Robust Kalman filter

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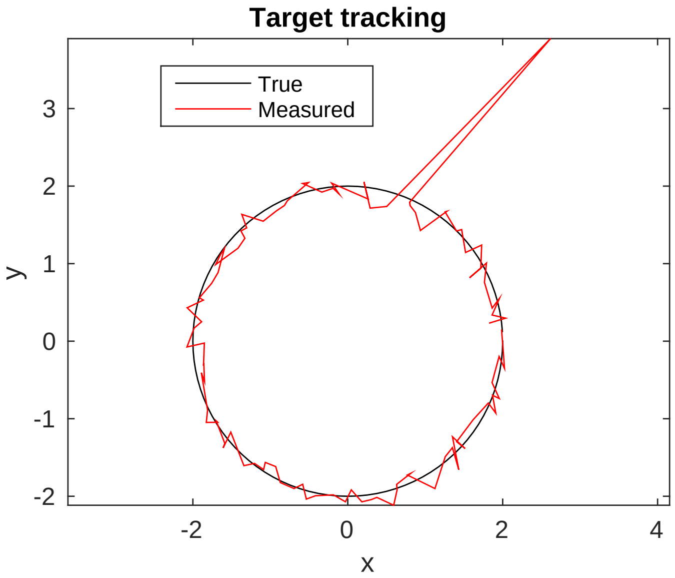
Why a robust version of the Kalman filter?

Why a robust version of the Kalman filter?

The Kalman filter assumes Gaussian measurement noise and so it is very sensitive to outliers.

Example 1

The target moves in a circle. Observations are noisy with one outlier



Example 1 contd.

The model of the dynamics: *Nearly constant velocity model*

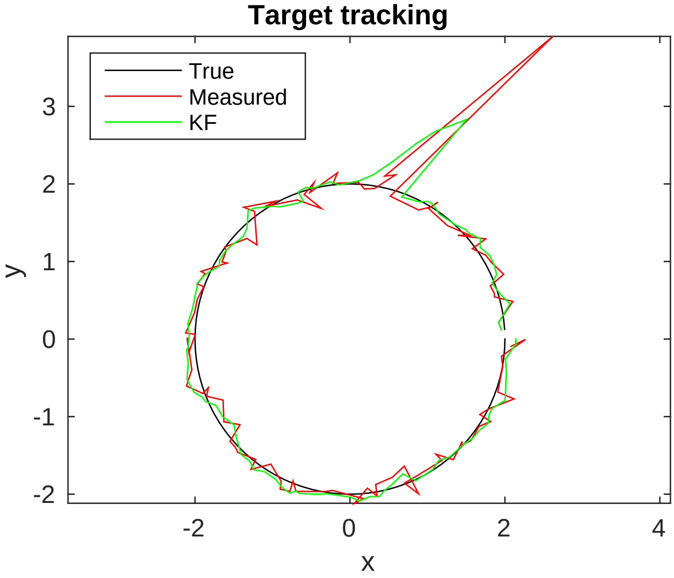
$$x(k+1) = \begin{bmatrix} I & hI \\ 0 & I \end{bmatrix} x(k) + \begin{bmatrix} \frac{h^2}{2} I \\ hI \end{bmatrix} v(k),$$

where the state vector contains the position and velocity of the target

$$x = \begin{bmatrix} p \\ \dot{p} \end{bmatrix}.$$

Example 1 contd.

Result of tracking using standard Kalman filter



Convex optimization

Convex Optimization

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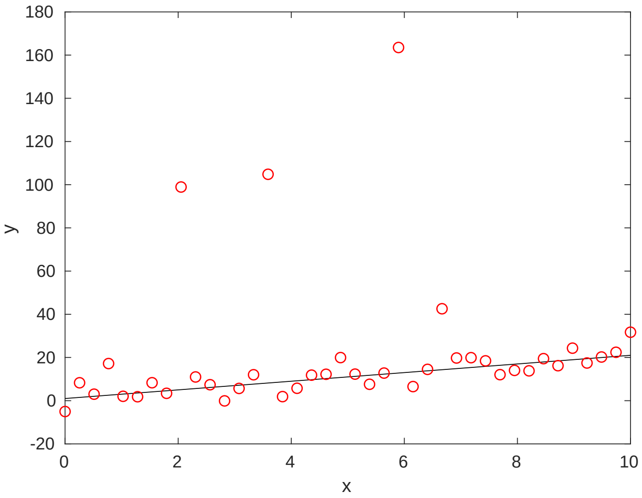
Preparation example

Linear regression model

$$y(k) = ax(k) + b + e(k) + w(k),$$

where $e(k)$ is Gaussian noise and $w(k)$ is a sparse vector of outliers.



Preparation example, contd

Least squares estimation:

$$\text{minimize } ||y - ax - b||_2$$

Or, equivalently

$$\begin{aligned} &\text{minimize } ||\epsilon||_2 \\ &\text{subject to } \epsilon = y - ax - b \end{aligned}$$

Preparation example, contd

Least squares estimation:

$$\text{minimize } ||y - ax - b||_2$$

Solved by forming

$$A = \begin{bmatrix} x(1) & 1 \\ x(2) & 1 \\ \vdots & \vdots \\ x(N) & 1 \end{bmatrix}$$

and

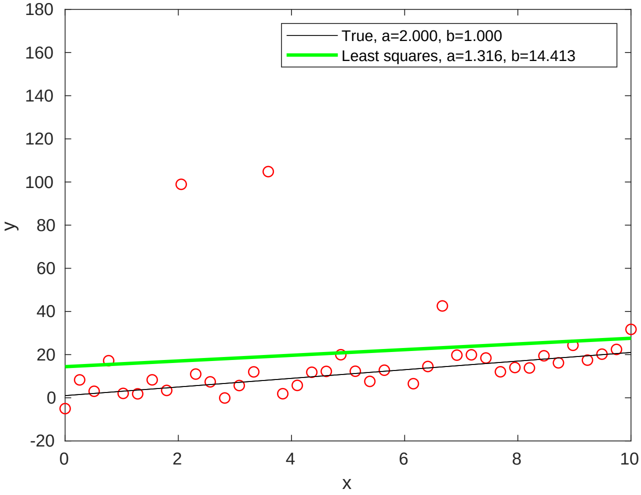
$$z = \begin{bmatrix} a \\ b \end{bmatrix},$$

and solving for z in the (over-determined) system of equations

$$Az = y.$$

Preparation example, contd

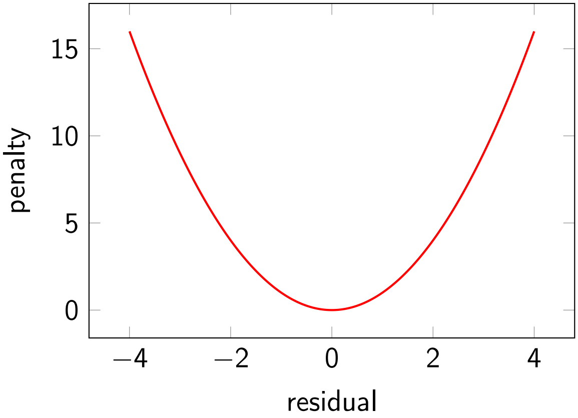
$$\text{minimize } ||y - ax - b||_2$$



The problem with least squares

$$\text{minimize} \sum_k \phi_S(\epsilon_k)$$

where $\phi_S(u) = u^2$

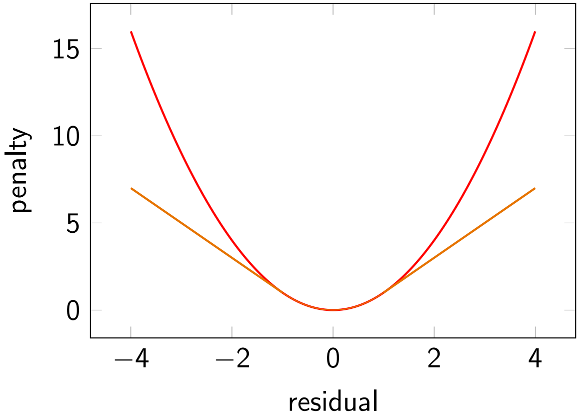


More robust: The Huber penalty function

Also known as robust least squares

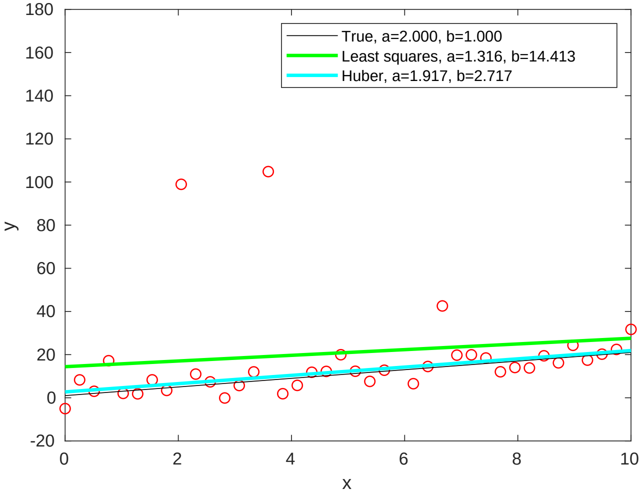
$$\text{minimize} \sum_k \phi_{hub}(\epsilon_k)$$

where $\phi_{hub}(u) = \begin{cases} u^2 & |u| \leq M \\ M(2|u| - M) & |u| > M \end{cases}$



Preparation example: Robust least squares

$$\text{minimize} \sum_k \phi_{hub}(\epsilon_k)$$



Robustifying the Kalman filter

The measurement update of the Kalman filter

We have the state space model

$$\begin{aligned} x(k+1) &= Fx(k) + v(k) \\ y(k) &= Hx(k) + e(k) + z(k) \\ e &\sim \mathcal{N}(0, R) \\ v &\sim \mathcal{N}(0, Q) \end{aligned}$$

The measurement update of the Kalman filter can be shown to be equivalent to solving the problem

$$\text{minimize } (y - Hx)^{\mathrm{T}} R^{-1} (y - Hx) + (x - \hat{x}_{k|k-1})^{\mathrm{T}} P_{k|k-1}^{-1} (x - \hat{x}_{k|k-1})$$

The optimal solution is $x^* = \hat{x}_{k|k} = \hat{x}_{k|k} + K(y - H\hat{x}_{k|k-a})$, where K is the Kalman gain.

The measurement update of the Kalman filter

Introduce $\tilde{x} = x - \hat{x}_{k|k-1}$ and $\tilde{y} = y - H\hat{x}_{k|k-1}$. The minimization problem can then be written

$$\begin{aligned} \text{minimize} \quad & (y - Hx)^{\mathrm{T}} R^{-1} (y - Hx) + (x - \hat{x}_{k|k-1})^{\mathrm{T}} P_{k|k-1}^{-1} (x - \hat{x}_{k|k-1}) \\ & = (\tilde{y} - H\tilde{x})^{\mathrm{T}} R^{-1} (\tilde{y} - H\tilde{x}) + \tilde{x}^{\mathrm{T}} P_{k|k-1}^{-1} \tilde{x} \end{aligned}$$

The measurement update of the Kalman filter

We now define the residuals ϵ for the system of equations

$$\begin{bmatrix} Z_R & 0 \\ 0 & Z_P \end{bmatrix} \begin{bmatrix} (\tilde{y} - H\tilde{x}) \\ \tilde{x} \end{bmatrix} = \epsilon,$$

where $Z_R^{\mathrm{T}} Z_R = R^{-1}$ and $Z_P^{\mathrm{T}} Z_P = P_{k|k-1}^{-1}$.

The minimization problem can now be written

$$\begin{aligned} \text{minimize} \quad & \epsilon^{\mathrm{T}} \epsilon \\ \text{subject to} \quad & \begin{bmatrix} Z_R & 0 \\ 0 & Z_P \end{bmatrix} \begin{bmatrix} (\tilde{y} - H\tilde{x}) \\ \tilde{x} \end{bmatrix} = \epsilon, \end{aligned}$$

which is a least-squares problem.

Robustifying the measurement update

The idea is to use the Huber penalty function ϕ_{hub} instead of the quadratic criterion $\epsilon^{\mathrm{T}}\epsilon$.

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{n+m} \phi_{hub}\big(\epsilon(i)\big) \\
\text{subject to} \quad & \begin{bmatrix} Z_R & 0 \\ 0 & Z_P \end{bmatrix} \begin{bmatrix} (\tilde{y} - H\tilde{x}) \\ \tilde{x} \end{bmatrix} = \epsilon,
\end{aligned}
$$

Tracking example again

10% chance of outlier with 10 times normal standard deviation