Lecture 6: Observability

Recall that the goal of state estimation is to determine the state of a dynamic system from its (possibly noisy) observations. However, before designing a state estimator it is natural to ask: *is it possible to infer the internal state of a system from its input/output data?* Intuitively, a system should be observable if the output is "sufficiently informative". Obviously, if the output contains all of the state variables (e.g., $\boldsymbol{y} = \boldsymbol{x}$) then it provides the system state directly. But what if the output vector $\boldsymbol{y} \in \mathbb{R}^m$ is much smaller than the state vector $\boldsymbol{x} \in \mathbb{R}^n$ (i.e., $m < n$)? Moreover, the output could contain (possibly nonlinear) transformation of the state variables that depends on the choice of sensors. To answer whether we can infer the state from outputs we can check a property of the system called *observability*.

Loosely speaking, a system is *observable* if we can uniquely determine the initial state $\boldsymbol{x}_0$ using knowledge of the input $u(\tau)$ and the output $y(\tau)$ for all $\tau \in [0, t]$ where $t > 0$ is some final time. Of course, if we can determine $\boldsymbol{x}_0$ and we know the control input history $\boldsymbol{u}(\tau)$ then we can also determine the state at any other time $\boldsymbol{x}(\tau)$ (assuming a deterministic system).

Example. Consider the motion of a particle in one dimension under an acceleration input $u(t)$. The states of the system are position s and velocity v, giving the state vector $\boldsymbol{x} = [s \;\; v]^T$ and dynamics:

$$\dot{\boldsymbol{x}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \tag{1}$$

where $\boldsymbol{x}(t_0) = [s_0 \;\; v_0]^T$ is the initial condition. Suppose we can measure the position of the particle only. Then the output equation is the scalar:

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \tag{2}$$

Intuitively, this system is observable since we can infer the unmeasured state x_2 by differentiating the output x_1 (since the dynamics are $x_2 = \dot{x}_1 - u(t)$ and we also know $u(t)$). Now, consider instead that the only output is velocity:

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \, . \tag{3}$$

We can use our observations of $x_2(t)$ and knowledge of the input $u(t)$ to obtain *change in position* (i.e., $x_1(t) - x_1(0) = \int_0^t [x_2(\tau) + u(\tau)] d\tau$) but we cannot infer the *initial position* $x_1(0)$ this way. Intuitively, the system with velocity measurement (only) is unobservable.

Continuous LTV System Observability [Stengel, 1994, Sec. 2.5]

Consider the LTV system (ignoring the control input):

$$\dot{\boldsymbol{x}} = \boldsymbol{A}(t)\boldsymbol{x}(t), \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{4}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) \tag{5}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, and $\boldsymbol{C} \in \mathbb{R}^{m\times n}$. The system (5) is said to be *observable* on $[t_0, t_f]$ if any initial state $\boldsymbol{x}_0$ is uniquely determined by the corresponding response $\boldsymbol{y}(t)$ for $t \in [t_0, t_f]$. We know that the state of this system evolves according to

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) \tag{6}$$

and hence the output is

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) \tag{7}$$

If at some time $t \in [t_0, t_f]$ we could invert $\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)$ then the initial condition could be recovered by rearranging (7):

$$\boldsymbol{x}(t_0) = (\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0))^{-1}\boldsymbol{y}(t) \ . \tag{8}$$

Unfortunately, this is rarely possible. It is certainly is not possible whenever the output vector is smaller than the state vector, $m < n$ (because then $\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)$ is a $m \times n$ matrix which is not invertible). However, if we multiply both sides of (7) by $\boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}$

$$\boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}\boldsymbol{y}(t) = \boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) \tag{9}$$

and integrate from t_0 to t_f we obtain

$$\int_{t_0}^{t_f} \boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}\boldsymbol{y}(t)dt = \int_{t_0}^{t_f} \boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)dt\ \boldsymbol{x}(t_0) \ . \tag{10}$$

Now define the *observability Gramian* as the square $n \times n$ matrix:

$$\boldsymbol{M}(t_0, t_f) = \int_{t_0}^{t_f} \boldsymbol{\Phi}^{\mathrm{T}}(t, t_0)\boldsymbol{C}^{\mathrm{T}}(t)\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)dt \tag{11}$$

to simplify the right-hand side of (10)as

$$\int_{t_0}^{t_f} \boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}\boldsymbol{y}(t)dt = \boldsymbol{M}(t_0, t_f)\boldsymbol{x}(t_0) \ . \tag{12}$$

The initial state can be recovered by inverting $\boldsymbol{M}(t_0, t_f)$ as:

$$\boldsymbol{x}(t_0) = \boldsymbol{M}(t_0, t_f)^{-1}\int_{t_0}^{t_f} \boldsymbol{\Phi}(t, t_0)^{\mathrm{T}}\boldsymbol{C}(t)^{\mathrm{T}}\boldsymbol{y}(t)dt \tag{13}$$

Consequently, the observability of the system relates to the ability to invert $\boldsymbol{M}(t_0, t_f)$ on the interval in question. Our discussion above suggests this is only a sufficient condition (i.e., *if $\boldsymbol{M}(t_0, t_f)$ is invertible then the system is observable*), but a more careful analysis (not presented here for the LTV case) can confirm this condition is both necessary and sufficient condition (i.e., *if and only if*) for a LTV system.

Continuous LTI System Observability [Ogata, Ch.9]

Now, let's again examine observability but for an LTI system:

$$\begin{aligned} \dot{\boldsymbol{x}} &= \boldsymbol{A}\boldsymbol{x} \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \\ \boldsymbol{y} &= \boldsymbol{C}\boldsymbol{x} \end{aligned} \tag{14}$$

where $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{y} \in \mathbb{R}^m$, $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, and $\boldsymbol{C} \in \mathbb{R}^{m\times n}$. Assume $t_0 = 0$, then the solution to the system is

$$\boldsymbol{x}(t) = e^{\boldsymbol{A}t}\boldsymbol{x}_0 \tag{15}$$

and the output is

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) = \boldsymbol{C}e^{\boldsymbol{A}t}\boldsymbol{x}_0 \tag{16}$$

From the Cayley-Hamilton Theorem we have

$$e^{\boldsymbol{A}t} = \sum_{k=0}^{n-1} \alpha_k(t)\boldsymbol{A}^k \tag{17}$$

which means the output can be expanded as a sum of n terms:

$$\begin{aligned}
\boldsymbol{y}(t) &= \boldsymbol{C}\sum_{k=0}^{n-1} \alpha_k(t)\boldsymbol{A}^k\boldsymbol{x}_0 &(18)\\
&= \boldsymbol{C}[\alpha_0(t)\mathbf{1} + \alpha_1(t)\boldsymbol{A} + \alpha_2(t)\boldsymbol{A}^2 + \cdots + \alpha_{n-1}(t)\boldsymbol{A}^{n-1}]\boldsymbol{x}_0 &(19)\\
&= [\alpha_0(t)\boldsymbol{C}\boldsymbol{x}_0 + \alpha_1(t)\boldsymbol{C}\boldsymbol{A}\boldsymbol{x}_0 + \alpha_2(t)\boldsymbol{C}\boldsymbol{A}^2\boldsymbol{x}_0 + \cdots + \alpha_{n-1}(t)\boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{x}_0] &(20)
\end{aligned}$$

Note: Recall that a square matrix $\boldsymbol{A}$ is invertible matrix if it has a non-zero determinant, $\det(\boldsymbol{A}) = 0$. Further, if it is invertible then the only value for which $\boldsymbol{A}\boldsymbol{x} = 0$ is the zero-vector, $\boldsymbol{x} = \mathbf{0}$, since its nullspace is trivial.

In the following, we will show that the system (14) is observable if and only if the $mn \times n$ *observability matrix*

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix} \tag{21}$$

is rank n. First we prove that is a necessary condition by contradiction. Suppose $\mathcal{O}$ is not full rank. In general, any matrix that is not full rank has a non-trivial nullspace, that is, there must exist a $\boldsymbol{x}_0 \neq 0$ such that

$$\mathcal{O}\boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{C}\boldsymbol{x}_0 \\ \boldsymbol{C}\boldsymbol{A}\boldsymbol{x}_0 \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1}\boldsymbol{x}_0 \end{bmatrix} = \mathbf{0}\,. \tag{22}$$

This means that each term in (22) is zero. Notice also that all of these terms also appear in (20). Hence, for this choice of $\boldsymbol{x}_0 \neq 0$ the output (20) is zero for all time, $\boldsymbol{y}(t) = \mathbf{0}$. This output is indistinguishable from the output with zero initial condition $\boldsymbol{x}_0 = 0$ and therefore such a case is unobservable.

We have shown above that $\text{rank}(\mathcal{O}) < n$ implies unobservability. Thus a necessary condition for observability is that $\mathcal{O}$ must be full rank. (*If* the system is observable then $\mathcal{O}$ is full rank.) Note the above statement is not a sufficient condition. (In other words, we have not yet shown that *if* $\mathcal{O}$ is full rank then the system is observable.)

For a sufficient condition we will show that if the observability Gramian is invertible then $\mathcal{O}$ is full rank and the system is observable. Under the LTI assumption (and with $t_0 = 0$) the observatility Gramian (11) simplifies to

$$\boldsymbol{M}(t_0, t_f) = \int_0^{t_f} e^{\boldsymbol{A}^{\mathrm{T}}\tau}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}e^{\boldsymbol{A}\tau}d\tau \tag{23}$$

Again, the proof is by contradiction. Suppose $\boldsymbol{M}(t_0, t_f)$ is not invertible then there exists an initial condition $\boldsymbol{x}_0$ such that

$$\boldsymbol{M}(t_0, t_f)\boldsymbol{x}_0 = \boldsymbol{0}$$

Pre-multiplying the above by $\boldsymbol{x}_0^{\mathrm{T}}$ gives a quadratic form which simplifies using (16)

$$\begin{aligned}
\boldsymbol{0} &= \boldsymbol{x}_0^{\mathrm{T}}\boldsymbol{M}(t_0, t_f)\boldsymbol{x}_0 && (24)\\
&= \boldsymbol{x}_0^{\mathrm{T}}\int_0^{t_f} e^{\boldsymbol{A}^{\mathrm{T}}\tau}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}e^{\boldsymbol{A}\tau}d\tau\boldsymbol{x}_0 && (25)\\
&= \int_0^{t_f} \boldsymbol{x}_0^{\mathrm{T}}e^{\boldsymbol{A}^{\mathrm{T}}\tau}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}e^{\boldsymbol{A}\tau}\boldsymbol{x}_0 d\tau && (26)\\
&= \int_0^{t_f} ||\boldsymbol{C}e^{\boldsymbol{A}\tau}\boldsymbol{x}_0||^2 d\tau && (27)\\
&= \int_0^{t_f} ||\boldsymbol{y}(\tau)||^2 d\tau && (28)
\end{aligned}$$

This shows that the output is exactly zero for all $t \in [t_0, t_f]$ if the observability Gramian is not invertible. If the output is zero then (20) is zero and (and using similar arguments as before) the terms in the matrix $\mathcal{O}\boldsymbol{x}_0$ in (22) must also all be zero so that $\mathcal{O}$ is not full rank. Consequently, the system is observable if and only if $\mathcal{O}$ is full rank. The initial condition can be recovered by (13) as

$$\boldsymbol{x}_0 = \boldsymbol{M}(t_0, t_f)^{-1}\int_{t_0}^{t_f} (e^{\boldsymbol{A}t})^{\mathrm{T}}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{y}(t)dt \tag{29}$$

Example: RLC Circuit, Adapted from [Simon, 2006, Ex. 1.9]. Consider the RLC circuit

$$\begin{bmatrix} \dot{v} \\ \dot{i} \end{bmatrix} = \begin{bmatrix} -2/(RC) & 1/C \\ -1/L & 0 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} + \begin{bmatrix} 1/(RC) \\ 1/L \end{bmatrix} u \tag{30}$$

$$y = \begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} v \\ i \end{bmatrix} = -v \tag{31}$$

where v is the voltage across the capacitor, i is the current through the inductor, and u is the applied voltage. The constants R, C, and L, are parameters for the resistance, capacitance, and inductance of the circuit. Since $n = 2$ the observability matrix is

$$\begin{aligned}
\mathcal{O} &= \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \end{bmatrix} && (32)\\
&= \begin{bmatrix} \begin{bmatrix} -1 & 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 \end{bmatrix}\begin{bmatrix} -2/(RC) & 1/C \\ -1/L & 0 \end{bmatrix} \end{bmatrix} && (33)\\
&= \begin{bmatrix} -1 & 0 \\ 2/(RC) & -1/C \end{bmatrix} && (34)
\end{aligned}$$

The observability matrix above is full rank by inspection, since the rows are clearly linearly independent. To confirm mathematically, we can make use of the special case that $\mathcal{O}$ is a square matrix and the full rank condition is satisifed if the determinant is nonzero, i.e., $\det(\mathcal{O}) \neq 0$. In this case,

$$\det(\mathcal{O}) = 1/C\,. \tag{35}$$

The observability matrix is therefore full rank for all physical values of capacitance, $C > 0$. Now, suppose that $R = L = C = 1$ and the output is instead the difference between the current and the voltage, i.e.,

$$y = \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} v \\ i \end{bmatrix} = i - v \tag{36}$$

The observability matrix is

$$\mathcal{O} = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{37}$$

and

$$\det(\mathcal{O}) = 0 \tag{38}$$

indicating that the system is not observable. By measuring the quantity $y = i - v$ we cannot uniquely infer the state of the system $\boldsymbol{x} = [v \;\; i]^T$ but by measuring just $y = -v$ we can.

The above example shows that a system may be observable depending on the output equations — removing/adding or changing the sensors that measure the system can alter observability. It also may be the case that a system is observable for only certain parameter values. Moreover, even if the system is unobservable, a subset of the states may still be observable. Such partial observability may be sufficient in some application (e.g., the observable states might be sufficient to form a stable output-feedback controller).

Unobservable Subspace [Hespanha, 2018, Sec. 15.2]

The *unobservable subspace* $\mathcal{U}$ is the set of initial conditions $\boldsymbol{x}_0 \in \mathbb{R}^n$ for which the output $\boldsymbol{y}(t)$ is equal to zero. Formally, given two times $t_1 > t_0 \geq 0$, the unobservable subspace is the set

$$\mathcal{U}[t_0, t_1] = \{\boldsymbol{x}_0 \in \mathbb{R}^n \mid \boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_0 = 0 \; \forall t \in [t_0, t_1]\} \tag{39}$$

Accordingly, an initial state $\boldsymbol{x}_* \notin \mathcal{U}[t_0, t_1]$ will produce a non-zero output $\boldsymbol{y}_*(t) = \boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_* \neq 0$ and an initial state $\boldsymbol{x}_u \in \mathcal{U}[t_0, t_1]$ will produce an output $\boldsymbol{y}(t) = 0$. Because of the linearity of the system it follows that for the initial state $\boldsymbol{x}_0 = \boldsymbol{x}_* + \boldsymbol{x}_u$ the output is indistinguishable from $\boldsymbol{y}_*(t)$

$$\begin{aligned} \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)(\boldsymbol{x}_* + \boldsymbol{x}_u) && (40) \\ &= \boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_* + \underbrace{\boldsymbol{C}(t)\boldsymbol{\Phi}(t, t_0)\boldsymbol{x}_u}_{=0} && (41) \\ &= \boldsymbol{y}_*(t) && (42) \end{aligned}$$

The unobservable space can be found by identifying the nullspace of $\mathcal{O}$.

Example: Returning to our 1D particle example. In the case with position measurement, the observability matrix is:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \end{bmatrix} \tag{43}$$

$$= \begin{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{44}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{45}$$

This matrix is full rank (the rows/columns are linearly independent) and the system is observable. Alternatively, with velocity measurement only:

$$\mathcal{O} = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \end{bmatrix} \tag{46}$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \tag{47}$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{48}$$

and the matrix is not full rank so the system is not observable. The unobservable subspace is the set of all vectors $\boldsymbol{x}$ for which

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The above is satisified for any vector $\boldsymbol{x} = [c \quad 0]^T$ where $c \in \mathbb{R}$. This indicates that the first component of any state (i.e., position) is unobservable.

Discrete-Time LTI Systems [Crassidis and Junkins, 2004, Sec. 3.5]

The observability criterion for discrete-time LTI systems is very similar to the criterion for continuous-time LTI systems. Consider the system

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} \tag{49}$$

$$\boldsymbol{y}_k = \boldsymbol{H}\boldsymbol{x}_k \tag{50}$$

where (as before) we ignore the control input. Given the initial condition $\boldsymbol{x}_0$ the initial observation is $\boldsymbol{y}_0 = \boldsymbol{H}\boldsymbol{x}_0$ and the $k = 1$ step is determined as $\boldsymbol{x}_1 = \boldsymbol{F}\boldsymbol{x}_0$. At the $k = 1$ step the observation is

$$\boldsymbol{y}_1 = \boldsymbol{H}\boldsymbol{x}_1 = \boldsymbol{H}\boldsymbol{F}\boldsymbol{x}_0 \tag{51}$$

Then, at the $k = 2$ step, the state is $\boldsymbol{x}_2 = \boldsymbol{F}\boldsymbol{x}_1$ or, equivalently, $\boldsymbol{x}_2 = \boldsymbol{F}(\boldsymbol{F}\boldsymbol{x}_0)$the observation is therefore

$$\boldsymbol{y}_2 = \boldsymbol{H}\boldsymbol{x}_2 = \boldsymbol{H}\boldsymbol{F}^2\boldsymbol{x}_0 \tag{52}$$

It is evident that the the sequence of observations will follow the pattern:

$$y_0 = Hx_0 \tag{53}$$
$$y_1 = HFx_0 \tag{54}$$
$$y_2 = HF^2x_0 \tag{55}$$
$$y_3 = HF^3x_0 \tag{56}$$
$$\vdots \tag{57}$$
$$y_{n-1} = HF^{(n-1)}x_0 \tag{58}$$

which can be written in matrix form as

$$\underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix}}_{mn\times 1} = \underbrace{\begin{bmatrix} H \\ HF \\ HF^2 \\ HF^3 \\ \vdots \\ HF^{n-1} \end{bmatrix}}_{mn\times n} \underbrace{x_0}_{n\times 1} \tag{59}$$

where the size of each matrix is indicated below. If the output is a scalar (i.e., $m = 1$) then we can solve for x_0 as

$$x_0 = \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^3 \\ \vdots \\ HF^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \end{bmatrix} \tag{60}$$

as long as the $n \times n$ matrix

$$\mathcal{O} = \begin{bmatrix} H \\ HF \\ HF^2 \\ HF^3 \\ \vdots \\ HF^{n-1} \end{bmatrix} \tag{61}$$

is invertible. If there are multiple outputs $m > 1$ then the $\mathcal{O}$ is no longer invertible (it is no longer square) and the requirement for observability is that it should have rank n (proof not shown).

Final Remarks

Observability is useful for helping understand if the state estimation problem is well posed, however observability provides only a binary answer—either a system is observable or it is not observable. Other metrics such as the *local unobservability index* or *local estimation condition number* can be used to measure the degree of observability (or unobservability) of a system. This can be useful if we wish to improve the observability properties of the system, for example, by deciding the most useful place to add additional sensors.

References

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