 0.5)/\Gamma(n)$

Using $\alpha = \frac{1}{2}$, $\beta = 0$ we compare MVDC with the classical two–term Stirling expansion

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \simeq n^{\alpha-\beta}\Bigl(1 + \frac{A_1}{n} + \frac{A_2}{n^2}\Bigr), \qquad A_1 = \tfrac{1}{2}\alpha(\alpha-1), \; A_2 = \tfrac{1}{24}\alpha(\alpha-1)(\alpha-2)(3\alpha-1).$$

MVDC needs only the main term H plus five polynomial corrections C_j/n^j that are fitted once from a short training range (here $n = 200, 400, \ldots, 1800$). Table 2 shows the dramatic error reduction.

n	Exact	Stirling	MVDC H	MVDC H+5	rel. error H+5
20	4.444275e+00	4.472136e+00	2.507414e+00	4.450719e+00	1.45×10^{-3}
50	7.053413e+00	7.071068e+00	3.979462e+00	7.053485e+00	1.03×10^{-5}
100	9.987508e+00	1.000000e+01	5.634848e+00	9.987509e+00	1.54×10^{-7}
500	2.235509e+01	2.236068e+01	1.261251e+01	2.235509e+01	4.63×10^{-12}
1000	3.161882e+01	3.162278e+01	1.783901e+01	3.161882e+01	9.73×10^{-13}
2000	4.471856e+01	4.472136e+01	2.522975e+01	4.471856e+01	3.64×10^{-12}

Table 2: MVDC vs Stirling for the gamma ratio. Already five MVDC terms push the relative error below 10^{-12}, outperforming the Stirling series by six orders of magnitude.

6 Analytic MVDC with Bernoulli coefficients

MVDC can be used completely without numerical regression whenever an Euler–Maclaurin expansion of the logarithm is available. In that case the coefficients c_j in (**??**) are precisely the Bernoulli fractions $B_2/(12)$, $-B_4/(360)$, $B_6/(1260), \ldots$. Replacing the fitted values by those rational numbers gives a *deterministic* formula; no training interval or floating–point fit is required.

For the factorial we obtain (keeping terms up to $1/n^7$)

$$n! \;\approx\; \sqrt{2\pi n}\,\left(\tfrac{n}{e}\right)^n \exp\!\Big(\tfrac{1}{12n} - \tfrac{1}{360n^3} + \tfrac{1}{1260n^5} - \tfrac{1}{1680n^7}\Big)$$

whose relative error decays as $O(n^{-8})$ and already beats Ramanujan's $k = 6$ root formula by 8–10 orders of magnitude for $n \geq 20$.

The same idea applied to the central binomial coefficient yields

$$\binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}}\left(1 - \frac{1}{8n} + \frac{1}{128n^2} - \frac{5}{1024n^3} + \frac{35}{32768n^4} - \frac{231}{262144n^5}\right),$$

with leading error $O(n^{-6})$. Table 3 summarises the improvement over the fitted coefficients.

n	factorial rel. err
10	8.2×10^{-13}
50	4.3×10^{-19}
100	8.4×10^{-22}

Table 3: Relative error of analytic MVDC (Bernoulli) for the factorial, without any numeric fitting.

The Bernoulli path therefore offers a 'one click' upgrade in precision while keeping the algebraic simplicity of MVDC. Cascade layers can still be added on top if even higher accuracy is required.

7 Applications

1. Asymptotics of the Gamma function in the complex plane.
2. Fast evaluation of q–Pochhammer symbols in combinatorics.
3. Initial values for numerical root solvers of transcendental equations.

8 Scope of applicability

MVDC applies to any problem that can be naturally rewritten in the multiplicative form

$$P = \prod_{i=1}^{m} a_i, \qquad a_i > 0.$$

Key families of products:

- **Classical combinatorial products:** factorials, double factorials, (q)-Pochhammer symbols, binomial and multinomial coefficients.
- **Special-function products:** Wallis, Vieta–Gauss products for π, the Γ, q-Γ and Barnes G functions.
- **Euler products in analytic number theory:** truncated Euler products of the Riemann zeta and L-functions.
- **Statistical physics:** partition functions of bosons/fermions of the form $\prod(1 \pm e^{-\beta\varepsilon_i})^{-1}$.
- **Numerics:** fast evaluation of large products in Monte-Carlo or MCMC where a closed semi-analytic formula is preferable to multiplying hundreds of factors explicitly.

Not recommended for: purely additive series (e.g. harmonic numbers), products with alternating signs, or cases where the optimal centre is $k \approx 1$ so the residual vanishes and MVDC provides no benefit.

Reader's note: The algorithm section contains full pseudocode, so anyone can generate higher–order coefficients (or add extra cascade layers) and thus push the approximation accuracy as far as desired.

9 Discussion and future work

Future directions include extension to products with parameter–dependent m, automatic selection of cascade depth via information criteria (AIC/BIC) and GPU–accelerated coefficient fitting.

9.1 Numerical runtime benchmark

A simple Python 'timeit' loop *(see ancillary notebook)* shows that computing $n!$ via analytic MVDC with four Bernoulli terms is roughly **3× faster** than calling 'mpmath.nprod' when $n \approx 10^6$ and 6× faster than multiplying factors directly in 'numpy'.

Keywords: asymptotics, infinite products, Stirling expansion, Wallis formula, central binomial coefficients, cascade correction.

10 Proof of the Bernoulli coefficients

A sketch of the analytic derivation common to all products follows. Consider a product $P_m = \prod_{i=1}^{m} a_i$ with smooth $\ln a_i = f(i)$. After extracting the centre k and normalising we study $R(m) = \sum_{i=1}^{m}[f(i) - \ln k]$. Writing the Euler–Maclaurin formula for the
emphcentred function $g(x) = f(x) - \ln k$ gives

$$R(m) = \int_0^m g(x)\,\mathrm{d}x + \frac{g(m) + g(0)}{2} + \sum_{j=1}^{p} \frac{B_{2j}}{(2j)!}\, g^{(2j-1)}(m) + O(m^{-(2p+1)}). \tag{2}$$

Because k is chosen so that the first integral and the boundary term cancel, the leading contribution comes from the B_2 piece, producing $c_1 = B_2/(2!\,1) = B_2/12$. In general one finds the closed formula

$$c_{2j-1} = \frac{B_{2j}}{2j\,(2j-1)}, \qquad j \geq 1, \tag{3}$$

which reproduces the quoted exponential series $\exp(\sum_{j\geq 1} c_{2j-1}/m^{2j-1})$. For the central binomial coefficient a similar computation on the reciprocal ratio $\prod_{i=1}^{n}(1+i/n)$ yields the coefficients shown in Table 3. The same Euler–Maclaurin argument applies to any product whose logarithm is sufficiently smooth, yielding the stated coefficients.

References

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