Homework 5 (due at the start of class October 25th, 2022)

1 Problem

Sketch by hand the following probability density functions (p.d.f.s) corresponding to the three random variables X, Y, and Z on a single set of axes. The relative shape of the curves and location of the means should be accurate. Provide the numerical value of the apex of each p.d.f. on your plot.

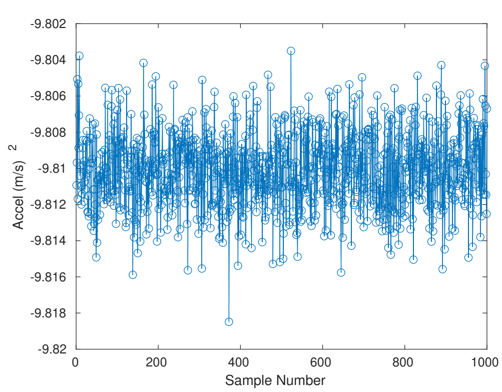
- $X \sim \mathcal{N}(0, 2^2)$
- $Y \sim \mathcal{N}(2, 2^2)$
- $Z \sim \mathcal{N}(1, 0.8^2)$.

For each of the following statements, identify the mean μ and standard deviation σ.

- The random variable X is between 2.0 and 3.0 about 95 % of the time.
- The random variable X is between 1.0 and 4.0 about 68 % of the time.
- The random variable X is between -8.0 and -2.0 about 99.7 % of the time.

2 Problem

A 3-axis accelerometer onboard a robot sitting at rest in the lab registers the following reading along the downward-facing axis over 10 seconds (sampled at 100 Hz):



The data was exported to MATLAB and is provided in the file `grav_sens.mat` (which you can load into your workspace with `load grav_sens.mat` command). You wish to model the noise of your sensor by considering the output as a random variable $X \sim \mathcal{N}(\mu, \sigma^2)$. The variance you determine can be used to define the measurement noise in a state-space model (i.e., v_k)that includes this acceleration component as an output.

From the data provided, compute the sample mean

$$\mu = \sum_{i=1}^{N} x_i / N \tag{1}$$

where N is the number of data points and the standard deviation

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N-1}} \tag{2}$$

where $N - 1$ is used in the denominator instead of N to give an unbiased estimate (known as Bessel's correction). Plot a p.d.f. curve of $p(x)$ over a range from $x \in [\mu - 4\sigma, \mu + 4\sigma]$.

3 Problem

Suppose that $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$ are continuous random variables that are correlated (i.e., $E[(X - \mu_x)(Y - \mu_y)] = \sigma_{xy} \neq 0$). We can define a third random variable as the sum of the first two: $Z = \alpha X + \beta Y$. What is the expacted value μ_z and variance σ_z^2 of Z? Express your answers in terms of the variables $\mu_x, \mu_y, \sigma_x, \sigma_y, \sigma_{xy}$. (Hint: use the definition of mean and variance applied to Z as well as linearity of the expected value operator.)

4 Problem

The (x, y) position of a robot is computed using an onboard SLAM (Simulatenous Localization and Mapping) algorithm as a random vector $\boldsymbol{x} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{P})$ with mean $\boldsymbol{\mu} = [5, 15]^{\mathrm{T}}$ and covariance

$$\boldsymbol{P} = \begin{bmatrix} 1 & 4 \\ 4 & 20 \end{bmatrix} \tag{3}$$

1. In MATLAB, plot the mean of the robot with a "+" marker and draw a 95 % confidence ellipse around the robot's position (see Lecture 9).
2. Two external tracking cameras, A and B, each detect an April Tag on the robot and report their own estimates of the robot's position: $\boldsymbol{x}_A = [13, 20]^{\mathrm{T}}$ and $\boldsymbol{x}_B = [10, 23]^{\mathrm{T}}$. What is Euclidean distance distance between $\boldsymbol{\mu}$ and each of the camera's estimates?
3. What is the Mahalanobis distance between $\boldsymbol{\mu}$ and each of the camera's estimates?
4. Draw $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ on your plot and explain the results above by referencing the plot. Make sure all markers/lines on your plot are clearly labeled with a legend or text overlay.

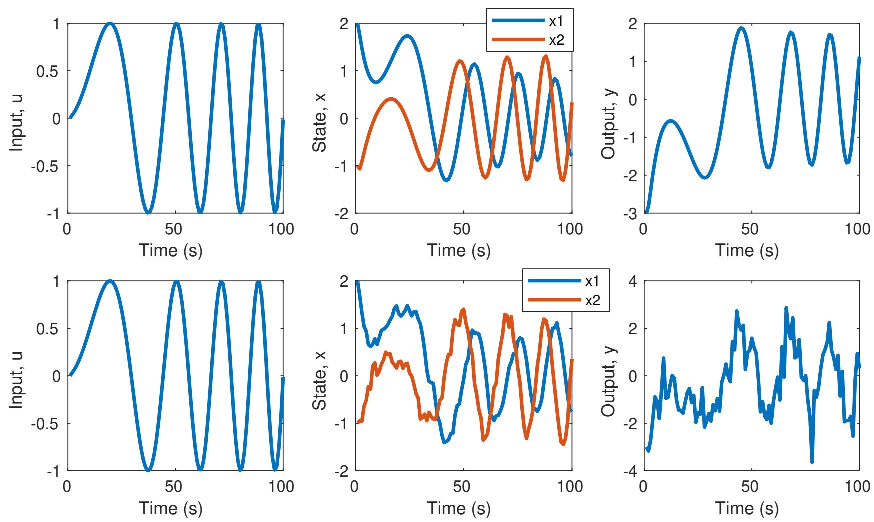
5 Problem

Consider the following discrete-time LTI system:

$$\boldsymbol{x}_k = \begin{bmatrix} 1 & 0.25 \\ -0.5 & 0.1 \end{bmatrix} \boldsymbol{x}_{k-1} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_k \tag{4}$$

$$y_k = \begin{bmatrix} -1 & 1 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{v}_k \tag{5}$$

with initial condition $x(t_0) = [2, -1]^{\mathrm{T}}$ and discrete-time control input $u = \sin(\omega \cdot k^{\beta})$ where $\omega = 0.01$ and $\beta = 1.7$. The process noise is $\boldsymbol{w}_k \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{Q})$ with $\boldsymbol{Q} = \boldsymbol{1}_{2\times 2}0.01$ and the measurement noise is $v_k \sim \mathcal{N}(0, R)$ where $R = 0.5$. Simulate the system for $N = 100$ steps first without noise and then with noise. Plot the input, internal system states, and output in both cases (similar to the plots shown below).



6 Problem

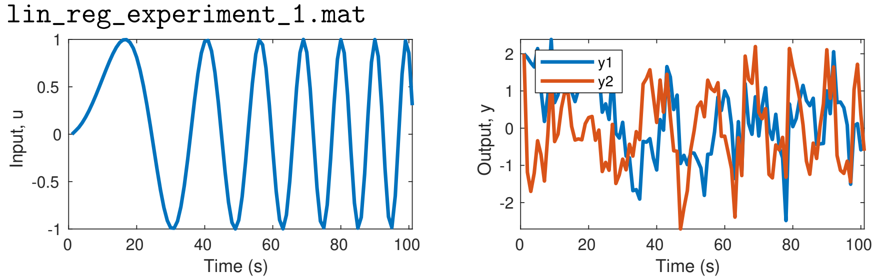
You are tasked with identifying a model for a dynamical system and suspect that the following discrete-time LTI system model will be appropriate:

$$\boldsymbol{x}_k = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{\boldsymbol{F}} \boldsymbol{x}_{k-1} + \underbrace{\begin{bmatrix} e \\ f \end{bmatrix}}_{\boldsymbol{G}} u_k \tag{6}$$

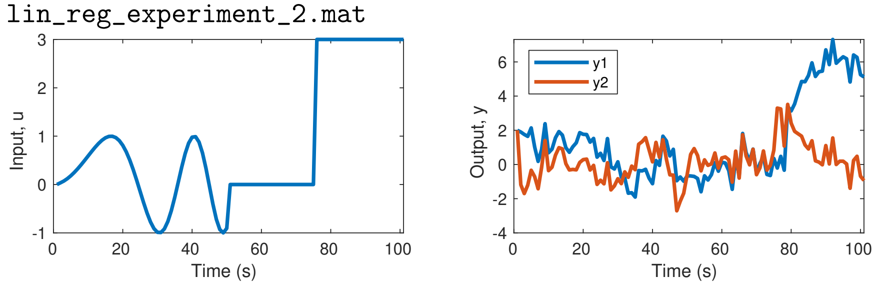
$$y_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \boldsymbol{x}_k + \boldsymbol{v}_k \tag{7}$$

To identify the model parameters you conduct two experiments and record 100 samples of the following data:

- `lin_reg_experiment_1.mat`



- `lin_reg_experiment_2.mat`



Since the output vector $\boldsymbol{y}$ contains the entire state $\boldsymbol{x}$ we can easily compute $\boldsymbol{x}_{k-1}$ and $\boldsymbol{x}_k$ for each k. The data in the provided .mat files provides these time-histories in the form of `xkm1` and `xk`, respectively. The .mat files also provide the input `u` along with the correct values for the matrices $\boldsymbol{F}$ and $\boldsymbol{G}$ (they are the same as in the previous problem).

1. Predict the values $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{e}$, $\hat{f}$ for each experiment and state your estimated system matrices $\hat{\boldsymbol{F}}$ and $\hat{\boldsymbol{G}}$. (Note: you can ignore the bias term θ_0 in your estimated $\hat{\boldsymbol{\theta}}$.)

2. Is one experiment better than the other at predicting the parameters? Explain why or why not.