## Lecture 9: Solving LTI ODEs with Laplace Transforms

Now that we can compute Laplace transforms and their inverse, we are ready to apply these tools to the problem of solving LTI ODEs. An outline of the process is as follows:

Given an ODE in the time domain
$$f(t, x, \dot{x}, \ddot{x}) = u(t)$$
with appropriate initial conditions (e.g., for a second-order system $x(0) = x_0$ and $\dot{x}(0) = \dot{x}_0$), we perform the following steps:

1. Compute the Laplace transform $F(s, X(s)) = \mathcal{L}[f(t, x, \dot{x}, \ddot{x})] = \mathcal{L}[u(t)]$
2. Rearrange this expression into the form $X(s) = \underline{\quad}$ by isolating $X(s)$ to the left-hand side.
3. Compute the inverse Laplace transform $x(t) = \mathcal{L}^{-1}[X(s)]$ using PFE.

*Example (Complex Poles):* Solve the initial value problem

$$3\ddot{x} + 12\dot{x} + 60x = 0$$

with $x(0) = 0$ and $\dot{x}(0) = 3$. First we take the Laplace transform and rearrange for $X(s)$

$$\begin{aligned}
\mathcal{L}[3\ddot{x} + 12\dot{x} + 60x] &= \mathcal{L}[0] \\
3\mathcal{L}[\ddot{x}] + 12\mathcal{L}[\dot{x}] + 60\mathcal{L}[x] &= 0 \\
3(s^2X(s) - sx(0) - \dot{x}(0)) + 12(sX(s) - x(0)) + 60X(s) &= 0 \\
3(s^2X(s) - 3) + 12sX(s) + 60X(s) &= 0 \\
X(s)(3s^2 + 12s + 60) &= 9 \\
X(s) &= \frac{3}{s^2 + 4s + 20}
\end{aligned}$$

Next, we compute the poles of the system to determine what type of partial fraction is needed. Using the quadratic formula

$$p_{1,2} = \frac{-4 \pm \sqrt{4^2 - 4(20)}}{2} = -2 \pm 4i\ .$$

Since the poles are complex we expand as exponential sine/cosine terms with $\alpha = 2$ and $\omega = 4$:

$$\begin{aligned}
X(s) = \frac{3}{s^2 + 4s + 20} &= a_1 \frac{4}{(s+2)^2 + 4^2} + a_2 \frac{(s+2)}{(s+2)^2 + 4^2} \\
&= \frac{a_2 s + (4a_1 + 2a_2)}{(s+2)^2 + 4^2}
\end{aligned}$$

Recall that the denominators are already equal (by our choice of $\alpha$ and $\omega$). Then, equating the numerators

$$\begin{aligned}(\text{Equating } s \text{ coefficient}) &: 0 = a_2 \\ (\text{Equating constants}) &: 3 = 4a_1 + a_2 \implies a_1 = 3/4\end{aligned}$$

Thus, the partial fraction expansion is

$$X(s) = \frac{3}{4}\left(\frac{4}{(s+2)^2+4^2}\right)$$

Taking the inverse Laplace transform yields the solution

$$\begin{aligned}x(t) = \mathcal{L}^{-1}[X(s)] &= \frac{3}{4}\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2+4^2}\right] \\ &= \frac{3}{4}e^{-2t}\sin 4t\end{aligned}$$

*Example (Real Distinct Poles):* Solve the initial value problem

$$10\ddot{x} + 90\dot{x} + 200x = -400$$

with $x(0) = 0$ and $\dot{x}(0) = 0$. First we take the Laplace transform and rearrange for $X(s)$

$$\begin{aligned}\mathcal{L}[10\ddot{x} + 90\dot{x} + 200x] &= \mathcal{L}[-400u(t)] \\ 10\mathcal{L}[\ddot{x}] + 90\mathcal{L}[\dot{x}] + 200\mathcal{L}[x] &= -400\mathcal{L}[u(t)] \\ 10s^2X(s) + 90sX(s) + 200X(s) &= -400\left(\frac{1}{s}\right) \\ X(s) &= \frac{-40}{s(s^2+9s+20)}\end{aligned}$$

Next, we compute the poles of the system to determine what type of partial fraction is needed. The first pole is clearly $p_1 = 0$. Using the quadratic formula the second and third poles are

$$p_{2,3} = \frac{-9 \pm \sqrt{81 - 4(20)}}{2} = \{-4, -5\}\ .$$

Since the poles are real and distinct we expand as:

$$X(s) = \frac{-40}{s(s^2+9s+20)} = \frac{-40}{s(s+4)(s+5)} = \frac{a_1}{s} + \frac{a_2}{s+4} + \frac{a_3}{s+5}$$

Multiply both sides by each denominator and evaluate at each corresponding pole.

$$
\begin{aligned}
a_1 &= \frac{-40}{s(s+4)(s+5)}s\bigg|_{s=0} = \frac{-40}{20} = -2 \\
a_2 &= \frac{-40}{s(s+4)(s+5)}(s+4)\bigg|_{s=-4} = \frac{-40}{-4} = 10 \\
a_3 &= \frac{-40}{s(s+4)(s+5)}(s+5)\bigg|_{s=-5} = \frac{-40}{5} = -8
\end{aligned}
$$

Thus, the partial fraction expansion is

$$X(s) = -2\left(\frac{1}{s}\right) + 10\left(\frac{1}{s+4}\right) - 8\left(\frac{1}{s+5}\right) \; .$$

Taking the inverse Laplace transform yields the solution

$$
\begin{aligned}
x(t) &= = -2\mathcal{L}^{-1}\left[\frac{1}{s}\right] + 10\mathcal{L}^{-1}\left[\frac{1}{s+4}\right] - 8\mathcal{L}^{-1}\left[\frac{1}{s+5}\right] \\
\implies x(t) &= -2 + 10e^{-4t} - 8e^{-5t} \; .
\end{aligned}
$$

*Example (Repeated Poles):* Solve the initial value problem

$$\ddot{x} - 4\dot{x} + 4x = 0$$

with $x(0) = 12$ and $\dot{x}(0) = -3$. First we take the Laplace transform and rearrange for $X(s)$

$$
\begin{aligned}
(s^2X(s) - s(12) + 3) - 4(sX(s) - 12) + 4X(s)0 \\
X(s)(s^2 - 4s + 4) - 12s + 3 + 48 = 0 \\
X(s) = \frac{12s - 51}{s^2 - 4s + 4}
\end{aligned}
$$

Next, we compute the poles of the system

$$p_{1,2} = \frac{-3 \pm \sqrt{16 - 4(4)}}{2} = \{2, 2\} \; .$$

Since the poles are repeated we expand as:

$$X(s) = \frac{12s - 51}{s^2 - 4s + 4} = \frac{a_1}{s-2} + \frac{a_2}{(s-2)^2}$$

Multiply both sides by $(s-2)^2$ to give

$$12s - 51 = a_1 s - (2a_1 + a_2)$$

which implies that

$$\begin{aligned}(\text{Equating constants}) : &\implies 12 = a_1 \\ (\text{Equating } s \text{ coefficient}) : &-51 = 2(12) + a_2 \implies a_2 = -27\end{aligned}$$

Thus, the partial fraction expansion is

$$X(s) = 12\left(\frac{1}{s-2}\right) - 27\left(\frac{1}{(s-2)^2}\right)$$

and taking the inverse Laplace transform yields the solution

$$\implies x(t) = 12e^{2t} - 27te^{2t}$$

*Example (Mixed Poles):* Solve the initial value problem

$$\ddot{x} + 2\dot{x} + 5x = 3$$

with $x(0) = 0$ and $\dot{x}(0) = 0$. Recall that the term on the right-hand side implies a step input, $3H(s)$. First we take the Laplace transform and rearrange for $X(s)$

$$\begin{aligned}\mathcal{L}[\ddot{x}] + 2\mathcal{L}[\dot{x}] + 5\mathcal{L}[x] &= 3\mathcal{L}[H(t)] \\ s^2X(s) + 2sX(s) + 5X(s) &= 3\frac{1}{s} \\ X(s)(s^2 + 2s + 5) &= 3\frac{1}{s} \\ X(s) &= \frac{3}{s(s^2 + 2s + 5)}\end{aligned}$$

One of the poles is $p_1 = 0$ (real and distinct), and the other two poles are complex conjugates

$$p_{2,3} = \frac{-2 \pm \sqrt{2^2 - 4(5)}}{2} = -1 \pm 2i$$

which implies $\alpha = 1$ and $\omega = 2$. Thus, we use a mixed PFE:

$$X(s) = \frac{3}{s(s^2 + 2s + 5)} = a_1\left(\frac{1}{s}\right) + a_2\left(\frac{2}{(s+1)^2 + 2^2}\right) + a_3\left(\frac{(s+1)}{(s+1)^2 + 2^2}\right)$$

First, solve for $a_1$ the usual way (by multiplying by denominator and evaluating at the pole):

$$\implies a_1 = \left.\frac{3}{s^2 + 2s + 5}\right|_{s=0} = 3/5 \,.$$

Now, move the $a_1$ term to the left-hand-side and create a common denominator for both sides

$$X(s) = \frac{3}{s(s^2+2s+5)} - \left(\frac{3}{5s}\right) = a_2\left(\frac{2}{(s+1)^2+2^2}\right) + a_3\left(\frac{(s+1)}{(s+1)^2+2^2}\right)$$
$$\frac{15-3(s^2+2s+5)}{5s(s^2+2s+5)} = \frac{2a_2+a_3(s+1)}{(s+1)^2+2^2}\cdot\frac{5s}{5s}$$
$$-3s^2-6s = 5a_3s^2+5(2a_2+a_3)s$$

Then, equating the numerators

$$(\text{Equating } s^2 \text{ coefficient}) : -3 = 5a_3 \implies a_3 = -3/5$$
$$(\text{Equating } s \text{ coefficient}) : -6 = 5(2a_2+(-3/5)) \implies a_2 = -3/10$$

Thus, the partial fraction expansion is

$$X(s) = \frac{3}{5}\left[\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{(s+1)^2+2^2}\right) - \left(\frac{(s+1)}{(s+1)^2+2^2}\right)\right]$$

Taking the inverse Laplace transform yields the solution

$$\implies x(t) = \frac{3}{5}\left[1 - \frac{e^{-t}\sin 2t}{2} - e^{-t}\cos 2t\right]$$

For more complex systems we may encounter mixed pole types and can (optionally) rely on MATLAB and matrix algebra to solve for coefficients, as illustrated by the following example.

*Example (Mixed poles):* Solve the initial value problem

$$\ddot{x} + 6\dot{x} + 9x = \sin t$$

with $x(0) = 0$ and $\dot{x}(0) = 0$. First we take the Laplace transform and rearrange for $X(s)$

$$\mathcal{L}[\ddot{x}+6\dot{x}+9x] = \mathcal{L}[\sin t]$$
$$X(s)(s^2+6s+9) = \frac{1}{s^2+1}$$
$$X(s) = \frac{1}{(s^2+1)(s^2+6s+9)}$$

Next we compute the poles of the system to determine what type of partial fraction is needed. The poles for $s^2+1$ are $p_{1,2} = \pm i$ complex conjugates. The poles for $s^2+6s+9$ are repeated

$$p_{3,4} = \frac{-6 \pm \sqrt{36-4(9)}}{2} = \{-3, -3\}\ .$$

Since this is a case of mixed pole types we use a combination of PFEs from Case III (for $p_{1,2}$

with $\alpha = 0$ and $\omega = 1$) and Case II for $p_{3,4}$ (refer to cases discussed in Lecture 9).

$$X(s) = \frac{1}{(s^2+1)(s+3)^2} = \underbrace{\frac{a_1}{(s^2+1)} + \frac{a_2 s}{(s^2+1)}}_{\text{PFE for } p_{1,2}} + \underbrace{\frac{a_3}{(s+3)} + \frac{a_4}{(s+3)^2}}_{\text{PFE for } p_{3,4}}$$

To eliminate the denominators, multiply both sides by $(s^2+1)(s+3)^2$

$$\begin{aligned}
1 &= a_1(s+3)^2 + a_2 s(s+3)^2 + a_3(s+3)(s^2+1) + a_4(s^2+1) \\
1 &= a_1(s^2+6s+9) + a_2 s(s^2+6s+9) + a_3(s^3+s+3s^2+3) + a_4 s^2 + a_4 \\
1 &= s^3(a_2+a_3) + s^2(a_1+6a_2+3a_4) + s(6a_1+9a_2+a_3) + 9a_1 + 3a_3 + a_4
\end{aligned}$$

This yields the following system of equations:

$$\begin{aligned}
(\text{Equating } s^3 \text{ coefficient}) &: 0 = a_2 + a_3 \\
(\text{Equating } s^2 \text{ coefficient}) &: 0 = a_1 + 6a_2 + 3a_3 + a_4 \\
(\text{Equating } s \text{ coefficient}) &: 0 = 6a_1 + 9a_2 + a_3 \\
(\text{Equating constants}) &: 1 = 9a_1 + 3a_3 + a_4
\end{aligned}$$

The above system contains 4 equations with 4 unknowns that can be solved algebraically. It can also be written equivalently in matrix form by defining the 4 x 1 vector $\boldsymbol{x} = [a_1\ a_2\ a_3\ a_4]^{\mathrm{T}}$ and using the coefficients from the right-hand side above to define a 4 x 4 matrix $\boldsymbol{A}$ and the coefficients form the left-hand side above to define a 4 x 1 vector $\boldsymbol{b}$. The linear system of equations is written in matrix form as

$$\underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\boldsymbol{b}} = \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 6 & 3 & 1 \\ 6 & 9 & 1 & 0 \\ 9 & 0 & 3 & 1 \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}}_{\boldsymbol{x}}$$

and the solution (for the $c$ values) is given by

$$\boldsymbol{x} = \boldsymbol{A}^{-1}\boldsymbol{b}$$

where $\boldsymbol{A}^{-1}$ is the *matrix inverse*. In MATLAB,

```
>> A = [0 1 1 0;
        1 6 3 1;
        6 9 1 0;
        9 0 3 1];
>> b = [0; 0; 0; 1];
>> x = inv(A) * b
x =
    0.0800
```

```
-0.0600
 0.0600
 0.1000
```

Thus, the coefficients are $a_1 = 0.08$, $a_2 = -0.06$, $a_3 = 0.06$, and $a_4 = 0.1$ and the partial fraction expansion is

$$X(s) = 0.08\left(\frac{1}{s^2+1}\right) - 0.06\left(\frac{2}{s^2+1}\right) + 0.06\left(\frac{1}{s+3}\right) + 0.1\left(\frac{1}{s+3^2}\right)$$

Taking the inverse Laplace transform (using rows 10, 11, 6, and 7) yields the solution

$$\implies x(t) = 0.08\sin t - 0.06\cos t + 0.06e^{-3t} + 0.1te^{-3t}$$

## References and Further Reading

- Davies: Sec. 2.7-2.9
- Ogata: Sec. 2.4, 2.5, 4.4