

Homework 4

1 Problem

For each of the following differential equations, use Laplace transforms to find the solution to the IVP.

1. $3\ddot{x} + 12\dot{x} + 60x = \delta(t)$; $x(0) = 0$; $\dot{x}(0) = 0$ where $\delta(t)$ is the impulse or dirac delta function (row 1 in the Laplace transform table).
2. $\ddot{x} + 10\dot{x} + 25x = 0$; $x(0) = 1$; $\dot{x}(0) = 0$
3. $\ddot{x} + 5\dot{x} + 6x = 2e^{-t}$; $x(0) = 1$; $\dot{x}(0) = 0$
4. $\ddot{x} + 2\dot{x} = 8t$; $x(0) = 0$; $\dot{x}(0) = 0$

Show all your work/intermediate steps. Other solution methods besides Laplace transforms will not receive any credit.

Solution

Problem 1.1

Take the Laplace transform of the system (note $\mathcal{L}[\delta(t)] = 1$):

$$3s^2X(s) + 12sX(s) + 60X(s) = 1$$

and rearrange

$$X(s) = \frac{(1/3)}{s^2 + 4s + 20}$$

There are two complex-valued poles:

$$p_{1,2} = \frac{-4 \pm \sqrt{16 - 4(20)}}{2} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i$$

Since the poles are complex in the form $p_{1,2} = -\alpha \pm i\omega$, expand as damped sinusoid and cosines (PFE):

$$X(s) = \frac{(1/3)}{s^2 + 4s + 20} = \frac{c_1(s + \alpha)}{(s + \alpha)^2 + \omega^2} + \frac{c_2\omega}{(s + \alpha)^2 + \omega^2} \quad (1)$$

with $\alpha = 2$ and $\omega = 4$. Equating the numerators gives two equations (one for coefficient of s and one for the constant):

$$s^2 : \quad 0 = c_1s \quad (2)$$

$$(\text{constant}) : \quad (1/3) = c_1\alpha + c_2\omega \quad (3)$$

The first equation implies that $c_1 = 0$ and the second equation implies that

$$(1/3) = c_2(4) \implies c_2 = 1/12$$

So the partial fraction expansion is:

$$X(s) = \frac{1}{12} \left(\frac{4}{(s+2)^2 + 16} \right)$$

and taking the inverse Laplace transform

$$\begin{aligned} x(t) &= \frac{1}{12} \mathcal{L}^{-1} \left[\frac{4}{(s+2)^2 + 16} \right] \\ &= \frac{1}{12} e^{-2t} \sin 4t \end{aligned}$$

1.2

$$\ddot{x} + 10\dot{x} + 25x = 0$$

$$x(0) = 1$$

$$\dot{x}(0) = 0$$

Laplace Transform: $[s^2X(s) - \underbrace{s x(0)}_{=1} - \underbrace{\dot{x}(0)}_{=0}] + 10[sX(s) - \underbrace{x(0)}_{=1}] + 25X(s) = 0$

$$s^2X(s) - s + 10sX(s) - 10 + 25X(s) = 0$$

$$X(s) = \frac{s+10}{s^2+10s+25}$$

$$\text{poles: } p_{1,2} = \frac{-10 \pm \sqrt{100 - 4(25)}}{2} = -5$$

$$\text{Thus, } X(s) = \frac{s+10}{(s+5)^2}$$

Since poles are repeated, we expand as powers of the denom.

$$X(s) = \frac{(s+10)}{(s+5)^2} = \frac{C_1}{(s+5)} + \frac{C_2}{(s+5)^2}$$

Multiply both sides by highest power denom $(s+5)^2$

$$\begin{aligned} \underline{(s+10)} &= C_1(s+5) + C_2 \\ &= \underline{C_1 s} + \underline{(5C_1 + C_2)} \end{aligned}$$

Equate coefficients:

$$\boxed{1 = C_1}$$

$$10 = 5(1) + C_2 \Rightarrow \boxed{C_2 = 5}$$

$$\text{Thus, } X(s) = \frac{1}{(s+5)} + \frac{5}{(s+5)^2}$$

$$\text{Inverse L.T. } x(t) = \mathcal{L}^{-1}\left[\frac{1}{s+5}\right] + 5 \mathcal{L}^{-1}\left[\frac{1}{(s+5)^2}\right]$$

row 6 row 7

$$\boxed{x(t) = e^{-5t} + 5te^{-5t}}$$

1.3

$$\ddot{x} + 5\dot{x} + 6x = 2e^{-t} \quad x(0) = 1 \quad \dot{x}(0) = 0$$

Take L.T. $\left[\underbrace{s^2 X(s)}_{=1} - \underbrace{s x(0)}_{=1} - \underbrace{\dot{x}(0)}_{=0} \right] + 5 \left[\underbrace{s X(s)}_{=1} - \underbrace{x(0)}_{=1} \right] + 6X(s) = 2 \left(\frac{1}{s+1} \right)$ row 6
↓

$$s^2 X(s) - s + 5sX(s) - 5 + 6X(s) = 2 \left(\frac{1}{s+1} \right)$$

$$X(s)(s^2 + 5s + 6) = 2 \left(\frac{1}{s+1} \right) + (s+5)$$

$$X(s) = \frac{2}{(s+1)(s^2+5s+6)} + \frac{(s+5)}{(s^2+5s+6)}$$
} make common denom.

$$= \frac{2 + (s+5)(s+1)}{(s+1)(s^2+5s+6)}$$

$$= \frac{2 + s^2 + 6s + 5}{(s+1)(s^2+5s+6)}$$

poles: $p_1 = -1$ $p_{2,3} = \frac{-5 \pm \sqrt{25-4(6)}}{2}$

$$= \frac{-5 \pm 1}{2}$$

$$p_2 = -2$$

$$p_3 = -3$$

Thus, $X(s) = \frac{s^2 + 6s + 7}{(s+1)(s+2)(s+3)}$

Since all poles are distinct, expand as partial fractions with corresponding distinct denominators

$$X(s) = \frac{s^2 + 6s + 7}{(s+1)(s+2)(s+3)} = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)} + \frac{C_3}{(s+3)}$$

To solve for coefficients we multiply both sides by denominator and set s equal to the pole. (see topic 11)

$$C_1 = \left. \frac{(s^2 + 6s + 7)}{(s+2)(s+3)} \right|_{s=-1} = \frac{1 - 6 + 7}{1 - 2} = 1$$

$$C_2 = \frac{(s^2 + 6s + 7)}{(s+1)(s+3)} \bigg|_{s=-2} = \frac{4 - 12 + 7}{-1 \cdot 1} = 1$$

$$C_3 = \frac{(s^2 + 6s + 7)}{(s+1)(s+2)} \bigg|_{s=-3} = \frac{9 - 18 + 7}{-2 \cdot -1} = -1$$

Taking I.L.T. $x(t) = \mathcal{Z}^{-1} \left[\frac{1}{s+1} \right] + \mathcal{Z}^{-1} \left[\frac{1}{s+2} \right] - \mathcal{Z}^{-1} \left[\frac{1}{s+3} \right]$

↓ row 6 ↙ ↘

$$x(t) = e^{-t} + e^{-2t} - e^{-3t}$$

1.4

$$\ddot{x} + 2\dot{x} = 8t \quad \dot{x}(0) = 0 \quad x(0) = 0$$

L.T. $s^2 X(s) + 2s X(s) = 8 \frac{1}{s^2}$

$$X(s) = 8 \frac{1}{s^2} \frac{1}{(s^2 + 2s)}$$

$$= \frac{8}{s^3(s+2)}$$

$\hookrightarrow p_4 = -2$ (distinct)
 $\hookrightarrow p_{1,2,3} = 0$ (repeated poles)

This case involves mixed poles, hence we expand using both methods (for distinct and for real)

$$X(s) = \frac{8}{s^3(s+2)} = \underbrace{\frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s^3}}_{\text{method for repeated}} + \underbrace{\frac{C_4}{(s+2)}}_{\text{method for distinct}}$$

Multiply both sides by $s^3(s+2)$:

$$\begin{aligned} 8 &= C_1 s^2(s+2) + C_2 s(s+2) + C_3(s+2) + C_4 s^3 \\ &= C_1 s^3 + 2C_1 s^2 + C_2 s^2 + 2C_2 s + C_3 s + 2C_3 + C_4 s^3 \\ &= (C_1 + C_4) s^3 + (2C_1 + C_2) s^2 + (2C_2 + C_3) s + 2C_3 \end{aligned}$$

Equating coefficients

const.: $8 = 2C_3 \Rightarrow \boxed{C_3 = 4}$

s : $0 = 2C_2 + C_3 \Rightarrow C_2 = -\frac{C_3}{2}$

$$\boxed{C_2 = -2}$$

s^2 : $0 = 2C_1 + C_2 \Rightarrow \boxed{C_1 = 1}$

s^3 : $0 = C_1 + C_4 \Rightarrow \boxed{C_4 = -1}$

L.T.: $x(t) = \mathcal{Z}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{Z}^{-1}\left[\frac{1}{s^2}\right] + 4\mathcal{Z}^{-1}\left[\frac{1}{s^3}\right] - \mathcal{Z}^{-1}\left[\frac{1}{s+2}\right]$

$x(t) = u(t) - 2t + \frac{4t^2}{2} - e^{-2t}$
or
 $x(t) = 1 - 2t + 2t^2 - e^{-2t}$

2 Problem

Use the MATLAB function `dsolve` to verify your answer for Problem 1.4. Generate a plot of the solution over the time interval $t \in [0, 3]$ seconds. Submit your code.

Solution

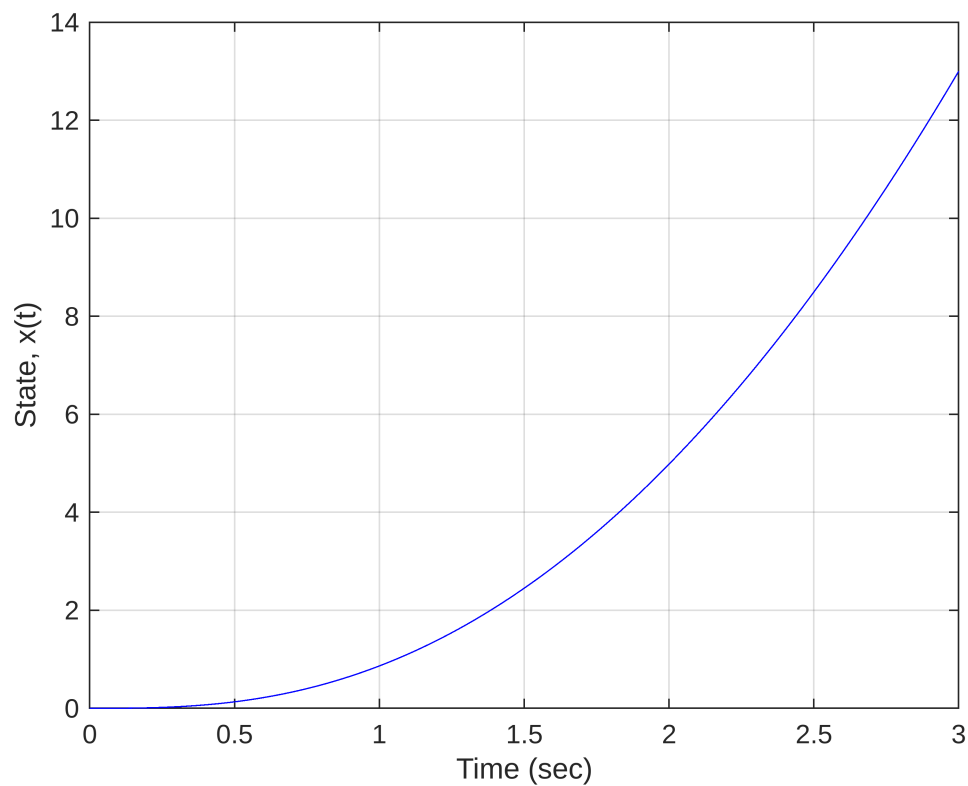
Using the symbolic toolbox with dsolve to solve Problem 1.4

```
syms x(t);  
xdot = diff(x,t);  
xddot = diff(x,t,2);  
assume(t>=0)  
eqn = xddot == -2*xdot + 8*t;  
cond = [x(0)==0, xdot(0)==0];  
xsol = dsolve(eqn,cond)
```

$$xsol = 2t^2 - e^{-2t} - 2t + 1$$

Now to plot the solution

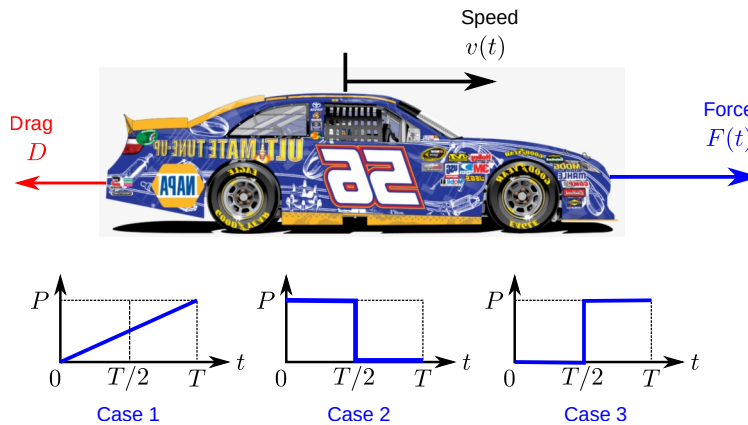
```
tvals = linspace(0,3);  
xvals = double(subs(xsol,tvals));  
figure;  
plot(tvals,xvals,'b-')  
grid on;  
xlabel('Time (sec)')  
ylabel('State, x(t)');
```



3 Problem

Suppose the racecar below has a mass of $m = 750$ kg and is moving down a track with an initial speed of $v(t_0) = 45$ m/s at time $t_0 = 0$ sec. The drag on the car is modeled as a linear function of velocity: $D = bv$, where $b = 20$ N/(m/s).

- Using the free-body diagram below, where $F(t)$ is an applied force, apply Newton's 2nd Law to find the equations of motion. Since $a(t) = \dot{v}(t)$ you can write this equation as a first-order ODE in speed (i.e., $\sum F = m\dot{v}$).



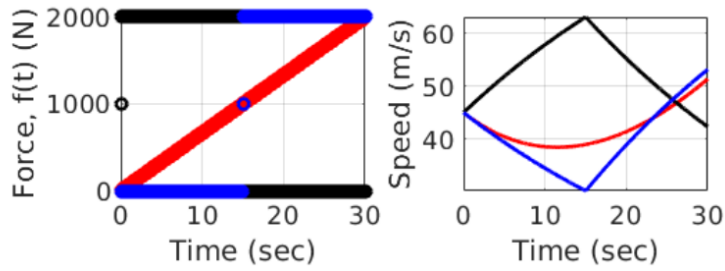
Suppose that over the next $T = 30$ seconds the driver can choose from the three possible force profiles, $F(t)$, shown above, where $P = 2000$ N is the same maximum force reached during each profile.

- Write down an expression for each of the force profiles $F_1(t)$, $F_2(t)$, $F_3(t)$ as a function of the magnitude P and time. You can construct the force profiles from a combination of Heaviside functions and ramps (straight lines) with appropriate slope. Reviewing the doublet example (Lecture 7 PDF, p.2) may be helpful.

Interestingly, each profile has the same impulse (area under the force-time curve) but results in a different final displacement and velocity. Determine the velocity profile $v(t)$ that results from each case by following these steps:

- Solve for the velocity profile in each of the three cases using MATLAB (following the methods of Lecture 10 e.g., using `laplace`, `solve`, and `ilaplace` OR `dsolve`). and plot the three solutions on the same axes. Which case results in the largest final speed? Label your axes, add a legend for each line, and use a thick line type for clarity.

Note that MATLAB defines the step function as: `heaviside(t)`. Your solution should look similar to the one below:



Bonus: Which case results in the furthest distance traveled at time T ? Justify your answer with a plot of distance traveled in MATLAB.

Solution

Answers may vary. Solution below uses `laplace`, `solve`, `ilaplace`, `eval`. Another approach may use `dsolve`.

```

clear; close all; clc; % prepare workspace

% constants
b = 20; % N/(m/s)
m = 750; % kg
v0 = 45; % x(0) IC, initial position
P = 2000;
T = 30;
tvals = [0:1/50:T]; % 50 frames per second

% v' + b/m*v = f

% case 1
fprintf('-----\nCase 1:\n');
syms V s v t f; % define symbolic variables
f = t*P/T;
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist1 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist1 = eval(subs(int(v,0,t),tvals));
fhist1 = eval(subs(f,tvals));

% case 2
fprintf('-----\nCase 2:\n');
f = P*heaviside(t) - P*heaviside(t-T/2);
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist2 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist2 = eval(subs(int(v,0,t),tvals));
fhist2 = eval(subs(f,tvals));

% case 3
fprintf('-----\nCase 3:\n');
f = P*heaviside(t-T/2);
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist3 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist3 = eval(subs(int(v,0,t),tvals));
fhist3 = eval(subs(f,tvals));

figure;
subplot(2,2,1)
plot(tvals,fhist1,'ro','linewidth',2); hold on;

```

```

plot(tvals,fhist2,'ko','linewidth',2)
plot(tvals,fhist3,'bo','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Force, f(t) (N)')
grid on;
hold on;
axis tight;

subplot(2,2,2)
plot(tvals,vhist1,'r','linewidth',2); hold on;
plot(tvals,vhist2,'k','linewidth',2)
plot(tvals,vhist3,'b','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Speed (m/s)')
grid on;
hold on;
axis tight;

subplot(2,2,[3:4])
plot(tvals,dhist1,'r','linewidth',2); hold on;
plot(tvals,dhist2,'k','linewidth',2)
plot(tvals,dhist3,'b','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Distance (m)')
legend('Case 1','Case 2','Case 3','location','southwest');
grid on;
hold on;
axis tight;

fprintf('*****\n')
fprintf('Case 3 leads to the greatest speed at t = 30 sec\n')
fprintf('Case 2 leads to the greatest distance at t = 30 sec\n')
fprintf('Case 1 has intermediate performance compared to Case 2 and\n')
fprintf('3\n')
fprintf('*****\n')

-----
Case 1:
10 t      /      2 t \
---- + exp| - --- | 170 - 125
      3      \      75 /

-----
Case 2:
                        /      / 2      2 t \      \
                        | exp| - - --- | 75      |
                        |      \ 5      75 /      75 |
heaviside(t - 15) | ----- - -- | 8
                  \      2      2 /
-----
                  3
                        - exp| - --- | 55 + 100
                        \      75 /

```

$$\frac{\exp\left(\frac{2t}{75}\right) \operatorname{heaviside}(t-15) \sqrt{\frac{\exp\left(\frac{2t}{75}\right)}{5} - \frac{75}{2}}}{45 \sqrt{\frac{\exp\left(\frac{2t}{75}\right)}{75} - \frac{75}{2}}} - \frac{8}{3}$$

Figure 10 consists of three subplots comparing the proposed control strategy (red line) with the reference control strategy (black line). The bottom subplot shows Distance (m) vs Time (sec) from 0 to 30 seconds. Case 1 (red) starts at 0m, Case 2 (black) starts at 400m, and Case 3 (blue) starts at 0m. All cases reach approximately 1200m at 30 seconds. The top-left subplot shows Force $f(t)$ (N) vs Time (sec). Case 1 (red) is a diagonal line from (0,0) to (30,2000). Case 2 (black) is 2000N from 0 to 15s and 0N from 15 to 30s. Case 3 (blue) is 0N from 0 to 15s and 2000N from 15 to 30s. The top-right subplot shows Speed (m/s) vs Time (sec). Case 1 (red) is a parabolic curve starting at 45, dipping to 38 at 15s, and rising to 52 at 30s. Case 2 (black) is a triangular wave starting at 45, peaking at 65 at 15s, and falling to 42 at 30s. Case 3 (blue) is a triangular wave starting at 45, dipping to 32 at 15s, and rising to 55 at 30s.