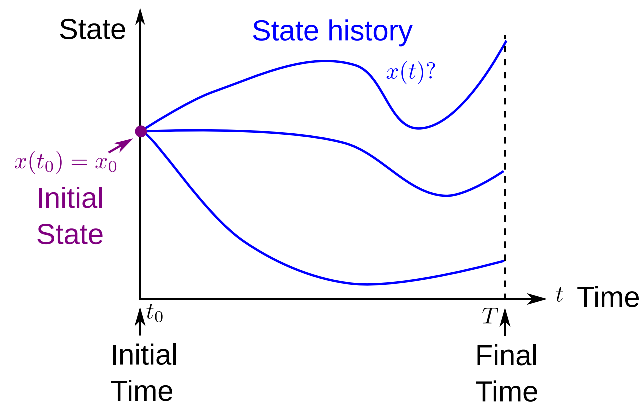
Lecture 3: Solving Linear First-Order ODEs

Initial Value Problems.

In this class we will learn techniques for solving initial value problems (IVPs) for first and second-order linear time-invariant ODEs. That is, we will seek to find the unknown function $x(t)$ (sometimes called a *state history* or *trajectory*) that evolves over time as from an initial time t_0 and initial state $x(t_0) = x_0$. Possible solutions to an IVP are depicted as the blue curves below.



To solve an IVP for a first or second-order LTI system we need the following:

1. an initial time t_0 (usually taken to be zero)

2. an initial state $x(t_0) = x_0$

3. the system dynamics, either:

$$\dot{x}(t) + ax(t) = u(t) \qquad \text{(first-order LTI system)}$$

or

$$\ddot{x}(t) + a\dot{x}(t) + bx = u(t) \qquad \text{(second-order LTI system)}$$

Below, we review solution techiques for first-order LTI systems you've probably already seen in a previous course on differential equations. Later, we will introduce Laplace transforms that can be used to solve similar IVPs.

Examples of Real-World First-Order Systems

A first-order system model is the simplest model of a dynamic physical system. Any systems whose state-rate is proportional to the state itself is first-order. Examples include:

- Newton's Law of Heating/Cooling: $\dot{T} + k(T - T_{\text{ambient}}) = 0$ where T is the temperature of an object, T_{ambient} is the ambient room temperature, and k is a coefficient of heat transfer. This equations describes how the temperature equalizes to ambient conditions.

- An RC circuit (a resistor connected to a capacitor with no voltage source): $\dot{V} + [1/(RC)]V = 0$ where V is voltage, R is resistance, and C is capacitance. This equation describes how the voltage of the capacitor decays.

- Stoke's Drag: $\dot{v} + bv = 0$ where v is the velocity of an object and b is a drag coefficient. This equation describes hows an object comes to rest under drag/friction that is proportional to its velocity.

All of these systems have only one "storage" element that can absorb or dissipate energy — i.e., the thermal energy, the electrical energy, or the kinetic energy.

Homogeneous First-Order LTI ODEs.

The standard form of a homogeneous first-order LTI ODE is

$$\dot{x}(t) + ax(t) = 0 \,.$$

Rearrange this equation as

$$\dot{x}(t) = -ax(t)$$

and re-write the derivative in more conventional notation

$$\frac{dx}{dt} = -ax(t) \,.$$

Seperate variables such that the left-hand side includes only terms in $x(t)$ and the right-hand side includes only terms in t

$$\frac{1}{x(t)}dx = -adt \,.$$

Now, integrate both sides to obtain

$$\begin{aligned}\int \frac{1}{x(t)}dx &= \int -adt \\ \ln x(t) &= -at + \tilde{C} \,,\end{aligned}$$

where $\tilde{C}$ is a constant of integration and ln is the natural logarithm (the logarithm with base e). Proceed by exponentiating each side:

$$\begin{aligned}e^{\ln x(t)} &= e^{-at+\tilde{C}} \\ x(t) &= e^{-at}e^{\tilde{C}} \,.\end{aligned}$$

Define a new constant $C \triangleq e^{\tilde{C}}$:

$$x(t) = Ce^{-at} \,. \tag{1}$$

Equation (1) is the *general solution* since it holds for any initial condition (we still have the unknown variable C). Whereas, the *particular solution* is the solution for a specific initial condition $x(t_0) = x_0$ with $t_0 = 0$. To solve for the unknown constant C given an initial condition we simply evaluate the general solution (1) at this initial point

$$x(0) = Ce^{-a\cdot 0} = C = x_0 \,.$$

Using this value of C, the particular solution is

$$x(t) = x_0 e^{-at} \tag{2}$$

The solution (2) indicates that $x(t)$ begins at x_0 and varies exponentially, depending on the value of a. If $a = 0$ the particular solution is $x(t) = x_0 e^{-0 \cdot t} = x_0$ and the solution remains at x_0 for all time. If $a > 0$, then the solution can be rewritten as

$$x(t) = \frac{x_0}{e^{at}}$$

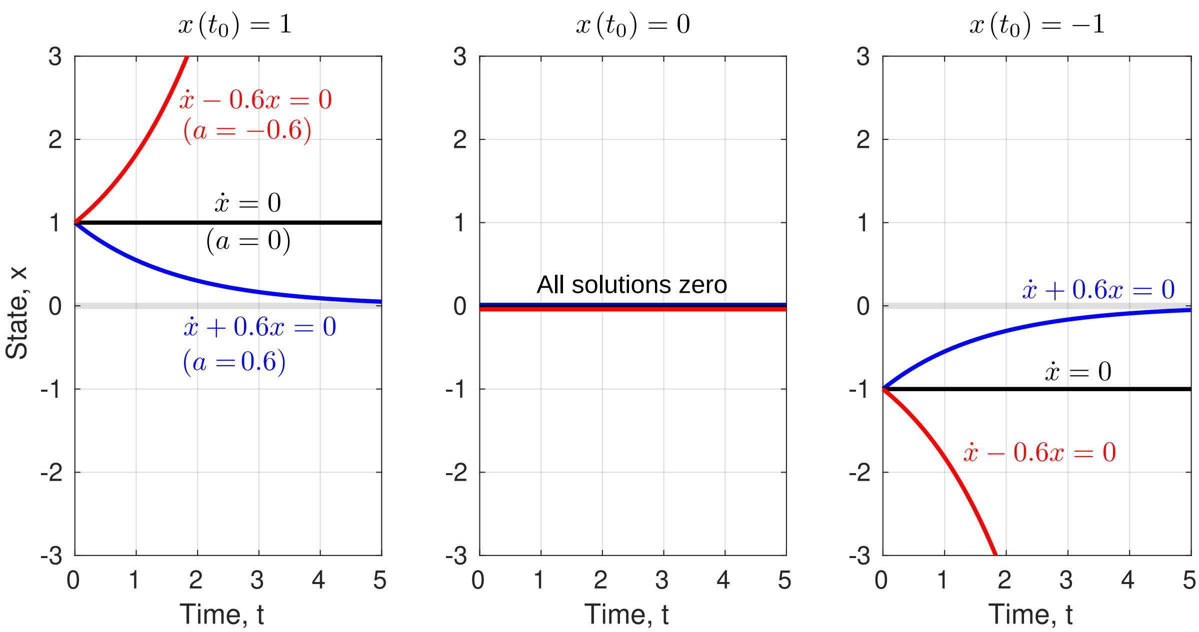
and as time grows large, $t \to \infty$, the denominator $e^{at} \to \infty$ so that the solution $x(t) \to 0$. Thus, if $a > 0$ the solution always decays to zero. Lastly, consider the case $a < 0$. The solution can be written as

$$x(t) = x_0 e^{-at} = x_0 e^{|a|t} \qquad \text{(Note: since } a \text{ is negative, then } -a = |a|)$$

which makes it clear that as $t \to \infty$ the solution $|x(t)| \to \infty$. In other words, the solution diverges away from the origin (either towards positive or negative infinity, depending on the initial condition).

Example. Consider the ODE $\dot{x} + 2x = 0$ with initial condition $x(t_0) = 10$ and $t_0 = 0$. Using (2) the solution is $x(t) = 10e^{-2t}$.

Example. Consider a set of ODEs of the form $\dot{x} + ax = 0$ with either $a = 0.6$, $a = 0$, or $a = -0.6$ (corresponding to the blue, black, and red curves shown below). Each panel below corresponds to a different initial condition (as indicated in the panel title). Notice that in the middle panel the response is always zero, regardless of a. For the left and right panels (non-zero initial conditions), the blue curves with $a < 0$ converge towards $x = 0$ while the red curves with $a > 0$ diverge away from the origin.



Time Constant

For first-order LTI homogeneous ODEs that are stable (i.e., with $a > 0$) the solution always decays to zero from the the initial condition. We can gain some intuitive sense for how long this decay will take by determining the *time constant*

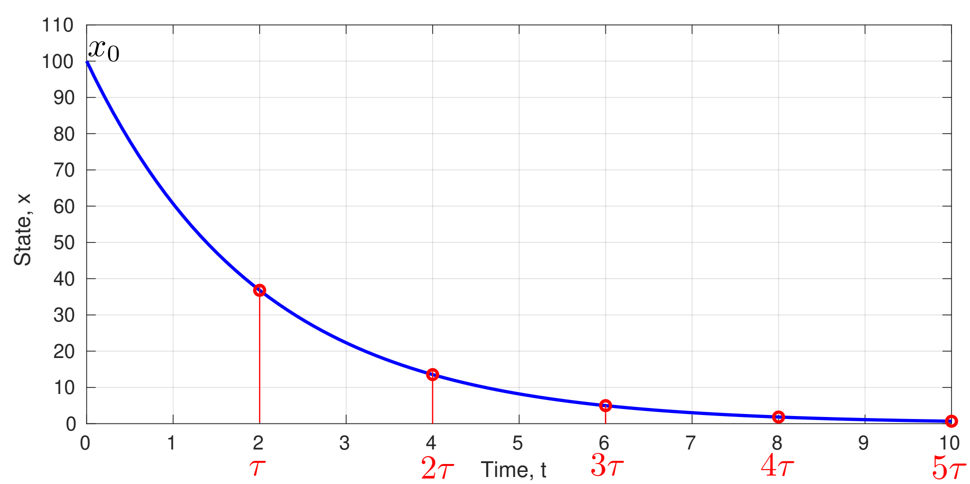
$$\tau = \frac{1}{a} \,. \tag{3}$$

Now consider evaluating the particular solution, $x(t) = x_0 e^{-at}$ at times equal to multiples of the time constant $t = \tau, 2\tau, 3\tau, 4\tau$. We find that

$$\begin{aligned}
&\text{after one time constant} \quad && x(\tau) = x_0 e^{-a\tau} = x_0 e^{-a(\frac{1}{a})} = x_0 e^{-1} \approx 0.368 x_0 && (4)\\
&\text{after two time constants} \quad && x(2\tau) = x_0 e^{-a\tau} = x_0 e^{-a(2\frac{1}{a})} = x_0 e^{-2} \approx 0.135 x_0 && (5)\\
&\text{after three time constants} \quad && x(3\tau) = x_0 e^{-a\tau} = x_0 e^{-a(3\frac{1}{a})} = x_0 e^{-1} \approx 0.050 x_0 && (6)\\
&\text{after four time constants} \quad && x(4\tau) = x_0 e^{-a\tau} = x_0 e^{-a(4\frac{1}{a})} = x_0 e^{-4} \approx 0.018 x_0 && (7)\\
& && && (8)
\end{aligned}$$

After four time constants the solution $x(4\tau)$ is at 1.8% of the initial condtion x_0 and has almost entirely decayed to zero.

Example: Consider the system $\dot{x} + 0.5x = 0$ with $x(0) = 100$. The particular solution is $x(t) = 100e^{-0.5t}$ and the time constant is $\tau = 1/0.5 = 2$ sec. Since the system has $a > 0$ it will decay to zero. At about four time constants, $4\tau = 10$ sec, the response has almost entirely decayed.



Inhomogeneous First-Order LTI ODEs.

Now we will consider the inhomogeneous case of the form:

$$\dot{x}(t) + ax(t) = u(t) \tag{9}$$

where the inhomogeneous term $u(t)$ is known. Suppose there is an "integrating factor" $q(t)$ that has the property that

$$q(t)ax(t) = \dot{q}(t)x(t) \tag{10}$$

which an be re-written to eliminate x as

$$q(t)a = \dot{q}(t) = \frac{dq(t)}{dt} \; .$$

Seperating variables and integrating:

$$\begin{aligned} adt &= \frac{1}{q(t)}dq \\ \int adt &= \int \frac{1}{q}dq \\ at + \tilde{D} &= \ln q(t) \; . \end{aligned}$$

Now exponentiating each side

$$\begin{aligned} e^{at+\tilde{D}} &= e^{\ln q(t)} \\ e^{at}e^{\tilde{D}} &= q(t) \; . \end{aligned}$$

Define $D = e^{\tilde{D}}$ so that the integrating factor satisfying (10) is

$$q(t) = De^{at} \tag{11}$$

Multiply both sides of (9) by the integrating factor q

$$q(t)\dot{x}(t) + q(t)ax(t) = q(t)u(t) \; .$$

Substitute (10)

$$q(t)\dot{x}(t) + \dot{q}(t)x(t) = q(t)u(t)$$

and then substitute (11) and eliminate D

$$\begin{aligned} (De^{at})\dot{x}(t) + \frac{d}{dt}(De^{at})x(t) &= (De^{at})u(t) \\ De^{at}\dot{x}(t) + Dae^{at}x(t) &= De^{at}u(t) \\ e^{at}\dot{x}(t) + ae^{at}x(t) &= e^{at}u(t) \end{aligned}$$

which can be further simplified by recognizing that $\frac{d}{dt}\left(e^{at}x(t)\right) = e^{at}\dot{x}(t) + ae^{at}x(t)$ (from the product rule). Thus,

$$\frac{d}{dt}\left(e^{at}x(t)\right) = e^{at}u(t) \; .$$

and integrate the above equation

$$\begin{aligned} \int \frac{d}{dt}\left(e^{at}x(t)\right) &= \int e^{at}u(t)dt \\ e^{at}x(t) &= \int e^{at}u(t)dt + C \end{aligned}$$

then multiply both sides by e^{-at} to yield the solution

$$
\begin{aligned}
e^{-at}e^{at}x(t) &= e^{-at}\int e^{at}u(t)dt + Ce^{-at} \\
x(t) &= \underbrace{e^{-at}\int e^{at}u(t)dt}_{\text{inhomogeneous component}} + \underbrace{Ce^{-at}}_{\text{homogeneous component}} .
\end{aligned} \tag{12}
$$

We see that the second term in (12) is the same as our general solution (1) for the homogeneous case, whereas the first term accounts for the inhomogeneous solution (and it goes away if $g = 0$). The *principle of linear superposition* allows us to add solutions in linear combinations.

Example. Consider the ODE $\dot{x} + 2x = 5$ with initial condition $x(t_0) = 10$ and $t_0 = 0$. Using (12) the general solution is

$$
\begin{aligned}
x(t) &= e^{-2t}\left[\int e^{2t}\cdot 5\cdot dt\right] + Ce^{-2t} \\
&= e^{-2t}\left[\frac{1}{2}e^{2t}\right] + Ce^{-2t} \\
&= \frac{5}{2} + Ce^{-2t} .
\end{aligned}
$$

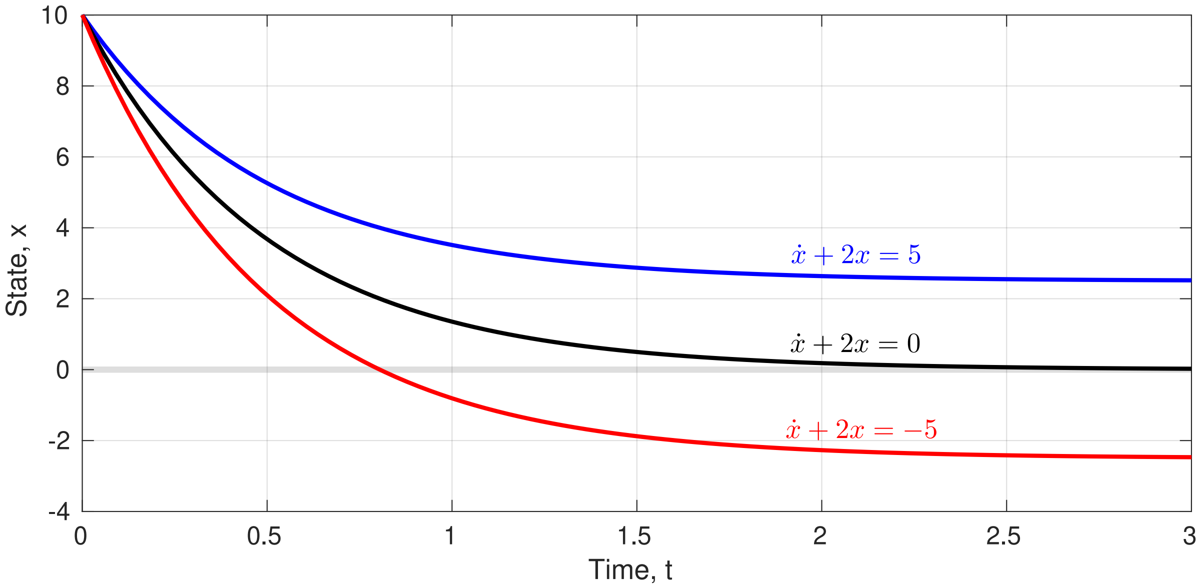
Then, applying the initial condition

$$
\begin{aligned}
x(0) = \frac{5}{2} + Ce^{-2\cdot 0} &= 10 \\
\frac{5}{2} + C &= 10
\end{aligned}
$$

we can rearrange for the constant $C = 7.5$. The particular solution is

$$x(t) = 2.5 + 7.5e^{-2t} .$$

As $t \to \infty$ the second term in this solution goes to zero, and $x(t) \to 2.5$. This is shown as the blue curve below.



Example (changing inhomogeneous term) Now, let's briefly consider a more general form of this ODE: $\dot{x} + 2x = g$, where the inhomogeneous term g is left as a variable. Consider three possible values, $g = 5$ (the original case above), $g = 0$, or $g = -5$ (corresponding to the blue, black, and red curves shown below). When $g = 0$ we have the homogeneous case and the curve converges to $x = 0$. When g is non-zero we see that the curve has a similar shape, but it converges to a different steady-state value. Solving the IVP for more general, non-constant, forms of $u(t)$ using (12) is possible only in some other special cases.

References and Further Reading

- `https://tutorial.math.lamar.edu/Classes/DE/IntroFirstOrder.aspx`