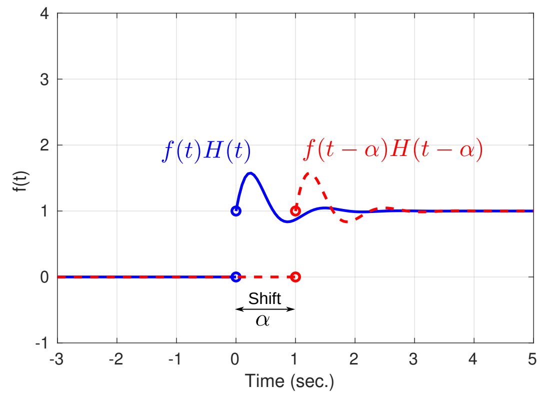
Lecture 7: Additional Properties of The Laplace Transform

In this lecture we continue our discussion of Laplace transforms by introducing several additional properties.

Laplace Transform of a Translated Function

Consider a translate function as shown below



where

$$H(t-\alpha) = \begin{cases} 0 & \text{for } t \leq \alpha \\ 1 & \text{for } t > \alpha \end{cases}$$

is the unit step/Heaviside function shifted by α. It is useful to translate a function in this way when the signal is delayed by some amount of time. For example, suppose we were modeling the throttle input to a car in a simulation and the throttle goes from 0% to 100% at exactly 10 seconds. We could model this as a translated step input $H(t-10)$ which would be zero for all time $t < 10$ and one for all time $t \geq 10$.

Given a translated function, we wish to find its Laplace transform. Applying the Laplace transform integral to $f(t-\alpha)H(t-\alpha)$ gives

$$\mathcal{L}[f(t-\alpha)H(t-\alpha)] = \int_0^\infty f(t-\alpha)H(t-\alpha)e^{-st}dt$$

Defining $\tau = t - \alpha$. Then, $d\tau/dt = 1$ and $t = \tau + \alpha$. Changing variables by substitution:

$$\mathcal{L}[f(t-\alpha)H(t-\alpha)] = \int_{-\alpha}^\infty f(\tau)H(\tau)e^{-s(\tau+\alpha)}d\tau$$

where the lower bound on the integral, $t = 0$, becomes $\tau = -\alpha$. Since $H(\tau)$ is zero for any $\tau < 0$ we can change the lower limit (again, back to zero). Pulling out the exponential term that does not depend on τ and using the definition of a Laplace transform:

$$
\begin{aligned}
\mathcal{L}[f(t-\alpha)H(t-\alpha)] &= \underbrace{\int_{-\alpha}^{0} f(\tau)H(\tau)e^{-s\tau}e^{-s\alpha}d\tau}_{=0} + \int_{0}^{\infty} f(\tau)H(\tau)e^{-s\tau}e^{-s\alpha}d\tau \\
&= \int_{0}^{\infty} f(\tau)H(\tau)e^{-s\tau}e^{-s\alpha}d\tau \\
&= e^{-s\alpha}\underbrace{\int_{0}^{\infty} f(\tau)H(\tau)e^{-s\tau}d\tau}_{\mathcal{L}[f(t)H(t)]} \\
&= e^{-s\alpha}\mathcal{L}[f(t)H(t)] \\
\implies \mathcal{L}[f(t-\alpha)H(t-\alpha)] &= e^{-s\alpha}F(s)
\end{aligned}
$$

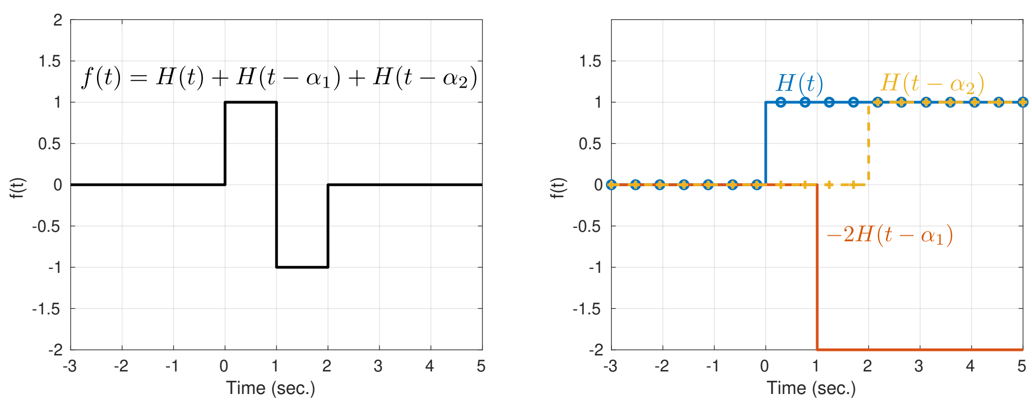
Several delayed functions can be combined to create a more complex signal, as shown in the following example.

Example: Consider

$$f(t) = H(t) - 2H(t-\alpha_1) + H(t-\alpha_2)$$

with $\alpha_1 = 1$ and $\alpha_2 = 2$ and recall that $\mathcal{L}[H(t)] = \frac{1}{s}$. Thus,

$$
\begin{aligned}
F(s) &= \mathcal{L}[f(t)] \\
&= \frac{1}{s} - 2e^{-s}\frac{1}{s} + e^{-2s}\frac{1}{s} \\
&= \frac{1 - 2e^{-s} + e^{-2s}}{s}
\end{aligned}
$$



This signal is sometimes called a doublet and the example illustrates how we can create more complex signals by combining shifted Heaviside functions.

Differentiation Theorem

Our ultimate goal is to be able to take the Laplace transform of an ODE, solve for $X(s)$ and then compute the inverse Laplace transform to yield the solution $x(t)$. To do so we first discuss the

Laplace transform of the derivative of a function $f(t)$, which is the following (stated without proof):

$$\mathcal{L}[\frac{d}{dt}f(t)] = sF(s) - f(0)$$

where $f(0)$ is the initial value of $f(t)$ at $t = 0$. For the second derivative:

$$\mathcal{L}[\frac{d^2}{dt^2}f(t)] = s^2F(s) - sf(0) - \dot{f}(0)$$

where $\dot{f}(0)$ is the initial value of df/dt at $t = 0$. For the *nth* derivative

$$\mathcal{L}[\frac{d^n}{dt^n}f(t)] = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - f^{n-1}(0)$$

For example, with $n = 3$

$$\mathcal{L}[\dddot{f}] = s^3F(s) - s^2f(0) - s\dot{f}(0) - \ddot{f}(0)$$

Example: Find the Laplace transform $X(s)$ given the ODE

$$\ddot{x} + 3\dot{x} + 2x = 0$$

with initial conditions $x(0) = a$ and $\dot{x}(0) = b$. In this equation, $x(t)$ is the unknown function whose Laplace transform is denoted

$$X(s) = \mathcal{L}[x(t)] \ .$$

According to the differentiation theorem, the Laplace transform of the first and second derivatives of $x(t)$ are given by

$$\begin{aligned}
\mathcal{L}[\dot{x}(t)] &= sX(s) - x(0) \\
&= sX(s) - a \\
\mathcal{L}[\ddot{x}(t)] &= s^2X(s) - sx(0) - \dot{x}(0) \\
&= s^2X(s) - sa - b
\end{aligned}$$

Note that the Laplace transform of the right-hand side of the ODE $\mathcal{L}[0] = 0$. Thus, by linearity of the Laplace transform, we may write the Laplace transform of the ODE as

$$\begin{aligned}
\mathcal{L}[\ddot{x} + 3\dot{x} + 2x] &= \mathcal{L}[0] \\
\mathcal{L}[\ddot{x}] + 3\mathcal{L}[\dot{x}] + 2\mathcal{L}[x] &= 0 \\
(s^2X(s) - sa - b) + 3(sX(s) - a) + 2X(s) &= 0 \\
s^2X(s) + 3sX(s) + 2X(s) - sa - b - 3a &= 0 \\
X(s)(s^2 + 3s + 2) &= sa + 3a + b \\
X(s) &= \frac{sa + 3a + b}{(s^2 + 3s + 2)}
\end{aligned}$$

Initial/Final Value Theorem

Given the Laplace transform $F(s) = \mathcal{L}[f(t)]$ the following theorems allow us to determine the initial value (at time $t = 0$) and the final value (as $t \to \infty$) of $f(t)$.

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{(Initial Value Theorem)}$$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{(Final Value Theorem)}$$

This can be very useful—if we know the Laplace transform of a particular function in the s-domain we can find out what the function doe sin the time t-domain by applying the above two limits. That is, we can determine where the function starts out at time $t = 0$ and where it ends up as $t \to \infty$.

Example: Given the Laplace transform

$$F(s) = 0.25\frac{(s+2)}{(s+2)^2 + (3\pi)^2}$$

find the initial and final value of $f(t)$.

$$\begin{aligned}
\lim_{t\to 0} f(t) &= \lim_{s\to\infty} sF(s) && \text{(From the initial value theorem)} \\
&= \lim_{s\to\infty} s\left(0.25\frac{(s+2)}{(s+2)^2 + (3\pi)^2}\right) && \text{(Substitute } F(s)\text{)} \\
&= \lim_{s\to\infty} \left(0.25\frac{(s^2+2s)}{s^2 + 4s + 4 + (3\pi)^2}\right) && \text{(Expand)} \\
&= \lim_{s\to\infty} \left(0.25\frac{\left(\frac{s^2}{s^2} + \frac{2s}{s^2}\right)}{\frac{s^2}{s^2} + \frac{4s}{s^2} + \frac{4+(3\pi)^2}{s^2}}\right) && \text{(Divide top and bottom by } s^2\text{)} \\
&= \lim_{s\to\infty} \left(0.25\frac{\left(1 + \frac{2}{s}\right)}{1 + \frac{4}{s} + \frac{4+(3\pi)^2}{s^2}}\right) && \text{(Simplify)}
\end{aligned}$$

Notice that all the terms involving s are in the form $1/s$ or $1/s^2$. As $s \to \infty$ these terms tend to zero. Thus, we may evaluate the limit as

$$\lim_{t\to 0} f(t) = 0.25$$

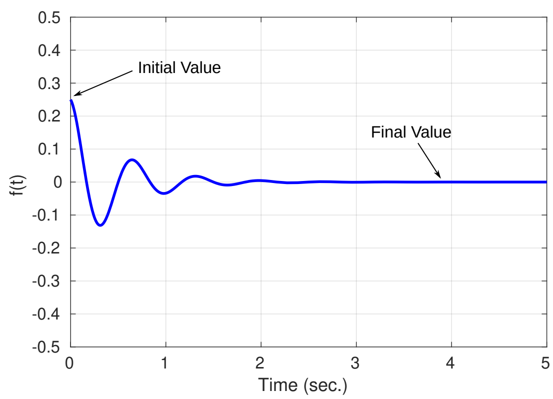
To apply the final value theorem:

$$\begin{aligned}
\lim_{t\to\infty} f(t) &= \lim_{s\to 0} sF(s) \qquad \text{(From the final value theorem)}\\
&= \lim_{s\to 0} s\left(0.25\frac{(s+2)}{(s+2)^2+(3\pi)^2}\right) \qquad \text{(Substitute } F(s)\text{)}\\
&= \lim_{s\to 0}\left(0.25\frac{(s^2+2s)}{s^2+4s+4+(3\pi)^2}\right) \qquad \text{(Expand)}\\
&= \left(0.25\frac{(0)^2+2(0))}{(0)^2+4(0)+4+(3\pi)^2}\right) \qquad \text{(Substitute } s=0\text{)}\\
&= 0
\end{aligned}$$

The results from above can be confirmed by computing the corresponding time-domain function

$$f(t) = 0.25e^{-2t}\cos(3\pi t)$$

and inspecting its graph below



References and Further Reading

- Davies: Sec. 2.6
- Ogata: Sec. 2.3