Homework 1

1 Problem

For each of the ODEs in the 1st column, indicate whether it is:

1. linear time-invariant (LTI), linear time-varying (LTV), or nonlinear
2. 1st order, 2nd order, or higher order
3. homogeneous or inhomogeneous (only for the case of linear systems—for nonlinear systems you can leave the last column blank)

by marking the appropriate column. The unknown function $x(t)$ represents the state of some mechanical system and t represents time. Hint: An ODE is considered nonlinear only if the nonlinearity involves the unknown function $x(t)$.

	Linearity?			Order?			Homogeneity?	
System ODE	LTI	LTV	NL	1st	2nd	Higher	Homog.	Inhomog.
$\ddot{x} + 3tx = 0$								
$(\dot{x} - x)^2 + 1 = 0$								
$t^2 x + bx + c\dot{x} = 0$								
$\dddot{x} = 0$								
$\ddot{x} + \dot{x} + x - 2 =$								
$\ddot{x} + \sin(x) = 0$								
$e^t x + \dot{x} = \sin t$								
$\dot{x} + x = 0$								
$\dot{x}x + a + bt = 0$								
$\ddot{x} - b\dot{x}^2 = 0$								

Solution

	Linearity?			Order?			Homogeneity?	
System ODE	LTI	LTV	NL	1st	2nd	Higher	Homog.	Inhomog.
$\ddot{x} + 3tx = 0$		x			x		x	
$(\dot{x} - x)^2 + 1 = 0$			x	x			–	–
$t^2 x + bx + c\dot{x} = 0$		x		x			x	
$\dddot{x} = 0$	x					x	x	
$\ddot{x} + \dot{x} + x - 2 =$	x				x			x
$\ddot{x} + \sin(x) = 0$			x		x		–	–
$e^t x + \dot{x} = \sin t$		x		x				x
$\dot{x} + x = 0$	x			x			x	
$\dot{x}x + a + bt = 0$			x	x			–	–
$\ddot{x} - b\dot{x}^2 = 0$			x		x		–	–

Note to graders: For systems that are nonlinear the last column regarding homogeneity can be ignored.

2 Problem

Consider the following IVP:

$$\dot{x} + 2x = 0$$

with initial condition $x(t_0) = -10$ and $t_0 = 0$.

1. What is the particular solution, $x(t)$?
2. What is the value of x as time $t \to \infty$?

A. $x \to -\infty$
B. $x \to -10$
C. $x \to 0$
D. $x \to +10$
E. $x \to +\infty$

Solution

The solution to the IVP for this homogeneous first-order ODE is

$$x(t) = -10e^{-2t} \ .$$

As $t \to \infty$ grows large the term e^{-2t} approaches zero. Thus, the solution is C.

3 Problem

Consider the ODE $\dot{x} + 2x = e^{-2t}$ with initial condition $x(t_0) = 10$ and $t_0 = 0$. What is the particular solution, $x(t)$?

Solution

The ODE is first-order and inhomogeneous. Thus, the general solution is:

$$x(t) = \underbrace{e^{-at}\int e^{at} g dt}_{\text{inhomogeneous component}} + \underbrace{Ce^{-at}}_{\text{homogeneous component}} \ .$$

For the problem given $a = 2$ and $g = e^{-2t}$ and

$$\begin{aligned} x(t) &= e^{-2t}\int e^{2t}e^{-2t}dt + Ce^{-2t} \\ &= e^{-2t}\int dt + Ce^{-2t} \\ &= e^{-2t}t + Ce^{-2t} \ . \end{aligned}$$

To solve for the particular solution, evaluate the above expression at the initial condition $x(0) = 10$.

$$x(0) = 10 = e^{-2\cdot 0}0 + Ce^{-0\cdot t} = C$$

Thus, the particular solution is

$$\implies x(t) = e^{-2t}t + 10e^{-2t}$$

4 Problem

For each of the following ODEs determine if the eigenvalues are (a) real and distinct, (b) repeated, (c) complex conjugate pairs:

1. $\ddot{x} + 2\dot{x} + 3x = 0$
2. $\ddot{x} + 4\dot{x} + x = 0$
3. $\ddot{x} + 4\dot{x} + 4x = 0$
4. $\ddot{x} + 3x = 0$

Solution

The solution is found by evaluating $\sqrt{b^2 - 4ac}$ in each case.

1. $\sqrt{b^2 - 4ac} = 2.82i \implies$ Complex conjugate pairs
2. $\sqrt{b^2 - 4ac} = 3.4641 \implies$ Real
3. $\sqrt{b^2 - 4ac} = 0 \implies$ Repeated
4. $\sqrt{b^2 - 4ac} = 3.4641i \implies$ Complex conjugate pairs

5 Problem

For each of the following linear, time-invariant, 2nd order homogeneous ODEs solve for the particular solution that satisfies the initial values given.

1. $\ddot{x} - 4\dot{x} + 4x = 0$, Initial Values: $x(0) = 12$, $\dot{x}(0) = -3$
2. $\ddot{x} + 3\dot{x} - 10x = 0$, Initial Values: $x(0) = 4$, $\dot{x}(0) = -2$
3. $\ddot{x} - 8\dot{x} + 17x = 0$, Initial Values: $x(0) = -4$, $\dot{x}(0) = -1$

Solution

1. $\ddot{x} - 4\dot{x} + 4x = 0$, Initial Values: $x(0) = 12$, $\dot{x}(0) = -3$

Example 1 Solve the following IVP.

$$y'' - 4y' + 4y = 0 \quad y(0) = 12 \quad y'(0) = -3$$

Hide Solution ▾

The characteristic equation and its roots are.

$$r^2 - 4r + 4 = (r-2)^2 = 0 \quad r_{1,2} = 2$$

The general solution and its derivative are

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{2t} + c_2 t\mathbf{e}^{2t} \\ y'(t) &= 2c_1\mathbf{e}^{2t} + c_2\mathbf{e}^{2t} + 2c_2 t\mathbf{e}^{2t} \end{aligned}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$\begin{aligned} 12 &= y(0) = c_1 \\ -3 &= y'(0) = 2c_1 + c_2 \end{aligned}$$

This system is easily solved to get $c_1 = 12$ and $c_2 = -27$. The actual solution to the IVP is then.

$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

2. $\ddot{x} + 3\dot{x} - 10x = 0$, Initial Values: $x(0) = 4$, $\dot{x}(0) = -2$

Example 2 Solve the following IVP

$$y'' + 3y' - 10y = 0 \quad y(0) = 4 \quad y'(0) = -2$$

Hide Solution ▾

The characteristic equation is

$$\begin{aligned} r^2 + 3r - 10 &= 0 \\ (r+5)(r-2) &= 0 \end{aligned}$$

Its roots are $r_1 = -5$ and $r_2 = 2$ and so the general solution and its derivative is.

$$\begin{aligned} y(t) &= c_1\mathbf{e}^{-5t} + c_2\mathbf{e}^{2t} \\ y'(t) &= -5c_1\mathbf{e}^{-5t} + 2c_2\mathbf{e}^{2t} \end{aligned}$$

Now, plug in the initial conditions to get the following system of equations.

$$\begin{aligned} 4 &= y(0) = c_1 + c_2 \\ -2 &= y'(0) = -5c_1 + 2c_2 \end{aligned}$$

Solving this system gives $c_1 = \frac{10}{7}$ and $c_2 = \frac{18}{7}$. The actual solution to the differential equation is then

$$y(t) = \frac{10}{7}\mathbf{e}^{-5t} + \frac{18}{7}\mathbf{e}^{2t}$$

3. $\ddot{x} - 8\dot{x} + 17x = 0$, Initial Values: $x(0) = -4$, $\dot{x}(0) = -1$

Example 2 Solve the following IVP.

$$y'' - 8y' + 17y = 0 \quad y(0) = -4 \quad y'(0) = -1$$

Hide Solution ▾

The characteristic equation this time is.

$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^{4t} \cos(t) + c_2 \mathbf{e}^{4t} \sin(t) \\ y'(t) &= 4c_1 \mathbf{e}^{4t} \cos(t) - c_1 \mathbf{e}^{4t} \sin(t) + 4c_2 \mathbf{e}^{4t} \sin(t) + c_2 \mathbf{e}^{4t} \cos(t) \end{aligned}$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

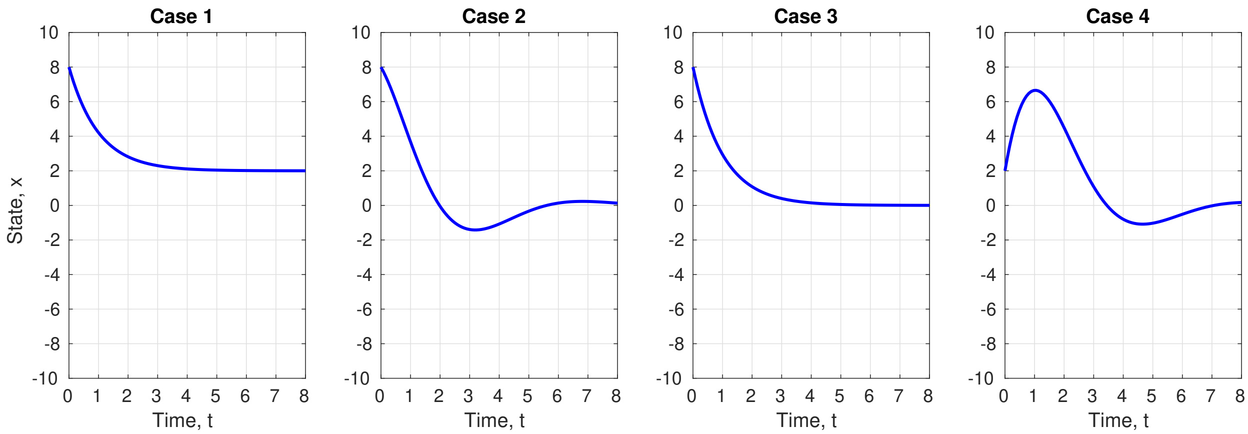
$$\begin{aligned} -4 &= y(0) = c_1 \\ -1 &= y'(0) = 4c_1 + c_2 \end{aligned}$$

Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.

$$y(t) = -4\mathbf{e}^{4t} \cos(t) + 15\mathbf{e}^{4t} \sin(t)$$

6 Problem

Match each one of the responses shown below (Cases 1-4) with one of the following IVPs



(a) $\ddot{x} + \dot{x} + x = 0, x(0) = 8, \dot{x}(0) = 0$

(b) $\dot{x} - x = 2,\ x(0) = 8$

(c) $\dot{x} + x = 2,\ x(0) = 8$

(d) $\ddot{x} + \dot{x} + x = 2, x(0) = 0, \dot{x}(0) = -8$

(e) $\ddot{x} + \dot{x} + x = 2, x(0) = 2, \dot{x}(0) = 0$

(f) $\dot{x} + x = 8,\ x(0) = 2$

(g) $\dot{x} + x = 0,\ x(0) = 8$

(h) $\ddot{x} + \dot{x} + x = 0, x(0) = 8, \dot{x}(0) = -3$

(i) $\ddot{x} + \dot{x} + x = 0, x(0) = 2, \dot{x}(0) = 10$

(j) $\dot{x} + 8x = 0,\ x(0) = 0$

Solution

- By inspection, Case 1 is a first-order ODE with initial condition $x(0) = 8$. Since the solution decays to $x = 2$ it must have $a > 0$ and be inhomogeneous. Answer: (c)
- By inspection, Case 2 is a second-order ODE with initial condition $x(0) = 8$ and $\dot{x}(0) < 0$. Answer: (h)
- By inspection, Case 3 is a first-order ODE with initial condition $x(0) = 8$. Since the solution decays to zero it must have $a > 0$ and be homogeneous. Answer: (g)
- By inspection, Case 4 is a second-order ODE with initial condition $x(0) = 2$ and $\dot{x}(0) > 0$. Answer: (i)