Lecture 2: The Terminology of ODEs

Recall that an *ordinary differential equation (ODE)* is an equation that contains one or more derivatives of an unknown function, $x(t)$. A scalar ODE with one input can always be rearranged into the form:

$$F(t, x, \dot{x}, \ddot{x}, \cdots) = g(t) \tag{1}$$

where

$$\begin{aligned} t &: \text{is the independent variable (e.g., time)} \\ x = x(t) &: \text{is the dependent variable/unknown function (e.g., the state of a mechanical system)} \\ \dot{x} = \dot{x}(t) &: \text{is the first derivative of the state} \\ \ddot{x} = \ddot{x}(t) &: \text{is the second derivative of the state} \\ g(t) &: \text{is called a } \textit{forcing function}, \textit{input}, \text{ or } \textit{inhomogeneous term} \text{ that lumps all constants and} \\ &\quad \text{time-varying terms not involving } x, \dot{x}, \text{ etc.} \end{aligned}$$

In (1) the time-dependence is suppressed for brevity (i.e., we wrote x instead of $x(t)$).

> *Example*: the ODE $\dot{x} - t\cos x = 0$ is first-order because the highest derivative is $\dot{x}$. We can re-write this equation in the form: $F(t, x, \dot{x}) = \dot{x} - t\cos x = 0$ to emphasize that it only depends on the variables t, x and $\dot{x}$ (and possibly other constant coefficients).

> *Example*: the ODE $-\ddot{x} - 9x\dot{x} + e^{-2t} = 0$ is second-order because the highest derivative is $\ddot{x}$. We can re-write this equation in the form as $F(t, x, \dot{x}, \ddot{x}) = \ddot{x} + 9x\dot{x} = e^{-2t}$ by moving all the terms involving x to the left-hand side except, for the time-varying term $g(t) = e^{-2t}$.

Linearity. A linear ODE can be written as a linear combination of the independent variable and its derivatives on the left-hand side (i.e., for the $F(t, x, \dot{x}, \cdots)$ term) and an inhomogeneous function (i.e., one that does not depend on x) on the right-hand side, $g(t)$.

> *Example*: $a(t)\dot{x} + b(t)x = g(t)$ is a first-order linear ODE. The coefficients $a(t)$ and $b(t)$ may be constants or time-varying.

> *Example*: $a(t)\ddot{x} + b(t)\dot{x} + c(t)x(t) = g(t)$ is a second-order linear ODE.

In the above examples the unknown function $x(t)$ and its derivatives, $\dot{x}(t)$ and $\ddot{x}(t)$, appear as separate terms that multiple a (possibly) time-varying coefficient, such as $a(t)$, $b(t)$, or $c(t)$. If an equation can be written in this form it is called linear. Otherwise, it is called nonlinear. Note, however, that a linear ODE may still have coefficients (e.g., $a(t)$, $b(t)$, or $c(t)$) that are nonlinear in the independent variable t.

> *Example*: $t^3\dot{x} + 1 = 0$ is a linear first-order ODE with coefficients $a(t) = t^3$, $b(t) = 0$, and inhomogeneous constant term $g(t) = 1$.

ODEs that are nonlinear will contain terms that have powers, quotients, square roots, exponentials, trigonometric functions, and other special functions with $x(t)$ as the argument.

> *Example*: the following terms are all nonlinear terms in x: $x^2, x\dot{x}, \sqrt{x}, \sin x, e^x$, and 2^x

> *Example*: the following terms are all linear in x, despite containing nonlinearities in the independent variable t: $xt^2, t\dot{x}, \sqrt{t}\ddot{x}, x\sin t, (1-x)e^t$, and $2^t(t+1)x$

Time-variance. If the coefficients of a linear ODE (e.g., $a(t)$, $b(t)$, or $c(t)$) are time-varying we refer to the system as a *linear time-varying (LTV)* system:

> *Example*: $tx + \dot{x} = 0$ is an first-order LTV system of the form $a(t)\dot{x} + b(t)x = 0$ with time-varying coefficient $b(t) = t$ and constant coefficient $a(t) = 1$.

On the other hand, a system with all constant coefficients is called *linear time-invariant (LTI)*. In this case we can drop the time-dependence from the coefficients and simply write: $a\dot{x} + bx + c = g$ in the case of a first-order LTI system.

> *Example*: $\ddot{x} + 2\dot{x} + x = 3$ is an LTI system with constant coefficients $a = 1$, $b = 2$, $c = 1$, and $g = 3$.

Linear Standard Form. In the case of a linear equation we can divide by the coefficient multiplying the highest-order term so that it is normalized to one. For example, consider a first-order linear system written as $c_2(t)\dot{x} + c_1(t)x = h(t)$ (we've used different variables for the coefficients, but don't let that bother you). If we divide each term by $c_2(t)$ we obtain:

$$\dot{x} + \frac{c_1(t)}{c_2(t)}x = \frac{h(t)}{c_2(t)}$$

which can then be re-written as

$$\dot{x} + a(t)x = g(t)$$

by defining $a(t) \triangleq c_1(t)/c_2(t)$ and $g(t) \triangleq h(t)/c_2(t)$.

Similarly, a second-order linear system of the form $c_3(t)\ddot{x} + c_2(t)\dot{x} + c_1(t)x = h(t)$ can always be rewritten as:

$$\ddot{x} + a(t)\dot{x} + b(t)x = g(t)$$

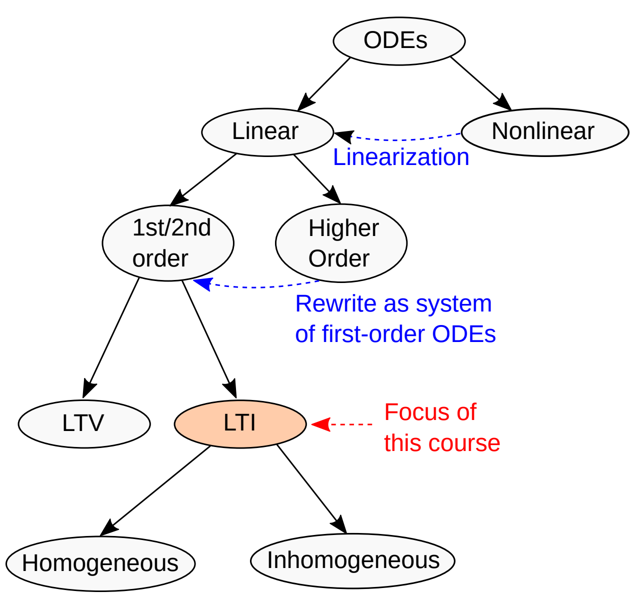
by defining $a(t) \triangleq c_2(t)/c_3(t)$, $a(t) \triangleq c_1(t)/c_3(t)$, and $g(t) \triangleq h(t)/c_3(t)$. Linear ODEs written this way are in *standard form*.

Example: The equation $2\dot{x} + 4x - \sin t = 0$ can be rewritten in standard form as $\dot{x} + 2x = \left(\frac{1}{2}\right)\sin t$.

Example: The equation $t^2\dot{x} + x\sin t + t\ddot{x} + 10 = 0$ can be rewritten in standard form as $\ddot{x} + t\dot{x} + \left(\frac{\sin t}{t}\right)x = \left(\frac{-10}{t}\right)$.

Homogeneity. Linear ODEs in standard form that have $g(t) = 0$ are referred to as *homogeneous* ODES, and those with $g(t) \neq 0$ are referred to as *inhomogeneous*. When $g(t) = 0$ there is no external input into the system, that is, it evolves only due to the initial conditions.

Summary. The flowchart below summarizes the terminology introduced in this section. The focus of this course is on linear time-invariant (LTI) systems. While not covered in this course, it is possible to approximate a nonlinear system around a particular operating point through *linearization*. Also, all higher-order linear systems can be written as a system of multiple coupled first-order ODEs. LTI systems therefore represent a wide class of systems that have applications in mechanical and electrical engineering and beyond.



References and Further Reading

- Ogata: Section 1.1
- P. Dawnkins, "Paul's Online Notes: Differential Equations: Basic Concepts", Lamar University, URL: `https://tutorial.math.lamar.edu/Classes/DE/Definitions.aspx`