Attitude estimation using an IMU

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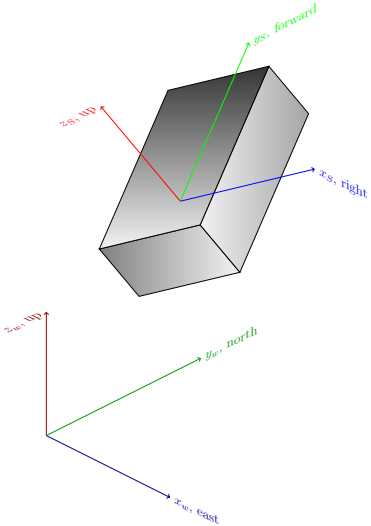
The problem



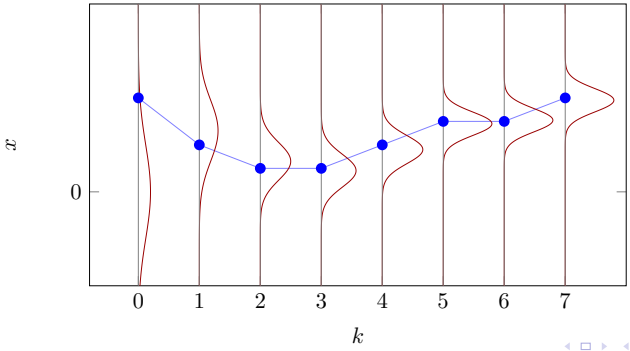
Find the orientation of the IMU (cell phone in our case) with respect to the world fixed frame!

$$v^W = Rv^S, \qquad v^S = R^T v^W$$

The Kalman Filter

$$x_{k+1} = Fx_k + Bu_k + v_k$$
$$y_k = Hx_k + e_k$$

The (filtering) KF provides the pdf $p(x_k|y_{1:k})$. Linear model and Gaussian noise leads to gaussian pdf, completely specified by mean $\hat{x}_{k|k}$ and error covariance $P_{k|k}$.



The KF algorithm

Prediction step / time update

$$\hat{x}_{k+1|k} = F\hat{x}_{k|k} + Bu_k \quad \text{predicted state estimate}$$
$$P_{k+1|k} = FP_{k|k}F^T + Q_k \quad \text{predicted error covariance}$$

Measurement update

$$\tilde{y}_k = y_k - H\hat{x}_{k|k-1} \quad \text{innovation}$$
$$S_k = HP_{k|k-1}H^T + R_k \quad \text{innovation covariance}$$
$$K_k = P_{k|k-1}H^T S_k^{-1} \quad \text{the Kalman gain}$$
$$x_{k|k} = x_{k|k-1} + K_k\tilde{y}_k \quad \text{updated state estimate}$$
$$P_{k|k} = (I - K_kH)P_{k|k-1} \quad \text{Updated error covariance}$$

The Extended Kalman Filter

Useful for *nonlinear* state-space models

$$x_{k+1} = f(x_k, u_k) + v(k)$$
$$y_k = h(x_k) + e(k)$$

The (filtering) KF provides an *approximation* of the pdf $p(x_k|y_{1:k})$.

The EKF algorithm

Prediction step / time update

$$F_k = \frac{\partial}{\partial x} f(x,u)|_{x=x_k,u=u_k} \qquad B_k = \frac{\partial}{\partial u} f(x,u)|_{x=x_k,u=u_k}$$

$$\hat{x}_{k+1|k} = f(\hat{x}_{k|k}, u_k) \quad \text{predicted state estimate, using nonlinear model}$$

$$P_{k+1|k} = F_k P_{k|k} F_k^T + Q_k \quad \text{predicted error covariance}$$

Measurement update

$$H_k = \frac{\partial}{\partial x} h(x)|_{x=x_k}$$

$$\tilde{y}_k = y_k - h(\hat{x}_{k|k-1}) \quad \text{innovation, using nonlinear model}$$

$$S_k = H_k P_{k|k-1} H_k^T + R_k \quad \text{innovation covariance}$$

$$K_k = P_{k|k-1} H_k^T S_k^{-1} \quad \text{the Kalman gain}$$

$$x_{k|k} = x_{k|k-1} + K_k \tilde{y}_k \quad \text{updated state estimate}$$

$$P_{k|k} = (I - K_k H_k) P_{k|k-1} \quad \text{Updated error covariance}$$

Describing orientations using unit quaternions

The unit quaternion $q = \begin{bmatrix} q_0 & q_1 & q_2 & q_3 \end{bmatrix}^T$, can represent a 3D rotation of angle α about the direction $v = \begin{bmatrix} v_x & v_y & v_z \end{bmatrix}^T$

$$q = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} \cos(\frac{1}{2}\alpha) \\ \sin(\frac{1}{2}\alpha) \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \cos(\frac{1}{2}\alpha) \\ \sin(\frac{1}{2}\alpha) v \end{bmatrix}$$

From quaternion to rotation matrix

There are straight-forward formulas to convert a unit quaternion to a rotation matrix and vice versa

$$R = Q(q), \qquad q = Q^{-1}(R),$$

where

$$Q(q) = \begin{bmatrix} 2q_0^2 - 1 + 2q_1^2 & 2q_1q_2 - 2q_0q_3 & 2q_1q_3 + 2q_0q_2 \\ 2q_1q_2 + 2q_0q_3 & 2q_0^2 - 1 + 2q_2^2 & 2q_2q_3 - 2q0q_1 \\ 2q_1q_3 - 2q_0q_2 & 2q_2q_3 + 2q_0q_1 & 2q_0^2 - 1 + 2q_3^2 \end{bmatrix}$$

The time-derivative of a unit quaternion

$$\dot{q} = \frac{1}{2} S(\omega) q, \quad \text{where}$$

$$S(\omega) = \begin{bmatrix} 0 & -\omega_x & -\omega_y & -\omega_z \\ \omega_x & 0 & \omega_z & -\omega_y \\ \omega_y & -\omega_z & 0 & \omega_x \\ \omega_z & \omega_y & -\omega_x & 0 \end{bmatrix}$$

The state-vector

There are a two main alternatives

1. $$x_k = q_k$$

2. $$x_k = \begin{bmatrix} q_k \\ w_k \end{bmatrix},$$

   where w_k is the gyro bias.

Integrating angular velocities

The orientation quaternion can be updated from gyro measurements, ω_k

$$q_{k+1} = \mathrm{e}^{\frac{1}{2}S(\omega_k)h} q_k$$

$$\underbrace{q_{k+1}}_{x_{k+1}} = \underbrace{\left(\cos(\frac{||\omega_k||h}{2})I + \frac{h}{2} \cdot \frac{\sin(\frac{||\omega_k||h}{2})}{\frac{||\omega_k||h}{2}} S(\omega_k)\right)}_{F_k} \underbrace{q_k}_{x_k} + v_k,$$

where h is the sampling period (which need not be constant).

The accelerometer measurement

The accelerometer measures both the gravitational acceleration g^0 and the acceleration of the IMU itself wrt the earth, a_k^f, called the *specific force*.

$$y_k^a = R^T(g^0 + a_k^f) + e_k^a = Q^T(q_k)\big(g^0 + a_k^f\big) + e_k^a$$

Assuming negligable specific force, $a_k^f \approx 0$:

$$y_k^a = \underbrace{Q^T(q_k)g^0}_{h_a(x)} + e_k^a$$

For the EKF we need the derivative of $h^a(x)$ wrt x

$$H_k^a = \frac{\partial}{\partial x} h^a(x)|_{x_k} = \begin{bmatrix} H_{k,0}^a & H_{k,1}^a & H_{k,2}^a & H_{k,3}^a \end{bmatrix},$$

where

$$H_{k,0}^a = \left(\frac{\partial}{\partial q_0} Q^T(q)\right)\Big|_{q_k} g^0, \; H_{k,1}^a = \left(\frac{\partial}{\partial q_1} Q^T(q)\right)\Big|_{q_k} g^0, \text{ etc.}$$

The magnetometer measurement

The magnetometer measures the local magnetic field (hopefully dominated by the earth's magnetic field)

$$y_k^m = R^T(m^0) + e_k^m = \underbrace{Q^T(q)m^0}_{h^m(x)} + e_k^m$$

$$H_k^m = \frac{\partial}{\partial x} h^m(x)|_{x_k} = \begin{bmatrix} H_{k,0}^m & H_{k,1}^m & H_{k,2}^m & H_{k,3}^m \end{bmatrix},$$

where

$$H_{k,0}^m = \left(\frac{\partial}{\partial q_0} Q^T(q)\right)\Big|_{q_k} m^0, \; H_{k,1}^m = \left(\frac{\partial}{\partial q_1} Q^T(q)\right)\Big|_{q_k} m^0, \text{ etc.}$$

Hands-on

Set-up Wireless IMU

Plot some data

Implement time update

Implement measurement updates