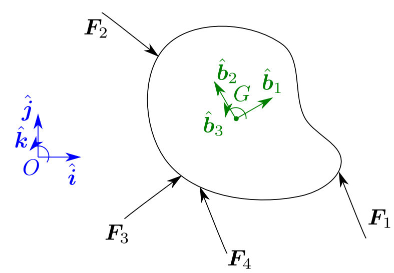
Lecture 15: Lumped Parameter Models

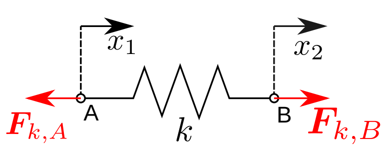
Distributed-parameter models. Mechanical systems are often most accurately described by *distributed-parameter models* in which the parameters/characteristics (e.g., mass, thickness, stiffness) and the state (e.g. position, velocity) of the system is continuous over the spatial extent of the body. Consider the vibration of a flexible beam of length L. To accurately describe the state of the entire beam (e.g., its deflection from a nominal cantilevered horizontal position) we cannot simply use one state variable $x(t)$ and its derivative. Rather, we must specify the state for *each point* s along the beam, where $0 \leq s \leq L$. In this case, a function $x(s,t)$ and its derivative might accurately capture the state. Since there are infinitely many points along the beam we say this is an *infinite-dimensional system*, which is another term for a distributed-parameter model. You may recall from Lecture 1 that infinite-dimensional systems are governed by partial differential equations (PDEs). For example, the Navier-Stokes equation (fluid systems), heat equation (thermal systems), wave equation (in acoustics), or dynamic beam equation (structural mechanics).

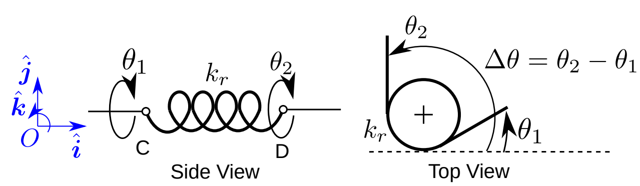
Lumped-parameter models. Engineers generally seek to find the simplest model that retains the essential characteristics of problem and is tractable for design, simulation, and analysis. Therefore, in some circumstances, it may be appropriate to sacrifice the accuracy/detail of a distributed-parameter model by making a *lumped parameter assumption* that "lumps" the distributed parameters of a mechanical system into a finite number of idealized elements. For mechanical systems, the lumped-parameter assumption is that a mechanical system consists of the following idealized elements:

- **Inertial elements**: point masses or rigid bodies that can store kinetic energy (rotational and/or translational) and gravitational energy.



- **Spring elements**: translational or rotational (torsional) springs that store elastic potential energy.





The translational spring force acting on an object connected at the leftmost point depends on the relative displacement:

$$F_{k,A} = k(x_2 - x_1)\hat{\imath} \qquad F_{k,B} = -F_{k,A}$$

and the force acting on an object connected to the right point B is opposite. The rotational spring torque acting at point C depends on the relative angles:

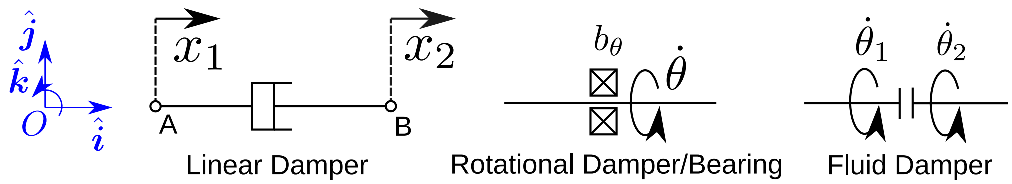
$$\boldsymbol{M}_{k,C} = k_r(\theta_2 - \theta_1)\hat{\boldsymbol{\imath}} \qquad \boldsymbol{M}_{k,D} = -\boldsymbol{M}_{k,C}$$

and the force acting on the right point B is opposite.

- **Damping elements**: dashpots, friction, air resistance or other components that dissipate energy. For a linear damper, the damping force acting at the leftmost point in the diagram below depends on the relative velocity

$$\boldsymbol{F}_{b,A} = -b(\dot{x}_2 - \dot{x}_1)\hat{\boldsymbol{\imath}} \qquad \boldsymbol{F}_{b,B} = \boldsymbol{F}_{b,A}$$

and the force acting on the right point B is equal. For a rigid shaft connected through a



rotational damper the shaft velocity before and after the damper are equal. The damping torque is then

$$\boldsymbol{M}_{b_r} = -b_r\dot{\theta}\hat{\boldsymbol{\imath}} \; .$$

For a fluid damper the two shafts may have different velocities, and the damping torque depends on their relative magnitudes:

$$\boldsymbol{M}_{b_r} = -b_r(\dot{\theta}_2 - \dot{\theta}_1)\hat{\boldsymbol{\imath}} \; .$$

- **Transmission elements:** transform one type of motion/force/moment into another. Examples include gears, levers, pulleys, belts, wheels, leadscrews, rack and pinion systems etc.

The resulting *lumped-parameter model* is a *finite-dimensional system* (i.e., an ODE that contains only a finite number of states) and is usually much easier to work with than the PDEs that arise from distributed-parameter models. Similar lumped-parameter assumptions exist for other types of systems. For electrical systems, wires are assumed to be perfectly conducting and circuits are modeled by idealized resistors, capacitors, and inductors. For thermal systems, the spatially distributed temperature of objects is replaced by lumps of mass that are assumed to have uniform temperature. Examples of some lumped-parameter models for mechanical systems are shown below.

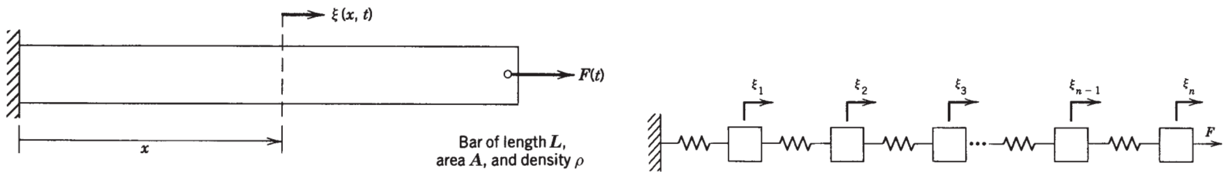


Figure 1: Distributed-parameter model of a beam in axial compression/extension (left) and analogous lumped-parameter model (right). Image Source: Karnopp, Chapter 10.

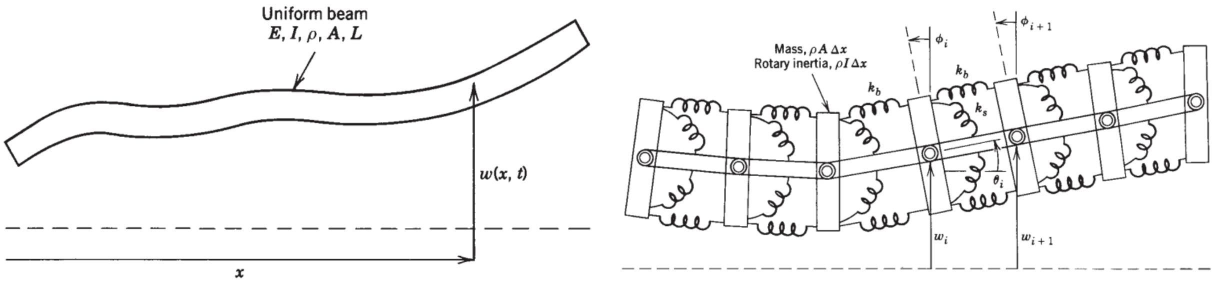


Figure 2: Distributed-parameter model of a beam in bending (left) and analogous lumped-parameter model (right). Image Source: Karnopp, Chapter 10.

Lumping Inertial Elements

Kinetic energy is stored in the mass or inertia of a mechanical system. For translational motion of a mass m with velocity v the kinetic energy is

$$T = \frac{1}{2}mv^2$$

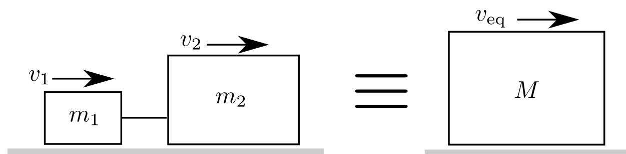
and for rotational motion

$$T = \frac{1}{2}I\dot{\theta}^2$$

where $\dot{\theta}$ is the angular velocity and I is the inertia. To lump inertial elements we seek to find the equivalent mass or inertia, m_{eq} or I_{eq}, that gives the equivalent kinetic energy as a system of masses (m_1, m_2, etc.) or inertias (I_1, I_2, etc.).

Lumping two rigidly connected masses. Consider the simplest example of 1D motion of two masses, m_1 and m_2, connected by a massless rigid bar. Since the masses are connected by a *rigid* bar their velocities are equivalent $v_{\text{eq}} = v_1 = v_2$. To lump the masses we consider the kinetic energy of a single equivalent mass, m_{eq} with velocity v_{eq} and set it equal to the total kinetic energy of the system:

$$T_{\text{total}} = \frac{1}{2}m_1 v_1^2 + m_2 v_2^2 = \frac{1}{2}(m_1 + m_2)v_1^2 = \frac{1}{2}m_{\text{eq}}v_{\text{eq}}^2 \implies m_{\text{eq}} = m_1 + m_2$$

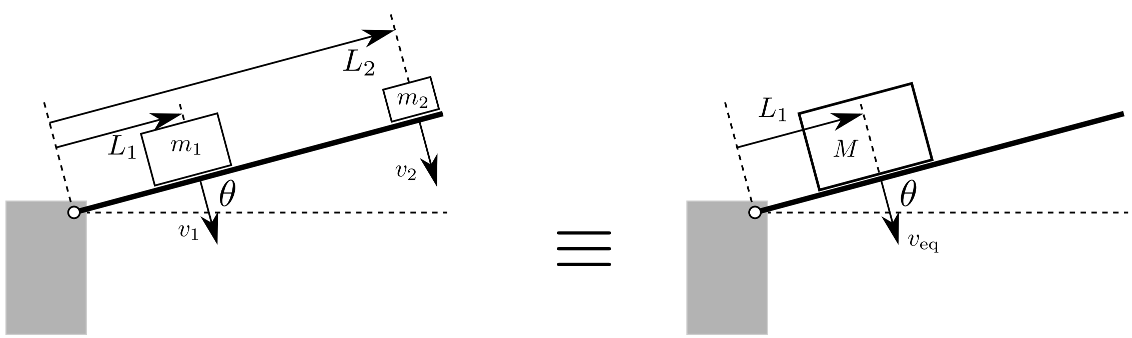


Lumping two masses on a lever. Consider another example of two masses m_1 and m_2 resting on a massless bar (lever) at positions L_1 and L_2 as shown below. Assume that the lever rotates around a hinge point with small amplitude angular motions. Since the lever arm is rigid the velocities of the masses are related by:

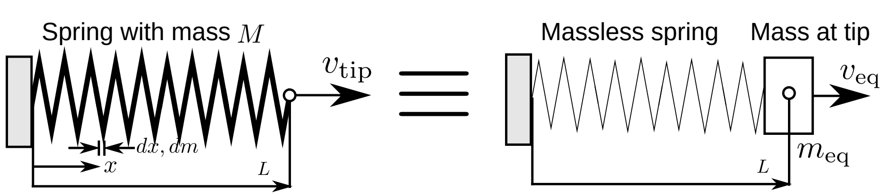
$$v_2 = v_1(L_2/L_1)$$

Suppose we wish to lump the masses into a single mass at position L_1 with equivalent velocity $v_{\text{eq}} = v_1$. Then, equating the kinetic energy of the distributed and lumped systems:

$$\begin{aligned} T_{\text{total}} &= \frac{1}{2}m_1v_1^2 + m_2v_2^2 \\ &= \frac{1}{2}m_1v_1^2 + m_2[v_1(L_2/L_1)]^2 \\ &= \frac{1}{2}(m_1 + m_2(L_2/L_1)^2)v_{\text{eq}}^2 = \frac{1}{2}m_{\text{eq}}v_{\text{eq}}^2 \implies m_{\text{eq}} = m_1 + m_2(L_2/L_1)^2 \end{aligned}$$



Equivalent mass of a spring. Consider a spring of length L with mass M whose tip is moving at a speed v_{tip}. The velocity of an infinitesimal mass dm at position x is $v(x) = v_{\text{tip}}(x/L)$. The



relationship between the infinitesimal mass dm and the infinitesimal length dx is

$$dm = dx(M/L)$$

The kinetic energy can then be expressed as:

$$T_{\text{total}} = \frac{1}{2}\int_0^M v(x)^2 dm = \frac{1}{2}m_{\text{eq}}v_{\text{eq}}^2$$

Choose the equivalent speed to be the tip speed $v_{\text{eq}} = v_{\text{tip}}$. Then, equate the total energy of the distributed and lumped systems by substituting in the previous expressions for dm and $v(x)$ so the integral is with respect to x and has integral bounds from $x = 0$ to $x = L$:

$$\begin{aligned} T_{\text{total}} &= \frac{1}{2}\int_0^L \left[v_{\text{eq}}\left(\frac{x}{L}\right)\right]^2 \left[dx\left(\frac{M}{L}\right)\right] \\ &= \frac{Mv_{\text{eq}}^2}{2L^3}\int_0^L x^2 dx = \frac{1}{2}m_{\text{eq}}v_{\text{eq}}^2 \end{aligned}$$

For the above equation to hold we require

$$m_{\text{eq}} = \frac{M}{L^3}\int_0^L x^2 dx = \frac{M}{L^3}\left[\frac{L^3}{3}\right] \implies m_{\text{eq}} = \frac{M}{3}$$

Equivalent lumped mass tables. The process described above can be repeated for various other configurations of mechanical elements. Consider the following two tables that illustrate the equivalence of a mass-spring-damper for translational systems, and its analog for rotational systems.

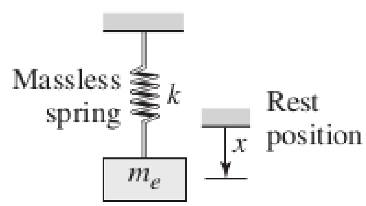
Nomenclature:

m_c = concentrated mass
m_d = distributed mass
m_e = equivalent lumped mass

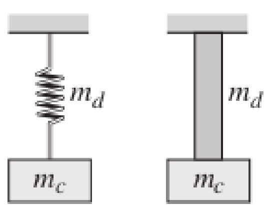
System model:

$m_e\ddot{x} + kx = 0$

Equivalent system

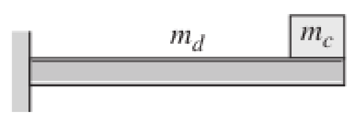


Helical spring, or rod in tension/compression



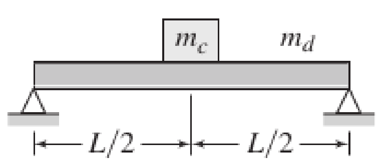
$m_e = m_c + m_d/3$

Cantilever beam



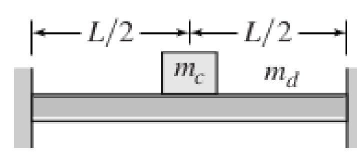
$m_e = m_c + 0.23m_d$

Simply supported beam



$m_e = m_c + 0.50m_d$

Fixed-end beam



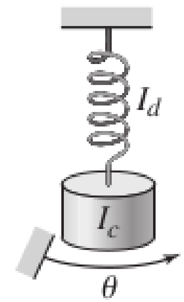
$m_e = m_c + 0.38m_d$

Rotational systems

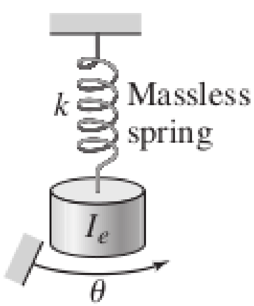
Nomenclature:
- I_c = concentrated inertia
- I_d = distributed inertia
- I_e = equivalent lumped inertia

System model:
- $I_e\ddot{\theta} + k\theta = 0$

Equivalent system

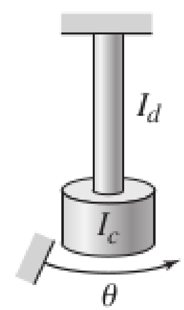


Helical spring



$I_e = I_c + I_d/3$

Rod in torsion



$I_e = I_c + I_d/3$

Equivalent Springs

The deflection or rotation of solid bodies, such as columns or rods, is related to the force applied by the stiffness, k. Such bodies can be approximated by an idealized spring element as a function of their properties: e.g., elastic modulus E, cross-section area A, length L, and and second moment of area J_A for the column (shown below), and the shear modulus G_s and polar moment of inertia of the cross-section J_P for the shaft.

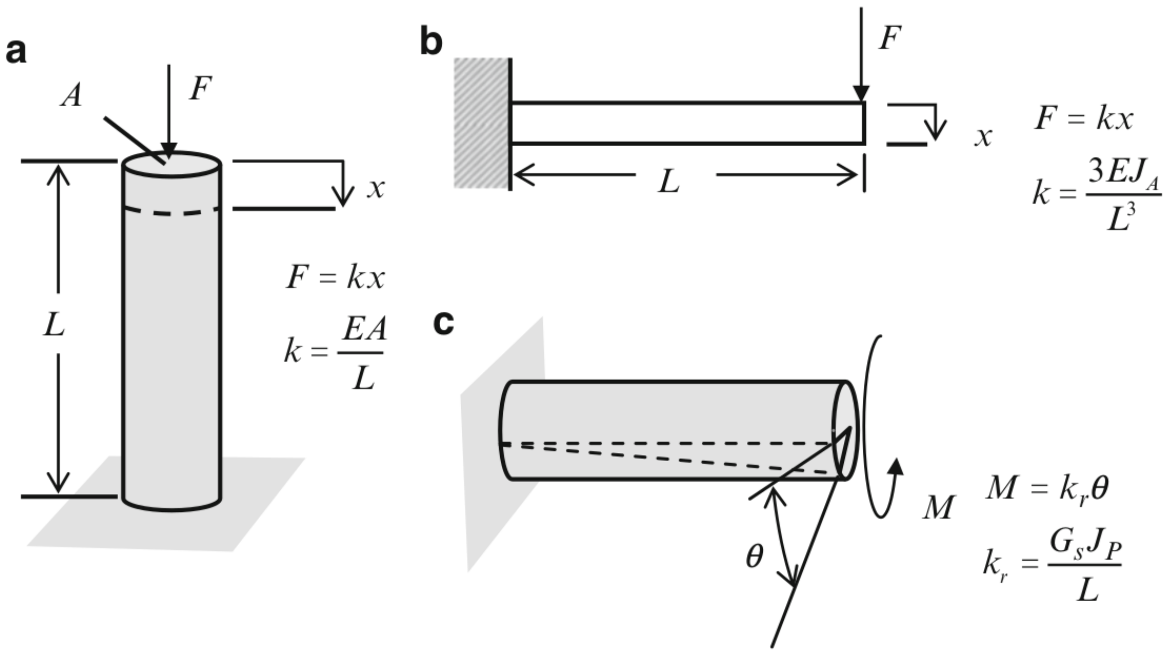
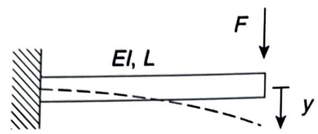


Figure 3: Example spring elements (Image: Davies and Schmitz, p. 58)

Table 3.2.1 Effective spring coefficients of certain continua

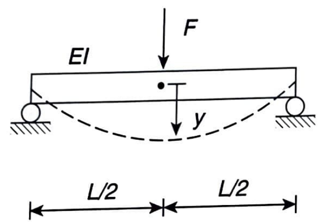
Cantilever beam



$F = k_{eff}\, y,\ k_{eff} = \dfrac{3EI}{L^3}$

$EI =$ bending stiffness

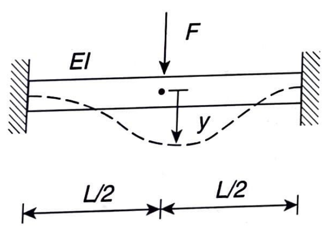
Simply supported beam



$F = k_{eff}\, y,\ k_{eff} = \dfrac{48EI}{L^3}$

$EI =$ bending stiffness

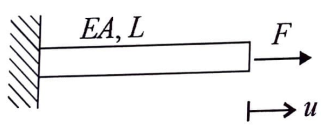
Fixed-end beam



$F = k_{eff}\, y,\ k_{eff} = \dfrac{192EI}{L^3}$

$EI =$ bending stiffness

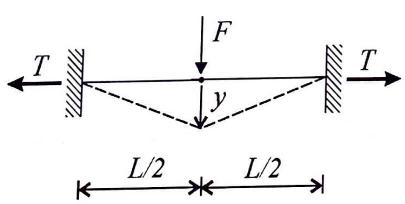
Rod in longitudinal deformation



$F = k_{eff}\, u,\ k_{eff} = \dfrac{EA}{L}$

$EA =$ longitudinal rigidity

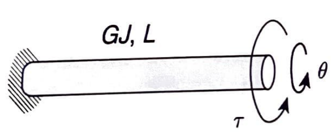
Taut string



$F = k_{eff}\, y,\ k_{eff} = \dfrac{4T}{L}$

$T =$ tension

Rod in torsion



$\tau = k_{eff}\, \theta,\ k_{eff} = \dfrac{GJ}{L}$

$GJ =$ torsional rigidity

Figure 4: Effective Spring (Image: Yang and Abramova, p. 124)

References

- Hallauler, Sec. 7.6
- Palm, Chapter 4
- Karnopp, Chapter 10