Homework 3 Anindrima Datta

1.1 $M_1 = \begin{bmatrix}1&0&3\\1&1&0\\1&0&3\end{bmatrix} \longrightarrow \det M_1 = 0$ $\therefore M_1$ is singular

1.2 $M_2 = \begin{bmatrix}1&0&3\\1&1&0\\1&2&1\end{bmatrix} \longrightarrow \det M_2 = 4 \Rightarrow M_2$ is invertible

2 Case $n=2$ $M = \begin{bmatrix}a_1&b_1\\0&a_2\end{bmatrix} \Rightarrow \det M = a_1a_2$

Case $n=3$ $M = \begin{bmatrix}a_1&x&b_1\\0&a_2&b_2\\0&0&a_3\end{bmatrix} \Rightarrow \det M = a_1\begin{vmatrix}a_2&b_2\\0&a_3\end{vmatrix} - x\begin{vmatrix}0&b_2\\0&a_3\end{vmatrix} + b_1\begin{vmatrix}0&a_2\\0&0\end{vmatrix}$

$= a_1\begin{vmatrix}a_1&b_1\\0&a_2\end{vmatrix} = a_1a_2a_3$

Continuing we can prove that if

$$M = \begin{bmatrix}a_1&\cdots&b_1\\0\ a_2&\cdots&b_2\\\cdots&\cdots&b_n\\0&\cdots&a_n\end{bmatrix} \Rightarrow \det M = a_1\begin{bmatrix}a_2&\cdots&b_2\\\vdots&&\\0&\cdots&a_n\end{bmatrix} + \sum 0$$

$= a_1(a_2 \ldots a_n)$ (by recursion)

$= a_1 \ldots a_n$

Case A $\det A \neq 0$

3. Nonsingular matrix is product of elementary matrix

$\therefore A = E_1 \dots E_t$

Lemma: $\det(A.A) = \det(E_1 \dots E_t . A) = \det(E_1) \dots \det(E_t) \det(A)$

Proof: We know,

1) If A is a square matrix & B is a matrix obtained by swapping rows or colums of A then $\det(A) = \det(B)$

2) If A is square matrix & B is a matrix obtained by adding a scalar multiple of a row/column to another row/column, then $\det(A) = \det(B)$

Using this $\det(E_k A) = \det(E_k) \det(A)$

$$\therefore \det(A.A) = \det(E_1) \dots \det(E_t) \det(A) = \det(E_1 \dots E_t) \det(A) = \det^2(A)$$

Case 2 $\det A = 0 \Rightarrow A$ is singular $\Rightarrow A.B$ is singular $(\forall B)$
$\Rightarrow A^2$ is singular $\Rightarrow \det(A^2) = 0 = \det(A)$

4) Using 1) & 2) from lemma above, any square matrix A can be transformed into a UTM such that $\det(A) = \det(M) = m_1 \dots m_n$ where $M = \begin{vmatrix} m_1 & x \\ 0 & m_n \end{vmatrix}$

Using the same transformation $\det(2A) = \det(2M) = (2m_1) \dots (2m_n)$

$$= 2^n (m_1 \dots m_n) = 2^n \det(A)$$

5. $A \cdot x = \lambda \cdot x$

$$A^2 \cdot x = A \cdot (A \cdot x) = A(\lambda x) = \lambda(A \cdot x) = \lambda(\lambda \cdot x)$$
$$= \lambda^2 \cdot x$$

$\therefore$ eigenvalues of A^2 are λ^2 & eigenvectors are x.

6. $Q^T = Q^{-1}$ let $Q = \begin{bmatrix} q_{11} & \cdots & q_{1n} \\ q_{m1} & \cdots & q_{mn} \end{bmatrix}$

$$(Q Q^T)_{ij} = \sum_{k=1}^{m} q_{ik} q_{kj} =$$

(1) $\|q_i\|^2 = \sum_{j=1}^{m} (QQ^T)_{ij} = \sum_{j=1}^{m} I_{ij} = 1$

2) $q_i^T q_j = \sum_{k=1}^{m} q_{ik} q_{kj} = (QQ^T)_{ij} = I_{ij} = \begin{cases} 0 \text{ if } i \neq j \\ 1 \text{ if } i = j \end{cases}$

Since given $i \neq j \Rightarrow q_i^T q_j = 0$

3) $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $QQ^T = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \cos^2\theta + \sin^2\theta \end{bmatrix}$

$= I_2$

7. $M = \begin{vmatrix} 0.3 & 0.5 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.4 & 0.1 & 0.7 \end{vmatrix}$

$$[x \; y \; z] \begin{bmatrix} 0.3 & 0.5 & 0.1 \\ 0.3 & 0.4 & 0.2 \\ 0.4 & 0.1 & 0.7 \end{bmatrix} = [x \; y \; z)$$

$x + y + z = 1$

$0.3x + 0.3y + 0.4z = x$

$0.5x + 0.4y + 0.1z = y$

$0.1x + 0.2y + 0.7z = z$

$\Rightarrow \quad 3x + 3y + 4z = 10x \quad \Rightarrow \quad 7x - 3y - 4z = 0$

$5x + 4y + z = 10y \quad\quad 5x - 6y + z = 0$

$x + 2y + 7z = 10z \quad\quad x + 2y - 3z = 0$

$15x - 18y + 3z = 0$

$16x - 16y = 0$

$\Rightarrow x = y$

$-y + z = 0 \quad \Rightarrow \quad y = z$

$x = y = z = 1/3$

$\therefore$ Steady state of M is $[1/3, 1/3, 1/3]$

8. $M = \begin{vmatrix} 4 & 3 & 2 \\ 1 & 2 & 6 \\ 5 & 8 & 1 \end{vmatrix}$

$\therefore \begin{vmatrix} 4-\lambda & 3 & 2 \\ 1 & 2-\lambda & 6 \\ 5 & 8 & 1-\lambda \end{vmatrix} = 0$

$\Rightarrow (4-\lambda)\begin{vmatrix} 2-\lambda & 6 \\ 8 & 1-\lambda \end{vmatrix} - 3 \begin{vmatrix} 1 & 6 \\ 5 & 1-\lambda \end{vmatrix} + 2 \begin{vmatrix} 1 & 2-\lambda \\ 5 & 8 \end{vmatrix}$

$\Rightarrow 4-\lambda\left[(2-\lambda)(1-\lambda) - 42\right] - 3(1-\lambda-30) + 2(8-5(2-\lambda))$

$= (4-\lambda)\left[2-3\lambda+\lambda^2-42\right) - 3(1-\lambda-30) + 2(8-10+5\lambda)$

$= 8 - 2\lambda - 12\lambda + 3\lambda^2 + 4\lambda^2 - \lambda^3 - 84 + 42\lambda - 3 + 3\lambda + 90$

$+16 - 20 + 10\lambda$

$\Rightarrow -\lambda^3 + 7\lambda^2 + 47\lambda - 101$

$\lambda = \begin{Bmatrix} 1.792828 \\ -5.34086 \\ 10.548032 \end{Bmatrix}$

Case 1 $\lambda = 1.792828$

The equations are $4x_1 + 3x_2 + 2x_3 = 1.7928x_1$

$x_1 + 2x_2 + 6x_3 = 1.79..x_2$

$5x_1 + 8x_2 + x_3 = 1.79..x_3$

This records to $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -0.548449 \\ 0.53469985 \\ -0.63554801 \end{bmatrix}$

Case 2

$\lambda = -5.34086$ $\begin{bmatrix} -0.69903633 \\ 0.12092837 \\ 0.7710746 \end{bmatrix}$

Case 3

$\lambda = 10.548032$ $\begin{bmatrix} -0.46313273 \\ -0.83634467 \\ 0.03902155 \end{bmatrix}$