Lecture 8: Inverse Laplace Transform

In the previous lecture, we discussed how to apply the Laplace transform to a time-domain function $f(t)$ to obtain $F(s) = \mathcal{L}[f(t)]$. Now, we consider the opposite case. Given $F(s)$, how can we recover $f(t)$? This will be an important step when in solving LTI ODEs (to be discussed in the next lecture). Here, we'll discuss two methods for obtaining the inverse Laplace transform—using the Laplace transform table and using partial fraction expansion.

Method 1: Laplace Transform Table

When the function $F(s)$ appears as a linear combination of simple functions that are found in the Laplace transform table we can read the table directly to find the inverse Laplace transform of each simple function then add them together to give the overall solution.

Example: Find the inverse Laplace transform of

$$F(s) = \frac{3}{s+1} + \frac{2\pi}{s^2 + (2\pi)^2} \ .$$

By the linearity of the inverse Laplace transform,

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] \\ &= 3\underbrace{\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]}_{\text{Row 6}} + \underbrace{\mathcal{L}^{-1}\left[\frac{2\pi}{s^2+(2\pi)^2}\right]}_{\text{Row 10}} \ . \end{aligned}$$

Consulting the table, we find that the two Laplace transforms above appear on rows 6 and 10. Sometimes it is necessary to manipulate the terms such that they match the table's entries (e.g., by pulling out the scalar multiplier in the first term or rearranging). The inverse Laplace transform is thus

$$f(t) = 3e^{-t} + \sin(2\pi t) \ .$$

In the example above the two terms in $F(s)$ were easily found in the Laplace transform table. In some cases it may be necessary to manipulate the equations so that they occur in the correct form as shown in the Laplace transform table.

Method 2: Partial Fraction Expansion

When $F(s)$ does not contain expression that are readily found in the Laplace transform table we rely on partial fraction expansion (PFE). The basic idea of PFE is simple: since $F(s)$ is not in the required form we will expand it as a sum of terms that *are* in the Laplace transform table. Once this is done we can find the inverse Laplace transform in the same way as we did above in Method 1. PFE is thus merely an approach to re-write $F(s)$ in a convenient way.

Zero-Pole-Gain Form. Every $F(s)$ can always be written as

$$F(s) = \frac{Q(s)}{R(s)} \,, \tag{1}$$

where the numerator

$$Q(s) = d_m s^m + d_{m-1} s^{m-1} + \cdots + d_1 s + d_0$$

is a polynomial in s with highest power m and $d_m, d_{m-1}, \cdots, d_0$ are all constant coefficients, and similarily the denominator

$$R(s) = a_n s^n + c_{n-1} s^{n-1} + \cdots + a_1 s + c_0$$

is a polynominal in s with highest power n and $a_n, c_{n-1}, \cdots, c_0$ are all constant coefficients. Moreover, since a polynominal can always be factored, we can rewrite (1) in zero-pole-gain form by factoring the numerator and demoniator as

$$F(s) = \frac{k(s - z_1)(s - z_1) \cdots (s - z_m)}{(s - p_1)(s - p_2) \cdots (s - p_n)} = \frac{A(s)}{B(s)} \,, \tag{2}$$

where the values $z_1, z_2, \ldots, z_m$ are called the *zeros* and the values $p_1, p_2, \ldots, p_n$ are called the *poles* of $F(s)$. The new (factored) numerator is denoted $A(s)$ and the new (factored) denominator is denoted $B(s)$.

Generally, we assume that the the highest power in the denominator $B(s)$ is greater than the highest power in the numerator $A(s)$ (i.e., that $n > m$). The scalar multiplier k is the *gain*. The term "zero" refers to the fact that if s is equal to any of the z values, $s = z_i$ for $i = 1, \ldots, m$, then $F(s) = 0$. Alternatively, if s is equal to any of the p values "poles", $s = p_i$ for $i = 1, \ldots, n$, then $F(s) = \infty$ (since the denominator becomes zero).

Tent Analogy. The above observations have a geometric interpretation. Note tha $F(s)$ is, after all, just a complex number whose value depends on the point $s = x + iy$ in the complex plane. Viewing the magnitude/modulus $|F(s)|$ as a surface plotted over the complex s domain, the value of $|F(s)| = \infty$ occurs whenever s is equal to any of the poles — reminiscent of a tent "pole" lifting the surface $|F(s)|$ up toward infinity. On the other hand, whenever s is equal to the zeros the surface $|F(s)| = 0$ like the point where a tent is staked to the ground. We will return to this analogy in a future lecture.

Partial Fraction Expansion. Now, to compute the inverse Laplace transform we will use partial fraction expansion (PFE) of (2) to expand the expression $F(s)$ into a summation of fractions that are found in the Laplace transform table. To reiterate: the benefit is that once we have done the algebra to re-write $F(s)$ in this form, we can simply consult the table to find the inverse Laplace transform of each term (as in Method 1 describe above). The form of the expansion depends on the value of the poles. In the following, we consider three cases: (I) distinct, real poles, (II) complex poles, and (III) repeated poles. (Just as we did in Lecture 4 for second order systems.) Lastly, we will briefly discuss the case (IV) of mixed poles.

Case I (Distinct, Real Poles). Suppose that the poles $p_1, p_2, \ldots, p_n$ of the denominator in (2) are all unique and real. That is $p_i \in \mathbb{R}$ for all $i = 1, 2, \ldots, n$ and $p_i \neq p_j$ for all $i, j = 1, 2, \ldots, n$. In this case the PFE is of the form

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1}{s - p_1} + \cdots + \frac{a_k}{s - p_k} + \cdots + \frac{a_n}{s - p_n} \,, \tag{3}$$

which is a linear combination of terms that are found in the Laplace transform table. Thus, given some $F(s)$ in the form (1) or (2) with two real-valued poles we wish to re-write using PFE in the form (3) above.

We can solve for the coefficients in the numerator of (3), for example a_k, by multiplying by the corresponding demoniator, $(s - p_k)$, and evaluating at the corresponding pole, $s = p_k$. This process causes all of the terms on the right hand side to vanish, except for a_k:

$$\begin{aligned}
\left.\frac{A(s)(s-p_k)}{B(s)}\right|_{s=p_k} &= \left.\frac{a_1}{s-p_1}(s-p_k)\right|_{s=p_k} + \cdots + \left.\frac{a_k}{s-p_k}(s-p_k)\right|_{s=p_k} + \cdots + \left.\frac{a_n}{s-p_n}(s-p_k)\right|_{s=p_k} \\
&= \underbrace{\frac{a_1}{s-p_1}(s-p_k) + \cdots}_{=0 \quad \text{when } s = p_k} + a_k + \underbrace{\cdots + \frac{a_n}{s-p_n}(s-p_k)}_{=0 \quad \text{when } s = p_k} \\
&= a_k
\end{aligned}$$

On the left-hand side, the $(s - p_k)$ term in the numerator will cancel the same $(s - p_k)$ term in the denominator. On the right-hand side, all of the terms go to zero, except a_k. By repeating this process k times we obtain all the coefficients $a_1, a_2, \ldots, a_k$. Once the coefficients are found, we can substitute them into (3) and compute the inverse Laplace transform:

$$\begin{aligned}
f(t) &= \mathcal{L}^{-1}[F(s)] \\
&= \mathcal{L}^{-1}\left[\frac{a_1}{s-p_1}\right] + \cdots + \mathcal{L}^{-1}\left[\frac{a_n}{s-p_n}\right] \\
\implies f(t) &= a_1 e^{p_1 t} + \cdots + a_n e^{p_n t}
\end{aligned}$$

Example: Find the inverse Laplace transform of

$$F(s) = \frac{s+3}{s^2 + 3s + 3} \,.$$

First we re-write the expression in zero-pole-gain form by factoring the denominator

$$F(s) = \frac{s+3}{(s+1)(s+2)} \,.$$

From above, we note that there is one zero, $z_1 = -3$, and two real, distinct poles, $p_1 = -1$ and $p_2 = -2$. The gain is one, $k = 1$. Thus, our partial fraction expansion will be in the form:

$$F(s) = \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+2)}$$

and it remains to solve for the coefficients a_1 and a_2. Solving for a_1, we multiply both sides by the denominator of the corresponding partial fraction and evaluate at $s = p_1 = -1$:

$$\begin{aligned}
\frac{s+3}{(s+1)(s+2)}(s+1)\bigg|_{s=-1} &= \frac{a_1}{(s+1)}(s+1)\bigg|_{s=-1} + \frac{a_2}{(s+2)}(s+1)\bigg|_{s=-1} \\
\frac{s+3}{(s+2)}\bigg|_{s=-1} &= a_1 + \underbrace{\frac{a_2}{(s+2)}(s+1)\bigg|_{s=-1}}_{=0} \\
2 &= a_1
\end{aligned}$$

Repeating this process for a_2:

$$\begin{aligned}
\frac{s+3}{(s+1)(s+2)}(s+2)\bigg|_{s=-2} &= \frac{a_1}{(s+1)}(s+2)\bigg|_{s=-2} + \frac{a_2}{(s+2)}(s+2)\bigg|_{s=-2} \\
\frac{s+3}{(s+1)}\bigg|_{s=-2} &= \underbrace{\frac{a_1}{(s+1)}(s+2)\bigg|_{s=-2}}_{=0} + a_2 \\
-1 &= a_2
\end{aligned}$$

Note the right-hand side always reduces to the kth coefficient a_k. When solving this yourself you may skip the intermediate steps and simply evaluate the left-hand side. Now, substituting a_1 and a_2, $F(s)$ becomes:

$$F(s) = \frac{2}{s+1} - \frac{1}{s+2}\,,$$

and the inverse Laplace transform of each term above is found from the table to give

$$\begin{aligned}
&\mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{2}{s+1}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] \\
&\implies f(t) = 2e^{-t} - e^{-2t}\,.
\end{aligned}$$

Case II (Repeated Real Poles). A system with n repeated poles p implies that the denominator is of the form:

$$F(s) = \frac{A(s)}{B(s)} = \frac{A(s)}{(s-p)^n} \tag{4}$$

The PFE uses n summations of a fraction with a denominator $(s-p)$ whose exponent increases from 1 to n:

$$F(s) = \frac{A(s)}{(s-p)^n} = \frac{a_1}{(s-p)} + \cdots + \frac{a_k}{(s-p)^k} + \cdots + \frac{a_n}{(s-p)^n}\,. \tag{5}$$

The solution process to find the unknown coefficients involves multiplying by the highest order denominator to give two polynomials expressions on the left and right sides of the equation, then grouping like terms and equating the corresponding coefficients to give a system of equations for the coefficients. That is, multiply both sides by $(s-p)^n$:

$$F(s)(s-p)^n = \frac{a_1(s-p)^n}{(s-p)} + \cdots + \frac{a_k(s-p)^n}{(s-p)^k} + \cdots + \frac{a_n(s-p)^n}{(s-p)^n} \tag{6}$$

$$F(s)(s-p)^n = a_1(s-p)^{n-1} + \cdots + a_k(s-p)^{n-k} + \cdots + a_n \tag{7}$$

and re-write the LHS and RHS time grouping terms of s, s^2, ..., s^n. Equating coefficients gives a system of n equations for n unknowns. Consider the following example.

Example: Find the inverse Laplace transform of

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3}$$

Since the denominator is third order ($n = 3$) with a repeated pole $p_{1,2,3} = -1$, we expand $F(s)$ as three partial fractions with denominators $(s+1)$ that have an increasing exponent from 1 to n

$$F(s) = \frac{s^2 + 2s + 3}{(s+1)^3} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+1)^3} .$$

Multiply both sides by the highest order denominator, $(s+1)^3$, and simplify

$$\begin{aligned} s^2 + 2s + 3 &= a_1(s+1)^2 + a_2(s+1) + a_3 \\ &= a_1(s^2 + 2s + 1) + a_2 s + a_2 + a_3 \\ &= a_1 s^2 + (a_2 + 2a_1)s + (a_1 + a_2 + a_3) \end{aligned}$$

Equate coefficients on the left and right hand sides above, and solve

$$\begin{aligned} (\text{Equating } s^2 \text{ coefficient}) &: 1 = a_1 \\ (\text{Equating } s \text{ coefficient}) &: 2 = a_2 + 2a_1 \implies a_2 = 0 \\ (\text{Equating constants}) &: 3 = a_1 + a_2 + a_3 \implies a_3 = 2 \end{aligned}$$

Thus, the complete partial fraction expansion is

$$F(s) = \frac{1}{s+1} + 2 \cdot \frac{1}{(s+1)^3} ,$$

and the inverse Laplace transform is obtained from row 8 of the table (with $n = 3$) and from row 6

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{(s+1)^3}\right] \\ &= e^{-t} + 2\left(\frac{t^2}{2} e^{-t}\right) \\ &= (1 + t^2)e^{-t} \end{aligned}$$

Example: Find the inverse Laplace transform for the following expression using partial fraction expansion

$$F(s) = \frac{9s + 25}{(s+3)^2}$$

Since the denominator is second order with repeated poles we expand this as two partial fractions

$$F(s) = \frac{9s + 25}{(s+3)^2} = \frac{a_1}{(s+3)} + \frac{a_2}{(s+3)^2}$$

Multiply both sides by the denominator and simplify

$$\begin{aligned} 9s + 25 &= a_1(s+3) + a_2 \\ 9s + 25 &= a_1 s + (3a_1 + a_2) \end{aligned}$$

Equate coefficients and solve

$$(\text{Equating } s \text{ coefficient}) : \implies a_1 = 9$$
$$(\text{Equating constants}) : 25 = 3 \cdot 9 + a_2 \implies a_2 = -2$$

Thus, the complete partial fraction expansion is

$$F(s) = 2 \cdot \frac{9}{(s+3)} - \frac{2}{(s+3)^2} \,,$$

and the inverse Laplace transform is obtained from row 8 of the table (with $n = 2$) and from row 6

$$\begin{aligned} f(t) &= 9\mathcal{L}^{-1}\left[\frac{1}{(s+3)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right] \\ &= 9e^{-3t} - 2te^{-3t} \end{aligned}$$

Case III (Complex Poles). Complex poles occur in complex conjugate pairs. We could use the approach described in Case I by converting the the complex valued exponentials in the solution using Euler's formula. However, this is not recommended since it adds additional complexity required by working with complex numbers. A simpler approach is to recognize that the solutions will consist of sinusoids and cosines multiplied by exponential terms. That is, we expect the solution to have damped sine/cosine terms $e^{-\alpha t} \sin \omega t$ and $e^{-\alpha t} \cos \omega t$ for each complex conjugate pair of poles. From the Laplace transform table, we know that:

$$\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s+\alpha)^2 + \omega^2}$$

and

$$\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

and therefore we seek to find a partial fraction expansion in the form

$$F(s) = a_1 \frac{\omega}{(s+\alpha)^2 + \omega^2} + a_2 \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2} \,. \tag{8}$$

As before, we must solve for the unknown coefficients a_1 and a_2 and the values α and ω. Recall that $F(s) = A(s)/B(s)$ is a ratio of polynomials, and notice that (8) can be rewritten as:

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1\omega + a_2(s+\alpha)}{(s+\alpha)^2 + \omega^2} \,. \tag{9}$$

Aside: If the poles of the $F(s)$ are $p_{1,2} = -\alpha \pm i\omega$ then the denominator of the $F(s)$ can be factored as $(s+\alpha)^2 + \omega^2$ required above since:

$$
\begin{aligned}
(s-p_1)(s-p_2) &= (s - [-\alpha + i\omega])(s - [-\alpha - i\omega]) && (10)\\
&= (s^2 - s(-\alpha - i\omega) - s(-\alpha + i\omega) + (-\alpha + i\omega)(-\alpha - i\omega)) && (11)\\
&= s^2 + 2s\alpha + \alpha^2 + \omega^2 && (12)\\
&= (s+\alpha)^2 + \omega^2 && (13)
\end{aligned}
$$

Thus, α and ω can be found by either equating denominators $R(s) = (s+\alpha)^2 + \omega^2$ and solving for α and ω or, alternatively, by finding the poles since they always appear in the form $p_{1,2} = -\alpha \pm i\omega$. Next, by equating the numerators $Q(s) = a_1\omega + a_2(s+\alpha)$ we can solve for a_1 and a_2. With the coefficients known, and returning to the form (8), the inverse Laplace transform is:

$$
\begin{aligned}
& f(t) = a_1 \mathcal{L}^{-1}\left[\frac{\omega}{(s+\alpha)^2 + \omega^2}\right] + \mathcal{L}^{-1} a_2 \left[\frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}\right] && (14)\\
\implies & f(t) = e^{-\alpha t}[a_1 \sin \omega t + a_2 \cos \omega t] && (15)
\end{aligned}
$$

Example: Find the inverse Laplace transform of

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

Since the denominator is second-order we expect there are two poles. Using the quadratic formula or by completing the square they are found to be $p_{1,2} = -1 \pm 2i$ (see note below for reminder on how to complete the square). Since the poles are complex, we use the partial fraction expansion for complex poles

$$F(s) = \frac{2s+12}{s^2+2s+5} = a_1 \frac{\omega}{(s+\alpha)^2 + \omega^2} + a_2 \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

Equating the denominators:

$$s^2 + 2s + 5 = s^2 + 2\alpha s + (\alpha^2 + \omega^2)\ .$$

For this equation to hold, the coefficients of each polynomial in s must be equal. That is,

$$
\begin{aligned}
\text{(Equating } s \text{ coefficient)} : 2 = 2\alpha \implies \alpha = 1\\
\text{(Equating constants)} : 5 = (\alpha^2 + \omega^2) \implies \omega = 2
\end{aligned}
$$

Notice that these α and ω are related to the poles found earlier as $p_{1,2} = -\alpha \pm \omega$. (We can skip ahead next time and just use α and ω directly from the poles.) Thus, our partial fraction expansion is

$$F(s) = \frac{2s+12}{s^2+2s+5} = \frac{2a_1 + a_2(s+1)}{(s+1)^2 + 2^2}$$

Equating the numerators

$$2s + 12 = a_2 s + (2a_1 + a_2)$$

For this equation to hold, the coefficients of each polynomial in s must be equal. That is,

$$\begin{aligned}
(\text{Equating } s \text{ coefficient}) &: 2 = a_2 \\
(\text{Equating constants}) &: 12 = 2a_1 + a_2 \implies a_1 = 5
\end{aligned}$$

Thus, the complete partial fraction expansion is

$$F(s) = \frac{2s+12}{s^2+2s+5} = 5\frac{2}{(s+1)^2+2^2} + 2\frac{(s+1)}{(s+1)^2+2^2} \,,$$

and the inverse Laplace transform is found for each term from the Laplace transform table

$$\begin{aligned}
f(t) &= 5\mathcal{L}^{-1}\left[\frac{2}{(s+1)^2+2^2}\right] + 2\mathcal{L}^{-1}\left[\frac{(s+1)}{(s+1)^2+2^2}\right] \\
\implies f(t) &= 5e^{-t}\sin 2t + 2e^{-t}\cos 2t \,.
\end{aligned}$$

Aside: The poles in the above example can be found from the denominator by completing the square: we split the constant term such that the left-hand-side is a perfect square and the right-hand-side has a constant

$$\begin{aligned}
s^2 + 2s + 5 &= 0 \\
s^2 + 2s + 1 &= -4 \\
(s+1)^2 &= -4 \\
\implies p_{1,2} &= -1 \pm 2i
\end{aligned}$$

The quadratic formula for $ax^2 + bx + c = 0$ with $a = 1$, $b = 2$ and $c = 5$ gives the same result:

$$p_{1,2} = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm \sqrt{-16}/2 = -1 \pm 2i$$

Case IV (Mixed Poles). In the case when the poles are mixed (e.g., some are repeated, some are complex, and/or some are real and distinct) the partial fraction expansion will also be mixed and a combination of the previous approaches is used.

References and Further Reading

- Davies: Sec. 2.7-2.9
- Ogata: Sec. 2.4, 2.5, 4.4