Lecture 15: Continuous-Time Kalman Filter

In this section we will introduce the *Kalman-Bucy filter*, named after Rudolph Kalman and Richard S. Bucy, as a continuous-time variant of the discrete-time Kalman filter. We will base our derivation on the results of the discrete-time Kalman filter in the limit as the discretization time-step goes to zero [Simon, 2006, Ch.8]. Alternative derivations of the continuous-time Kalman filter take a different approach (without starting from the discrete-time version). For example, see [Crassidis and Junkins, 2004, Sec. 5.4].

Derivation. Suppose we have the continuous-time, linear, time-invariant system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{w} \tag{1}$$
$$\dot{\boldsymbol{y}} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{v} \tag{2}$$
$$\boldsymbol{w} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_c) \tag{3}$$
$$\boldsymbol{v} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_c) \tag{4}$$

where $\boldsymbol{w}(t)$ and $\boldsymbol{v}(t)$ are process and measurement noise, respectively. The above notation implies that the noise signals have covariance $E[\boldsymbol{w}(t)\boldsymbol{w}^{\mathrm{T}}(\tau)] = \boldsymbol{Q}_c\delta(t-\tau)$ and $E[\boldsymbol{v}(t)\boldsymbol{v}^{\mathrm{T}}(\tau)] = \boldsymbol{R}_c\delta(t-\tau)$ as discussed in our lecture on stochastic systems. Discretizing this system with a sample period of Δt leads to

$$\boldsymbol{x}_k = \boldsymbol{F}\boldsymbol{x}_{k-1} + \boldsymbol{G}\boldsymbol{u}_{k-1} + \boldsymbol{L}\boldsymbol{w}_{k-1} \tag{5}$$
$$\boldsymbol{y}_k = \boldsymbol{H}\boldsymbol{x}_k + \boldsymbol{v}_k \tag{6}$$

where for small Δt the discrete-time system matrices can be approximated as

$$\boldsymbol{F} \approx (\boldsymbol{I} + \boldsymbol{A}\Delta t) \tag{7}$$
$$\boldsymbol{G} \approx \boldsymbol{B}\Delta t \tag{8}$$
$$\boldsymbol{L} \approx \boldsymbol{I}\Delta t \tag{9}$$
$$\boldsymbol{H} = \boldsymbol{C} \tag{10}$$

Recall that the continuous time and discrete-time noise covariances are related by

$$\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}), \qquad \boldsymbol{Q} = \boldsymbol{Q}_c\Delta t \tag{11}$$
$$\boldsymbol{v}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}), \qquad \boldsymbol{R} = \boldsymbol{R}_c/\Delta t \tag{12}$$

and that for the discrete-time system the Kalman gain was

$$\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T\boldsymbol{S}_k^{-1} \tag{13}$$

with $\boldsymbol{S}_k = \boldsymbol{H}\boldsymbol{P}_{k|k-1}\boldsymbol{H}_k^T + \boldsymbol{R}$. Using (10) and $\boldsymbol{R}$ from (12) the Kalman gain becomes

$$\boldsymbol{K}_k = \boldsymbol{P}_{k|k-1}\boldsymbol{C}^{\mathrm{T}}(\boldsymbol{C}\boldsymbol{P}_{k|k-1}\boldsymbol{C}^{\mathrm{T}} + \boldsymbol{R}_c/\Delta t)^{-1} \tag{14}$$

Now, divide both sides by Δt and take the limit as Δt goes to zero

$$\lim_{\Delta t \to 0} \frac{\boldsymbol{K}_k}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} (\boldsymbol{C} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} + \boldsymbol{R}_c / \Delta t)^{-1} \tag{15}$$

$$= \lim_{\Delta t \to 0} \frac{1}{\Delta t} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} ([\boldsymbol{C} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} \Delta t + \boldsymbol{R}_c] / \Delta t)^{-1} \tag{16}$$

$$= \lim_{\Delta t \to 0} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} (\boldsymbol{C} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} \Delta t + \boldsymbol{R}_c)^{-1} \tag{17}$$

$$= \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} \boldsymbol{R}_c^{-1} \tag{18}$$

The above limit demonstrates that $\boldsymbol{K}_k / \Delta t$ approaches a finite constant as $\Delta t \to 0$. Rearranging the limit (18) we obtain another fact,

$$\lim_{\Delta t \to 0} \boldsymbol{K}_k = \lim_{\Delta t \to 0} \boldsymbol{P}_{k|k-1} \boldsymbol{C}^{\mathrm{T}} \boldsymbol{R}_c^{-1} \Delta t = 0 \,, \tag{19}$$

which we will use momentarily. Recall that the estimation error covariances from the discrete-time Kalman filter are

$$\boldsymbol{P}_{k|k} = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{H}) \boldsymbol{P}_{k|k-1} \tag{20}$$

$$\boldsymbol{P}_{k+1|k} = \boldsymbol{F} \boldsymbol{P}_{k|k} \boldsymbol{F}^{\mathrm{T}} + \boldsymbol{Q}_c \Delta t \tag{21}$$

For small Δt we can use (7) and (21) becomes

$$\boldsymbol{P}_{k+1|k} = (\boldsymbol{I} + \boldsymbol{A} \Delta t) \boldsymbol{P}_{k|k} (\boldsymbol{I} + \boldsymbol{A}^{\mathrm{T}} \Delta t) + \boldsymbol{Q}_c \Delta t \tag{22}$$

$$= (\boldsymbol{I} + \boldsymbol{A} \Delta t)(\boldsymbol{P}_{k|k} + \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t) + \boldsymbol{Q}_c \Delta t \tag{23}$$

$$= (\boldsymbol{P}_{k|k} + \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t) + \boldsymbol{A} \Delta t (\boldsymbol{P}_{k|k} + \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t) + \boldsymbol{Q}_c \Delta t \tag{24}$$

$$= \boldsymbol{P}_{k|k} + \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t + \boldsymbol{A} \boldsymbol{P}_{k|k} \Delta t + \boldsymbol{A} \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t^2 + \boldsymbol{Q}_c \Delta t \tag{25}$$

$$= \boldsymbol{P}_{k|k} + [\boldsymbol{A} \boldsymbol{P}_{k|k} + \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c] \Delta t + \boldsymbol{A} \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t^2 \tag{26}$$

Then, substitute $\boldsymbol{P}_{k|k}$ from (20) in each term above (except for the last term since we shall see it vanishes)

$$\boldsymbol{P}_{k+1|k} = (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}) \boldsymbol{P}_{k|k-1} + [\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}) \boldsymbol{P}_{k|k-1} + (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}) \boldsymbol{P}_{k|k-1} \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c] \Delta t + \boldsymbol{A} \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t^2 \tag{27}$$

Subtract $\boldsymbol{P}_{k|k-1}$ from both sides

$$\begin{aligned} \boldsymbol{P}_{k+1|k} - \boldsymbol{P}_{k|k-1} &= (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}) \boldsymbol{P}_{k|k-1} + [\boldsymbol{A}(\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}) \boldsymbol{P}_{k|k-1} + (\boldsymbol{I} - \boldsymbol{K}_k \boldsymbol{C}) \boldsymbol{P}_{k|k-1} \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c] \Delta t \\ &\quad + \boldsymbol{A} \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t^2 - \boldsymbol{P}_{k|k-1} \qquad (28) \\ &= -\boldsymbol{K}_k \boldsymbol{C} \boldsymbol{P}_{k|k-1} + [\boldsymbol{A} \boldsymbol{P}_{k|k-1} - \boldsymbol{A} \boldsymbol{K}_k \boldsymbol{C} \boldsymbol{P}_{k|k-1} + \boldsymbol{P}_{k|k-1} \boldsymbol{A}^{\mathrm{T}} - \boldsymbol{K}_k \boldsymbol{C} \boldsymbol{P}_{k|k-1} \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c] \Delta t \\ &\quad + \boldsymbol{A} \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t^2 \qquad (29) \end{aligned}$$

then divide by Δt

$$\frac{\boldsymbol{P}_{k+1|k} - \boldsymbol{P}_{k|k-1}}{\Delta t} = -\frac{\boldsymbol{K}_k \boldsymbol{C} \boldsymbol{P}_{k|k-1}}{\Delta t} + [\boldsymbol{A} \boldsymbol{P}_{k|k-1} - \boldsymbol{A} \boldsymbol{K}_k \boldsymbol{C} \boldsymbol{P}_{k|k-1} + \boldsymbol{P}_{k|k-1} \boldsymbol{A}^{\mathrm{T}} - \boldsymbol{K}_k \boldsymbol{C} \boldsymbol{P}_{k|k-1} \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c] + \boldsymbol{A} \boldsymbol{P}_{k|k} \boldsymbol{A}^{\mathrm{T}} \Delta t \tag{30}$$

and take the limit as $\Delta t \to 0$

$$\lim_{\Delta t \to 0} \frac{\boldsymbol{P}_{k+1|k} - \boldsymbol{P}_{k|k-1}}{\Delta t} = \lim_{\Delta t \to 0} -\frac{\boldsymbol{K}_k}{\Delta t}(\boldsymbol{C}\boldsymbol{P}_{k|k-1}) + [\boldsymbol{A}\boldsymbol{P}_{k|k-1} - \boldsymbol{A}\boldsymbol{K}_k\boldsymbol{C}\boldsymbol{P}_{k|k-1} + \boldsymbol{P}_{k|k-1}\boldsymbol{A}^{\mathrm{T}} - \boldsymbol{K}_k\boldsymbol{C}\boldsymbol{P}_{k|k-1}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c] + \boldsymbol{A}\boldsymbol{P}_{k|k}\boldsymbol{A}^{\mathrm{T}}\Delta t \tag{31}$$

In this limit, the covariance $\boldsymbol{P}_{k|k-1}$ becomes a continuous variable $\boldsymbol{P}$ and the LHS of (31) is the definition of $\dot{\boldsymbol{P}}$. Equation (18) is substituted into the first term on the RHS as the limit is evaluated. Moreover, we know from (19) that the second and fourth terms containing $\boldsymbol{K}_k$ in square brackets vanish in the limit. The last term also vanishes. Thus (31) becomes

$$\dot{\boldsymbol{P}} = -\boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}\boldsymbol{C}\boldsymbol{P} + \boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c \tag{32}$$

Equation (32) is known as the *differential Ricatti equation (DRE)* and it gives the time-evolution of the state covariance matrix $\boldsymbol{P}$. You may notice that this equation is independent of the actual output $\boldsymbol{y}(t)$, thus the covariance decays in a way that is irrespective of the initial condition or output. The matrix $\boldsymbol{P}$ is a valid covariance matrix, that is, it is symmetric and positive definite.

Aside: Equation (32) includes both the motion and measurement update steps of a Kalman filter. Notice that the first term (i.e., $-\boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}\boldsymbol{C}\boldsymbol{P}^{-1}$) is negative and involves the measurement equation matrices. This term corresponds to the measurement update step that "removes uncertinaty". The remaining terms "add uncertainty". In fact, The equation

$$\dot{\boldsymbol{P}} = \boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c \tag{33}$$

can be derived seperately from the Kalman filter by just considering the propagation of the covariance matrix under linear system dynamics [Simon, 2006, Sec. 4.3]. The equation is known as the *continuous-time Lyapunov equation* or the *Sylvester equation*.

Recall from our previous lecture that the discrete-time Kalman filter equations included a motion update step

$$\hat{\boldsymbol{x}}_{k|k-1} = \boldsymbol{F}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{G}\boldsymbol{u}_{k-1} \tag{34}$$

and the measurement update

$$\hat{\boldsymbol{x}}_{k|k} = \hat{\boldsymbol{x}}_{k|k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{H}\hat{\boldsymbol{x}}_{k|k-1}) \tag{35}$$

Now, assuming Δt is small allows the use of (7), and substiuting (34) into (35) gives

$$\begin{aligned}
\hat{\boldsymbol{x}}_{k|k} &= (\boldsymbol{F}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{G}\boldsymbol{u}_{k-1}) + \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{H}(\boldsymbol{F}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{G}\boldsymbol{u}_{k-1})) && (36)\\
&\approx (\{\boldsymbol{I} + \boldsymbol{A}\Delta t\}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{B}\Delta t\boldsymbol{u}_{k-1}) + \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{C}(\{\boldsymbol{I} + \boldsymbol{A}\Delta t\}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{B}\Delta t\boldsymbol{u}_{k-1})) && (37)\\
&= (\boldsymbol{I} + \boldsymbol{A}\Delta t)\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{B}\Delta t\boldsymbol{u}_{k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{A}\Delta t\hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{B}\Delta t\boldsymbol{u}_{k-1}) && (38)
\end{aligned}$$

Now subtract $\hat{\boldsymbol{x}}_{k-1|k-1}$ from both sides and divide by Δt.

$$\frac{\hat{\boldsymbol{x}}_{k|k} - \hat{\boldsymbol{x}}_{k-1|k-1}}{\Delta t} = \frac{(\boldsymbol{I} + \boldsymbol{A}\Delta t)\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{B}\Delta t \boldsymbol{u}_{k-1} + \boldsymbol{K}_k(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{A}\Delta t \hat{\boldsymbol{x}}_{k-1|k-1}}{} \tag{39}$$

$$\frac{-\boldsymbol{C}\boldsymbol{B}\Delta t \boldsymbol{u}_{k-1}) - \hat{\boldsymbol{x}}_{k-1|k-1}}{\Delta t} \tag{40}$$

$$= \boldsymbol{A}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{B}\boldsymbol{u}_{k-1} + \frac{\boldsymbol{K}_k}{\Delta t}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{A}\Delta t \hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{B}\Delta t \boldsymbol{u}_{k-1}) \tag{41}$$

Then taking the limit as $\Delta t \to 0$ and substituting $\boldsymbol{K}_k$ from (18)

$$\lim_{\Delta t \to 0} \frac{\hat{\boldsymbol{x}}_{k|k} - \hat{\boldsymbol{x}}_{k-1|k-1}}{\Delta t} = \lim_{\Delta t \to 0} \boldsymbol{A}\hat{\boldsymbol{x}}_{k-1|k-1} + \boldsymbol{B}\boldsymbol{u}_{k-1} + \frac{\boldsymbol{K}_k}{\Delta t}(\boldsymbol{y}_k - \boldsymbol{C}\hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{A}\Delta t \hat{\boldsymbol{x}}_{k-1|k-1} - \boldsymbol{C}\boldsymbol{B}\Delta t \boldsymbol{u}_{k-1}) \tag{42}$$

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{K}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}) \tag{43}$$

where $\boldsymbol{K} = \boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}$ as per (18) and the last two terms vanish since they contain Δt as it goes to zero. The expressions (43) and (32) give the time-evolution of the state estimate and the state covariance, respectively.

Summary

We can summarize the continuous-time Kalman filter as follows. Consider the system

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{w} \tag{44}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{v} \tag{45}$$

where $\boldsymbol{w}(t) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_c)$ and $\boldsymbol{v}(t) \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_c)$ are zero-mean white-noise process and measurement noise, respectively, with $E[\boldsymbol{w}(t)\boldsymbol{w}^{\mathrm{T}}(\tau)] = \boldsymbol{Q}_c\delta(t-\tau)$ and $E[\boldsymbol{v}(t)\boldsymbol{v}^{\mathrm{T}}(\tau)] = \boldsymbol{R}_c\delta(t-\tau)$. Suppose the initial state estimate and state estimate covariance are:

$$\hat{\boldsymbol{x}}(0) = E[\boldsymbol{x}(0)] \tag{46}$$

$$\boldsymbol{P}(0) = E[(\boldsymbol{x}(0) - \hat{\boldsymbol{x}}(0))(\boldsymbol{x}(0) - \hat{\boldsymbol{x}}(0))^{\mathrm{T}}] \tag{47}$$

then the continuous-time Kalman filter equations are

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{K}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}) \tag{48}$$

$$\dot{\boldsymbol{P}} = -\boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}\boldsymbol{C}\boldsymbol{P}^{-1} + \boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c \tag{49}$$

where in the first equation $\boldsymbol{K} = \boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}$ and the second equation is called the differential Ricatti equation.

Algorithm: Kalman-Bucy Filter

1. Ensure your system is in the form:

$$\dot{\boldsymbol{x}} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{w} \tag{50}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{C}\boldsymbol{x} + \boldsymbol{v} \tag{51}$$

$$\boldsymbol{w} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q}_c) \tag{52}$$

$$\boldsymbol{v} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{R}_c) \tag{53}$$

where the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and the control input $\boldsymbol{u}(t)$ are known. The process and measurement noise covariances $\boldsymbol{Q}_c$ and $\boldsymbol{R}_c$ are also known where

$$E[\boldsymbol{w}(t)\boldsymbol{w}(\tau)^T] = \boldsymbol{Q}_c\delta(t-\tau) \quad (54)$$

$$E[\boldsymbol{v}(t)\boldsymbol{v}(\tau)^T] = \boldsymbol{R}_c\delta(t-\tau) \quad (55)$$

and it is assumed that $E[\boldsymbol{w}(t)] = 0$ and $E[\boldsymbol{v}(t)] = 0$ for all $t \geq t_0$.

2. Select an initial guess and an initial covariance

$$\hat{\boldsymbol{x}}(t_0) = E[\boldsymbol{x}(t_0)] \quad (56)$$

$$\boldsymbol{P}(t_0) = E[(\boldsymbol{x}(t_0) - \hat{\boldsymbol{x}}(t_0))(\boldsymbol{x}(t_0) - \hat{\boldsymbol{x}}(t_0))^{\mathrm{T}}] \quad (57)$$

In the absence of other information you may choose $\hat{\boldsymbol{x}}_0 = \boldsymbol{0}$ and $\boldsymbol{P}_0 = \boldsymbol{1}_{n\times n}\sigma_0^2$ where σ_0^2 is a large number (e.g., 10E6).

3. Numerically solve the differential Ricatti equations for $t > t_0$

$$\dot{\hat{\boldsymbol{x}}} = \boldsymbol{A}\hat{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{u} + \boldsymbol{K}(\boldsymbol{y} - \boldsymbol{C}\hat{\boldsymbol{x}}) \quad (58)$$

$$\dot{\boldsymbol{P}} = -\boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}\boldsymbol{C}\boldsymbol{P}^{-1} + \boldsymbol{A}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{Q}_c \quad (59)$$

where $\boldsymbol{K} = \boldsymbol{P}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{R}_c^{-1}$.

Remarks

Notice that in the continuous-time Kalman filter equations we don't keep track of the motion update and measurement update separately. That is, the differential equations are for the *posterior* state estimate and covariance. Also, $\boldsymbol{P}$ is a $n \times n$ matrix and to simulate the differential equation $\dot{\boldsymbol{P}}$ it is convenient to vectorize this expression. In MATLAB, a matrix can be vectorized using the `reshape` command and use the following scheme with a solver that expects a vector ODE:

1. Reshape $\boldsymbol{P}$ from a $n^2 \times 1$ vector into a $n \times n$ matrix
2. Compute (58)–(59) to obtain the $n \times n$ matrix $\dot{\boldsymbol{P}}$
3. Reshape $\dot{\boldsymbol{P}}$ from a $n \times n$ matrix into a $n^2 \times 1$ vector

Also, since $\boldsymbol{P}$ is symmetric it is only necessary to simulate $n(n+1)/2$ (i.e., the number of elements in the upper triangle, including the diagonal) if we desire a more efficient implementation.

Recall from Lecture 10 that simulating a stochastic system using a variable time-step solver such as `ode45` can be problematic due to the random vector generator producing different values at the same time-step when called multiple times. To overcome this challenge during simulation one might either implement fixed-time step solver or generate the noise signals before hand so they are random but deterministic (and can be interpolated from).

References

[Crassidis and Junkins, 2004] Crassidis, J. L. and Junkins, J. L. (2004). *Optimal Estimation of Dynamic Systems*. Chapman and Hall/CRC.

[Simon, 2006] Simon, D. (2006). *Optimal State Estimation: Kalman, H infinity, and Nonlinear Approaches*. John Wiley & Sons.