Homework 4

1 Problem

For each of the following differential equations, use Laplace transforms to find the solution to the IVP.

1. $3\ddot{x} + 12\dot{x} + 60x = \delta(t);\ x(0) = 0;\ \dot{x}(0) = 0$ where $\delta(t)$ is the impulse or dirac delta function (row 1 in the Laplace transform table).
2. $\ddot{x} + 10\dot{x} + 25x = 0;\ x(0) = 1;\ \dot{x}(0) = 0$
3. $\ddot{x} + 5\dot{x} + 6x = 2e^{-t};\ x(0) = 1;\ \dot{x}(0) = 0$
4. $\ddot{x} + 2\dot{x} = 8t;\ x(0) = 0;\ \dot{x}(0) = 0$

Show all your work/intermediate steps. Other solution methods besides Laplace transforms will not receive any credit.

Solution

1.1

ODE $\quad 3\ddot{x} + 12\dot{x} + 60x = \delta(t) \qquad$ ICs $\quad x(0) = 0 \quad \dot{x}(0) = 0$

$\downarrow$ Laplace Diff. Thm. w/ I.Cs $\qquad \downarrow$ Table row 1

$$3s^2X(s) + 12sX(s) + 60X(s) = 1$$

$$X(s) = \frac{1}{3}\left(\frac{1}{\underbrace{s^2+4s+20}_{\text{poles}}}\right)$$

$$p_{1,2} = \frac{-4 \quad \sqrt{16-4(20)}}{2}$$

$$= -4 \; ' \; 8 \quad = -\frac{1}{2} \pm i$$

Since poles are complex, expand as damped sinusoid

$$X(s) = \frac{(1/3)}{s^2+4s+20} = \underbrace{\frac{C_1(s+\alpha)^2}{(s+\alpha)^2+\omega^2}}_{= C_1 \mathcal{L}[e^{-\alpha t}\cos\omega t]} + \underbrace{\frac{C_2\,\omega}{(s+\alpha)^2+\omega^2}}_{= C_2 \mathcal{L}[e^{-\alpha t}\sin\omega t]}$$

$$= \frac{C_1(s^2+2\alpha s + \alpha^2) + C_2\omega}{s^2+2\alpha s + (\alpha^2+\omega^2)}$$

$$= \frac{C_1 s^2 + 2C_1\alpha s + (C_1\alpha^2 + C_2\omega)}{s^2+2\alpha s + (\alpha^2+\omega^2)}$$

Equate denominators of LHS / RHS

$$s^2 + 4s + 20 = s^2 + 2\alpha s + (\alpha^2+\omega^2)$$

$$4 = 2\alpha \implies \boxed{\alpha = 2}$$

$$20 = (\alpha^2+\omega^2) \implies \boxed{\begin{aligned}\omega &= \sqrt{20-4} \\ &= 4\end{aligned}}$$

Equate numerators of LHS/RHS

$$\frac{1}{3} = C_1 s^2 + 2C_1\alpha + (C_1\alpha^2 + C_2\omega)$$

$$\boxed{0 = C_1}$$

$$\frac{1}{3} = \underbrace{C_1\alpha^2}_{=0} + C_2(4) \implies \boxed{C_2 = \frac{1}{12}}$$

Taking inverse laplace, $\quad x(t) = \frac{1}{12}\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+4^2}\right]$

$$\boxed{x(t) = \frac{1}{12}e^{-2t}\sin 4t}$$

1.2 $\ddot{x} + 10\dot{x} + 25x = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 0$

Laplace Transform: $\left[s^2X(s) - s\underbrace{x(0)}_{=1} - \cancel{\dot{x}(0)}_{=0}\right] + 10\left[sX(s) - \underbrace{x(0)}_{=1}\right] + 25X(s) = 0$

$$s^2X(s) - s + 10sX(s) - 10 + 25X(s) = 0$$

$$X(s) = \frac{s+10}{\underbrace{s^2+10s+25}}$$

poles: $p_{1,2} = \frac{-10 \pm \sqrt{100-4(25)}}{2}$

$= -5$

Thus, $X(s) = \frac{s+10}{(s+5)^2}$

Since poles are repeated, we expand as powers of the denom.

$$X(s) = \frac{(s+10)}{(s+5)^2} = \frac{c_1}{(s+5)} + \frac{c_2}{(s+5)^2}$$

Multiply both sides by highest power denom $(s+5)^2$

$$(s+10) = c_1(s+5) + c_2$$

$$= c_1 s + (5c_1 + c_2)$$

Equate coefficients: $\boxed{1 = c_1}$

$10 = 5(1) + c_2 \Rightarrow \boxed{c_2 = 5}$

Thus, $X(s) = \frac{1}{(s+5)} + \frac{5}{(s+5)^2}$

Inverse L.T. $x(t) = \mathcal{L}^{-1}\left[\frac{1}{s+5}\right] + 5\,\mathcal{L}^{-1}\left[\frac{1}{(s+5)^2}\right]$

row 6 row 7

$$\boxed{x(t) = e^{-5t} + 5te^{-5t}}$$

1.3

$$\ddot{x} + 5\dot{x} + 6x = 2e^{-t} \qquad x(0) = 1 \qquad \dot{x}(0) = 0$$

row 6

Take L.T. $[s^2X(s) - s\underbrace{x(0)}_{=1} - \underbrace{\cancel{\dot{x}(0)}}_{=0}] + 5[sX(s) - \underbrace{x(0)}_{=1}] + 6X(s) = 2\left(\frac{1}{s+1}\right)$

$$s^2X(s) - s + 5sX(s) - 5 + 6X(s) = 2\left(\frac{1}{s+1}\right)$$

$$X(s)(s^2+5s+6) = 2\left(\frac{1}{s+1}\right) + (s+5)$$

$$X(s) = \frac{2}{(s+1)(s^2+5s+6)} + \frac{(s+5)}{(s^2+5s+6)}$$

make common denom.

$$= \frac{2 + (s+5)(s+1)}{(s+1)(s^2+5s+6)}$$

$$= \frac{2 + s^2 + 6s + 5}{(s+1)(s^2+5s+6)}$$

poles: $p_1 = -1$ $\quad p_{2,3} = \frac{(-5 \pm \sqrt{25 - 4(6)})}{2}$

$$= \frac{-5 \pm 1}{2}$$

$$p_2 = -2$$
$$p_3 = -3$$

Thus, $X(s) = \dfrac{s^2+6s+7}{(s+1)(s+2)(s+3)}$

Since all poles are distinct, expand as partial fractions with corresponding distinct denominators

$$X(s) = \frac{s^2+6s+7}{(s+1)(s+2)(s+3)} = \frac{c_1}{(s+1)} + \frac{c_2}{(s+2)} + \frac{c_3}{(s+3)}$$

To solve for coefficients we multiply both sides by denominator and set s equal to the pole. (see topic 11)

$$c_1 = \left.\frac{(s^2+6s+7)}{(s+2)(s+3)}\right|_{s=-1} = \frac{1-6+7}{1 \cdot 2} = 1$$

$$C_2 = \left.\frac{(s^2+6s+7)}{(s+1)(s+3)}\right|_{s=-2} = \frac{4-12+7}{-1\cdot 1} = 1$$

$$C_3 = \left.\frac{(s^2+6s+7)}{(s+1)(s+2)}\right|_{s=-3} = \frac{9-18+7}{-2\cdot -1} = -1$$

Taking I.L.T. $x(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$

$\downarrow$ row 6

$$\boxed{x(t) = e^{-t} + e^{-2t} - e^{-3t}}$$

1.4 $\ddot{x} + 2\dot{x} = 8t \qquad \dot{x}(0) = 0 \qquad x(0) = 0$

row 3

L.T. $s^2X(s) + 2sX(s) = 8\frac{1}{s^2}$

$$X(s) = 8\frac{1}{s^2}\frac{1}{(s^2+2s)}$$

$$= \frac{8}{s^3(s+2)}$$

$p_4 = -2$ (distinct)

$p_{1,2,3} = 0$ (repeated poles)

This case involves mixed poles, hence we expand using both methods (for distinct and for real)

$$X(s) = \frac{8}{s^3(s+2)} = \underbrace{\frac{c_1}{s} + \frac{c_2}{s^2} + \frac{c_3}{s^3}}_{\text{method for repeated}} + \underbrace{\frac{c_4}{(s+2)}}_{\text{method for distinct}}$$

Multiply both sides by $s^3(s+2)$:

$$8 = c_1s^2(s+2) + c_2s(s+2) + c_3(s+2) + c_4s^3$$

$$= c_1s^3 + 2c_1s^2 + c_2s^2 + 2c_2s + c_3s + 2c_3 + c_4s^3$$

$$= (c_1+c_4)s^3 + (2c_1+c_2)s^2 + (2c_2+c_3)s + 2c_3$$

Equating coefficients

const. : $8 = 2c_3 \Rightarrow \boxed{c_3 = 4}$

s : $0 = 2c_2 + c_3 \Rightarrow c_2 = -\frac{c_3}{2}$

$\boxed{c_2 = -2}$

s^2 : $0 = 2c_1 + c_2 \Rightarrow \boxed{c_1 = 1}$

s^3 : $0 = c_1 + c_4 \Rightarrow \boxed{c_4 = -1}$

I.L.T: $x(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$

row 2 3 4 6

$$\boxed{\begin{array}{l} x(t) = u(t) - 2t + 4\frac{t^2}{2} - e^{-2t} \\ \text{or} \\ x(t) = 1 - 2t + 2t^2 - e^{-2t} \end{array}}$$

2 Problem

Use the MATLAB function `dsolve` to verify your answer for Problem 1.4. Generate a plot of the solution over the time interval $t \in [0, 3]$ seconds. Submit your code.

Solution

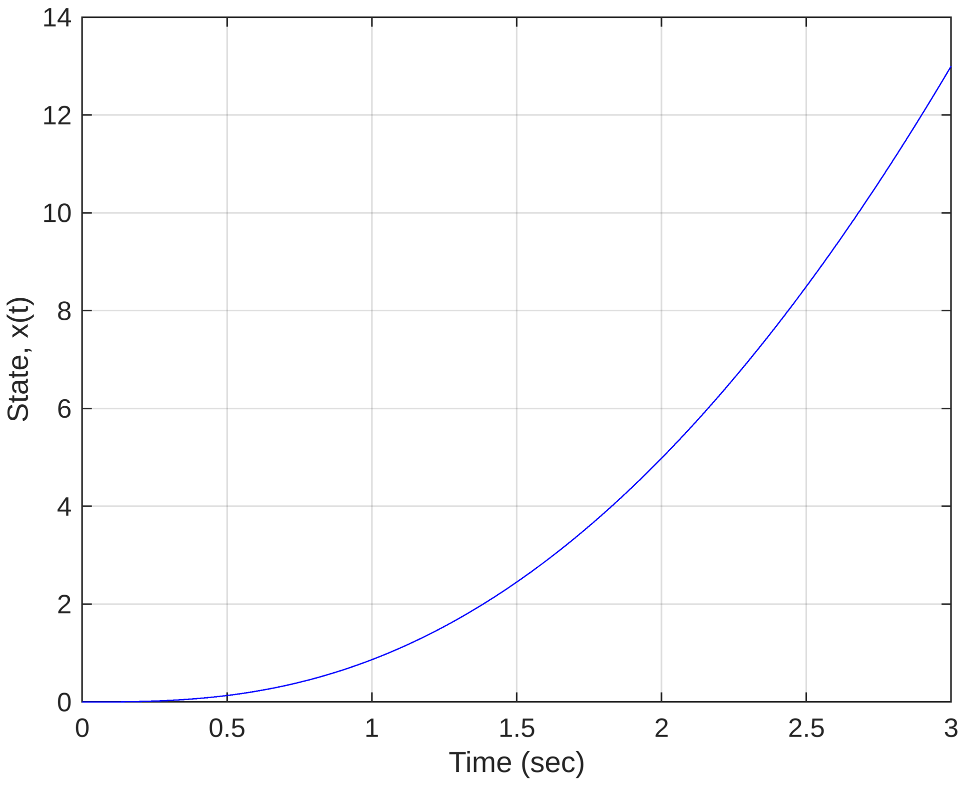
Using the symbolic toolbox with dsolve to solve Problem 1.4

```
syms x(t);
xdot = diff(x,t);
xddot = diff(x,t,2);
assume(t>=0)
eqn = xddot == -2*xdot + 8*t;
cond = [x(0)==0, xdot(0)==0];
xsol = dsolve(eqn,cond)
```

xsol = $2\,t^2 - e^{-2\,t} - 2\,t + 1$

Now to plot the solution

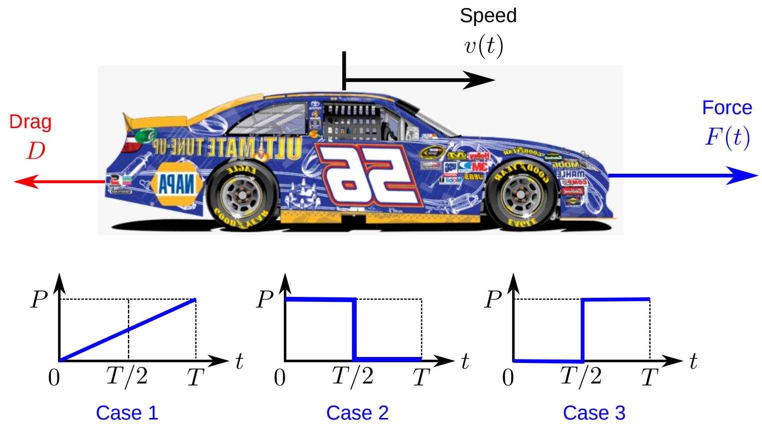
```
tvals = linspace(0,3);
xvals = double(subs(xsol,tvals));
figure;
plot(tvals,xvals,'b-')
grid on;
xlabel('Time (sec)')
ylabel('State, x(t)');
```



3 Problem

Suppose the racecar below has a mass of $m = 750$ kg and is moving down a track with an initial speed of $v(t_0) = 45$ m/s at time $t_0 = 0$ sec. The drag on the car is modeled as a linear function of velocity: $D = bv$, where $b = 20$ N/(m/s).

- Using the free-body diagram below, where $F(t)$ is an applied force, apply Newton's 2nd Law to find the equations of motion. Since $a(t) = \dot{v}(t)$ you can write this equation as a first-order ODE in speed (i.e., $\sum F = m\dot{v}$).



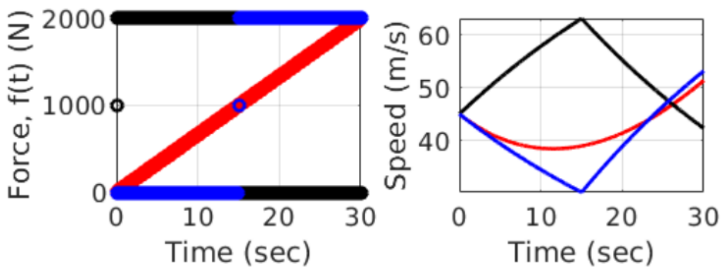
Suppose that over the next $T = 30$ seconds the driver can choose from the three possible force profiles, $F(t)$, shown above, where $P = 2000$ N is the same maximum force reached during each profile.

- Write down an expression for each of the force profiles $F_1(t)$, $F_2(t)$, $F_3(t)$ as a function of the magnitude P and time. You can construct the force profiles from a combination of Heaviside functions and ramps (straight lines) with appropriate slope. Reviewing the doublet example (Lecture 7 PDF, p.2) may be helpful.

Interestingly, each profile has the same impulse (area under the force-time curve) but results in a different final displacement and velocity. Determine the velocity profile $v(t)$ that results from each case by following these steps:

- Solve for the velocity profile in each of the three cases using MATLAB (following the methods of Lecture 10 e.g., using `laplace`, `solve`, and `ilaplace` OR `dsolve`). and plot the three solutions on the same axes. Which case results in the largest final speed? Label your axes, add a legend for each line, and use a thick line type for clarity.

Note that MATLAB defines the step function as: `heaviside(t)`.Your solution should look similar to the one below:



Bonus: Which case results in the furthest distance traveled at time T? Justify your answer with a plot of distance traveled in MATLAB.

Solution

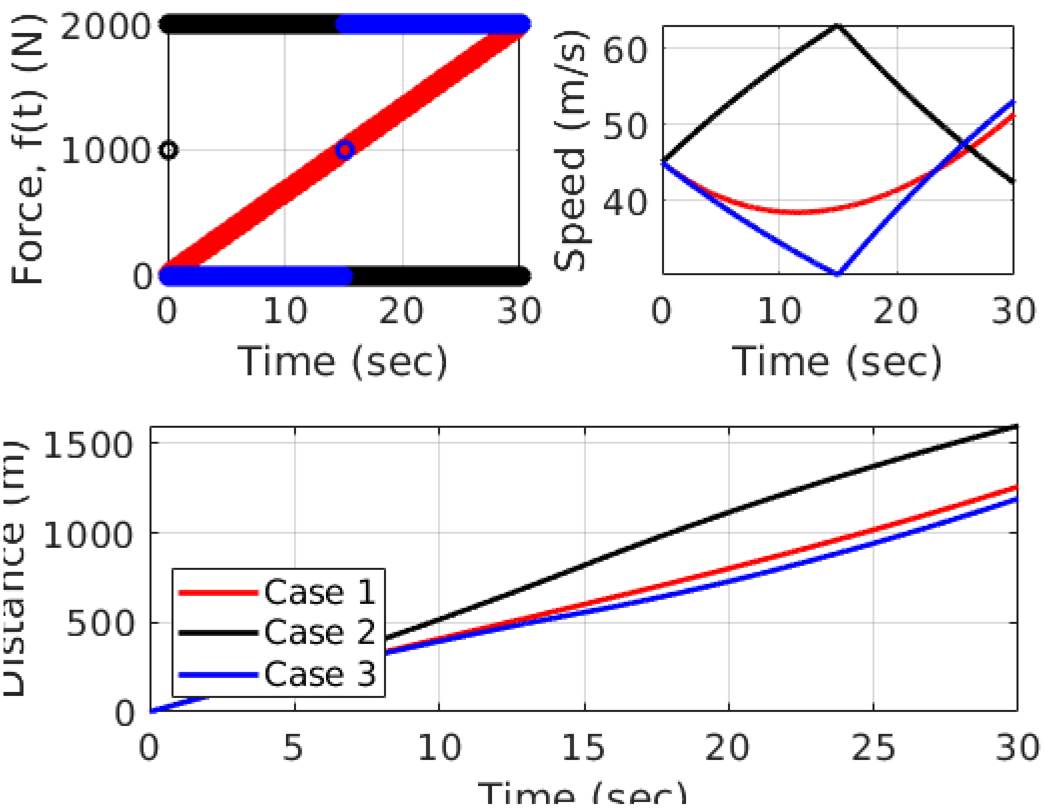
Answers may vary. Solution below uses `laplace`, `solve`, `ilaplace`, `eval`. Another approach may use `dsolve`.

```
clear; close all; clc; % prepare workspace

% constants
b = 20; % N/(m/s)
m = 750; % kg
v0 = 45; % x(0) IC, initial position
P = 2000;
T = 30;
tvals = [0:1/50:T]; % 50 frames per second

% v' + b/m*v = f

% case 1
fprintf('---------------\nCase 1:\n');
syms V s v t f; % define symbolic variables
f = t*P/T;
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist1 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist1 = eval(subs(int(v,0,t),tvals));
fhist1 = eval(subs(f,tvals));

% case 2
fprintf('---------------\nCase 2:\n');
f = P*heaviside(t) -P*heaviside(t-T/2);
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist2 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist2 = eval(subs(int(v,0,t),tvals));
fhist2 = eval(subs(f,tvals));

% case 3
fprintf('---------------\nCase 3:\n');
f = P*heaviside(t-T/2);
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist3 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist3 = eval(subs(int(v,0,t),tvals));
fhist3 = eval(subs(f,tvals));

figure;
subplot(2,2,1)
plot(tvals,fhist1,'ro','linewidth',2); hold on;
```

```
plot(tvals,fhist2,'ko','linewidth',2)
plot(tvals,fhist3,'bo','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Force, f(t) (N)')
grid on;
hold on;
axis tight;

subplot(2,2,2)
plot(tvals,vhist1,'r','linewidth',2); hold on;
plot(tvals,vhist2,'k','linewidth',2)
plot(tvals,vhist3,'b','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Speed (m/s)')
grid on;
hold on;
axis tight;

subplot(2,2,[3:4])
plot(tvals,dhist1,'r','linewidth',2); hold on;
plot(tvals,dhist2,'k','linewidth',2)
plot(tvals,dhist3,'b','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Distance (m)')
legend('Case 1','Case 2','Case 3','location','southwest');
grid on;
hold on;
axis tight;

fprintf('*******************************************************\n')
fprintf('Case 3 leads to the greatest speed at t = 30 sec\n')
fprintf('Case 2 leads to the greatest distance at t = 30 sec\n')
fprintf('Case 1 has intermediate performance compared to Case 2 and
 3\n')
fprintf('*******************************************************\n')
```

```
---------------
Case 1:
10 t      /   2 t \
---- + exp| - --- | 170 - 125
  3       \    75 /

---------------
Case 2:
                  /    / 2   2 t \         \
                  | exp| - - --- | 75      |
                  |    \ 5    75 /      75 |
heaviside(t - 15) | ----------------- - -- | 8
                  \         2            2 /        /   2 t \
---------------------------------------------- - exp| - --- | 55 + 100
                       3                            \    75 /
```

```
---------------
Case 3:
                                    /    / 2    2 t \         \
                                    | exp| - - --- | 75       |
                                    |    \ 5    75 /      75 |
                  heaviside(t - 15) | ----------------- - -- | 8
   /   2 t \                        \         2            2 /
exp| - --- | 45 - ----------------------------------------------
   \    75 /                             3

******************************************************
Case 3 leads to the greatest speed at t = 30 sec
Case 2 leads to the greatest distance at t = 30 sec
Case 1 has intermediate performance compared to Case 2 and 3
******************************************************
```



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