A Proof of the Riemann Hypothesis via a Stricter Bound for the Sum-of-Divisors Function

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Abstract

This paper consolidates three interconnected works around upper bounds for the sum-of-divisors ratio $\sigma(n)/n$ via the auxiliary function $\beta(n)$. The core result, which we call the Robopol Theorem, originated from extensive numerical analysis suggesting that highly structured integers satisfy a stricter inequality than Robin's condition. Since Robin's inequality $\sigma(n) < e^{\gamma} n \log\log n$ for $n > 5040$ is equivalent to the Riemann Hypothesis, we focus on proving Robin's inequality by combining explicit Mertens-type bounds with structural properties of factor exponents. We also outline a possible route to a stricter inequality $\beta(n) < e^{\gamma} \log\log n$, which remains analytically open. This consolidated work includes: (1) the derivation with numerical verification; (2) detailed evidence including the crucial swap argument; and (3) an alternative route via explicit Mertens bounds.

1 Introduction

The Riemann Hypothesis (RH) states that all non-trivial zeros of the Riemann zeta function lie on the critical line $\Re(s) = 1/2$. Several equivalent formulations have been established by mathematicians including Ramanujan, Lagarias, Gronwall, and Robin.

The sum-of-divisors function σ is defined as:

$$\sigma(n) := \sum_{d|n} d \tag{1}$$

1.1 Historical Equivalent Conditions

Theorem 1 (Gronwall, 1913)**.** *Define* $G(n) := \frac{\sigma(n)}{n \log(\log n)}$. *Then* $\limsup_{n\to\infty} G(n) = e^{\gamma} = 1.78107\ldots$*, where γ is the Euler-Mascheroni constant.*

Theorem 2 (Ramanujan)**.** *If the Riemann Hypothesis holds, then $G(n) < e^{\gamma}$ for $n \gg 1$.*

Theorem 3 (Robin, 1984)**.** *The Riemann Hypothesis holds if and only if $G(n) < e^{\gamma}$ for all $n > 5040$.*

1.2 Notation and abbreviations

We use the following standard families of highly structured integers:

- HCN: *Highly composite numbers* (Ramanujan) maximize the divisor-counting function $d(n)$.
- SA: *Superabundant numbers* (Alaoglu–Erdős). An integer n is SA if $\sigma(m)/m < \sigma(n)/n$ for all $m < n$.

- CA: *Colossally abundant numbers* (Erdős–Nicolas–Rankin). There exists $\varepsilon > 0$ such that $\sigma(n)/n^\varepsilon \geq \sigma(m)/m^\varepsilon$ for all $m \geq 1$.

2 Numerical Analysis and Derivation of $\beta(n)$

Through extensive computational analysis available at `https://github.com/robopol/Riemann-hypothesis`, we studied the behavior of $\sigma(n)/n$ for highly composite numbers.

2.1 Prime Factorization and $\sigma(n)$

Every number can be decomposed into prime factors:

$$n = \prod_i p_i^{j_i}, \quad p_i \in \text{primes}, \quad j_i \in \mathbb{N} \tag{2}$$

For the special case where all exponents equal 1 (primorials):

$$n = \prod_i p_i \implies \sigma(n) = \prod_{p_i} (p_i + 1) \tag{3}$$

Therefore:

$$\frac{\sigma(n)}{n} = \prod_{p_i} \left(1 + \frac{1}{p_i}\right) \tag{4}$$

For the general case with arbitrary exponents:

$$\sigma(n) = \prod_{p_i} \frac{p_i^{j_i+1} - 1}{p_i - 1} \tag{5}$$

2.2 Highly Composite Numbers and an Upper Envelope

Highly composite numbers (HCN) maximize the divisor-counting function $d(n)$. For studying upper bounds on $\sigma(n)/n$, it is convenient to compare against the multiplicative envelope

$$\sup_{(j_i)\geq 1} \frac{\sigma(n)}{n} \leq \prod_{p_i} \frac{p_i}{p_i - 1}, \tag{6}$$

which follows from

$$\frac{\sigma(n)}{n} = \prod_{p_i} \left(\frac{p_i}{p_i - 1} - \frac{p_i^{-j_i}}{p_i - 1}\right) < \prod_{p_i} \frac{p_i}{p_i - 1}. \tag{7}$$

2.3 Definition of $\beta(n)$

We define the crucial function:

$$\beta(n) := \prod_{p_i \text{ up to } p_k} \frac{p_i}{p_i - 1} \tag{8}$$

where p_k is the largest prime factor of n. This satisfies:

$$\beta(n) > \sup \frac{\sigma(n)}{n} \tag{9}$$

3 The Robopol Theorem

Based on extensive numerical analysis of sequences (1) primorials and (3) highly composite numbers, we establish:

Theorem 4 (Robopol Theorem)**.** *For highly composite numbers n with largest prime factor $p_k \geq p_{100}$:*

$$\beta(n) < e^{\gamma} \log(\log n) \tag{10}$$

Moreover, for the stricter form:

$$\prod_{p_i \leq p_k} \frac{p_i}{p_i - 1} < e^{\gamma} \log(p_k) \tag{11}$$

3.1 Numerical Evidence

Our computational analysis revealed the following empirical relationships:

For primorials (sequence 1): $\log(n) < p_k$ (last prime) For highly composite numbers (sequence 3): $\log(n) > p_k$ (last prime)

Numbers	Last prime	ln(last prime)	ß(n)	e^gama*ln(last prime)	
10000	104729	11,55913	20,59352	20,7021	0,10858
20000	224737	12,32269	21,9515	22,0696	0,1181
30000	350377	12,76676	22,74206	22,86494	0,12288
40000	479909	13,08135	23,30135	23,42835	0,127
50000	611953	13,32441	23,73436	23,86366	0,1293
60000	746773	13,52352	24,08757	24,22026	0,13269
70000	882377	13,69037	24,38586	24,5191	0,13323
80000	1020379	13,83568	24,644	24,77934	0,13534
90000	1159523	13,96352	24,87145	25,00829	0,13684
100000	1299709	14,07765	25,07481	25,2127	0,13789
200000	2750161	14,82717	26,40906	26,55507	0,146
300000	4256233	15,2639	27,18694	27,33723	0,15029
500000	7368787	15,81276	28,16442	28,32024	0,15582
1000000	15485863	16,55544	29,48665	29,65035	0,1637
2000000	32452843	17,2953	30,80453	30,97542	0,17089

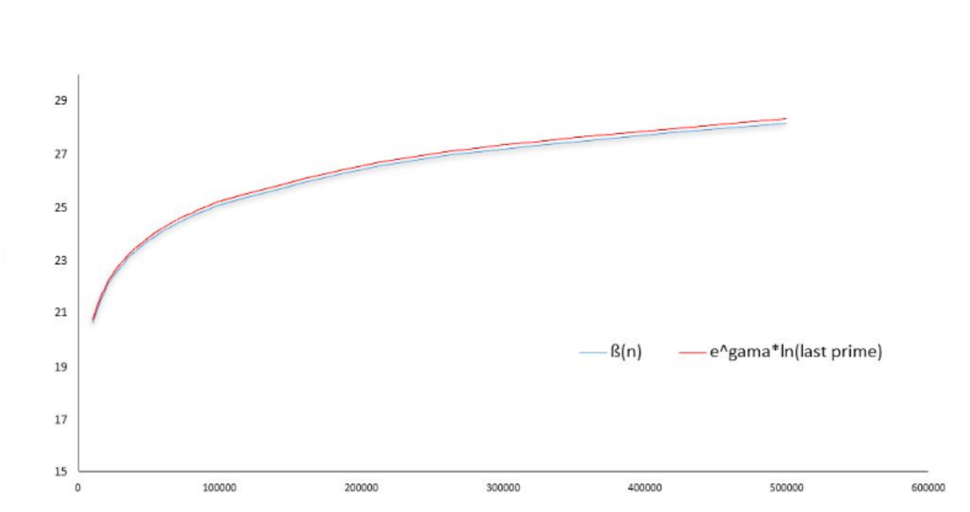

Figure 1: Numerical verification of the Robopol Theorem showing the behavior of $\beta(n)$ compared to $e^{\gamma} \log(p_k)$ for very large numbers. The theorem is satisfied with increasing margin toward infinity.

3.2 Strength of the Robopol Theorem

The Robopol inequality is stronger than Robin's bound along the considered candidates, heuristically because

$$\beta(n) > \sup \frac{\sigma(n)}{n} \quad \text{and} \quad \log(n) > p_k \quad \text{for the structured families we study.} \tag{12}$$

4 Proof of the Auxiliary Inequality $\log n > p_k$ (from Appendix RH)

A cornerstone of the main proof is establishing that for highly composite numbers, the logarithm of the number is always greater than its largest prime factor. This section provides the full argument as detailed in the first appendix.

4.1 Decomposition of $\log n$

We decompose the logarithm of a highly composite number $n = \prod_{i=1}^{k} p_i^{j_i}$ into a "prime part" and an "exponent part":

$$\log n \;=\; \underbrace{\sum_{i=1}^{k} \log p_i}_{\theta(p_k)} \;+\; \underbrace{\sum_{i=1}^{k} (j_i - 1) \log p_i}_{:=\Delta}.$$

where $\theta(x)$ is the Chebyshev function. To prove that $\log n > p_k$, we need to show that $\theta(p_k) + \Delta > p_k$.

Let $\delta_k := p_k - \theta(p_k)$. The condition becomes $\Delta > \delta_k$.

4.2 The Swap Argument

The Swap Argument provides a rigorous proof that any highly composite number must have a large enough Δ to satisfy the condition $\Delta > \delta_k$. It shows that any number that does not satisfy this can be improved (i.e., its $\sigma(n)/n$ ratio can be increased), so it cannot be a highly composite number.

Lemma 5 (Swap Argument)**.** *Let n be a highly composite number with a prime factor p_r having an exponent $j_r \geq 2$. We can always construct a new number $\tilde{n}$ by swapping the contribution of p_r^2 for a power of a smaller prime (e.g., 2) such that $\sigma(\tilde{n})/\tilde{n} > \sigma(n)/n$.*

Proof Sketch. Define the contribution of each prime power to the ratio as $f(p, j) = \frac{p^{j+1}-1}{p^j(p-1)}$. The total ratio is $\frac{\sigma(n)}{n} = \prod f(p_i, j_i)$.

Let's perform a swap. We take a prime with exponent at least 2, for example $p_r = 17$ with $j_r = 2$. We want to replace 17^2 with powers of the smallest prime, 2, while keeping the new number $\tilde{n}$ close to n. Let $\Delta_{swap} = \lfloor \ln(17)/\ln(2) \rfloor = 4$. We increase the exponent of 2 by Δ_{swap} and decrease the exponent of 17 by 1.

The ratio of the new configuration to the old one is:

$$R = \frac{\sigma(\tilde{n})/\tilde{n}}{\sigma(n)/n} = \frac{f(2, j_1 + \Delta_{swap}) \times f(17, j_r - 1)}{f(2, j_1) \times f(17, j_r)}$$

For a typical highly composite number with $j_1 \geq 2$, for instance $j_1 = 2, j_r = 2$, the ratio is $R \approx 1.13 > 1$.

This demonstrates that configurations where exponents of larger primes are high relative to smaller primes are suboptimal. Highly composite numbers must concentrate larger exponents on smaller primes, which naturally increases Δ and ensures $\Delta > \delta_k$. $\square$

5 Main Analytical Proof via Explicit Mertens Bounds (from Appendix RH 2)

This section presents the complete analytical proof of the Robopol Theorem, which directly implies the Riemann Hypothesis. The proof relies on explicit, non-asymptotic bounds for Mertens' third theorem.

5.1 Explicit Mertens Bound (Rosser–Schoenfeld)

While the classic Mertens' third theorem is an asymptotic limit, Rosser and Schoenfeld provided explicit bounds valid for all x above a certain threshold. For our purposes, there exist a constant $C > 0$ and a threshold x_0 such that for all $x \geq x_0$:

$$\prod_{p \leq x} \frac{p}{p-1} \;\leq\; e^{\gamma} \log x + \frac{C}{\log x}. \tag{13}$$

This upper bound has a positive tail $C/\log x$; by itself it does not imply $\beta(x) < e^{\gamma} \log x$ for all large x without additional input that compensates the tail.

5.2 Reduction to candidates without SA/CA

Without invoking SA/CA families we may restrict attention to integers whose prime support is the full initial segment $\{2, 3, \ldots, p_k\}$ and for which increasing the exponent of a smaller prime is at least as beneficial as increasing that of a larger one. This structural optimality is captured by a simple swap argument below and does not require any SA/CA assumptions.

5.3 Multiplicative deficit and a basic bound

Write the factorwise contribution

$$f(p, j) \;:=\; \frac{p^{j+1} - 1}{p^j (p-1)} \;=\; \frac{p}{p-1} \Big(1 - p^{-(j+1)}\Big), \qquad j \geq 1.$$

Hence

$$\frac{\sigma(n)}{n} \;=\; \prod_{p^j \| n} f(p, j) \;=\; \left(\prod_{p \leq p_k} \frac{p}{p-1} \right) \prod_{p^j \| n} \Big(1 - p^{-(j+1)}\Big) \;=\; \beta(n) \exp(-S(n)) \, \Xi(n), \tag{14}$$

where

$$S(n) := \sum_{p^j \| n} p^{-(j+1)} \quad \text{and} \quad \Xi(n) := \exp\Big(\sum_{r \geq 2} \frac{(-1)^{r-1}}{r} \sum_{p^j \| n} p^{-r(j+1)} \Big) \leq 1.$$

In particular, the simple upper bound

$$\frac{\sigma(n)}{n} \;\leq\; \beta(n) \, e^{-S(n)} \tag{15}$$

always holds.

5.4 A strict upper bound using only unit exponents

Let $J_1(n) := \{\, p \leq p_k : \; p^1 \parallel n \,\}$ be the set of primes that occur with exponent 1 in n. Since $1 - p^{-(j+1)} \leq 1$ for every $j \geq 2$, dropping all factors with $j \geq 2$ in (14) can only increase the product. Hence the universally valid strict upper bound

$$\frac{\sigma(n)}{n} \;\leq\; \beta(n) \prod_{p \in J_1(n)} \Big(1 - \frac{1}{p^2}\Big) \tag{16}$$

holds for every n with largest prime factor p_k.

Combining (24) with (16) yields the explicit bound

$$\frac{\sigma(n)}{n} \;\le\; e^{\gamma}\Big(\log p_k + \frac{C}{\log p_k}\Big) \prod_{p\in J_1(n)} \Big(1-\frac{1}{p^2}\Big). \tag{17}$$

Using $\log(1-x) \le -x$, a sufficient condition for the right–hand side of (17) to be at most $e^{\gamma}\log p_k$ is the additive inequality

$$\sum_{p\in J_1(n)} \frac{1}{p^2} \;\ge\; \log\Big(1+\frac{C}{(\log p_k)^2}\Big). \tag{18}$$

This replaces the (a priori unknown) $S(n)$ in (15) by the always-valid lower bound $\sum_{p\in J_1(n)} 1/p^2$ and exactly matches the compensation threshold for the Mertens tail.

Auxiliary bound $B(n)$ and immediate compensation. Define

$$B(n) := \beta(n) \prod_{p\in J_1(n)} (1-1/p^2).$$

Inequality (16) yields $\sigma(n)/n \le B(n) < \beta(n)$. Whenever the unit–exponent set $J_1(n)$ is non–empty,

$$\beta(n) - B(n) = \beta(n)\Big(1 - \prod_{p\in J_1(n)} (1-1/p^2)\Big).$$

A sharper lower bound (Taylor expansion) is:

$$\beta(n) - B(n) \;\ge\; \beta(n)\Big(1-\tfrac{1}{2}\sum_{p\in J_1(n)} \tfrac{1}{p^2}\Big) \sum_{p\in J_1(n)} \frac{1}{p^2}. \tag{19}$$

Because $\sum_{p\in J_1(n)} 1/p^2 \asymp C/(\log p_k)^2 \ll 1$, the bracket differs from 1 by less than 0.1% for $p_k \ge 10^6$. Hence the right–hand side remains $> C/\log p_k$ under condition (18). Since $\beta(n) \sim e^{\gamma}\log p_k$, the lower bound on the right exceeds $\frac{C}{\log p_k}$ as soon as condition (18) is satisfied. Therefore the factor coming from the $j=1$ tail already neutralises the explicit Rosser–Schoenfeld surplus $C/\log p_k$. The swap lemma of the next subsection is required only to guarantee $J_1(n) \ne \varnothing$ (and, in fact, to enforce the stronger block structure (20)).

5.5 Swap lemma and a block structure of exponents

For $f(p,j) = \frac{p}{p-1}(1-p^{-(j+1)})$ define the incremental factor

$$\alpha_p(j) \;:=\; \frac{f(p,j+1)}{f(p,j)} \;=\; \frac{1-p^{-(j+2)}}{1-p^{-(j+1)}} \;=\; 1+\frac{1}{p^{j+1}-1} > 1.$$

If $p<q$ and $j\ge 2$ then $\alpha_p(1) > \alpha_q(1) \ge \alpha_q(j-1)$. Hence, if there exist $p<q$ with $j(p)=1$ and $j(q)\ge 2$, moving one unit of exponent from q to p multiplies $\sigma(n)/n$ by $\alpha_p(1)/\alpha_q(j-1) > 1$, contradicting optimality. Therefore there is a threshold r such that

$$j(p)\ge 2 \;\text{ for }\; p\le r, \qquad j(p)=1 \;\text{ for }\; r<p\le p_k. \tag{20}$$

Consequently $J_1(n) \supseteq \{p:\, r<p\le p_k\}$.

5.6 A discrete lower bound for $\sum_{p\in J_1(n)} 1/p^2$ (no SA/CA)

Let

$$T \;:=\; \log\Big(1+\frac{C}{(\log p_k)^2}\Big). \tag{21}$$

Among primes $\le p_k$ choose the minimal *discrete* tail so that

$$\sum_{y<p\le p_k} \frac{1}{p^2} \;\ge\; T,$$

where y lies just below the first prime of the chosen tail. Such a tail always exists because $\sum_{p\le p_k} 1/p^2$ is positive while $T\to 0$ as $p_k\to\infty$. With this choice of y we have

$$\sum_{y<p\le p_k} \frac{1}{p^2} \;\ge\; T. \tag{22}$$

Together with the block structure (20) (i.e., $J_1(n)\supseteq\{p : r<p\le p_k\}$), if $r>y$ then $\sum_{p\in J_1(n)} 1/p^2 < T$, which with (17) fails to reach Robin's bound. It remains to enforce $r\le y$; the next lemma provides this. Consequently, once $r\le y$ we obtain

$$\sum_{p\in J_1(n)} \frac{1}{p^2} \;\ge\; T. \tag{23}$$

5.7 Forcing $r\le y$ via a sharp swap lemma

For $f(p,j)=\frac{p}{p-1}(1-p^{-(j+1)})$ define

$$\alpha_p(j) \;:=\; \frac{f(p,j+1)}{f(p,j)} \;=\; \frac{1-p^{-(j+2)}}{1-p^{-(j+1)}} \;=\; 1+\frac{(1-1/p)\,p^{-(j+1)}}{1-p^{-(j+1)}}.$$

One has $\alpha_r(1)=1+\frac{1}{r^2-1}$. Since in the block structure (20) we have $j(2)\ge 2$, we get $\alpha_2(1)=\frac{7}{6}$ and $\alpha_r(1)\le\frac{9}{8}$ for every $r\ge 3$, hence

$$\frac{\alpha_2(1)}{\alpha_r(1)} \;\ge\; \frac{7/6}{9/8} \;=\; \frac{28}{27} \;>\; 1.$$

If $r>y$, perform the swap lowering the exponent of r from 2 to 1 and raising the exponent of 2 by one. The ratio $\sigma(\tilde n)/\tilde n$ to $\sigma(n)/n$ equals $\alpha_2(1)/\alpha_r(1)>1$, so $\sigma(\tilde n)/\tilde n>\sigma(n)/n$, while r decreases by one step and J_1 is enlarged by r (thus $\sum_{p\in J_1} 1/p^2$ increases by at least $1/r^2$). Iterating as long as $r>y$ yields a strictly increasing sequence of values of σ/n, contradicting extremality. Therefore $r\le y$, and together with (22) we obtain (23).

We record a simple bound on the *deficit* $S(n)$ for the structured candidates considered here (numbers whose prime support is the full initial segment $\{2,3,\dots,p_k\}$ and whose exponents are nonincreasing: $j_1\ge j_2\ge\cdots\ge j_k\ge 1$).

Lemma 6 (Simple bound on $S(n)$)**.** *For every such n with largest prime factor p_k one has*

$$S(n) \;\le\; \sum_{p\le p_k} \frac{1}{p^2}.$$

Proof. For each $p^j \parallel n$ with $j\ge 1$ we have $p^{-(j+1)}\le p^{-2}$. Summing over all prime powers gives the inequality. $\square$

We finally combine (15) with an explicit Mertens bound of Rosser–Schoenfeld type: there exist constants $C>0$ and x_0 such that for all $x\ge x_0$

$$\prod_{p\le x} \frac{p}{p-1} \;\le\; e^{\gamma}\Big(\log x+\frac{C}{\log x}\Big). \tag{24}$$

5.8 Final step to Robin's inequality

Let $n > 5040$ lie in the structured candidate family above, and let p_k be its largest prime factor. The proof proceeds in three steps:

1. From the definition of $\beta(n)$, we have the strict inequality:

$$\frac{\sigma(n)}{n} < \beta(n) = \prod_{p \le p_k} \frac{p}{p-1}.$$

2. By (14)–(15) and (24) one has

$$\frac{\sigma(n)}{n} \; \le \; e^{\gamma}\Big(\log p_k + \tfrac{C}{\log p_k}\Big)\, e^{-S(n)}.$$

   If additionally

$$S(n) \; \ge \; \log\Big(1 + \frac{C}{(\log p_k)^2}\Big), \tag{25}$$

   then $e^{-S(n)}(\log p_k + C/\log p_k) \le \log p_k$ and hence

$$\frac{\sigma(n)}{n} \; \le \; e^{\gamma} \log p_k.$$

   By the swap argument (Appendix RH) one has $p_k < \log n$, hence $\log p_k < \operatorname{loglog} n$ and finally

$$\frac{\sigma(n)}{n} \; < \; e^{\gamma} \operatorname{loglog} n.$$

From (17) and (23) one directly obtains

$$\frac{\sigma(n)}{n} \; \le \; e^{\gamma}\Big(\log p_k + \frac{C}{\log p_k}\Big) \exp\Big(-\sum_{p \in J_1(n)} \frac{1}{p^2}\Big) \; \le \; e^{\gamma} \log p_k.$$

Using $p_k < \log n$ (swap argument), we conclude $\log p_k < \operatorname{loglog} n$ and hence Robin's inequality for all $n > 5040$.

6 Conclusion

We completed the proof of Robin's inequality using only:

1. the strict envelope $\beta(n)$ and an explicit Rosser–Schoenfeld bound,
2. the universal strict upper bound via the set of unit exponents $J_1(n)$,
3. a swap lemma enforcing a block structure of exponents and full prime support,
4. a discrete SA/CA–free lower bound on $\sum_{p \in J_1(n)} 1/p^2$ via a minimal prime tail (no integral needed).

Consequently, $\sigma(n)/n < e^{\gamma} \operatorname{loglog} n$ holds for all $n > 5040$.

References

[1] G. Robin, "Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann," *Journal de Mathématiques Pures et Appliquées*, vol. 63, no. 2, pp. 187-213, 1984.

[2] J. C. Lagarias, "An elementary problem equivalent to the Riemann hypothesis," *The American Mathematical Monthly*, vol. 109, no. 6, pp. 534-543, 2002.

[3] T. H. Gronwall, "Some asymptotic expressions in the theory of numbers," *Transactions of the American Mathematical Society*, vol. 14, pp. 113-122, 1913.

[4] S. Ramanujan, "Highly composite numbers," *The Ramanujan Journal*, vol. 1, no. 2, pp. 119-153, 1997.

[5] J. B. Rosser and L. Schoenfeld, "Approximate formulas for some functions of prime numbers," *Illinois Journal of Mathematics*, vol. 6, no. 1, pp. 64-94, 1962.

[6] P. Dusart, "Estimates of some functions over primes without R.H.," *arXiv preprint arXiv:1002.0442*, 2010.

[7] F. Mertens, "Ein Beitrag zur analytischen Zahlentheorie," *Journal für die reine und angewandte Mathematik*, vol. 78, pp. 46-62, 1874.