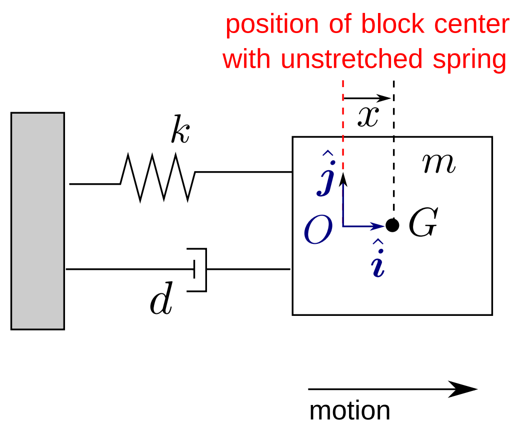
Lecture 4: Solving Linear Second-Order Homogeneous ODEs

Consider a linear time-invariant (LTI) second-order homogeneous ODE

$$\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0 \tag{1}$$

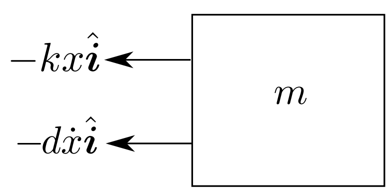
Second-order systems of this form have many anologies in mechanical and electrical engineering. Consider the following motivating example of a spring-mass-damper in the horizontal plane (ignoring gravity) with mass m, spring constant k, and damping coefficient d. Let x measure the displacement of the block from its unstretched position.



First consider the kinematics: the inertial acceleration of point G with respect to point the origin point O is

$$\boldsymbol{a}_{G/O} = \ddot{x}\hat{\boldsymbol{\imath}}\,.$$

Next, draw a free body diagram containing the spring and damper forces. The free body diagram arrows are drawn assuming a positive x and $\dot{x}$:



Summing forces

$$\sum \boldsymbol{F} = -(kx + d\dot{x})\hat{\boldsymbol{\imath}}$$

Equating the acceleration and forces using Newton's 2nd Law (in the $\hat{\boldsymbol{\imath}}$ direction)

$$\begin{aligned}
\sum \boldsymbol{F} &= m\boldsymbol{a}_{G/O} \\
\hat{\boldsymbol{\imath}}: \qquad -(kx + d\dot{x}) &= m\ddot{x} \\
m\ddot{x} + kx + d\dot{x} &= 0 \\
\ddot{x} + \underbrace{\left(\frac{d}{m}\right)}_{\triangleq a}\dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{\triangleq b}x &= 0
\end{aligned}$$

giving the second-order homogeneous system:

$$\ddot{x} + a\dot{x} + b\dot{x} = 0 \tag{2}$$

Recall that for our first-order LTI homogeneous system, $\dot{x}(t) + ax(t) = 0$, we found that the general solution was $x(t) = Ce^{-at}$. So, let's try a similar solution for (1) as a guess. Instead of using the variable a we will use λ and we will drop the constant C. That is, suppose

$$x(t) = e^{\lambda t} \tag{3}$$

is a solution of (1). Before we can plug (3) into (1) to check if it is valid, we compute the first and second derivative terms, $\dot{x}$ and $\ddot{x}$, which appear in (1):

$$\begin{aligned} \dot{x}(t) &= \lambda e^{\lambda t} \\ \ddot{x}(t) &= \lambda^2 e^{\lambda t} \; . \end{aligned}$$

Now, plugging into (1) and collecting like terms:

$$\begin{aligned} [\lambda^2 e^{\lambda t}] + a[\lambda e^{\lambda t}] + b[e^{\lambda t}] &= 0 \\ (\lambda^2 + a\lambda + b)e^{\lambda t} &= 0 \end{aligned}$$

This result suggests that our guess is valid if the left hand side $(\lambda^2 + a\lambda + b)e^{\lambda t}$ is equal to zero. But the term $e^{\lambda t}$ is never zero, thus the equation can only hold if

$$\lambda^2 + a\lambda + b = 0 \; . \tag{4}$$

Notice that (4) is a quadratic equation in λ with coefficients equal to those of the left-hand side in (1). We call (4) the *characteristic equation* of the system (1).

> *Recall*: For a quadratic equation of the form $\tilde{a}x^2 + \tilde{b}x + \tilde{c} = 0$, the quadratic formula
>
> $$x_{1,2} = \frac{-\tilde{b} \pm \sqrt{\tilde{b}^2 - 4\tilde{a}\tilde{c}}}{2\tilde{a}}$$
>
> yields the two roots (i.e., solutions) of the quadratic. The expression inside the square, $\tilde{b}^2 - 4\tilde{a}\tilde{c}$, is called the discriminant. Its value (negative, zero, or positive) determines whether the roots are imaginary, repeated, or distinct, respectively.

Applying the quadratic formula to (4) gives the roots:

$$\lambda_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} \tag{5}$$

For these special values of λ, called the *eigenvalues*, our guess (3) is a valid solution of (1). In the following, we examine the general solution for each case of discriminant sign. The particular solution is obtained by applying the initial conditions.

Case I: $a^2 - 4b > 0$ (real, distinct eigenvalues)

In this case, there are two distinct eigenvalues:

$$\lambda_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \qquad \text{and} \qquad \lambda_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}$$

Add back the integration constant (C_1 or C_2) and the two valid general solutions are

$$x_1(t) = C_1 e^{\lambda_1 t} \qquad \text{and} \qquad x_2(t) = C_2 e^{\lambda_2 t} \; .$$

In fact, any linear combination of these solutions is also a solution and the most general form of a solution to (1) in the case that $a^2 - 4b > 0$ is

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} \; . \tag{6}$$

To convince yourself that adding both general solutions as in (6) is also a solution (2) you can simply substitute it in and check that the LHS and RHS are equal. First note that:

$$\dot{x} = \lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t} \tag{7}$$

$$\ddot{x} = \lambda_1^2 C_1 e^{\lambda_1 t} + \lambda_2^2 C_2 e^{\lambda_2 t} \tag{8}$$

then

$$\ddot{x} + a\dot{x} + bx = 0 \tag{9}$$

$$[\lambda_1^2 C_1 e^{\lambda_1 t} + \lambda_2^2 C_2 e^{\lambda_2 t}] + a[\lambda_1 C_1 e^{\lambda_1 t} + \lambda_2 C_2 e^{\lambda_2 t}] + b[C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}] = 0 \tag{10}$$

$$(\lambda_1^2 + a\lambda_1 + b\lambda_1) C_1 e^{\lambda_1 t} + (\lambda_2^2 + a\lambda_2 + b\lambda_2) C_2 e^{\lambda_2 t} = 0 \tag{11}$$

where the LHS is zero since, by definition, both λ_1 and λ_2 satisfy (4).

For a second-order system it is not sufficient to just specify the initial state $x(t_0) = x_0$, we must also specify the initial state-rate, $\dot{x}(t_0) = \dot{x}_0$. Thus, we need to include both the C_1 and C_2 terms so that a particular solution can be found for any set of initial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$.

Example. Consider the initial value problem for the second-order, LTI, homogenoeus system:

$$\ddot{x} + \dot{x} - 2x = 0$$

with initial conditions $x(0) = 4$ and $\dot{x}(0) = -5$. In standard form, this equation has $a = 1$ and $b = -2$. The characteristic equation is

$$\lambda^2 + \lambda - 2 = 0$$

To solve for the roots we can use the quadratic equation, complete the square, or factor. This equation is simple enough to factor as:

$$(\lambda - 1)(\lambda + 2) = 0$$

Thus, we have $\lambda_1 = 1$ and $\lambda_2 = -2$. (Note, the order in which we define λ_1 and λ_2 is irrelevant. We could have chosen $\lambda_1 = -2$ and $\lambda_2 = 1$.) Since the eigenvalues are real and distinct, the general solution is:

$$x(t) = C_1 e^{t} + C_2 e^{-2t} \ .$$

To solve for the coefficients C_1 and C_2 we need to apply our initial conditions. But first, we compute the derivative of our general solution (so that we can apply the condition on $\dot{x}(t_0)$):

$$\dot{x}(t) = C_1 e^{t} + C_2 \cdot (-2) e^{-2t}$$

Now, evaluate at the initial conditions:

$$x(0) = 4 = C_1 e^{0} + C_2 e^{-2\cdot 0} = C_1 + C_2 \tag{12}$$

and

$$\dot{x}(0) = -5 = C_1 e^{0} - 2C_2 e^{-2\cdot 0} = C_1 - 2C_2 \tag{13}$$

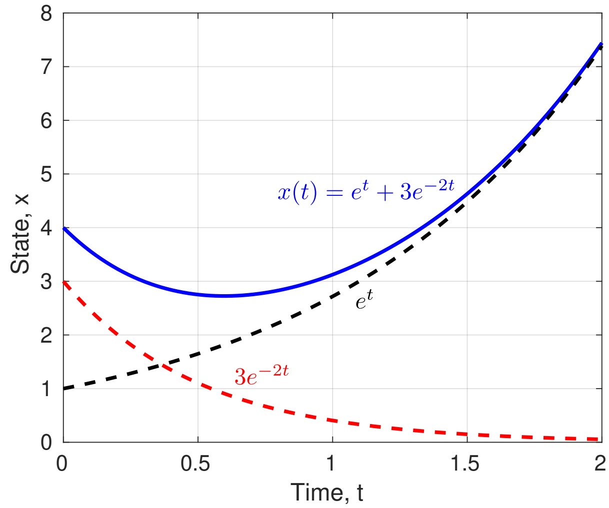
Rearranging (12), we have $C_1 = 4 - C_2$ and plugging this into (13) gives:

$$\begin{aligned} -5 &= (4 - C_2) - 2C_2 \\ -5 - 4 &= -3C_2 \\ C_2 &= -9/-3 = 3 \end{aligned}$$

and it follows that $C_1 = 1$. Thus, the particular solution is

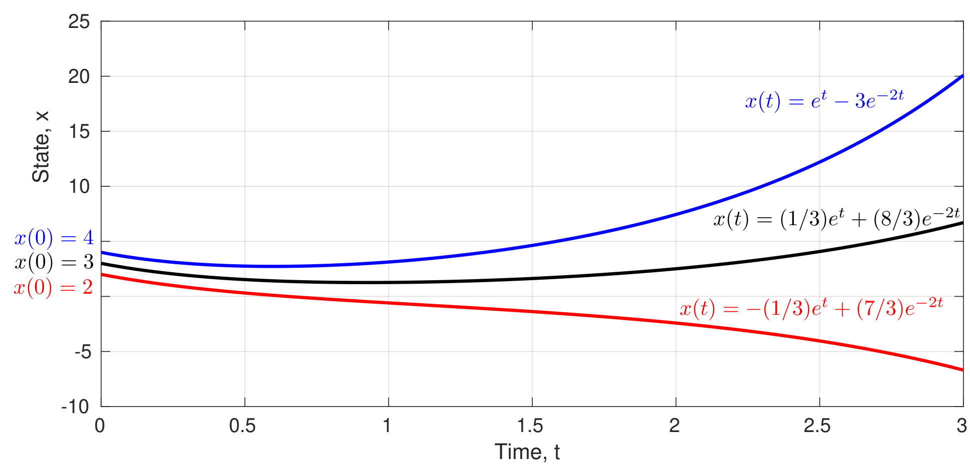
$$x(t) = e^{t} + 3e^{-2t} \ .$$

This solution is plotted below in blue. Notice that the first term e^{t} (black dashed line) tends towards infinity, while the second term $3e^{-2t}$ (red dashed line) tends towards zero. Unlike a first-order system, a second-order system can "change direction" as shown below (around 0.6 seconds on the blue curve).



Example (changing initial conditions): It is interesting to note that the behavior of the system can change depending on the initial condition. Remember, the constants C_1 and C_2 were determined entirely by the choice of x_0 and $\dot{x}_0$.

Consider the figure below with varying x_0. In blue, $x_0 = 4$ (as before), in black $x_0 = 3$, and in red $x_0 = 2$. In each case, the solution (shown next to each line) has different constants C_1 and C_2. In all cases, the second term with e^{-2t} always decays to zero as t grows large, whereas the first term e^t grows without bound (either towards $+\infty$ or $-\infty$, depending on the sign of C_1). It is evident that the sign of the eigenvalues, λ_1 and λ_2, plays an important role in determining if the solution will ultimately converge or diverge. As we will discuss later, if both eigenvalues are negative, then both terms decay and the solution always converges to a constant value.



Case II: $a^2 - 4b = 0$ (repeated eigenvalues)

In this case, with the discriminiant equal to zero, it is evident from (5) the eigenvalues are both $\lambda_{1,2} = -a/2$. Since both eigenvalues are the same the two terms in the general solution (6) are the same and we cannot find coefficients C_1 and C_2 to satisfy a generic set of intial conditions $x(t_0) = x_0$ and $\dot{x}(t_0) = \dot{x}_0$. It turns out that for repeated roots the general solution is instead:

$$x(t) = C_1 e^{-at/2} + C_2 t e^{-at/2} \tag{14}$$

For a detailed explanation see here. For our purposes, we simply need to recognize that when the eigenvalues of a second-order LTI system are repeated we can use (14) along with the initial conditions to obtain a particular solution.

Example. Consider the initial value problem for the second-order, LTI, homogeneous system:

$$\ddot{x} + \dot{x} + \left(\frac{1}{4}\right) x = 0$$

with initial conditions $x(0) = 3$ and $\dot{x}(0) = -3.5$. In standard form, this equation has $a = 3$ and $b = -3.5$. To verify that we have repeated roots we can check the discriminant:

$$a^2 - 4b = (1)^2 - 4\left(\frac{1}{4}\right) = 1 - 1 = 0\ .$$

Thus, the eigenvalues are $\lambda_{1,2} = -1/2$, and the characteristic equation could be factored as

$$\lambda^2 + \lambda + (1/4) = (\lambda + 1/2)^2 = 0$$

The general solution is of the form:

$$x(t) = C_1 e^{-0.5t} + C_2 t e^{-0.5t}$$

To solve for the coefficients C_1 and C_2 first compute the derivative of the general solution:

$$\dot{x}(t) = -0.5 C_1 e^{-0.5t} + C_2 e^{-0.5t} - 0.5 C_2 t e^{-0.5t}$$

Now, evaluate at the initial conditions:

$$x(0) = 3 = C_1 e^{-0.5 \cdot 0} + C_2 (0) e^{-0.5 \cdot 0} = C_1 \tag{15}$$

and

$$\begin{aligned} \dot{x}(0) = -3.5 &= -0.5 C_1 e^{-0.5 \cdot 0} + C_2 e^{-0.5 \cdot 0} - 0.5 C_2 (0) e^{-0.5 \cdot 0} && (16) \\ &= -0.5 C_1 + C_2 && (17) \end{aligned}$$

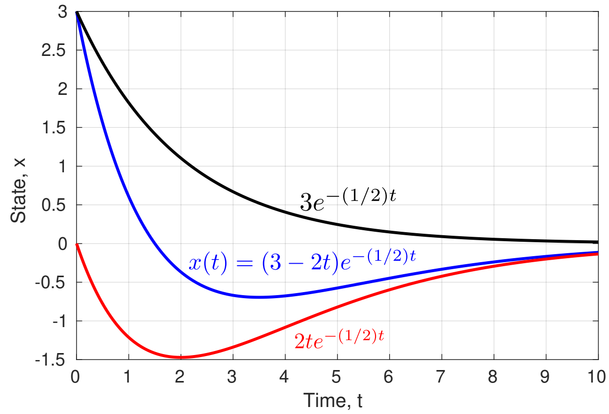
From (15), $C_1 = 3$ and plug this into (17) to give

$$C_2 = -3.5 + 0.5(3) = -2$$

Thus, the particular solution is

$$x(t) = 3e^{-0.5t} - 2te^{-0.5t}$$

This particular solution along with the individual terms is plotted below.



Case III: $a^2 - 4b < 0$ **(two complex conjugate pair eigenvalues)**

In this case the discriminant is negative, hence

$$\frac{\sqrt{a^2-4b}}{2} = \frac{\sqrt{-|a^2-4b|}}{2} = i\omega\ ,$$

where $i = \sqrt{-1}$ is the imaginary number and we introduced a new variable

$$\omega \triangleq \frac{\sqrt{|a^2-4b|}}{2}$$

which is the frequency (in radians/sec.) of the oscillation solution. It follows from (5) that the eigenvalues are

$$\lambda_{1,2} = -\left(\frac{a}{2}\right) \pm i\omega\ .$$

Since both eigenvalues are complex conjugate pairs the general solution of the form shown in (6) applies:

$$x(t) = C_1 e^{-[(\frac{a}{2})+i\omega]t} + C_2 e^{-[(\frac{a}{2})-i\omega]t} \tag{18}$$

and we can factor out the real term $e^{-at/2}$ to give

$$x(t) = e^{-at/2}(C_1 e^{-i\omega t} + C_2 e^{i\omega t})\ . \tag{19}$$

We can further simplify the expression above using Euler's formula:

$$e^{i\alpha} = \cos\alpha + i\sin\alpha \tag{20}$$

to give

$$x(t) = e^{-at/2}(C_1(\cos(-\omega t) + i\sin(-\omega t)) + C_2(\cos(\omega t) + i\sin(\omega t))) \tag{21}$$

$$= e^{-at/2}([C_1+C_2]\cos(\omega t) + [C_2 - C_1]i\sin(\omega t))\ . \tag{22}$$

Then defining the (possibly complex) constants $A = C_1 + C_2$ and $B = [C_2 - C_1]i$ we arrive at the general solution:

$$x(t) = e^{-at/2}(A\cos\omega t + B\sin\omega t)$$

The second term $(A \cos \omega t + B \sin \omega t)$ is periodic with period $1/\omega$ and the first term $e^{-at/2}$ determines whether the amplitude stays constants ($a = 0$), increases exponentially ($a < 0$), or approaches zero exponentially ($a > 0$).

Example. Consider the initial value problem for the second-order, LTI, homogenoeus system:

$$\ddot{x} + 0.4\dot{x} + 9.04x = 0$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 3$. In standard form, this equation has $a = 0.4$ and $b = 9.04$. Calculate the frequency

$$\omega = \frac{\sqrt{|0.4^2 - 4(9.04)|}}{2} = \frac{\sqrt{0.16 - 36.16}}{2} = 3$$

Thus, the eigenvalues are $\lambda_{1,2} = -0.2 \pm 3i$, and the characteristic equation is factored as

$$\lambda^2 + 0.4\lambda + 9.04 = (\lambda + (-0.2 + 3i))(\lambda + (-0.2 - 3i)) = 0$$

The general solution is:

$$x(t) = e^{-0.2t}(A \cos 3t + B \sin 3t)$$

To solve for the coefficients C_1 and C_2 compute the derivative of the general solution:

$$\dot{x}(t) = 0.2e^{-0.2t}(A \cos 3t + B \sin 3t) + e^{-0.2t}(-3A \sin 3t + 3B \cos 3t)$$

and evaluate at the initial conditions:

$$\begin{aligned} x(0) = 0 &= e^{-0.2 \cdot 0}(A \cos 3 \cdot 0 + B \sin 3 \cdot 0) \\ \implies A &= 0 \end{aligned}$$

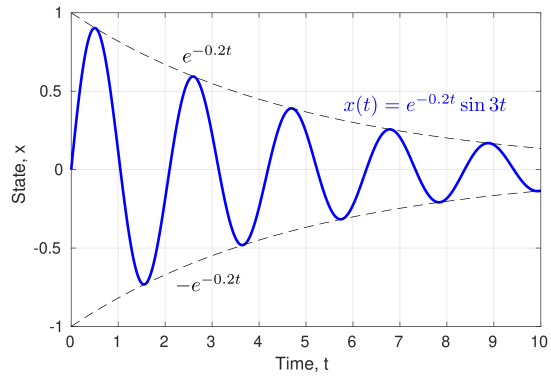
Then, with $A = 0$, the second initial condition becomes

$$\begin{aligned} \dot{x}(0) = 3 &= 0.2e^{-0.2 \cdot 0}(B \sin[3 \cdot 0]) + e^{-0.2 \cdot 0}(3B \cos[3 \cdot 0]) \\ &= 3B \\ \implies B &= 1 \end{aligned}$$

and the particular solution is

$$\implies x(t) = e^{-0.2t} \sin 3t$$

This solution is graphed below.



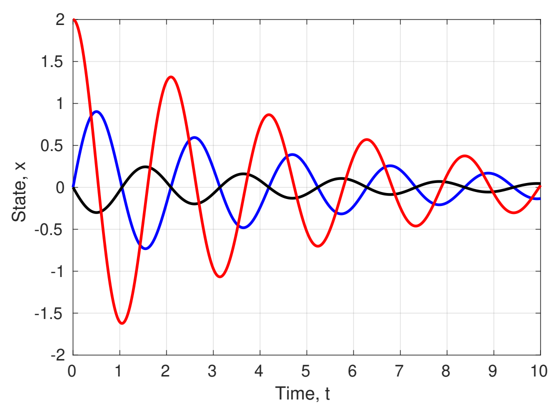
Example (Changing initial conditions). Consider again the system

$$\ddot{x} + 0.4\dot{x} + 9.04x = 0$$

under three sets of initial conditions corresponding to the figure below:

- $x(0) = 0$ and $\dot{x}(0) = 3$ (blue line, same conditions as before)
- $x(0) = 0$ and $\dot{x}(0) = -1$ (black line)
- $x(0) = 2$ and $\dot{x}(0) = 0$ (red line)

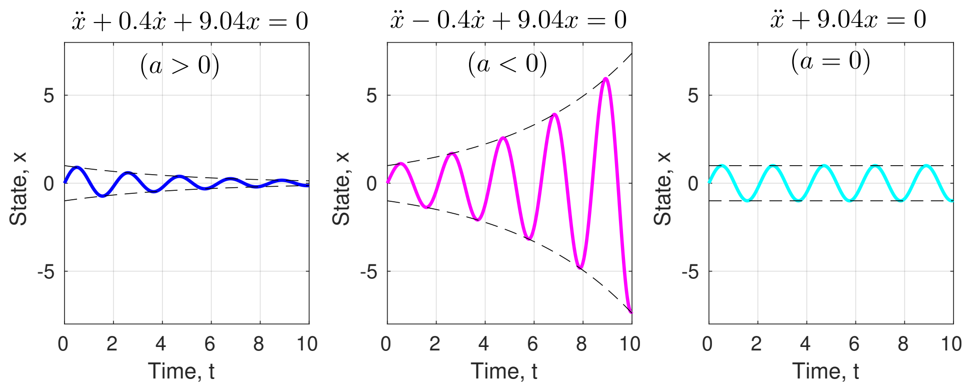
The state history for each case is plotted below.



Example (Changing the damping parameter). Consider yet another variant of the system as

$$\ddot{x} + a\dot{x} + 9.04x = 0$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 3$ and three possible values for the coefficient $a = \{0.4, -0.4, 0\}$. As illustrated below, the sign of the a coefficient determines the qualitative response of the system. This coefficient is the damping term in our mass-spring-damper analogy. For $a > 0$ we have our familiar case of a damper that removes energy from the system causing it to decay to rest. For $a < 0$ we have a (non-physical) type of damper that has a negative damping coefficient and adds energy into the system causing it to oscillate with increasing amplitude. For $a = 0$ we have no damping and the system oscillates with the same amplitude forever.



References and Further Reading

- `https://tutorial.math.lamar.edu/Classes/DE/IntroSecondOrder.aspx`