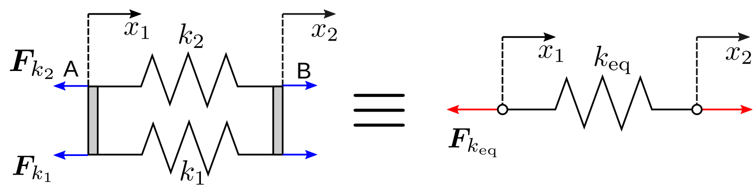
Lecture 14: Lumped Springs/Dampers and Multi-DOF Systems

When modeling systems that consist of multiple springs and dampers it is convenient to "lump" the parameters by defining an equivalent spring or equivalent damper.

Lumping Parallel Springs. Consider a pair of springs attached to massless bars in a parallel arrangement as shown in the left image below. The positions x_1 and x_2 refer to the displacement of the left and right bars from a nominal (unstretched) position. For the equivalent spring to



produce the same force as the two paralle springs, we require:

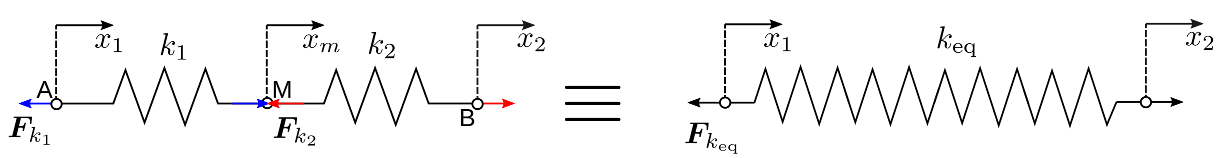
$$\boldsymbol{F}_{k_1} + \boldsymbol{F}_{k_2} = \boldsymbol{F}_{\text{eq}}$$
$$k_1(x_2 - x_1)\hat{\boldsymbol{\imath}} + k_2(x_2 - x_1)\hat{\boldsymbol{\imath}} = k_{\text{eq}}(x_2 - x_1)\hat{\boldsymbol{\imath}} \tag{1}$$

which implies that

$$\implies \quad k_{\text{eq}} = k_1 + k_2 \tag{2}$$

The above expression for k_{eq} can be used to replace any pair of parallel springs by a single equivalent spring with stiffness k_{eq}.

Lumping Springs in Series. Springs that are connected in series have an extra degree of freedom at their connection point. In the diagram below, the displacement of this middle point M is denoted by x_m. By "lumping" the two springs we wish to replace them with a single equivalent spring and eliminate the x_m degree of freedom from our model. That is, we assume x_m is not a variable of interest—if it is we can keep the original two-spring model and analyze the system as a multi degree-of-freedom system. To find the equivalent spring constant first consider the equilibrium of point M and the original system on the left as well as the desired equivalent system on the right:



Analyzing the original system the sum of forces at point M must be zero — since the point M has no mass ($m = 0$) then Newton's 2nd Law ($\sum \boldsymbol{F} = m\ddot{x}_m = 0$) dictates that it is always in equilibrium. Thus:

$$-\boldsymbol{F}_{k_1} + \boldsymbol{F}_{k_2} = 0 \tag{3}$$
$$-k_1(x_m - x_1)\hat{\boldsymbol{\imath}} + k_2(x_2 - x_m)\hat{\boldsymbol{\imath}} = 0 \tag{4}$$
$$(-k_1 x_m + k_1 x_1 + k_2 x_2 - k_2 x_m)\hat{\boldsymbol{\imath}} = 0 \implies \quad x_m = \frac{k_1 x_1 + k_2 x_2}{k_1 + k_2} \tag{5}$$

The expression for x_m is the amount by which x_m moves for different motions of x_1 and x_2. If the model on the right (above) is equivalent to the one on the left, then both models should produce the same force on the left-most side of the system. Mathematically, this means that:

$$\boldsymbol{F}_{k_1} = \boldsymbol{F}_{\text{eq}} \tag{6}$$
$$k_1(x_m - x_1)\hat{\boldsymbol{\imath}} = k_{\text{eq}}(x_2 - x_1)\hat{\boldsymbol{\imath}} \tag{7}$$
$$k_1\left(\frac{k_1x_1 + k_2x_2}{k_1 + k_2} - x_1\right) = k_{\text{eq}}(x_2 - x_1) \tag{8}$$
$$\left(\frac{k_1^2x_1 + k_2k_1x_2 - k_1x_1(k_1 + k_2)}{k_1 + k_2}\right) = k_{\text{eq}}(x_2 - x_1) \tag{9}$$
$$\left(\frac{k_2k_1x_2 - k_1x_1k_2}{k_1 + k_2}\right) = k_{\text{eq}}(x_2 - x_1) \tag{10}$$
$$\left(\frac{k_1k_2}{k_1 + k_2}\right)(x_2 - x_1) = k_{\text{eq}}(x_2 - x_1) \tag{11}$$
$$\implies k_{\text{eq}} = \frac{k_1k_2}{k_1 + k_2} \tag{12}$$

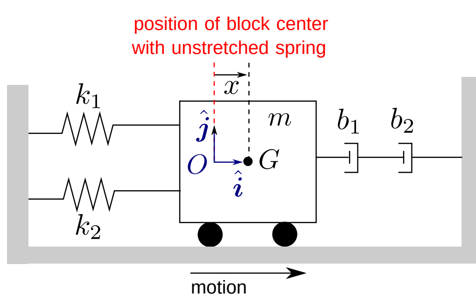
Lumping Dampers in Parallel or in Series. A similar derivation can be repeated to lump dampers that are in parallel:

$$b_{\text{eq}} = b_1 + b_2 \tag{13}$$

or in series:

$$b_{\text{eq}} = \frac{b_1b_2}{b_1 + b_2} \tag{14}$$

Example: Develop a lumped parameter model for the following system:



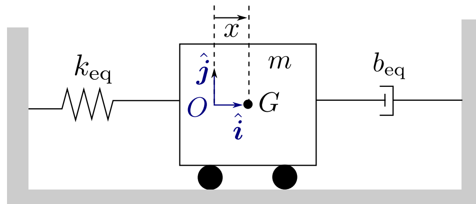
with parameters $m = 1$ kg, $k_1 = 200$ N/m, $k_2 = 450$ N/m, $b_1 = 30$ N/(m/s), and $b_2 = 15$ N/(m/s). First, combine the parallel springs into an equivalent spring

$$k_{\text{eq}} = k_1 + k_2 = 650N/m\ . \tag{15}$$

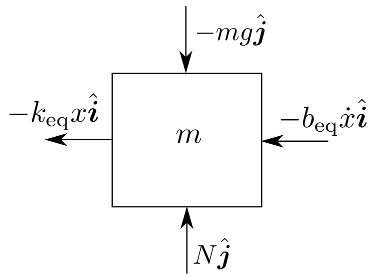
Next, combine the parallel dampers in series

$$b_{\text{eq}} = \frac{b_1b_2}{b_1 + b_2} = \frac{450}{45} = 10N\cdot(m/s) \tag{16}$$

Our simplified model becomes



The free body diagram is



Summing forces and applying Newton's 2nd Law in the $\hat{\imath}$ direction:

$$\hat{\imath}\ :\ \sum F = -k_{eq}x - b_{eq}\dot{x} = m\ddot{x} \tag{17}$$

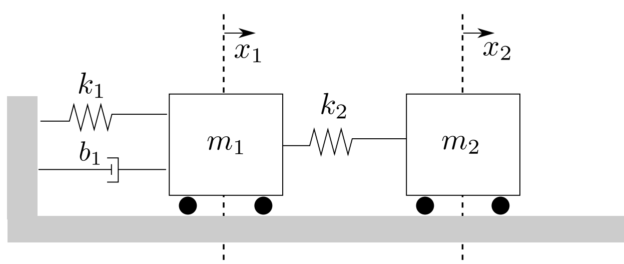
Rearranging we obtain the second-order LTI ODE (a damped harmonic oscillator)

$$\ddot{x} + \frac{b_{eq}}{m}\dot{x} + \frac{k_{eq}}{m}x = 0 \tag{18}$$

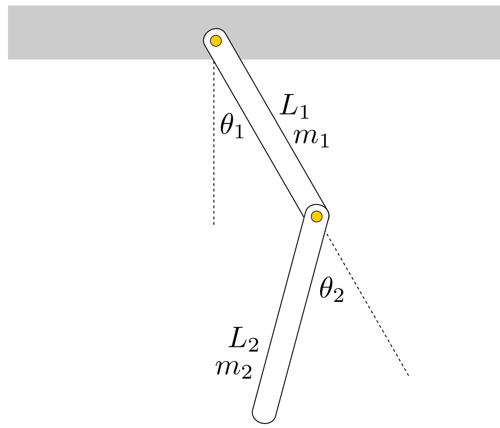
with initial position $x(0) = x_0$ and initial velocity $\dot{x}(0) = v_0$.

Multi Degree-of-Freedom Systems. Thus far we've studied systems with only one degree of freedom (DOF) where one variable and its derivatives (e.g., x and $\dot{x}$ or θ and $\dot{\theta}$) were sufficient to describe the system's state (e.g., its initial conditions). For *multi degree-of-freedom systems* there are two or more elements that can move independently. Therefore, describing the system dynamics requires additional variables. Consider the following examples:

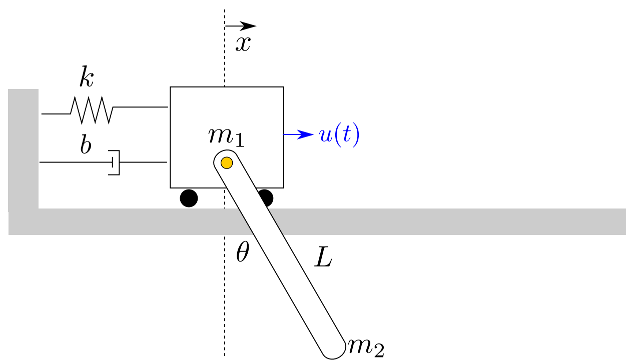
- e.g., A translational multi-DOF system. This system contains two translating masses and the system state is determined by $x_1, \dot{x}_1$ and $x_2, \dot{x}_2$.



- e.g., A rotational multi-DOF system. This system contains two rotating links and the system state is determined by $\theta_1, \dot{\theta}_1$ and $\theta_2, \dot{\theta}_2$.



- e.g., A combined translational/rotational multi-DOF system. This system contains a translating mass with an attached rigid link. The system state is determined by $x, \dot{x}$ and $\theta, \dot{\theta}$.



Multiple degree-of-freedom systems are *coupled* and represented by two or more ODEs that share variables. For example, the combined translational/rotational system above will have dynamics of the form:

$$\ddot{x} = f_1(x, \dot{x}, \theta, \dot{\theta}, u) \tag{19}$$

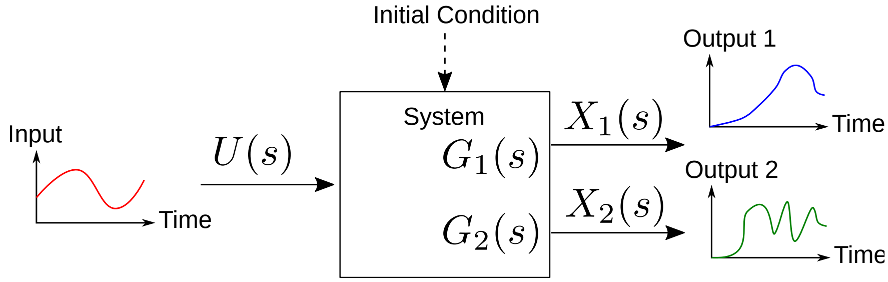
$$\ddot{\theta} = f_2(x, \dot{x}, \theta, \dot{\theta}, u) \tag{20}$$

where $u(t)$ is an externally applied input (such as a a force or displacement). For this coupled system we cannot solve for $x(t)$ without also incorporating the dynamics for $\theta(t)$. Together the two second order ODEs above form a 4th order system. Our general approach for solving such multi DOF systems is to:

1. Take the Laplace transform of each equation (they will be coupled).

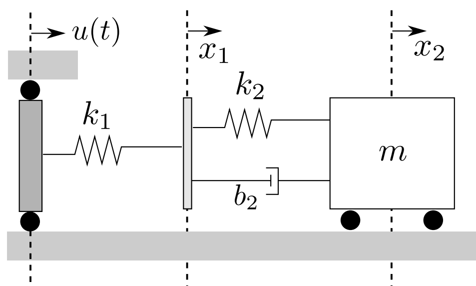
2. Rearrange the resulting equations such that each output is independently determined from the input (i.e., decouple the equations into separate transfer functions).

For the example given above, the desired transfer functions are $G_1(s) = X(s)/U(s)$ and $G_2(s) = \Theta(s)/U(s)$. The concept of a single input generating two outputs through some transfer function is sketched below:

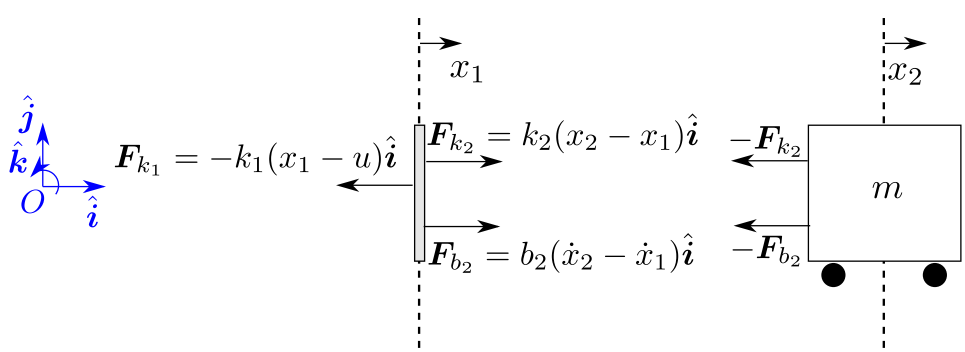


Using the transfer functions we can simulate the zero-state response to obtain $x_1(t)$ and $x_2(t)$ for some input $u(t)$.

Example: Consider the following system. On the far left is a movable wall that is an input into the system, this wall is actuated such that it produces a displacement according to $u(t)$. (Note: a common pitfall is to confuse whether the input is a force, displacement, or velocity — it can be any of them depending on the problem.) The movement of the left wall is transmitted via the spring k_1 to a massless bar whose position is given by $x_1(t)$. Lastly, the massless bar is connected to the block of mass m by a spring and damper. The position of the block is $x_2(t)$. We wish to find the transfer functions $G_1(s) = X_1(s)/U(s)$ and $G_2(s) = X_2(s)/U(s)$ so that we may simulate the system.



First, define an inertial reference frame at a point O and draw the free body diagram for the two moving components: the massless bar and the block. The direction of the force vectors may change depending on the state of the system, however it is important that the expression we write down for each force is consistent with the direction of the reference frame (i.e., the $\hat{\imath}$ unit vector) and consistent with how the coordinates $x_1(t)$ and $x_2(t)$ are defined. From Newton's 3rd Law we can write the forces on the block as the negative of the force on the massless rod for the spring and damper forces connecting the two elements.



Since the bar has no mass the right hand side of Newton's 2nd Law is zero. This reduces force balance to a equilibrium in the $\hat{\imath}$ direction:

$$\sum F = F_{k_1} + F_{k_2} + F_{b_2} = 0 \tag{21}$$

$$k(x_1 - u) = k_2(x_2 - x_1) + b_2(\dot{x}_2 - \dot{x}_1) \tag{22}$$

Then, applying Newton's 2nd law to the mass:

$$m\ddot{x}_2 = -F_{k_2} - F_{b_2} \tag{23}$$

$$m\ddot{x}_2 = -k_2(x_2 - x_1) - b_2(\dot{x}_2 - \dot{x}_1) \tag{24}$$

$$m\ddot{x}_2 + k_2 x_2 + b_2\dot{x}_2 = k_2 x_1 + b_2\dot{x}_1 \tag{25}$$

Take the Laplace transform (assuming zero initial conditions) of (22)

$$k_1X_1(s) - k_1U(s) = k_2X_2(s) - k_2X_1(s) + b_2sX_2(s) - b_2sX_1(s) \tag{26}$$

$$(k_1 + k_2 + b_2s)X_1(s) = (k_2 + b_2s)X_2(s) + k_1U(s) \tag{27}$$

then of (25)

$$m(s^2X_2(s) + k_2X_2(s) + b_2sX_2(s)) = k_2X_1(s) + b_2sX_1(s) \tag{28}$$

$$X_2(s)(ms^2 + k_2 + b_2s) = (k_2 + b_2s)X_1(s) \tag{29}$$

$$X_2(s) = \left(\frac{k_2 + b_2s}{ms^2 + k_2 + b_2s}\right)X_1(s) \tag{30}$$

Substitute (30) into (27)

$$(k_1 + k_2 + b_2s)X_1(s) = (k_2 + b_2s)\left(\frac{k_2 + b_2s}{ms^2 + k_2 + b_2s}\right)X_1(s) + k_1U(s) \tag{31}$$

and rearrange

$$(k_1 + k_2 + b_2s)(ms^2 + k_2 + b_2s)X_1(s) - (k_2 + b_2s)^2X_1(s) = K_1U(s)(ms^2 + k_2 + b_2s) \tag{32}$$

$$X_1(s)(b_2ms^3 + (k_1 + k_2)ms^2 + b_2k_1s + k_1k_2) = k_1(ms^2 + b_2s + k_2)U(s) \tag{33}$$

to give the transfer function

$$G_1(s) = \frac{X_1(s)}{U(s)} = \frac{k_1ms^2 + k_1b_2s + k_1k_2}{b_2ms^3 + (k_1 + k_2)ms^2 + b_2k_1s + k_1k_2} \tag{34}$$

Similarly, after some algebra,

$$G_2(s) = \frac{X_2(s)}{U(s)} = \frac{b_2k_1s + k_1k_2}{b_2ms^3 + (k_1 + k_2)ms^2 + b_2k_1s + k_1k_2} \tag{35}$$

In MATLAB we can define the transfer function for $G_2(s)$ as follows

```
num = [b_2*k_1 k_1*k_2];
den = [m_2*b_2 (k_1+k_2)*m b_2*k_1 k_1*k_2];
tf = sys(num,den);D
```

Then, to simulate a step response or a impulse response

```
step(tf)
impulse(tf)
```

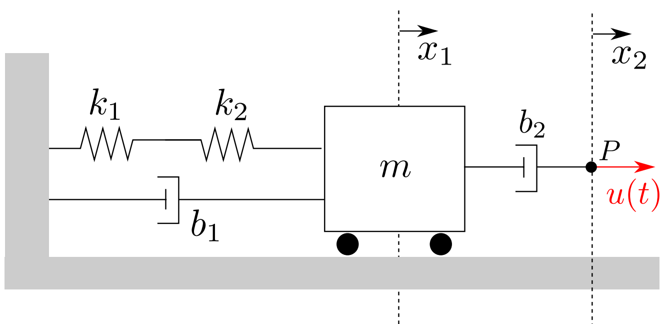
Similarily, we can define the transfer function for $G_1(s)$ and simulate the response.

Example: Find the transfer function $G(s) = X(s)/F(s)$ for the following system (assume zero initial conditions). Include a free body diagram.

Expand all terms and write $G(s)$ in the general transfer function format,

$$\text{e.g.,}\ \ G(s) = \frac{X(s)}{F(s)} = \frac{b_m s^m + b_{m-1}s^m + \cdots + b_0}{a_n s^n + a_{n-1}s^n + \cdots + a_0}\ .$$

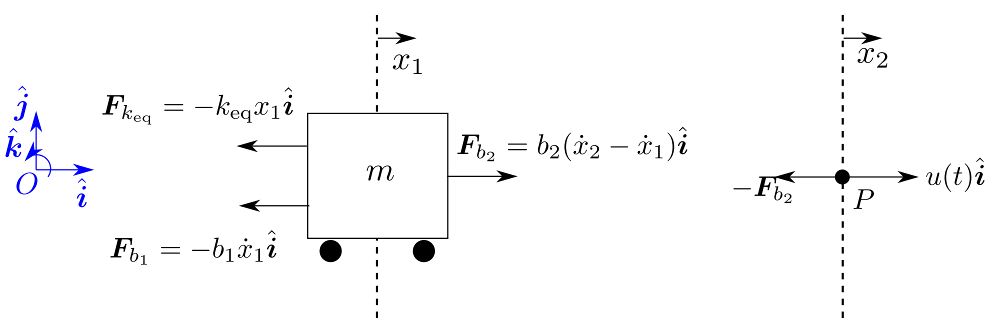
A force $u(t)$ (not a displacement) is applied at point P, and therefore point P has a degree-of-freedom.



The springs in series can be replaced by a single equivalent spring with stiffness:

$$k_{\text{eq}} = \frac{k_1 k_2}{k_1 + k_2}\ .$$

The free-body-diagrams for the mass and point P are:



Since point P has no mass it is in equilibrium and the sum of forces is zero

$$-\boldsymbol{F}_{b_2} + u(t) = 0 \implies \boldsymbol{F}_{b_2} = b_2(\dot{x}_2 - \dot{x}_1) = u(t)\ . \tag{36}$$

This implies that the point P moves with a velocity $\dot{x}_2$ such that the damping forces balance the applied force and is effectively transmitted to the cart. Summing the forces at the cart and applying Newton's 2nd Law:

$$\sum \boldsymbol{F} = \boldsymbol{F}_{k_{\text{eq}}} + \boldsymbol{F}_{b_1} + \boldsymbol{F}_{b_2} = -k_{\text{eq}}\boldsymbol{x}\hat{\boldsymbol{\imath}} - b_1\dot{x}\hat{\boldsymbol{\imath}} + \underbrace{b_2(\dot{x}_2 - \dot{x}_1)}_{=u(t)}\hat{\boldsymbol{\imath}} = m\ddot{x}_1\hat{\boldsymbol{\imath}}$$

Taking the Laplace transform of the scalar equation from above,

$$-k_{\text{eq}}X_1(s) - b_1 sX_1(s) + U(s) = ms^2X_1(s) \tag{37}$$

and the transfer function is

$$G_1(s) = \frac{X_1(s)}{U(s)} = \frac{1}{ms^2 + b_1 s + k_{\text{eq}}}$$

To obtain the transfer function for x_2 we take the Laplace transform of (36) and substitute the transfer function from above

$$b_2 sX_2(s) - b_2 sX_1(s) = U(s) \tag{38}$$

$$b_2 s\frac{X_2(s)}{U(s)} - b_2 s\frac{X_1(s)}{U(s)} = 1 \tag{39}$$

$$b_2 s\frac{X_2(s)}{U(s)} - b_2 s\left(\frac{1}{ms^2 + b_1 s + k_{\text{eq}}}\right) = 1 \tag{40}$$

Which gives the transfer function:

$$G_2(s) = \frac{X_2(s)}{U(s)} = \frac{1}{b_2 s} + \left(\frac{1}{ms^2 + b_1 s + k_{\text{eq}}}\right) \tag{41}$$

$$= \frac{ms^2 + b_1 s + (k_{\text{eq}} + 1)}{b_2 s(ms^2 + b_1 s + k_{\text{eq}})} \tag{42}$$

$$= \frac{ms^2 + b_1 s + (k_{\text{eq}} + 1)}{b_2 ms^3 + b_1 b_2 s^2 + k_{\text{eq}} b_2 s} \tag{43}$$

Example: Determine the transfer function $G(s) = X(s)/U(s)$ for the following system (below) with two degrees of freedom, $x(t)$ and $\theta(t)$ (assume zero initial conditions). The variables: m, b, k, L are all constants and $u(t)$ is an applied force. Hint: Take the Laplace transform of each equation first.

$$m\ddot{x}(t) + kx(t) - Lk\theta(t) = u(t) \tag{44}$$

$$\frac{1}{3}m\ddot{\theta}(t) - kL\dot{x}(t) = 0 \tag{45}$$

Expand all terms and write $G(s)$ in the general transfer function format,

$$\text{e.g.,}\;\; G(s) = \frac{X(s)}{F(s)} = \frac{b_m s^m + b_{m-1}s^m + \cdots + b_0}{a_n s^n + a_{n-1}s^n + \cdots + a_0}\,.$$

First, take the Laplace transform of (44)

$$X(s)(ms^2 + k) - Lk\Theta(s) = F(s) \tag{46}$$

and then of (45)

$$\frac{1}{3}ms^2\Theta(s) - kLX(s)s = 0 \tag{47}$$

Rearrange (47)

$$\Theta(s) = \frac{3kLsX(s)}{ms^2} = \frac{3kLX(s)}{ms} \tag{48}$$

and plug into (46)

$$X(s)\left(ms^2 + k - \frac{3k^2L^2}{ms}\right) = F(s)$$

to give the transfer function for $X(s)$

$$G_X(s) = \frac{X(s)}{F(s)} = \frac{ms}{m^2s^3 + kms - 3k^2L^2}\,. \tag{49}$$

Using (48) and (49)

$$\begin{aligned} G_\Theta(s) = \frac{\Theta(s)}{F(s)} = \frac{X(s)}{F(s)}\frac{\Theta(s)}{X(s)} &= G_X(s)\frac{\Theta(s)}{X(s)} \\ &= \frac{3kL}{m}\left(\frac{1}{s}\right)\left(\frac{ms}{m^2s^3 + kms - 3k^2L^2}\right) \\ &= \frac{3kL}{m}\left(\frac{ms}{m^2s^4 + kms^2 - 3k^2L^2s}\right) \end{aligned}$$