Oleksandr Romanko, Ph.D.
Senior Research Analyst, Risk Analytics, Watson Financial Services, IBM Canada
Adjunct Professor, University of Toronto

MIE1624H – Introduction to Data Science and Analytics

Lecture 4 – Linear Algebra and Matrix Computations

University of Toronto
October 9, 2018

Lecture outline

Matrix computations

- Matrix operations
- Computing determinants and eigenvalues

Linear algebra

- Solving systems of linear equations
- Solving non-linear equations (Bisection method, Newton’s method)
- Solving systems of non-linear equations
- Solving unconstrained non-linear optimization problems

Derivatives

- Gradients and Hessians
- Taylor series expansion

Functions and convexity

- Convex and concave functions
- Checking convexity
- Properties of convex functions

Systems of linear equations

- System of linear equations

$$\begin{aligned} 3 \cdot x_1 + 8 \cdot x_2 &= 46 \\ 10 \cdot x_1 - 7 \cdot x_2 &= -15 \end{aligned}$$

- To solve this system of equations we express one of the variables through the other from one of the equations and plug into the other equation:

$$x_2 = \frac{46 - 3 \cdot x_1}{8}$$

$$10 \cdot x_1 - 7 \cdot \frac{46 - 3 \cdot x_1}{8} = -1.5$$

- Therefore $x_1 = 2,\ x_2 = 5$
- Matrix notation:

$$A = \begin{pmatrix} 3 & 8 \\ 10 & -7 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} 46 \\ -15 \end{pmatrix}, \quad \boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- System of linear equations in matrix form:

$$A \cdot \boldsymbol{x} = \boldsymbol{b}$$

Gaussian elimination

System of equations	Row operations	Augmented matrix
$\begin{aligned} 2x + y - z &= 8 \\ -3x - y + 2z &= -11 \\ -2x + y + 2z &= -3 \end{aligned}$		$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ -3 & -1 & 2 & -11 \\ -2 & 1 & 2 & -3 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ 2y + z &= 5 \end{aligned}$	$L_2 + \frac{3}{2}L_1 \to L_2$ $L_3 + L_1 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 2 & 1 & 5 \end{array}\right]$
$\begin{aligned} 2x + y - z &= 8 \\ \frac{1}{2}y + \frac{1}{2}z &= 1 \\ -z &= 1 \end{aligned}$	$L_3 + -4L_2 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & -1 & 8 \\ 0 & 1/2 & 1/2 & 1 \\ 0 & 0 & -1 & 1 \end{array}\right]$
The matrix is now in echelon form (also called triangular form)		
$\begin{aligned} 2x + y &= 7 \\ \frac{1}{2}y &= \frac{3}{2} \\ -z &= 1 \end{aligned}$	$L_2 + \frac{1}{2}L_3 \to L_2$ $L_1 - L_3 \to L_1$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1/2 & 0 & 3/2 \\ 0 & 0 & -1 & 1 \end{array}\right]$
$\begin{aligned} 2x + y &= 7 \\ y &= 3 \\ z &= -1 \end{aligned}$	$2L_2 \to L_2$ $-L_3 \to L_3$	$\left[\begin{array}{ccc\|c} 2 & 1 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$
$\begin{aligned} x &= 2 \\ y &= 3 \\ z &= -1 \end{aligned}$	$L_1 - L_2 \to L_1$ $\frac{1}{2}L_1 \to L_1$	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{array}\right]$

Gaussian elimination

Back substitution