Linear Algebra HW I — June 30, 2016.

1) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 0 & 4 \end{bmatrix}$ $\quad x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ $\quad b = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$

$A \cdot x = b$: $\begin{bmatrix} 1 & 1 & 2 \\ 1 & -2 & 3 \\ 1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \\ 10 \end{bmatrix}$

2) $A = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 5 & 1 \\ 6 & 1 & 0 \end{bmatrix}$ $\quad \det A = 3(0-1) - 5(0-6) + (-1)(2-30)$

$= -3 + 30 - 2 + 30$

$= 60 - 5 = 55$

$\det A \neq 0 \Rightarrow A^{-1}$ exists

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} (0-1) & (6-0) & (2-30) \\ (-1-0) & (0+6) & (30-3) \\ (5+5) & (-2-3) & 15-10 \end{bmatrix}^T$$

$$= \frac{1}{55} \begin{bmatrix} -1 & 6 & -28 \\ -1 & 6 & 27 \\ 10 & -5 & 5 \end{bmatrix}^T = \frac{1}{55} \begin{bmatrix} -1 & -1 & 10 \\ 6 & 6 & -5 \\ -28 & 27 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1}{55} & \frac{-1}{55} & 0 \\ \frac{6}{55} & \frac{6}{55} & \frac{-5}{55} \\ \frac{-28}{55} & \frac{27}{55} & \frac{5}{55} \end{bmatrix}$$

3) $I_A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad I_A \cdot A = A$.

$A = \begin{bmatrix} 3 & 5 & -1 \\ 2 & 5 & 1 \\ 6 & 1 & 0 \end{bmatrix}$ $\quad A^T = \begin{bmatrix} 3 & 2 & 6 \\ 5 & 5 & 1 \\ -1 & 1 & 0 \end{bmatrix}$

4) $(R^T R)^T$. let $A = R^T$, $B = R$ $\therefore (R^T R)^T = (AB)^T = B^T A^T$

$= R^T (R^T)^T = \boxed{R^T R}$

5) $A = \begin{bmatrix} 5 & -1 & 1 \\ 1 & -2 & 2 \\ 6 & 1 & 3 \\ 9 & 1 & 5 \end{bmatrix}$

$A \cdot x = 0$

$$\underbrace{\begin{bmatrix} 5 & -1 & 1 \\ 1 & -2 & 2 \\ 6 & 1 & 3 \\ 9 & 1 & 5 \end{bmatrix}}_{4\times 3} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{3\times 1} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{4\times 1}$$

$\Rightarrow 5x - y + z = 0$

$x - 2y + 2z = 0$

$6x + y + 3z = 0$

$9x + y + 5z = 0$

$3x + 2z = 0$

$x + 2z = 0$

$\Rightarrow 2x = 0 \Rightarrow x = 0,\ z = 0,\ y = 0$

The nullspace contains only the zero vector $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

6) $A = \begin{bmatrix} 5 & -1 & 1 & 3 \\ 6 & 1 & 3 & 4 \\ 9 & 1 & 5 & 2 \end{bmatrix} \xrightarrow{L_1/5 \to L_1} \begin{vmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 6 & 1 & 3 & 4 \\ 9 & 1 & 5 & 2 \end{vmatrix} \xrightarrow{L_2 - 6L_1 \to L_2} \begin{vmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 11/5 & 9/5 & 2/5 \\ 9 & 1 & 5 & 2 \end{vmatrix}$

$\xrightarrow{L_3 - 9L_1 \to L_3} \begin{vmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 11/5 & 9/5 & 2/5 \\ 0 & -14/5 & 16/5 & -17/5 \end{vmatrix} \xrightarrow{\frac{5}{11}L_2 \to L_2} \begin{vmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 1 & 9/11 & 2/11 \\ 0 & -14/5 & 16/5 & -17/5 \end{vmatrix}$

$\xrightarrow{-\frac{14}{5}L_2 + L_3 \to L_3} \begin{vmatrix} 1 & -1/5 & 1/5 & 3/5 \\ 0 & 1 & 9/11 & 2/11 \\ 0 & 0 & 1 & \boxed{-43/22} \end{vmatrix}$ This is in row echleon form.

a) There are 3 pivot columns
b) $\therefore$ The Rank of matrix $= 3$.
c) The # free variable $=$ # zero rows $= 0$
d) <u>Nullspace</u>

$$A \cdot x = 0$$

$$\begin{bmatrix} 5 & -1 & 1 & 3 \\ 6 & 1 & 3 & 4 \\ 9 & 1 & 5 & 2 \end{bmatrix} \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$5w - x + y + 3z = 0$
$6w + x + 3y + 4z = 0$
$9w + x + 5y + 2z = 0$

$$\begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 37/22 \\ -43/22 \\ -5/11 \end{bmatrix} \cdot w$$

Solving (see: Wolframalpha)

$\therefore$ Nullspace is the vector $\begin{bmatrix} 1 \\ 37/22 \\ -43/22 \\ -5/11 \end{bmatrix}$

7) rank of matrix A $(m \times n)$ is full $\Rightarrow$ $\gamma = m$
$\gamma = n$

a) This means the system of linear equations are linearly independent of one another.

$\therefore$ If $Ax = b$ has a solution, # solution = 1
($Ax=b$ has a unique solution when rank(A) = rank$(A|b) = n$)
where $A|b$ is the reduced row echelon form of the augmented matrix.

b) If $\gamma < m$, then:
If there is a solution, there are generally many possible solutions & free variables.
Again, the system has a solution if rank(A) = rank$(A|b)$

c) If $\gamma < n$ then, if there is a solution, there are infinitly many solutions

Case 1 rank$(A) < m < n$ $\therefore$ # equations < # unknowns
$\therefore$ If system is consistent (rank(A) = rank$(A|b)$)

Then there are ∞ solutions

Case 2 rank$(A) < n < m$ $\therefore$ # unknowns < # equations.
$\therefore$ If system is consistent, there are ∞ solutions