Homework 3

1 Problem

Find the Laplace transform fraction for the following function and rearrange it such that $X(s)/F(s)$ is the only term on the left-hand-side:

$$\dddot{x}(t) + 2\zeta\omega\ddot{x}(t) + \omega^2\dot{x}(t) = f(t)$$

Assume the initial conditions are all zero, $x(t_0) = \dot{x}(t_0) = \ddot{x}(t_0) = \dddot{x}(t_0) = 0$ with initial time $t_0 = 0$. Hint: Use the differentiation theorem.

Solution

Since

$$\mathcal{L}[\dddot{x}] = s^3X(s) - \underbrace{s^2\ddot{x}(0) - s\dot{x}(0) - x(0)}_{=0}$$

$$\mathcal{L}[\ddot{x}] = s^2X(s) - \underbrace{s\dot{x}(0) - x(0)}_{=0}$$

By initial conditions

$$\mathcal{L}[\dot{x}] = sX(s) - \underbrace{x(0)}_{=0}$$

$$\mathcal{L}[f(t)] = F(s)$$

Then

$$\mathcal{L}[\dddot{x} + 2\zeta\omega_n\ddot{x} + \omega^2\dot{x}] = \mathcal{L}[f(t)]$$

$$\Big\downarrow \text{ By linearity.}$$

$$\mathcal{L}[\dddot{x}] + 2\zeta\omega_n\mathcal{L}[\ddot{x}] + \omega^2\mathcal{L}[\dot{x}] = F(s)$$

$$\Big\downarrow \text{ Plug in from above.}$$

$$s^3X(s) + 2\zeta\omega_n s^2X(s) + \omega^2 sX(s) = F(s)$$

$$\Big\downarrow \text{ Rearrange}$$

$$X(s)\left(s^3 + 2\zeta\omega_n s^2 + \omega^2 s\right) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{s(s^2 + 2\zeta\omega_n s + \omega^2)}$$

2 Problem

Compute the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of the following function:

$$F(s) = \frac{1}{s+\sigma}(e^{-as} - e^{-bs})$$

where a, b, and σ are constants.

Hint: Recall that the following property holds for translated functions $\mathcal{L}[f(t-\alpha)H(t-\alpha)] = e^{-s\alpha}F(s)$, which implies that

$$f(t) = \mathcal{L}^{-1}[e^{-s\alpha}F(s)] = f(t-\alpha)H(t-\alpha)$$

The expression above can be written as a sum of two functions of this form.

Solution

Expand $F(s)$:

$$F(s) = \frac{1}{s+\sigma}(e^{-as} - e^{-bs}) \tag{1}$$
$$= e^{-as}\frac{1}{s+\sigma} + e^{-as}\frac{1}{s+\sigma} \tag{2}$$

The above expression is in the form of a translated function. The first term delays the signal $\frac{1}{s+\alpha}$ until $t = a$ and the second term delays the signals $\frac{1}{s+\sigma}$ until $t = b$. Recall that $\mathcal{L}^{-1}[\frac{1}{s+\alpha}] = e^{-\sigma t}$ then

$$f(t) = \mathcal{L}^{-1}\left[e^{-as}\frac{1}{s+\sigma}\right] + \mathcal{L}^{-1}\left[e^{-as}\frac{1}{s+\sigma}\right] \tag{3}$$
$$= H(t-a)e^{-\sigma(t-a)} - H(t-b)e^{-\sigma(t-b)} \tag{4}$$

3 Problem

Compute the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of the following function:

$$F(s) = \frac{s+1}{s^2+6s+9}$$

Solution

The denominator can be factored as $(s+3)^2$ which indicates that there are two repeated poles $p_{1,2} = -3$

$$F(s) = \frac{s+1}{s^2+6s+9} = \frac{s+1}{(s+3)^2}$$

The partial fraction expansion is thus:

$$F(s) = \frac{c_1}{s+3} + \frac{c_1}{(s+3)^2} \tag{5}$$

Multiplying both sides by $(s+3)^2$:

$$F(s)(s+3)^2 = c_1(s+3) + c_2 \tag{6}$$
$$(s+1) = c_1 s + (3c_1 + c_2) \tag{7}$$

Equating coefficiens we see immediately that $c_1 = 1$ and can solve for c_2 as:

$$1 = 3c_1 + c_2 \tag{8}$$
$$1 = 3(1) + c_2 \tag{9}$$
$$\implies c_2 = -2 \tag{10}$$

Thus the partial fracton expansion is:

$$F(s) = \frac{1}{(s+3)} - 2\frac{1}{(s+3)^2} \tag{11}$$

and taking the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+3)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right] \tag{12}$$
$$= \tag{13}$$

4 Problem

Compute the inverse Laplace transform of

$$F(s) = \frac{s+1}{s(s+2)}$$

Solution

From the denominator it is clear that the poles are $p_1 = 0$ and $p_2 = -2$ hence real and distinct and thus we wish to expand $F(s)$ as

$$F(s) = \frac{a_1}{s} + \frac{a_2}{(s+2)}$$

The coefficients are found from

$$a_1 = \left[\frac{(s+1)s}{s(s+2)}\right]_{s=0} = \left[\frac{(s+1)}{(s+2)}\right]_{s=0} = \frac{1}{2} \tag{14}$$
$$a_2 = \left[\frac{(s+1)(s+2)}{s(s+2)}\right]_{s=-2} = \left[\frac{(s+1)}{s}\right]_{s=-2} = \frac{1}{2} \tag{15}$$

Thus, $F(s)$ is equivalent to

$$F(s) = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{(s+2)}$$

which is in a form for which we can easily compute the inverse Laplace by table lookup. Using rows 2 and 6 in the Laplace transform table:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right]$$
$$= \frac{1}{2}H(t) + \frac{1}{2}e^{-2t}$$

which, for $t \geq 0$ is equivalent to

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}$$

5 Problem

Compute the inverse Laplace transform of

$$F(s) = \frac{s+2}{(s+1)(s+4)^2}$$

Hint: Use a combination of partial fraction expansions for real distinct and repeated poles.

Solution

There are three poles of the system: $p_1 = -1$ and $p_{2,3} = -4$ which is a distinct pole and pair of repeated poles. Thus the partial fraction expansion we seek is of the form

$$F(s) = \frac{c_1}{s+1} + \frac{c_2}{s+4} + \frac{c_3}{(s+4)^2}s \tag{16}$$
$$\tag{17}$$

To find c_1 we can use the approach for distinct poles:

$$c_1 = \left[\frac{(s+2)(s+1)}{(s+1)(s+4)^2}\right]_{s=-1} \tag{18}$$
$$= \left[\frac{(s+2)}{(s+4)^2}\right]_{s=-1} = \frac{1}{9} \tag{19}$$

To find c_2 and c_3 we multiply both sides of (17) by $(s+1)(s+4)^2$

$$F(s)(s+4)^2(s+1) = \frac{1}{9}(s+4)^2 + c_2(s+4)(s+1) + c_3(s+1) \tag{20}$$
$$(s+2) = \frac{1}{9}(s^2+8s+16) + c_2(s^2+5s+4) + c_3(s+1) \tag{21}$$
$$(s+2) = s^2(\frac{1}{9}+c_2) + s(\frac{8}{9}+5c_2+c_3) + (\frac{16}{9}+4c_2+c_3) \tag{22}$$

Then, equating coefficients on the LHS and RHS:

$$s^2: \qquad 0 = \frac{1}{9} + c_2 \tag{23}$$

$$\implies c_2 = -\frac{1}{9} \tag{24}$$

$$s: \qquad 1 = \frac{8}{9} + 5(\frac{-1}{9}) + c_3 \tag{25}$$

$$\implies c_3 = \frac{9-8+5}{9} = \frac{2}{3} \tag{26}$$

The partial fraction expnasion is then:

$$F(s) = \frac{1}{9}\frac{1}{s+1} - \frac{1}{9}\frac{1}{s+4} + \frac{2}{3}\frac{1}{(s+4)^2}s \tag{27}$$

Taking the inverse Laplace transform (with rows 6 and 7):

$$f(t) = \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{(s+4)^2}s\right] \tag{28}$$

$$= \frac{1}{9}e^{-t} - \frac{1}{9}e^{-4t} + \frac{2}{3}te^{-4t} \tag{29}$$

6 Problem

Compute the inverse Laplace transform of

$$F(s) = \frac{3s+9}{s^2+4s+5}$$

Solution

Consider the example:

$$Y(s) = \frac{3s+9}{s^2+4s+5}$$

The roots of the denominator are -2+/-i. We can complete the square for the denominator. We have

$$s^2+4s+5 = s^2+4s+4+1 = (s+2)^2+1$$

Hence, we have

$$Y(s) = \frac{3s+9}{(s+2)^2+1}$$

Note the denominator (s+2)^2+1 is similar to that for Laplace transforms of exp(-2t)cos(t) and exp(-2t)sin(t). We need to manipulate the numerator. Note that in the formula in the table, we have a=-2 and b=1. We look for a decomposition of the form

$$\frac{3s+9}{(s+2)^2+1} = \frac{A(s+2)+B}{(s+2)^2+1}$$

If we can find A and B, then:

$$\frac{3s+9}{(s+2)^2+1} = A\frac{s+2}{(s+2)^2+1} + B\frac{1}{(s+2)^2+1}$$

The inverse transform is

$$y(t) = L^{-1}[Y(s)](t) = Ae^{-2t}\cos t + Be^{-2t}\sin t$$

We can determine A and B by equating numerators in the expression

$$\frac{3s+9}{(s+2)^2+1} = \frac{A(s+2)+B}{(s+2)^2+1} = \frac{As+2A+B}{(s+2)^2+1}$$

Comparing coefficients of s in the numerator we conclude 3=A. Comparing the constant terms we conclude 2A+B=9. Hence A=3 and B=3.