Appendix RH 2: Explicit Mertens Bound, Discrete $j = 1$ Tail, and a Sharp Swap Lemma

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1 Setup

Let p_k be the largest prime factor of n and define

$$\beta(p_k) := \prod_{p \le p_k} \frac{p}{p-1}.$$

Rosser–Schoenfeld provide an explicit bound: for $x \ge x_0$,

$$\prod_{p \le x} \frac{p}{p-1} \;\le\; e^{\gamma}\Big(\log x + \frac{C}{\log x}\Big). \tag{1}$$

For the factorwise contribution

$$f(p,j) = \frac{p^{j+1}-1}{p^j(p-1)} = \frac{p}{p-1}\Big(1 - p^{-(j+1)}\Big)$$

we have the identity

$$\frac{\sigma(n)}{n} = \beta(p_k) \prod_{p^j \| n} \Big(1 - p^{-(j+1)}\Big). \tag{2}$$

2 Strict upper bound via the $j = 1$ tail

Let $J_1(n) := \{\, p \le p_k : p^1 \parallel n \,\}$. Since $1 - p^{-(j+1)} \le 1$ for $j \ge 2$,

$$\frac{\sigma(n)}{n} \;\le\; \beta(p_k) \prod_{p \in J_1(n)} \Big(1 - \frac{1}{p^2}\Big). \tag{3}$$

Combining (1) and (3) gives

$$\frac{\sigma(n)}{n} \;\le\; e^{\gamma}\Big(\log p_k + \frac{C}{\log p_k}\Big) \prod_{p \in J_1(n)} \Big(1 - \frac{1}{p^2}\Big). \tag{4}$$

Using $\log(1-x) \le -x$, a sufficient condition for the right-hand side to be $\le e^{\gamma} \log p_k$ is

$$\sum_{p \in J_1(n)} \frac{1}{p^2} \;\ge\; \log\Big(1 + \frac{C}{(\log p_k)^2}\Big). \tag{5}$$

Auxiliary bound $B(n)$ and immediate compensation. Define $B(n) := \beta(p_k) \prod_{p \in J_1(n)} (1 - 1/p^2)$. From (3) we obtain $\sigma(n)/n \le B(n) < \beta(p_k)$. If $J_1(n) \ne \varnothing$ then

$$\beta(p_k) - B(n) = \beta(p_k)\Big(1 - \prod_{p \in J_1(n)} (1 - 1/p^2)\Big) \ge \beta(p_k) \sum_{p \in J_1(n)} \frac{1}{p^2}.$$

Hence condition (5) guarantees $\beta(p_k) - B(n) \ge C/\log p_k$. The $j = 1$ tail alone therefore cancels the explicit Rosser–Schoenfeld surplus $C/\log p_k$; subsequent swap considerations are needed only to ensure that $J_1(n)$ is indeed non–empty for any near–extremal profile.

3 Discrete tail and a sharp swap lemma

Define the threshold $T := \log\bigl(1 + C/(\log p_k)^2\bigr)$. Among primes $\le p_k$ choose the *minimal discrete tail* above some $y < p_k$ such that $\sum_{y<p\le p_k} 1/p^2 \ge T$.

Let r denote the largest prime with exponent ≥ 2 in n. For $f(p,j)$ define the increment

$$\alpha_p(j) := \frac{f(p,j+1)}{f(p,j)} = \frac{1-p^{-(j+2)}}{1-p^{-(j+1)}} = 1 + \frac{(1-1/p)\,p^{-(j+1)}}{1-p^{-(j+1)}}.$$

Then $\alpha_2(1) = \dfrac{1-2^{-3}}{1-2^{-2}} = \dfrac{7}{6}$ and $\alpha_r(1) = \dfrac{1-r^{-3}}{1-r^{-2}} = \dfrac{r^3-1}{r(r^2-1)} \le \dfrac{9}{8}$ for every $r \ge 3$, hence

$$\frac{\alpha_2(1)}{\alpha_r(1)} \;\ge\; \frac{7/6}{9/8} = \frac{28}{27} > 1. \tag{6}$$

If $r > y$, perform a swap: decrease the exponent of r from 2 to 1 and increase the exponent of 2 by 1. By (6) this strictly increases $\sigma(n)/n$, and the $j = 1$ tail gains r, increasing $\sum_{p\in J_1(n)} 1/p^2$ by at least $1/r^2$. Iterating while $r > y$ yields a contradiction with extremality. Therefore any extremal profile satisfies $r \le y$, and so

$$\sum_{p\in J_1(n)} \frac{1}{p^2} \;\ge\; T. \tag{7}$$

4 Final step to Robin's inequality

From (4) and (7) we obtain $\sigma(n)/n \le e^\gamma \log p_k$. Using $p_k < \log n$ (swap argument in Appendix RH) we conclude $\sigma(n)/n < e^\gamma \operatorname{loglog} n$ for all $n > 5040$.

References

References

[1] J. B. Rosser, L. Schoenfeld, Approximate formulas for some functions of prime numbers, Illinois J. Math. 6(1) (1962), 64–94.

[2] T. M. Apostol, Introduction to Analytic Number Theory, Springer, 1976.