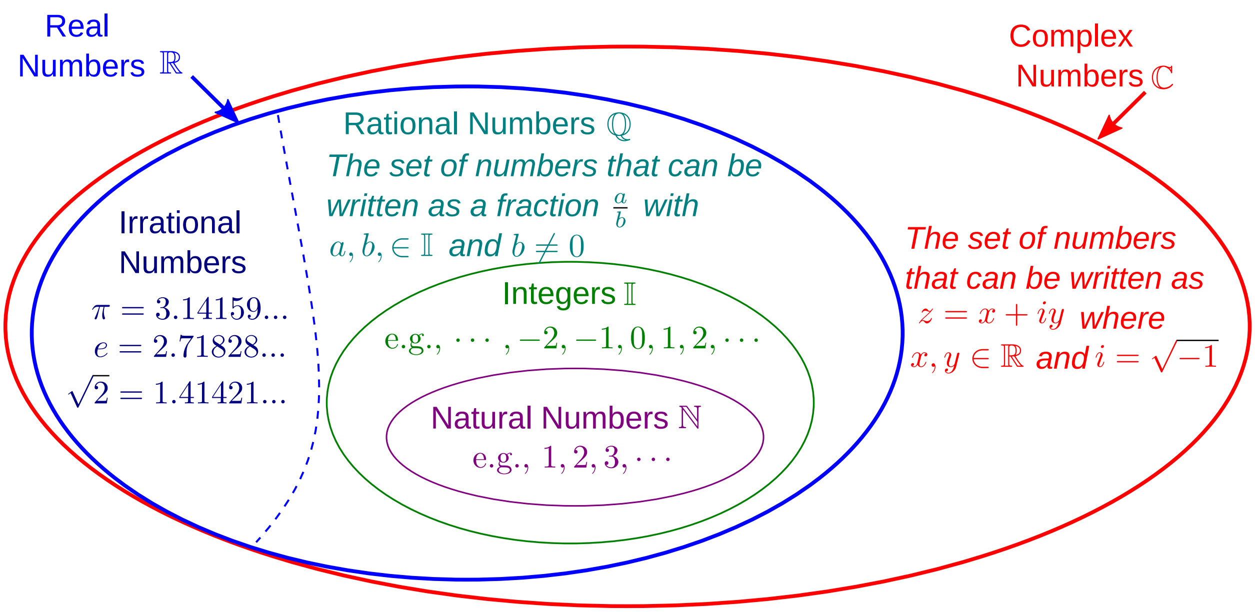
Lecture 5: Complex Arithmetic

Types of Numbers

Numbers can be categorized into particular types that depend on their value, as illustrate by the Venn diagram below. For example, the natural numbers are a subset of the integers, the set of real numbers consists of all rational numbers and irrational number, and complex numbers extend the real numbers to include the *imaginary number*. The imaginary number is usually denoted with the symbol i or j and is equal to $i = \sqrt{-1}$. From this definition it immediately follows that $i^2 = \sqrt{-1}\sqrt{-1} = -1$. To denote the type of a number we use the symbol "$\in$" which means "is an element of the set" or simply "in the set". For example if $z = 3 + 1i$ is a complex number, we write $z \in \mathbb{C}$ meaning "z is in the set of complex numbers $\mathbb{C}$".



> *Remark:* Equations that contain all real constants or coefficients may have complex-valued solutions. Consider the quadratic equation
>
> $$x^2 - 4x + 5 = 0 \,.$$
>
> This equation represents a parabola and has all integer coefficients. Yet the solution is a complex number, $x = 2 \pm i$, and indicates that the parabola does not intercept the x axis.

Rectangular and Polar Forms

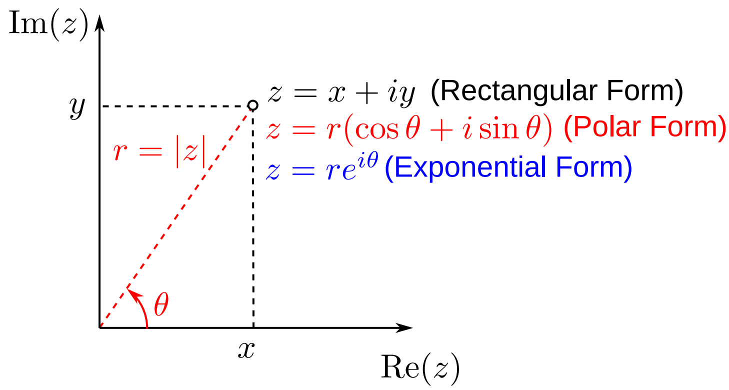
A *complex number* z can be represented either in rectangular or polar form. In *rectangular form*, z is written as:

$$z = \underbrace{x}_{\text{real part}} + i \underbrace{y}_{\text{imaginary part}} \,.$$

To denote the real and imaginary parts of z we use the notation

$$\text{Re}(z) = x \qquad \text{and} \qquad \text{Im}(z) = y$$

Complex numbers can be sketched as a point (or a ray extending from the origin) in the complex plane. The real part of a complex number gives the abscissa coordinate (the horizontal axis) in the complex plane, and the imaginary part gives the ordinate (vertical axis), as shown below.



The *modulus* (also called the absolute value or magnitude) of a complex number is its distance from the origin

$$|z| = \sqrt{x^2 + y^2}$$

and the *argument* is the angle that the ray joining the origin to z makes with the Re (real) axis

$$\theta = \arg(z) = \text{atan}\left(\frac{y}{x}\right) .$$

> *Remark*: The function atan(y/x) uses the ratio y/x to return an angle that is always between $-\pi/2$ and $\pi/2$. However, this angle is not unique, and it does distinguish between $(y = 1, x = 1)$ and $(y = -1, x = -1)$ since they both have the same ratio of $y/x = 1$. Instead, the four-quadrant arctangent function atan2(y, x) can be used as both a mathematical function (and a MATLAB function) to correctly distinguish between the two cases (the first one giving 45 deg. and the latter one giving 225 deg.

From the geometry, it is clear that

$$x = |z|\cos\theta$$
$$y = |z|\sin\theta$$

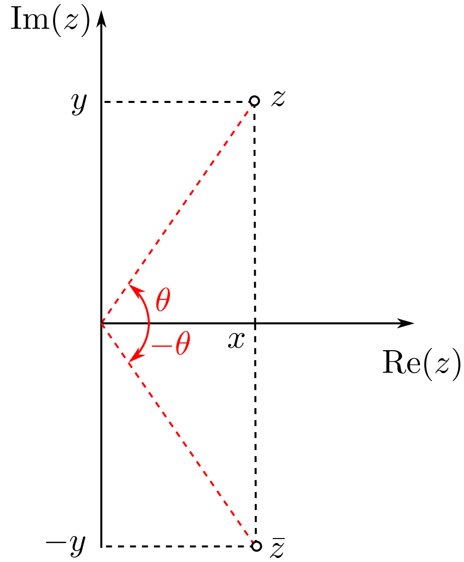
so that we may re-write the complex number in *polar form* as:

$$z = r(\cos\theta + i\sin\theta)$$

where it is understood that $r = |z|$. The polar form is sometimes abbreviated as (r, θ).

Complex Conjugate

The complex conjugate of z is the reflection of z about the Re axis as shown below.



If $z = x + iy$, the complex conjugate is denoted by an overbar and defined as $\bar{z} = x - iy$. In polar form, the complex number (r, θ) has conjugate $(r, -\theta)$. Thus, using trigonometric identities,

$$\begin{aligned} \bar{z} &= r(\cos(-\theta) + i\sin(-\theta)) \\ &= r(\cos\theta - i\sin\theta) \end{aligned}$$

Complex Arithmetic (Rectangular Form)

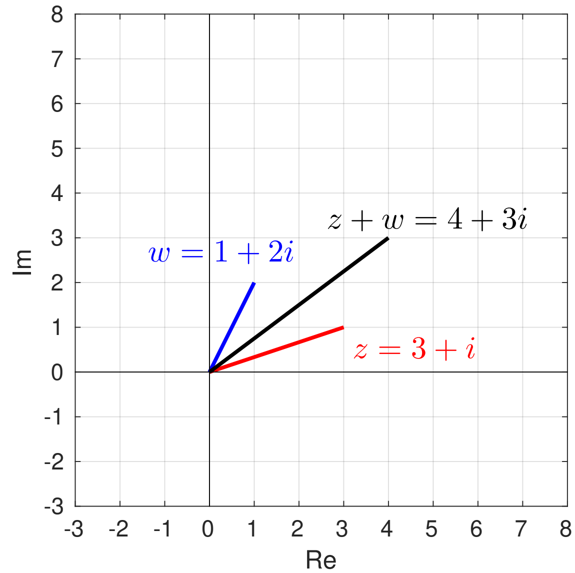
Consider the two complex numbers in rectangular form: $z = x + iy$ and $w = u + iv$. In the following, we develop expressions and show examples for the addition, subtraction, multiplication, and division of these complex numbers.

- Addition: The real and imaginary components are simply added together, similar to vector addition:

$$\begin{aligned} z + w &= (x + iy) + (u + iv) \\ &= (x + u) + i(y + v) \end{aligned}$$

Example: Let $z = 3 + i$ and $w = 1 + 2i$, then

$$z + w = (3 + 1) + (1 + 2)i = 4 + 3i$$

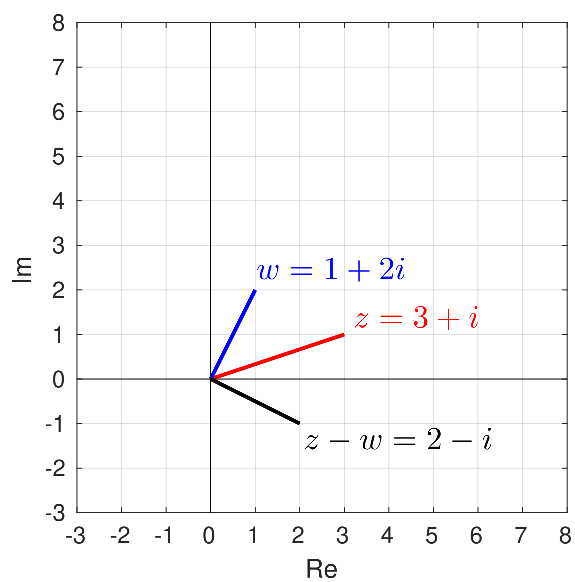


- Subtraction: The real and imaginary components are subtracted from one another, similar to vector subtraction:

$$\begin{aligned} z - w &= (x + iy) - (u + iv) \\ &= (x - u) + i(y - v) \end{aligned}$$

Example: Let $z = 3 + i$ and $w = 1 + 2i$, then

$$z - w = (3 - 1) + (1 - 2)i = 2 - i$$



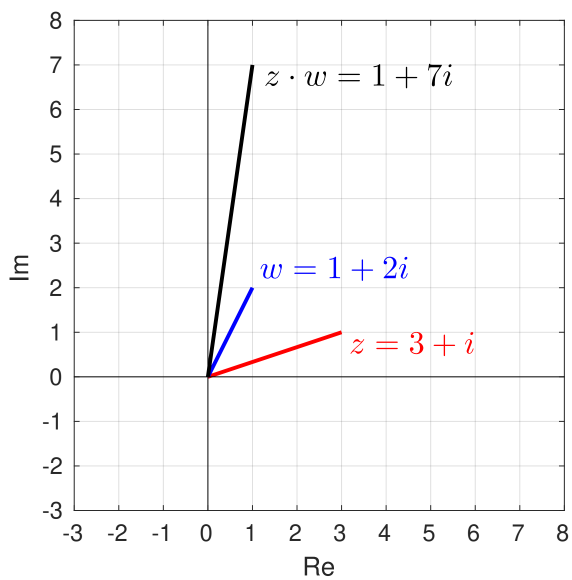
- Multiplication: We expand the two terms multiplied and simplify using the fact that $i^2 = -1$

$$\begin{aligned} z \cdot w &= (x+iy)(u+iv) \\ &= xu + ixv + iyu + \underbrace{i^2}_{=-1} yv \\ &= (xu - yv) + i(xv + yu) \end{aligned}$$

Example: Let $z = 3 + i$ and $w = 1 + 2i$, then

$$z \cdot w = (3 \cdot 1 - 1 \cdot 2) + (3 \cdot 2 + 1 \cdot 1)i = 1 + 7i$$

Notice that multiplication of w by z both scales and rotates the vector w.



- Multiplication by a complex conjugate gives the modulus squared

$$\begin{aligned} z \cdot \bar{z} &= (x+iy)(x-iy) \\ &= x^2 \underbrace{-ixy + iyx}_{=0} - i^2 y^2 \\ &= x^2 + y^2 = |z|^2 \end{aligned}$$

Example: Let $z = 3 + i$, then $\bar{z} = 3 - i$ and

$$z \cdot \bar{z} = (3+i)(3-i) = 9 - 3i + 3i - i^2 = 10$$

which is equal to the modulus since

$$|z|^2 = 3^2 + 1^2 = 10$$

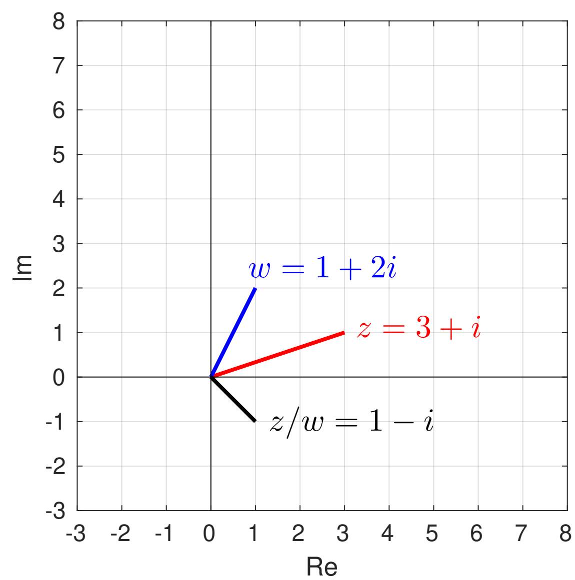
- Division: We first multiply the top and bottom by a complex conjugate of the denominator ($\bar{w} = u - iv$), then simplify using the fact that $w \cdot \bar{w} = (u+iv)(u-iv) = |w|^2 = u^2 + v^2$

$$\begin{aligned} \frac{z}{w} &= \frac{z}{w}\left(\frac{\bar{w}}{\bar{w}}\right) \\ &= \frac{(x+iy)(u-iv)}{(u+iv)(u-iv)} \\ &= \frac{xu - ixv + iyu - i^2 yv}{u^2+v^2} \\ &= \frac{(xu+yv) + i(yu - xv)}{u^2+v^2} \end{aligned}$$

Example: Let $z = 3+i$ and $w = 1+2i$. Then, $\bar{w} = 1-2i$ and

$$\frac{z}{w} = \frac{(3+i)}{(1+2i)}\frac{(1-2i)}{(1-2i)} = \frac{3-6i+i-2i^2}{1^2+2^2} = \frac{5-5i}{5} = 1-i$$

Notice, again, that division of z by w both scales and rotates the vector z.



Euler's Formula

To obtain expressions for the arithmetic operations described above with z and w expressed in polar form we could simply substitute their polar equivalents. But another representation using exponentials is quite useful and relies on *Euler's formula*:

$$e^{i\theta} = \cos\theta + i\sin\theta \, .$$

By using Euler's formula, the complex number $z = r(\cos\theta + i\sin\theta)$ is converted into *exponential form* as $z = re^{i\theta}$. Evaluating Euler's formula at $\theta = \pi$ we find that

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 + i(0) \end{aligned}$$

which can be rearranged to give *Euler's identity*:

$$e^{i\pi} + 1 = 0$$

For an explanation of how Euler's formula arises refer to the remark below.

Remark: At first glance, Euler's formula looks a bit unusual—what does cos and sin have to do with the exponential $e^{i\theta}$? To gain some insight, we can look at the Taylor series representations of these functions. Recall that a Taylor series can represent a function $f(x)$ as an infinite summation (a power series) of the form:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(x-b)^n &= a_0 \underbrace{(x-b)^0}_{=1} + a_1(x-b)^1 + a_2(x-b)^2 + \cdots \\ &= a_0 + a_1(x-b) + a_2(x-b)^2 + \cdots \end{aligned}$$

where the coefficients $a_0, a_1, \ldots, a_n$ are determined by evaluating the function $f(x)$ and its derivatives at a point $x = b$ where the approximation is centered:

$$a_0 = f(b) \quad a_1 = f'(b) \quad a_2 = f''(b) \quad \cdots \quad a_i f^{(i)}(x)\Big|_{x=b}$$

Using this method, the Taylor series approximation of the exponential, sine, and cosine functions around $x = 0$ are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \end{aligned}$$

Now, returning to our problem, substitute in $x = i\theta$ into the Taylor series for the exponential function above and replace $i^2 = -1$ everywhere it appears

$$\begin{aligned} e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^2 i\theta^3}{3!} + \frac{i^2 i^2\theta^4}{4!} + \frac{i i^2 i^2\theta^5}{5!} + \cdots \\ &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \cdots \\ &= \underbrace{[1 - \frac{\theta}{2!} + \frac{\theta^4}{4!} + \cdots]}_{\cos\theta} + i\underbrace{[\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} + \cdots]}_{\sin\theta} \end{aligned}$$

The simplified expressions can be rearranged into the Taylors series for sine and cosine that is related through Euler's formula.

Complex Arithmetic (Exponential Form)

Recall that a complex number in rectangular form is written as $z = x + iy$. Let $r = |z|$ be the distance of the complex-valued point from the origin (i.e., the modulus) and let θ be the angle with the horizontal, each of the rectangular components can then be written as:

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \ . \end{aligned}$$

So that, using Euler's formula, z becomes

$$\begin{aligned} z &= (r\cos\theta) + i(r\sin\theta) \\ &= r(\cos\theta + i\sin\theta) \\ &= re^{i\theta} \ . \end{aligned}$$

Now, consider two complex numbers represented in exponential form

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1} \\ z_2 &= r_2 e^{i\theta_2} \ , \end{aligned}$$

and the corresponding arithmetic operations. Addition and subtraction is not particularly simplified by using the exponential form:

- Addition:
$$z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$$
- Subtraction:
$$z_1 + z_2 = r_1 e^{i\theta_1} - r_2 e^{i\theta_2}$$

But using the exponential form for multiplication and division takes advantage of the power and quotient rule for exponents.

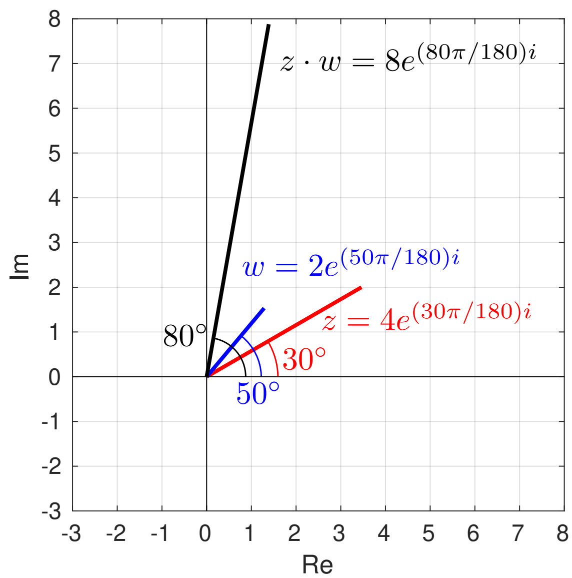
- Multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1+\theta_2)} \end{aligned}$$

Now it is much easier to interpret multiplying two complex numbers. If z_1 has length r_1 and z_2 has length r_2, the $z_1 \cdot z_2$ has length $r_1 r_2$. Moreover, the angle of the resulting vector $z_1 \cdot z_2$ is the addition of the two angles for z_1 and z_2.

Example: Let $z = 4e^{(30\pi/180)i}$ and $w = 2e^{(50\pi/180)i}$. Then,

$$z \cdot w = (4e^{(30\pi/180)i})(2e^{(50\pi/180)i}) = (4 \cdot 2)e^{[(30+50)\pi/180]i} = 8e^{(80\pi/180)i}$$



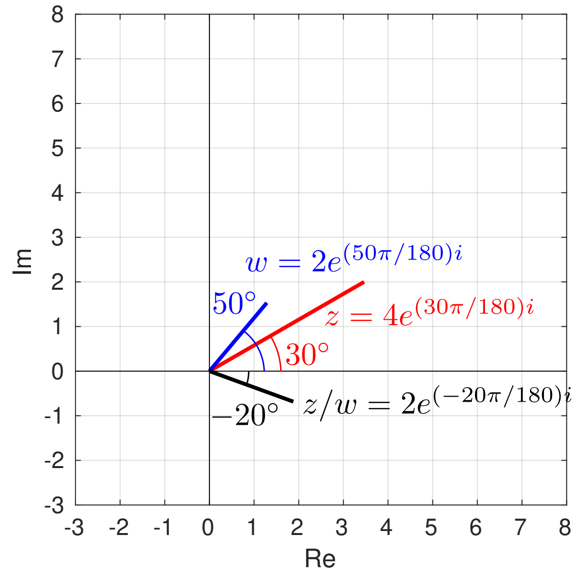
- Division:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)}$$

As with multiplication, it is now much easier to interpret division of two complex numbers when using exponential form. If z_1 has length r_1 and z_2 has length r_2, the z_1/z_2 has length r_1/r_2. Moreover, the angle of the resulting vector z_1/z_2 is the difference of the two angles for z_1 and z_2.

Example: Let $z = 4e^{(30\pi/180)i}$ and $w = 2e^{(50\pi/180)i}$. Then,

$$z/w = (4e^{(30\pi/180)i})(2e^{(50\pi/180)i}) = (4/2)e^{[(30-50)\pi/180]i} = 2e^{(-20\pi/180)i}$$



$w = 2e^{(50\pi/180)i}$
$50°$
$z = 4e^{(30\pi/180)i}$
$30°$
$-20°$
$z/w = 2e^{(-20\pi/180)i}$
Im
Re

Example. Part A: Consider the two complex numbers $z_1 = 5e^{i\pi/2}$ *and* $z_2 = 1 + i$*. Convert* z_2 *into exponential form and compute* z_1/z_2*.*

The modulus of z_2 is $|z_2| = \sqrt{2}$ and the polar angle is $\theta = \text{atan2}(1,1) = \pi/4$. Thus, $z_2 = \sqrt{2}e^{i\pi/4}$ and

$$\frac{z_1}{z_2} = \frac{5e^{i\pi/2}}{\sqrt{2}e^{i\pi/4}} = \frac{5}{\sqrt{2}}e^{i\pi/4}\,. \tag{1}$$

Part B: Convert z_1 *into rectangular form and compute* z_1/z_2*.*

In rectangular coordinates,

$$z_1 = (5\cos(\pi/2)) + i(5\sin(\pi/2)) = 5i \tag{2}$$

and

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{5i}{1+i}\frac{1-i}{1-i} \\ &= \frac{5i - 5i^2}{1+i+1-i} \\ &= \frac{5+5i}{2}\end{aligned}$$

Part C: Show that the two expressions are equivalent. Using Euler's formula:

$$\begin{aligned}\frac{5}{\sqrt{2}}e^{i\pi/4} &= \frac{5}{\sqrt{2}}(\cos(\pi/4) + i\sin(\pi/4)) \\ &= \frac{5}{\sqrt{2}}(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) \\ &= \frac{5+5i}{2}\end{aligned}$$

References and Further Reading

- Davies: Sec. 2.3, 2.4
- Ogata: Sec. 2.2