Homework 2

1 Problem

Let $z = x + iy$. Determine the value of x and y in each of the following cases below. Show all of your work for full credit. You can check your answer in MATLAB by typing in the expression with `1i` representing the imaginary number (e.g., `(3+1i)*(1+3i)`).

1. $z = (3 + i)(1 + 3i)$
2. $z = i^4 - 1$
3. $z = \frac{3+i}{1+3i}$
4. If $w = 1 + 2i$, then what is $|w|$ and $\theta = \arg(w)$?
5. If $z_1 = -i$ and $z_2 = e^{i\pi/2}$, then what is the sum $z_1 + z_2$?

Solution

Let $z = x + iy$. Determine the value of x and y in each of the following cases:

1. $x = 0$ and $y = 10$

$$\begin{aligned} z &= (3 + i)(1 + 3i) \\ &= 3 + 9i + i + 3i^2 \\ &= 3 + 9i + i - 3 \\ &= 10i \end{aligned}$$

2. $x = 0$ and $y = 0$

$$\begin{aligned} z &= i^4 - 1 \\ &= (i^2)(i^2) - 1 \\ &= (-1)(-1) - 1 \\ &= 0 \end{aligned}$$

3. $x = 3/5$ and $y = -4/5$

$$\begin{aligned} z &= \frac{3+i}{1+3i} \\ &= \frac{3+i}{1+3i}\left(\frac{1-3i}{1-3i}\right) \\ &= \frac{3-9i+i-3i^2}{1-3i+3i-9i^2} \\ &= \frac{3-8i+3}{1+9} \\ &= \frac{3-8i+3}{1+9} \\ &= \frac{6-8i}{10} \\ &= \frac{3-4i}{5} \end{aligned}$$

4. $|w| = \sqrt{1+2^2} = \sqrt{5}$ and $\theta = \arg(w) = \text{atan}(2) \approx 63.435$ deg.
5. Using Euler's Formula (or by inspection) $z_2 = i$ so that $z_1 + z_2 = -i + i = 0$.

2 Problem

Find the Laplace transform $F(s) = \mathcal{L}[f(t)]$ for each of the following functions. For full credit, show all of your work by expanding each Laplace transform into one or more Laplace transforms of the common functions listed in the provided Laplace Transform Table. Indicate which rows of the table you used to obtain your solution and fully simplify the result to a single term.

1. $f(t) = e^{at+b}$ where a, b are constant
2. $f(t) = \sin(\omega t - \phi)$ where ω, ϕ are constant
3. $f(t) = t^3 - 1/2$
4. $f(t) = (e^{-t}/4)[2 + t^2 + \cos(3t)]$. (Note: For this problem there is no need to simplify to a common denominator.)
5. $f(t) = (t-2)H(t-2)$, where $H(\cdot)$ is the unit step or Heaviside function

Solution

1. $f(t) = e^{at+b}$ where a, b are constant
   Using Laplace transform table (rows 2 and 6):

$$\begin{aligned} \mathcal{L}[e^{at+b}] &= \mathcal{L}[e^{at}e^{b}] \\ &= e^{b}\mathcal{L}[e^{at}] \\ &= e^{b}\left(\frac{1}{s-a}\right) \end{aligned}$$

2. $f(t) = \sin(\omega t - \phi)$ where ω, ϕ are constant
   Using Laplace transform table (rows 10 and 11) and the sine angle difference identity

$$\begin{aligned}
\mathcal{L}[\sin(\omega t - \phi)] &= \mathcal{L}[\sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)] \\
&= \mathcal{L}[\sin(\omega t)\cos(\phi)] - \mathcal{L}[\cos(\omega t)\sin(\phi)] \\
&= \cos(\phi)\mathcal{L}[\sin(\omega t)] - \sin(\phi)\mathcal{L}[\cos(\omega t)] \\
&= \cos(\phi)\left(\frac{\omega}{s^2+\omega^2}\right) - \sin(\phi)\left(\frac{s}{s^2+\omega^2}\right) \\
&= \left(\frac{\cos(\phi)\omega - \sin(\phi)s}{s^2+\omega^2}\right)
\end{aligned}$$

3. $f(t) = t^3 - 1/2$
   Using Laplace transform table (rows 2 and 5):

$$\begin{aligned}
\mathcal{L}[t^3 - 1/2] &= \mathcal{L}[t^3] - \mathcal{L}[1/2H(t)] \\
&= \frac{3!}{s^4} - \frac{1}{2}\frac{1}{s} \\
&= \frac{6}{s^4} - \frac{1}{2s} \\
&= \frac{-s^3+12}{2s^4}
\end{aligned}$$

4. $f(t) = \frac{e^{-t}}{4}[2 + t^2 + \cos(3t)]$
   First expand the function

$$f(t) = \frac{1}{2}e^{-t} + \frac{1}{4}e^{-t}t^2 + \frac{1}{4}e^{-t}\cos 3t$$

   then look in the Laplace transform table to find similar forms on row 6, row 8 (with $n = 3$) and on row 21 (with $a = 1$ and $\omega = 3$). The Laplace transform is then

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{1}{2}\mathcal{L}[e^{-t}] + \frac{1}{2}\mathcal{L}[\frac{t^2e^{-t}}{2}] + \frac{1}{4}\mathcal{L}[e^{-t}\cos 3t] \\
&= \frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{1}{(s+1)^3}\right) + \frac{1}{4}\left(\frac{s+1}{(s+1)^2+9}\right) \\
&= \frac{1}{4}\left(\frac{2[(s+1)^2+9](s+1)^2 + 2[(s+1)^2+9] + (s+1)^3}{(s+1)((s+1)^2+9)}\right) \\
&= \frac{1}{4}\left(\frac{2[(s+1)^2+9](s+1)^2 + 2[(s+1)^2+9] + (s+1)^3}{(s+1)(s^2+2s+10)}\right)
\end{aligned}$$

5. Notice that the function $(t-2)$ is just a ramp (t) shifted by $\alpha = 2$ units. The Laplace transform of a ramp is $\mathcal{L}[t] = 1/s^2$. Then using the LT transform theorem for translated functions:

$$F(s) = e^{-\alpha s}\mathcal{L}[t] = \frac{e^{-2s}}{s^2}$$

3 Problem

What is the final value of $x(t)$ as $t \to 0$ if the Laplace transform of $x(t)$ is the following?

$$X(s) = \frac{6}{s(s+2)}$$

Solution

From the initial value theorem,

$$\begin{aligned}
\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) && (1)\\
&= \lim_{s\to 0} s\frac{6}{s(s+2)} && (2)\\
&= \lim_{s\to 0} \frac{6}{(s+2)} && (3)\\
&= \frac{6}{2} = 3 && (4)
\end{aligned}$$

4 Problem

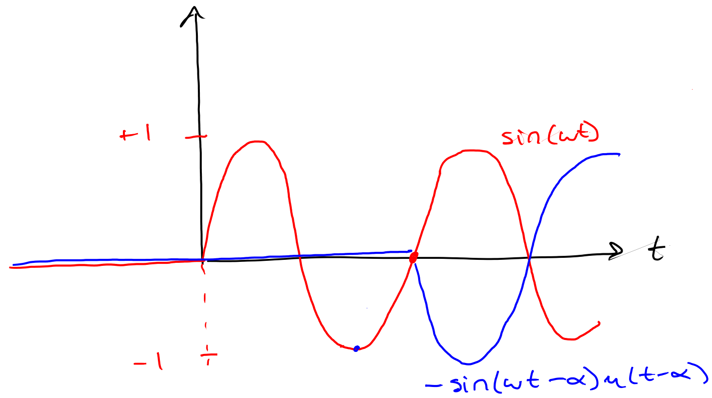
Sketch (by hand) the following function on the interval $t \in [-2\pi, 4\pi]$ and find its Laplace transform:

$$f(t) = \sin(t) - \sin(t-\alpha)H(t-\alpha)$$

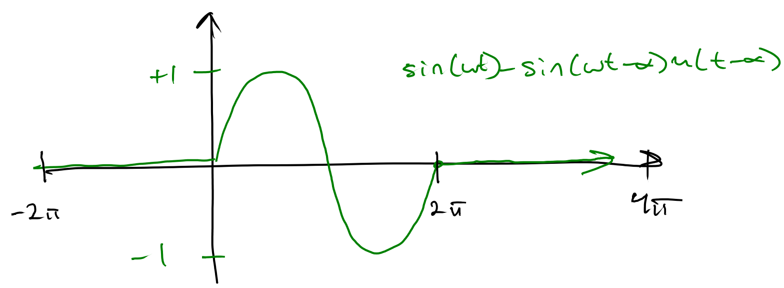
where $H(t-\alpha)$ is the unit step function delayed to start at time $\alpha = 2\pi$.

Hint: Use the Laplace transform for a translated function.

Solution



Adding together the two functions:



$$\mathcal{L}[f(t)] = \mathcal{L}[\sin(\omega t)] - \mathcal{L}[\sin(\omega t - \alpha)u(t-\alpha)]$$

row 10 of L.T. table

see topic 10

$$= \frac{\omega}{s^2+\omega^2} - e^{-\alpha s}\,\mathcal{L}^{-1}[\underbrace{\sin(\omega t - 2\pi)}_{=\sin(\omega t)}]$$

$\alpha = 2\pi$

$$= \frac{\omega}{s^2+\omega^2} - e^{-2\pi s}\frac{\omega}{s^2+\omega^2}$$

$$= \frac{\omega}{s^2+\omega^2}\left(1 - e^{-2\pi s}\right)$$