

# MEGR 3122 Dynamic Systems II: Formula Sheet

## 1st Order Differential Equations:

**Homogeneous**  $\dot{x}(t) + ax(t) = 0, \quad x(t_0) = x_0$

$$\Rightarrow x(t) = x_0 e^{-at}$$

**Time Constant**  $\tau = 1/a$

$$x(\tau) \approx 0.368x_0, \quad x(2\tau) \approx 0.135x_0$$

$$x(3\tau) \approx 0.050x_0, \quad x(4\tau) \approx 0.018x_0$$

**Inhomogeneous**  $\dot{x}(t) + ax(t) = u(t), \quad x(t_0) = x_0$

$$\Rightarrow x(t) = e^{-at} \int e^{at} u(t) dt + C e^{-at}$$

## 2nd Order Differential Equations:

**Homogeneous**  $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0$

**Characteristic Equation**  $\lambda^2 + a\lambda + b = 0$

**Quadratic Formula**  $\lambda_{1,2} = (-a \pm \sqrt{a^2 - 4b})/2$

**Case I:**  $a^2 - 4b > 0$  (real, distinct eigenvalues)

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

**Case II:**  $a^2 - 4b = 0$  (repeated eigenvalues)

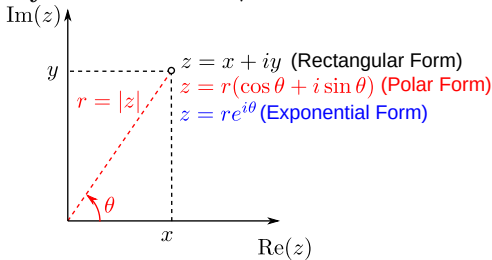
$$x(t) = C_1 e^{-at/2} + C_2 t e^{-at/2}$$

**Case III:**  $a^2 - 4b < 0$  (complex conjugate eigenvalues)

$$x(t) = e^{-at/2} (A \cos \omega t + B \sin \omega t)$$

## Complex Numbers:

**Imaginary number**  $i = \sqrt{-1}$



**Modulus**  $r = |z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$

**Argument**  $\theta = \arg(z) = \text{atan}(\text{Im}(z)/\text{Re}(z))$

**Euler's Formula**  $e^{i\theta} = \cos \theta + i \sin \theta$

## Properties

For  $z = (x + iy)$  and  $w = (u + iv)$

$$\bar{z} = x - iy$$

$$z \cdot w = (xu - yv) + i(xv + yu)$$

$$|z| = z \cdot \bar{z}$$

$$\frac{z}{w} = \frac{z}{w} \left( \frac{\bar{w}}{\bar{w}} \right)$$

For  $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$

$$\bar{z} = r e^{-i\theta}$$

$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

$$z_1 / z_2 = r_1 r_2 e^{i(\theta_1 - \theta_2)}$$

## Laplace Transform:

### Definition

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

### Properties

$$\mathcal{L}[\alpha f(t)] = \alpha \mathcal{L}[f(t)] \quad (\text{scalar multiplication})$$

$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] \quad (\text{addition})$$

$$\mathcal{L}[f(t - \alpha)H(t - \alpha)] = e^{-s\alpha} F(s) \quad (\text{translated function})$$

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \quad (\text{initial value})$$

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \quad (\text{final value})$$

$$\mathcal{L}[\dot{f}(t)] = sF(s) - f(0) \quad (\text{differentiation})$$

$$\mathcal{L}[\ddot{f}(t)] = s^2 F(s) - sf(0) - \dot{f}(0)$$

### Table of Common Laplace Transforms

Row	$f(t)$	$F(s)$
1	Unit impulse, $\delta(t)$	1
2	Unit step/Heaviside, $H(t)$	$\frac{1}{s}$
3	Ramp, $t$	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \dots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \dots$	$\frac{n!}{s^{n+1}}$
6	$e^{-at}$	$\frac{1}{(s+a)}$
7	$te^{-at}$	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!} e^{-at}, \quad n = 1, 2, 3, \dots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2 + \omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2 + \omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2 - \omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2 - \omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab} \left( 1 + \frac{1}{a-b} (be^{-bt} - ae^{-at}) \right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2} (1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2} (at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at} \sin(\omega t)$	$\frac{\omega}{(s+a)^2 + \omega^2}$
21	$e^{-at} \cos(\omega t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta \omega_n t} \sin(\omega_n \sqrt{1-\zeta^2} t)$	$\frac{s+a}{(s+a)^2 + \omega^2}$

## Inverse Laplace Transform

$$F(s) = \mathcal{L}(f(t)) \implies f(t) = \mathcal{L}^{-1}[F(s)]$$

## Partial Fraction Expansion:

### Polynomial form

$$F(s) = \frac{Q(s)}{R(s)} = \frac{d_m s^m + d_{m-1} s^{m-1} + \dots + d_1 s + d_0}{c_n s^n + c_{n-1} s^{n-1} + \dots + c_1 s + c_0}$$

### Zero-pole-gain form

$$F(s) = \frac{A(s)}{B(s)} = \frac{k(s - z_1)(s - z_2) \dots (s - z_m)}{(s - p_1)(s - p_2) \dots (s - p_n)},$$

### PF Form: Case I (Distinct, Real Poles)

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1}{s - p_1} + \dots + \frac{a_k}{s - p_k} + \dots + \frac{a_n}{s - p_n},$$

with coefficients solved by

$$a_k = \left. \frac{A(s)(s - p_k)}{B(s)} \right|_{s=p_k}$$

### PF Form: Case II (Repeated Real Poles).

$$F(s) = \frac{A(s)}{(s - p)^n} = \frac{a_1}{(s - p)} + \dots + \frac{a_k}{(s - p)^k} + \dots + \frac{a_n}{(s - p)^n}$$

with coefficients by multiplying by  $(s - p)^n$

$$F(s)(s - p)^n = a_1(s - p)^{n-1} + \dots + a_k(s - p)^{n-k} + \dots + a_n$$

equating coefficients, and solving the system of equations.

### PF Form: Case III (Complex Poles). $p_{1,2} = -\alpha \pm i\omega$

$$F(s) = a_1 \frac{\omega}{(s + \alpha)^2 + \omega^2} + a_2 \frac{(s + \alpha)}{(s + \alpha)^2 + \omega^2}$$

with coefficients solved by writing

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1 \omega + a_2(s + \alpha)}{(s + \alpha)^2 + \omega^2}$$

equating coefficients of numerator above and solving the system of equations.

### 2nd Order System (Damped Harmonic Oscillator):

$$\ddot{x} + \underbrace{\left(\frac{b}{m}\right)}_{=2\zeta\omega_n} \dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{=\omega_n^2} x = 0$$

$$\implies \ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$$

**Damping ratio:**  $\zeta = b/(2\sqrt{km}) \geq 0$

- Case  $\zeta > 1$  overdamped.
- Case  $\zeta = 1$  critically damped.
- Case  $0 < \zeta < 1$  underdamped.
- Case  $\zeta = 0$  undamped.

**Natural frequency**  $\omega_n = \sqrt{k/m}$  (rad/s)

**Damped natural frequency**  $\omega_d = \omega_n \sqrt{1 - \zeta^2}$

**Poles**  $p_{1,2} = -\zeta\omega_n \pm \omega_d i$

**Period**  $T = 2\pi/\omega$  (seconds)