Homework 8 (due at 11pm November 17th, 2022)

1 Problem: Extended Kalman Filter

Consider the following nonlinear discrete-time system [1]

$$\begin{bmatrix} x_k \\ z_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + z_{k-1}T \\ z_{k-1} + \mu(1 - x_{k-1}^2)z_{k-1}T - x_{k-1}T + u_{k-1} + w_{k-1} \end{bmatrix} \tag{1}$$

with a unknown parameter μ, where u is a control input, and $w \sim \mathcal{N}(0, \sigma_w^2)$, and $T = 0.1$. We may rewrite this as a parameter-augmented system (see Lecture 17) as

$$\boldsymbol{x}_k = \boldsymbol{f}(\boldsymbol{x}_{k-1}, u_{k-1}, w_{k-1}) \tag{2}$$

$$\begin{bmatrix} x_k \\ z_k \\ \mu_k \end{bmatrix} = \begin{bmatrix} x_{k-1} + z_{k-1}T \\ z_{k-1} + \mu(1 - x_{k-1}^2)z_{k-1}T - x_{k-1}T + u_{k-1} + w_{k-1} \\ \mu_{k-1} \end{bmatrix} \tag{3}$$

where we've assumed that the parameter μ is a constant and has trivial dynamics. Suppose that the sensor can only observe the square of the first state of the system. Thus, the measurement equation is:

$$y_k = x_k^2 + v_k \tag{4}$$

where $v_k \sim \mathcal{N}(0, \sigma_v^2)$. Your assignment is to implement an extended Kalman filter (EKF) to estimate the state of the nonlinear system and the value of the unknown parameter.

1. First, compute the Jacobians that you will need for the EKF and evaluate them around the estimate (leaving it in symbolic form). Provide these expressions.

$$\boldsymbol{F}_{k-1} = \boldsymbol{J}_{\boldsymbol{x}}\boldsymbol{f}\Big|_{\hat{\boldsymbol{x}}_{k-1|k-1}} \qquad \boldsymbol{L}_{k-1} = \boldsymbol{J}_{\boldsymbol{w}}\boldsymbol{f}\Big|_{\hat{\boldsymbol{x}}_{k-1|k-1}} \tag{5}$$

$$\boldsymbol{H}_{k} = \boldsymbol{J}_{\boldsymbol{x}}\boldsymbol{h}\Big|_{\hat{\boldsymbol{x}}_{k|k-1}} \qquad \boldsymbol{M}_{k} = \boldsymbol{J}_{\boldsymbol{v}}\boldsymbol{h}\Big|_{\hat{\boldsymbol{x}}_{k|k-1}} \tag{6}$$

2. Next, implement the EKF in MATLAB to process the log file of measurements (`y_noisy`) and control inputs (`u`) provided in `h8_p1.mat` (along with the true system state history `x_true`). Assume that $\sigma_w^2 = 0.05$ and $\sigma_v^2 = 0.05$. The true value of the parameter is $\mu = 1$. The true initial condition is $\boldsymbol{x}_0 = [3, 1, 1]^{\mathrm{T}}$.

```
data =
     x_true: [2×300 double]
    y_noisy: [9.0272 9.6530 9.5279 ... ]
          u: [0.0100 0.0100 0.0100 ... ]
    mu_true: 1
```

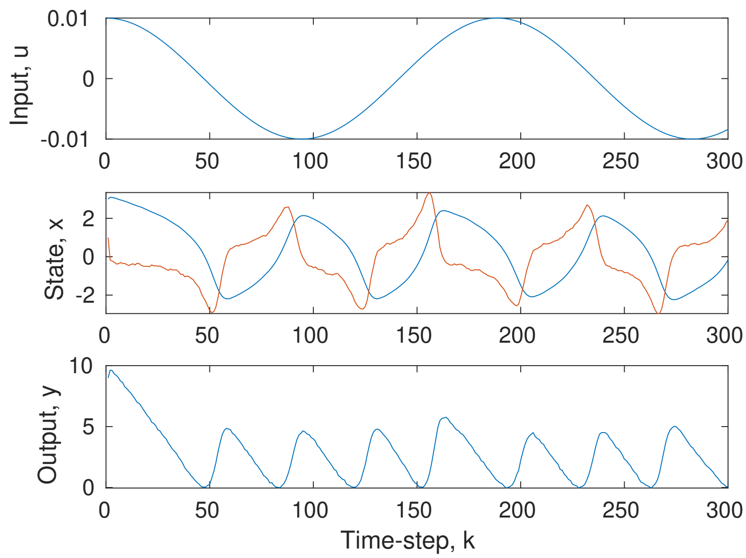
   Assume the following initial estimate:

$$\hat{\boldsymbol{x}}_{0|0} = [2, 2, 2]^{\mathrm{T}}$$
$$\boldsymbol{P}_{0|0} = \mathbf{1}_{3\times 3} \cdot 1E10;$$

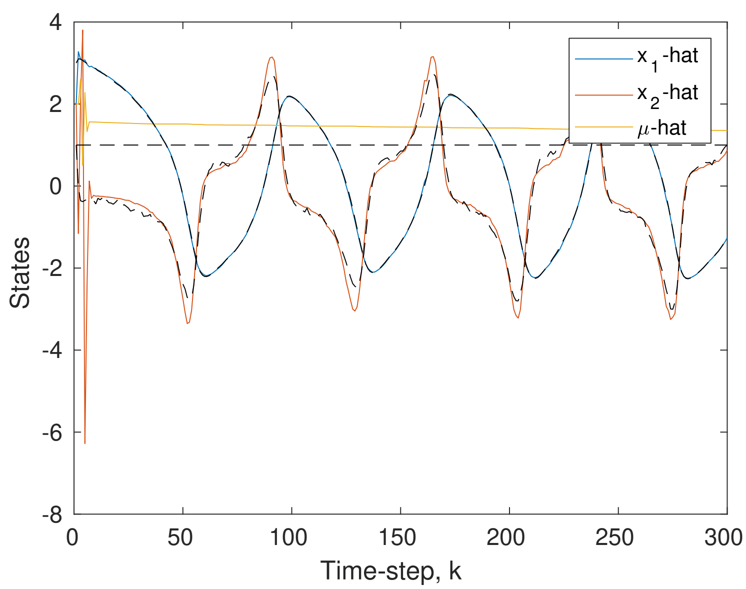
[1] This model resembles a discrete-time Van der Pol oscillator

3. Produce a plot similar to the reference solution given below to show your solution

The data set provided:



Reference solution (dashed-line represents true values):



2 Problem: Maximum Likelihood Estimation

A Bernoulli trial is a random experiment with two outcomes: either "success" or "failure". The probability of achieving success in each trial is p and the probability of failure is $1-p$. The *geometric distribution*

$$p(X=k) = (1-p)^{k-1}p \tag{7}$$

models the probability that the kth trial is the first success, where $k = \{1, 2, 3, \ldots\}$ is an integer number of trials.

Now suppose the engineers at Boston Dynamics have designed a controller to make their newest humanoid robot perform backflips. To quantify performance (each day for a week before an important demo) they tested how many attempts it needed to be successful. The data is as follows.

Trial	Attempts Until First Success
k_1	1
k_2	2
k_3	1
k_4	3
k_5	1
k_6	1
k_7	2

1. Derive an expression in terms of N data points k_i for the maximum likelihood estimate $\hat{p}_{MLE}$
2. What is the probability that the robot will perform a successful backflip on its first attempt for the VIP touring the facilities on Monday morning?