Homework 1 (due at the start of class September 01, 2022)

Homework should be submitted as a typeset document in LaTeX, see template on Canvas for details.

1 Problem

Suppose that $\boldsymbol{v}_1 = [1, 2, 0]^{\mathrm{T}}$, $\boldsymbol{v}_2 = [3, 1, 1]^{\mathrm{T}}$, and $\boldsymbol{w} = [4, -7, 3]^{\mathrm{T}}$. Is $\boldsymbol{w} \in \mathrm{span}(\boldsymbol{v}_1, \boldsymbol{v}_2)$?

Solution. If $\boldsymbol{w}$ belongs to $\mathrm{span}(\boldsymbol{v}_1, \boldsymbol{v}_2)$ then there exist two scalars α and β such that

$$\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2 = \boldsymbol{w} \tag{1}$$

$$\alpha \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 3 \end{bmatrix} \tag{2}$$

This is system of three equations with two unknowns and it can be solved algebraically, or by inspection, for $\alpha = -5$ and $\beta = 3$. For example, it is clear from the last equation that $\beta = 3$. The, using the first equation, $\alpha = 4 - 3\beta = -5$ which is consistent with the second equation $2\alpha + \beta = -7$.

2 Problem

Suppose that $\boldsymbol{x} \in \mathbb{R}^{n\times 1}$ and $\boldsymbol{y} \in \mathbb{R}^{m\times 1}$ are real-valued vectors and $\boldsymbol{U} \in \mathbb{R}^{n\times n}, \boldsymbol{V} \in \mathbb{R}^{m\times n}, \boldsymbol{W} \in \mathbb{R}^{n\times q}$ are real-valued matrices where $n \neq m \neq q$ are all positive integers. For each of the following expressions: 1) $\boldsymbol{y}^T \boldsymbol{V} \boldsymbol{x}$, 2) $\boldsymbol{W}^{-1} \boldsymbol{x}$, 3) $\boldsymbol{U} \boldsymbol{V}^{\mathrm{T}}$, and 4) $\boldsymbol{x} \boldsymbol{y}$, determine if the expression is well-defined. If it is, state the size of the resulting matrix product.

Solution.

1. The size of matrices multiplied is $(1 \times m)(m \times n)(n \times 1)$ thus $\boldsymbol{y}^T \boldsymbol{V} \boldsymbol{x} \in \mathbb{R}$
2. The size of matrices multiplied is $(n \times q)(n \times 1)$ and this expression is not well defined
3. The size of matrices multiplied is $(n \times n)(n \times m)$ and thus $\boldsymbol{U} \boldsymbol{V}^{\mathrm{T}} \in \mathbb{R}^{n\times m}$
4. The size of the matrices (vectors) multiplied is $(n \times 1)(m \times 1)$ and this expression is not well defined

3 Problem

Construct a matrix $\boldsymbol{W}$ for which

$$\boldsymbol{x}^{\mathrm{T}} \boldsymbol{W} \boldsymbol{x} = w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + w_4 x_1 x_2 \tag{3}$$

where $\boldsymbol{x} = [x_1, x_2, x_3]^{\mathrm{T}}$ and $w_i \in \mathbb{R}$ is a set of scalars for $i = 1, 2, 3, 4$.

Solution. The matrix $\boldsymbol{W}$ shown below produces the desired result:

$$
\begin{aligned}
\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} w_1 & w_4 & 0 \\ 0 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} && (4) \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} w_1 x_1 + w_4 x_2 \\ w_2 x_2 \\ w_3 x_3 \end{bmatrix} && (5) \\
&= w_1 x_1^2 + w_4 x_1 x_2 + w_2 x_2^2 + w_3 x_3^2 && (6)
\end{aligned}
$$

The transpose of the above $\boldsymbol{W}$ satisfies the property and so does the $\boldsymbol{W}$ below:

$$
\begin{aligned}
\boldsymbol{x}^{\mathrm{T}}\boldsymbol{W}\boldsymbol{x} &= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} w_1 & w_4/2 & 0 \\ w_4/2 & w_2 & 0 \\ 0 & 0 & w_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} && (7) \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} w_1 x_1 + w_4 x_2/2 \\ w_4 x_1/2 + w_2 x_2 \\ w_3 x_3 \end{bmatrix} && (8) \\
&= w_1 x_1^2 + w_4 x_1 x_2/2 + w_4 x_1 x_2/2 + w_2 x_2^2 + w_3 x_3^2 && (9) \\
&= w_1 x_1^2 + w_4 x_1 x_2 + w_2 x_2^2 + w_3 x_3^2 && (10)
\end{aligned}
$$

4 Problem

Suppose that

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{x}^{\mathrm{T}} \tag{11}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^n$, $\boldsymbol{B} \in \mathbb{R}^{n\times n}$, and $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ is a square symmetric matrix. Both $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible. Solve for $\boldsymbol{y}$.

Solution. To confirm this equation is well-defined: the RHS of the expression is a $1 \times n$ row vector and the LHS of the expression is a product of vectors/matrices: $(1 \times n)(n \times n)(n \times n)$. Since $\boldsymbol{A}$ is symmetric then $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$ and the expression $\boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A} = \boldsymbol{x}^{\mathrm{T}}$ is equal to

$$\boldsymbol{y}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}} \tag{12}$$

from which we can pull out the transpose as

$$(\boldsymbol{A}\boldsymbol{B}\boldsymbol{y})^{\mathrm{T}} = \boldsymbol{x}^{\mathrm{T}} \tag{13}$$

Now, take the transpose of both sides

$$\boldsymbol{A}\boldsymbol{B}\boldsymbol{y} = \boldsymbol{x} \tag{14}$$

Then pre-multiplying by $(\boldsymbol{A}\boldsymbol{B})^{-1}$

$$
\begin{aligned}
(\boldsymbol{A}\boldsymbol{B})^{-1}\boldsymbol{A}\boldsymbol{B}\boldsymbol{y} &= (\boldsymbol{A}\boldsymbol{B})^{-1}\boldsymbol{x} && (15) \\
\boldsymbol{I}\boldsymbol{y} &= (\boldsymbol{A}\boldsymbol{B})^{-1}\boldsymbol{x} && (16) \\
\implies \boldsymbol{y} &= (\boldsymbol{A}\boldsymbol{B})^{-1}\boldsymbol{x} && (17)
\end{aligned}
$$

5 Problem

Construct a matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ with $n = 3$ that is symmetric, has $\text{rank}(\boldsymbol{A}) = 2$, and $\boldsymbol{x} = [1, 0, 1]^{\text{T}} \in \text{null}(\boldsymbol{A})$.

Solution. Begin by defining a generic 3×3 symmetric matrix of coefficients

$$\boldsymbol{A} = \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \tag{18}$$

If $\boldsymbol{x} \in \text{null}(\boldsymbol{A})$ then this implies that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} a + c \\ b + e \\ c + f \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{20}$$

which implies that $a = -c$, $b = -e$ and $c = -f = -a$. Hence, we can rewrite our matrix as

$$\boldsymbol{A} = \begin{bmatrix} a & b & -a \\ b & d & -b \\ -a & -b & a \end{bmatrix} \tag{21}$$

It is clear that column 1 and column 3 are linear dependent since one is the negative of the other. Hence, it remains to choose d such that the column 2 is independent of columns 1 and 3. There are several cases

- If $a = b$ and both $a, b \neq 0$ then any choice $d \neq b$ (equivalently $d \neq a$) will make column 2 independent.

$$\boldsymbol{A} = \begin{bmatrix} a & a & -a \\ a & d & -a \\ -a & -a & a \end{bmatrix} \tag{22}$$

- If $a \neq b$ and $a, b \neq 0$ then any choice $d \neq b^2/a$ will make column 2 independent. (If $d = b^2/a$ then column 2 is equal to column 1 multiplied by a/b.)

$$\boldsymbol{A} = \begin{bmatrix} a & b & -a \\ b & d & -b \\ -a & -b & a \end{bmatrix} \tag{23}$$

- If $a = 0, b \neq 0$ then any choice $d \in \mathbb{R}$ will make column 2 independent.

$$\boldsymbol{A} = \begin{bmatrix} 0 & b & 0 \\ b & d & -b \\ 0 & -b & 0 \end{bmatrix} \tag{24}$$

- If $a \neq 0, b = 0$ then any choice $d \in \mathbb{R}$ will make column 2 independent.

$$\boldsymbol{A} = \begin{bmatrix} a & 0 & -a \\ 0 & d & 0 \\ -a & 0 & a \end{bmatrix} \tag{25}$$

- If $a = b = 0$ then it is impossible to make the matrix rank 2. If $d \neq 0$ the matrix rank is 1, if $d = 0$ the matrix is rank zero.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{26}$$

6 Problem

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{R}$. Then what are the eigenvalues of $\boldsymbol{B}$ where $\boldsymbol{B} = \alpha \boldsymbol{A}$?

Solution. The eigenvalue equation states that

$$\boldsymbol{A}\boldsymbol{q}_i = \lambda_i \boldsymbol{q}_i \tag{27}$$

where $\boldsymbol{q}_i$ and λ_i are the eigenvector and eigenvalue pairs for $i = 1, 2, \ldots, n$. Multiply both sides by α to obtain

$$\alpha \boldsymbol{A}\boldsymbol{q}_i = \alpha \lambda_i \boldsymbol{q}_i \tag{28}$$

$$\boldsymbol{B}\boldsymbol{q}_i = (\alpha \lambda_i)\boldsymbol{q}_i \tag{29}$$

This is in the form of an eigenvalue equation and it follows that the eigenvalues of $\boldsymbol{B}$ are $\alpha\lambda_i$ for $i = 1, 2, \ldots, n$. An argument can also be made by discussing the roots of the characterstic polynomials.