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MEGR 3122 Dynamic Systems II: Formula Sheet

1st Order Differential Equations:

Homogeneous $\dot{x} + ax = 0, \quad x(t_0) = x_0$

$$\implies x(t) = x_0 e^{-at}$$

Time Constant $\tau = 1/a$

$$x(\tau) \approx 0.368x_0, \quad x(2\tau) \approx 0.135x_0$$
$$x(3\tau) \approx 0.050x_0, \quad x(4\tau) \approx 0.018x_0$$

Inhomogeneous $\dot{x} + ax = g, \quad x(t_0) = x_0$

$$\implies x(t) = e^{-at} \int e^{at} g dt + Ce^{-at}$$

2nd Order Differential Equations:

Homogeneous $\ddot{x} + a\dot{x} + bx = 0, \;\; x(t_0) = x_0, \;\; \dot{x}(t_0) = \dot{x}_0$

Characteristic Equation $\lambda^2 + a\lambda + b = 0$

Case I: $a^2 - 4b > 0$ **(real, distinct eigenvalues)**

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Case II: $a^2 - 4b = 0$ **(repeated eigenvalues)**

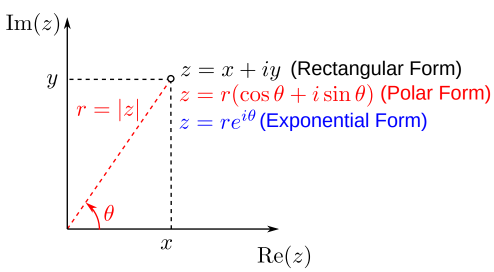
$$x(t) = C_1 e^{-at/2} + C_2 t e^{-at/2}$$

Case III: $a^2 - 4b < 0$ **(complex conjugate eigenvalues)**

$$x(t) = e^{-at/2}(A \cos \omega t + B \sin \omega t)$$

Complex Numbers:

Imaginary number $i = \sqrt{-1}$



Modulus $r = |z| = \sqrt{\mathrm{Re}(z)^2 + \mathrm{Im}(z)^2}$

Argument $\theta = \arg(z) = \mathrm{atan}(\mathrm{Im}(z)/\mathrm{Re}(z))$

Euler's Formula $e^{i\theta} = \cos \theta + i \sin \theta$

Properties

For $z = (x + iy)$ and $w = (u + iv)$

$$\bar{z} = x - iy$$
$$z \cdot w = (xu - yv) + i(xv + yu)$$
$$|z| = z \cdot \bar{z}$$
$$\frac{z}{w} = \frac{z}{w}\left(\frac{\bar{w}}{\bar{w}}\right)$$

For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$\bar{z} = re^{-i\theta}$$
$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$z_1 / z_2 = r_1 r_2 e^{i(\theta_1 - \theta_2)}$$

Laplace Transform:

Definition

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

Properties

$$\mathcal{L}[\alpha f(t)] = \alpha \mathcal{L}[f(t)] \qquad \text{(scalar multiplication)}$$
$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] \qquad \text{(addition)}$$
$$\mathcal{L}[f(t - \alpha)H(t - \alpha)] = e^{-s\alpha} F(s) \qquad \text{(translated function)}$$
$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \qquad \text{(initial value)}$$
$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s) \qquad \text{(final value)}$$
$$\mathcal{L}[\dot{f}(t)] = sF(s) - f(0) \qquad \text{(differentation)}$$
$$\mathcal{L}[\ddot{f}(t)] = s^2 F(s) - sf(0) - \dot{f}(0)$$

Table of Common Laplace Transforms

Row	$f(t)$	$F(s)$
1	Unit impulse, $\delta(t)$	1
2	Unit step/Heaviside, $H(t)$	$\frac{1}{s}$
3	Ramp, t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{(s+a)}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!} e^{-at}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left(1 + \frac{1}{a-b} \quad (be^{-bt} - ae^{-at})\right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2} \; (1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2} \; (at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} \; e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1 - \zeta^2 t})$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse Laplace Transform

$$F(s) = \mathcal{L}(f(t)) \implies f(t) = \mathcal{L}^{-1}[F(s)]$$

Partial Fraction Expansion:

Polynomial form

$$F(s) = \frac{Q(s)}{R(s)} = \frac{d_m s^m + d_{m-1}s^{m-1} + \cdots + d_1 s + d_0}{c_n s^n + c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}$$

Zero-pole-gain form

$$F(s) = \frac{A(s)}{B(s)} = \frac{k(s-z_1)(s-z_1)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} ,$$

PF Form: Case I (Distinct, Real Poles)

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1}{s-p_1} + \cdots + \frac{a_k}{s-p_k} + \cdots + \frac{a_n}{s-p_n} ,$$

with coefficients solved by

$$a_k = \left.\frac{A(s)(s-p_k)}{B(s)}\right|_{s=p_k}$$

PF Form: Case II (Repeated Real Poles).

$$F(s) = \frac{A(s)}{(s-p)^n} = \frac{a_1}{(s-p)} + \cdots + \frac{a_k}{(s-p)^k} + \cdots + \frac{a_n}{(s-p)^n}$$

with coefficients by multiplying by $(s-p)^n$

$F(s)(s-p)^n = a_1(s-p)^{n-1} + \cdots + a_k(s-p)^{n-k} + \cdots + a_n$

equating coefficients, and solving the system of equations.

PF Form: Case III (Complex Poles). $p_{1,2} = -\alpha \pm i\omega$

$$F(s) = a_1 \frac{\omega}{(s+\alpha)^2 + \omega^2} + a_2 \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

with coefficients solved by writing

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1\omega + a_2(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

equating coefficients of numertor above and solving the system of equations.

2nd Order System (Damped Harmonic Oscillator):

$$\ddot{x} + \underbrace{\left(\frac{b}{m}\right)}_{=2\zeta\omega_n} \dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{=\omega_n^2} x = 0$$

$$\implies \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Damping ratio: $\zeta = b/(2\sqrt{km}) \geq 0$

- Case $\zeta > 1$ overdamped.
- Case $\zeta = 1$ critically damped.
- Case $0 < \zeta < 1$ underdamped.
- Case $\zeta = 0$ undamped.

Natural frequency $\omega_n = \sqrt{k/m}$ (rad/s)

Damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Poles $p_{1,2} = -\zeta\omega_n \pm \omega_d i$

Misc

$$T = \frac{2\pi}{\omega} \quad \text{(seconds)}$$