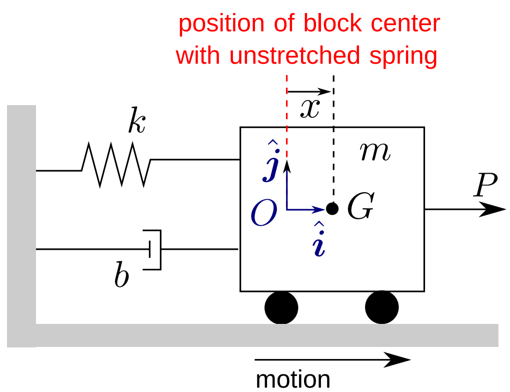
Lecture 11: Damped Harmonic Oscillator

Many mechanical systems exhibit oscillatory behavior, for example:

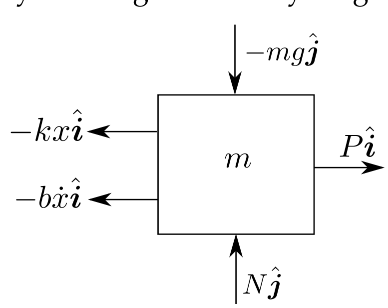
- A swinging pendulum on a grandfather clock
- A car suspension on a bumpy road
- Liquid sloshing in a tank
- A plucked guitar string

Oscillations correspond to energy being traded between different storage elements, from kinetic to potential (e.g., gravitational or elastic potential) and vice versa, in a periodic fashion. Real systems do not oscillate forever without energy input since there are always dissipating elements (dampers) that remove energy from the system. Oscillating systems can be understood by analogy to the mass-spring-damper, also called the damped harmonic oscillator (DHO). In the following, we analyze the equations of motion of the DHO to gain insight into other (e.g., electrical) oscillating systems.

Equations of Motion (Damped Harmonic Oscillator). Consider the diagram shown below with a block of mass m attached to a wall by a spring with constant k and a damper with damping constant b. We assume the mass moves on wheels over a friction-less surface and is acted upon by a (possibly time-varying) external force $P(t)$ in the horizontal direction. You may recognize this system from our earlier lecture on 2nd order ODEs. We've introduced an inertial reference frame $\mathcal{I} = \{O, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}\}$ that remains stationary at point O, where O is a fixed point in space corresponding to the center of the block when the spring is unstretched. Let G be the (moving) position of the center of the block. Then $x(t)$ is a the distance between O and G (positive when G is to the right of O).



To analyze this system we begin by drawing a free body diagram:



The applied force in vector notation is $\boldsymbol{P}(t) = P(t)\hat{\boldsymbol{\imath}}$ since it acts in the horizontal direction. To determine the sign of the spring and damping forces it is useful to conduct a thought experiment: in which direction would these forces act if the position is positive, $x(t) > 0$, and if the the velocity is positive, $\dot{x}(t) > 0$? (That is, assume the point G is to the right of O and moving further to the right.) Intuitively, both forces act to the left (negative $\hat{\boldsymbol{\imath}}$ directoin) to return the mass towards point O. Thus, we add the negative sign to our expressions for the spring force (Hooke's law) and damping (assume linear damping with velocity):

$$\boldsymbol{F}_{\text{damper}} = -b\dot{x}\hat{\boldsymbol{\imath}} \qquad \text{and} \qquad \boldsymbol{F}_{\text{spring}} = -kx\hat{\boldsymbol{\imath}} \tag{1}$$

where $\dot{x}$ is the speed, b is the damping coefficient, and k is the spring constant. The other two forces acting on the system are the weight and the normal force.

Since the cart slides only in the horizontal diretion, the inertial acceleration is $\boldsymbol{a}_{G/O} = \ddot{x}\hat{\boldsymbol{\imath}}$. Then, from Newton's 2nd Law,

$$\begin{aligned} \sum \boldsymbol{F} = (N - mg)\hat{\boldsymbol{\jmath}} + (P - b\dot{x} - kx)\hat{\boldsymbol{\imath}} &= m\boldsymbol{a}_{G/O} && (2)\\ &= (m\ddot{x})\hat{\boldsymbol{\imath}}\ . && (3) \end{aligned}$$

The weight force balances the normal force, so we only consider motion in the $\hat{\boldsymbol{\imath}}$ direction, which gives the scalar equation:

$$\hat{\boldsymbol{\imath}}\ : \qquad P - b\dot{x} - kx = m\ddot{x} \tag{4}$$

Finally, rearranging into the standard form for a second-order LTI ODE:

$$\implies \ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = \frac{P}{m} \tag{5}$$

We derived the above equation assuming $P \neq 0$ and will return to this form later. For the moment, suppose there is no applied force ($P = 0$) — this is the *unforced response* or the *free response*. The system exhibits an oscillatory behavior if either the initial displacement is non-zero $x(0) = x_0 \neq 0$ and/or if the initial velocity is non-zero $\dot{x}(0) = v_0 \neq 0$.

Free Response of a Damped Harmonic Oscillator Consider the system (5) with no applied force $P = 0$ and initial conditions $x(0) = x_0$ and $\dot{x}(0) = v_0 \neq 0$. Because it simplifies some of the algebra we also set $x_0 = 0$ and return to the general case later. Thus, we have the initial value problem

$$\ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0\ ,\ x(0) = x_0 = 0\ ,\ \dot{x}(0) = v_0 \neq 0 \tag{6}$$

Taking the Laplace transform,

$$(s^2X(s) - sx(0) - \dot{x}(0) + \frac{b}{m}(sX(s) - x(0)) + \frac{k}{m}X(s) = 0 \tag{7}$$

Substituting the initial conditions, simplifying, and rearranging:

$$\begin{aligned} s^2X(s) - v_0 + \frac{b}{m}sX(s) + \frac{k}{m}X(s) &= 0 && (8)\\ X(s)\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right) &= v_0 && (9)\\ X(s) &= \frac{v_0}{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} && (10) \end{aligned}$$

The poles of the system (roots of the denominator) are

$$p_{1,2} = \frac{\frac{-b}{m} \pm \sqrt{\frac{b^2}{m^2} - 4\frac{k}{m}}}{2} \tag{11}$$

DHO notation. The constants m, k, and b all have physical units of mass, force per unit length, and force per unit speed, respectively. The numerical values of these constants can vary depending on the particular problem at hand and the unit system used. To gain a more intuitive understanding it is helpful to introduce the following parameters:

- Damping ratio:

$$\zeta = \frac{b}{2\sqrt{km}} \geq 0 \qquad \text{(non-dimensional)}$$

- Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} \qquad \text{(rad/s)}$$

- Damped natural frequency:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \qquad \text{(rad/s)}$$

Recall that when a frequency ω is expressed in units of rad/s the period is given by

$$T = \frac{2\pi}{\omega} \quad \text{(seconds)} \,.$$

The above expressions also give the relation:

$$\frac{b}{m} = 2\zeta\omega_n \,.$$

Thus, the system (51) can be re-written as

$$\ddot{x} + \underbrace{\left(\frac{b}{m}\right)}_{=2\zeta\omega_n}\dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{=\omega_n^2} x = 0 \tag{12}$$

$$\implies \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \tag{13}$$

and the poles of the system become:

$$p_{1,2} = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4\omega_n^2}}{2} \tag{14}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2} \tag{15}$$

$$= \frac{-2\zeta\omega_n \pm \sqrt{4\omega_n^2(\zeta^2 - 1)}}{2} \tag{16}$$

$$= \frac{-2\zeta\omega_n \pm 2\omega_n\sqrt{\zeta^2 - 1}}{2} \tag{17}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \tag{18}$$

$$= -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\sqrt{-1} \tag{19}$$

$$\implies p_{1,2} = -\zeta\omega_n \pm \omega_d i \tag{20}$$

Working with the new notation in (13) and (20) we proceed to analyze the unforced response for different cases of the damping ratio ζ.

Case $\zeta < 1$ underdamped. If $\zeta < 1$ then $\omega_d = \sqrt{1-\zeta^2} > 0$ and the poles (20) are complex. This suggests we should expand $X(s)$ in (10) as a damped sinusoid using partial fraction expansion:

$$X(s) = \frac{v_0}{\left(s^2 + \frac{b}{m}s + \frac{k}{m}\right)} = \frac{v_0}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{c_1(s+\alpha)}{(s+\alpha)^2+\omega^2} + \frac{c_2\omega}{(s+\alpha)^2+\omega^2} \tag{21}$$

Equating the denominators gives the expression

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 2\alpha s + (\alpha^2 + \omega^2)\ . \tag{22}$$

Following the typical PFE approach we equate from this expression in s:

$$2\zeta\omega_n = 2\alpha \qquad \Longrightarrow \qquad \alpha = \zeta\omega_n \tag{23}$$

and constants

$$\omega_n^2 = (\alpha^2 + \omega^2) \tag{24}$$
$$= (\zeta^2\omega_n^2 + \omega^2) \tag{25}$$
$$\omega_n^2 - \zeta^2\omega_n^2 = \omega^2 \qquad \Longrightarrow \qquad \omega = \omega_n(1-\zeta^2) = \omega_d\ . \tag{26}$$

We could also have obtained the above result more directly from the poles. Now, equating the numerators of (21)

$$v_0 = c_1 s + (c_1\alpha + c_2\omega) \tag{27}$$

Since the left hand side has no terms in s it follows that

$$c_1 = 0$$

and, moreover, $v_0 = c_2\omega$ or

$$c_2 = \frac{v_0}{\omega_d}\ .$$

Substituting the constants into the PFE:

$$X(s) = \frac{v_0}{\omega_d}\left[\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}\right] \tag{28}$$

The inverse Laplace transform is the damped sinusoid

$$x(t) = \mathcal{L}^{-1}[X(s)] = \frac{v_0}{\omega_d}\mathcal{L}^{-1}\left[\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2}\right] \tag{29}$$
$$\Longrightarrow\ x(t) = \frac{v_0}{\omega_d}e^{-\zeta\omega_n t}\sin(\omega_d t) \tag{30}$$

The above equation shows us that the response of the system has frequency ω_d and the larger the ζ value the amplitude of the oscillation will decay.

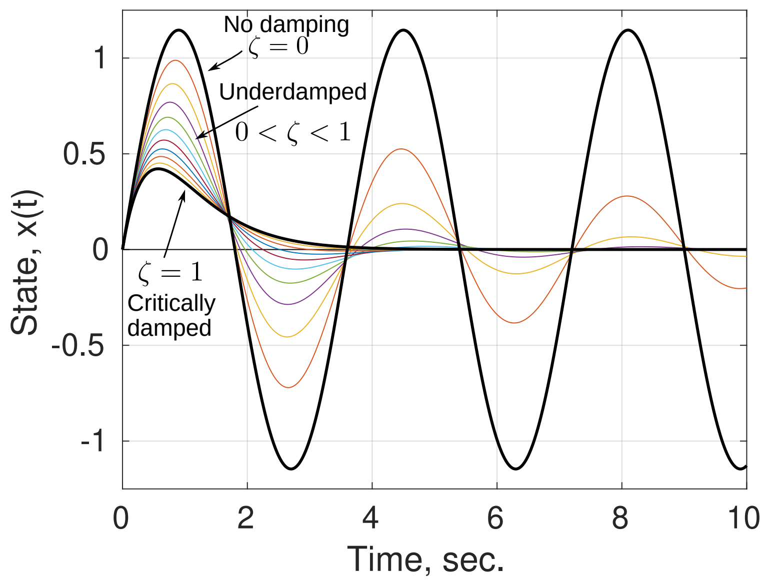
Case $\zeta = 0$ undamped. When $\zeta = 0$ the two terms in (30) become

$$e^{-\zeta\omega_n t} = 1 \qquad \text{and} \qquad \omega_d = \omega_n\sqrt{1-\zeta^2} = \omega_n$$

and (30) becomes

$$x(t) = \frac{v_0}{\omega_d}\sin(\omega_n t) \tag{31}$$

which corresponds to undamped oscillation that persists forever with a constant magnitude. This corresponds to the case where there is no damping ($b = 0$) and the DHO is referred to as a *simple harmonic oscillator* (SHO). The figure below shows the no damping case in a solid black line with increasing damping shown as colored lines. All of the underdamped cases exhibit oscillation to various degrees. When the damping ratio is increased and reaches the critical value $\zeta = 1$ the response has no oscillation and is called critically damped.



Case $\zeta = 1$ critically damped. In this case, the poles (20) become

$$p_{1,2} = -\underbrace{\zeta}_{=1}\omega_n \pm \underbrace{\omega_d}_{=\omega_n\sqrt{1-\zeta^2}=0} \tag{32}$$

$$p_{1,2} = -\omega_n \tag{33}$$

which is a repeated root and $X(s)$ can be factored as

$$X(s) = \frac{v_0}{s + 2\omega_n s + \omega_n^2} = \frac{v_0}{(s+\omega_n)^2} = \frac{c_1}{s+\omega_n} + \frac{c_2}{(s+\omega_n)^2}\,. \tag{34}$$

The above expression is already in a form found in the Laplace transform table. Taking the inverse Laplace Transform (row 8 in the table),

$$x(t) = \mathcal{L}^{-1}\left[\frac{v_0}{(s+\omega_n)^2}\right] \tag{35}$$

$$= v_0 t e^{-\omega_n t} \tag{36}$$

As mentioned above, the response (36) has no oscillatory component and clearly decays towards zero as time goes on.

Case $\zeta > 1$ overdamped. When the damping increasers further, beyone one, the system is overdamped and still has no oscillations but the response becomes increasingly sluggish and the decay is slower than for the critically damped case ($\zeta = 1$). Recall that the poles are given by

$$p_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}i \tag{37}$$

so with $\zeta > 1$ the term $(1-\zeta^2) < 0$ (equivalently, $\zeta^2 - 1 > 0$) and the poles no longer contain an imaginary component. They can be rewritten as:

$$\begin{aligned} p_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}i && (38)\\ &= -\zeta\omega_n \pm \omega_n\sqrt{1-\zeta^2}\sqrt{-1} && (39)\\ &= \omega_n(-\zeta \pm \sqrt{\underbrace{\zeta^2-1}_{\text{positive}}}) && (40) \end{aligned}$$

Since $\sqrt{\zeta^2 - 1} < \zeta$ the term in parentheses is always negative for both $\pm$ cases of the poles, so that both poles are also strictly negative:

$$\begin{aligned} p_1 &= \omega_n(-\zeta + \sqrt{\zeta^2-1}) < 0 && (41)\\ p_2 &= \omega_n(-\zeta - \sqrt{\zeta^2-1}) < 0 && (42) \end{aligned}$$

Using (10) with $\zeta = 1$ the partial fraction expansion with two real and distinct roots is

$$X(s) = \frac{v_0}{(s-p_1)(s-p_2)} = \frac{c_1}{(s-p_1)} + \frac{c_2}{(s-p_2)} \tag{43}$$

Then the coefficients are:

$$\begin{aligned} c_1 &= \left[\frac{v_0}{(s-p_2)}\right]_{s=p_1} = \frac{v_0}{\omega_n(-\zeta+\sqrt{\zeta^2-1}) - \omega_n(-\zeta-\sqrt{\zeta^2-1})} && (44)\\ &= \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}} && (45) \end{aligned}$$

and

$$\begin{aligned} c_2 &= \left[\frac{v_0}{(s-p_2)}\right]_{s=p_1} = \frac{v_0}{(\omega_n(-\zeta-\sqrt{\zeta^2-1}) - \omega_n(-\zeta+\sqrt{\zeta^2-1})} && (46)\\ &= \frac{v_0}{-2\omega_n\sqrt{\zeta^2-1}}\,. && (47) \end{aligned}$$

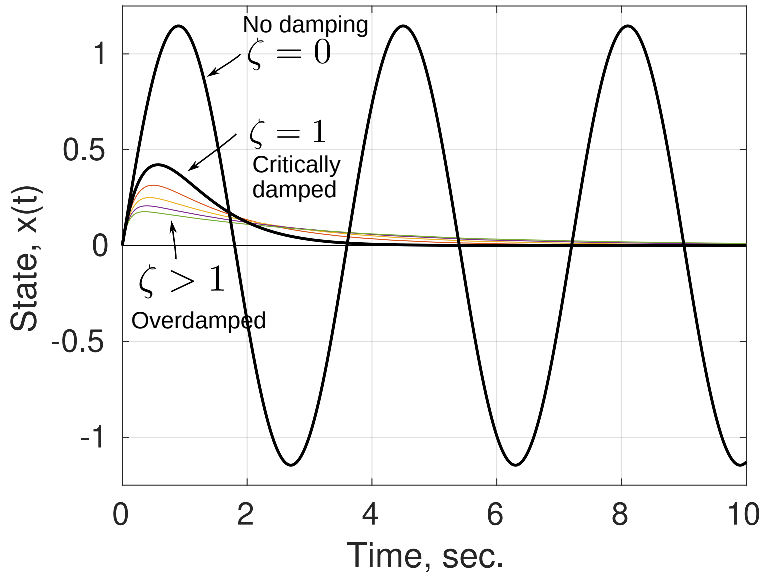
Substituting the coefficients into the PFE

$$X(s) = \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}\left(\frac{1}{s-p_1} - \frac{1}{s-p_2}\right) \tag{48}$$

Taking, the inverse Laplace transform:

$$\begin{aligned} x(t) = \mathcal{L}^{-1}\left[X(s)\right] &= \mathcal{L}^{-1}\left[\frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}\left(\frac{1}{s-p_1} - \frac{1}{s-p_2}\right)\right] && (49)\\ &= \frac{v_0}{2\omega_n\sqrt{\zeta^2-1}}(e^{p_1 t} - e^{p_2 t}) && (50) \end{aligned}$$

As shwon below, the response for an overdamped system is more sluggish and takes longer to decay to zero as ζ increases.



Free Response of a Damped Harmonic Oscillator (General Solution) We now return to the system (5) with no applied force $P = 0$ but relax the assumption that $x_0 = 0$. For general initial conditions the initial value problem is

$$\ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0 \; , \; x(0) = x_0 \; , \; \dot{x}(0) = v_0 \tag{51}$$

The IVP solutions can be derived (omitted here) using Laplace transforms:

- **Case** $\zeta > 1$ **overdamped.**

$$x(t) = e^{-\omega_n \zeta t}(C_1 e^{-\alpha t} + C_2 e^{\alpha t})$$

  where $\alpha = \omega_n \sqrt{\zeta^2 - 1}$ and

$$C_1 = \left[\frac{-\zeta + \sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}x_0 - \frac{1}{2\omega_n\sqrt{\zeta^2-1}}v_0\right] \quad \text{and} \quad C_2 = \left[\frac{\zeta + \sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}x_0 + \frac{1}{2\omega_n\sqrt{\zeta^2-1}}v_0\right]$$

- **Case** $\zeta = 1$ **critically damped.**

$$x(t) = e^{-\omega_n \zeta t}(C_1 + tC_2)$$

  where $C_1 = x_0$ and $C_2 = v_0 + \omega_n x_0$.

- **Case** $0 < \zeta < 1$ **underdamped.**

$$x(t) = e^{-\omega_n \zeta t}(C_1 \sin(\omega_d t) + C_2 \cos(\omega_d t))$$

  where

$$C_1 = \left[\frac{\zeta}{\sqrt{1-\zeta^2}}x_0 + \frac{1}{\omega_d}v_0\right] \qquad \text{and} \qquad C_2 = x_0$$

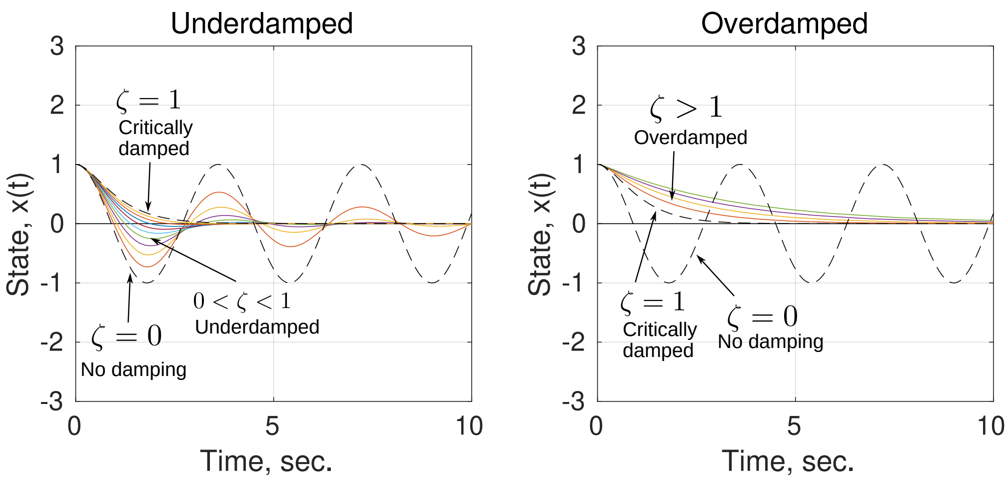
- **Case $\zeta = 0$ undamped.**

$$x(t) = C_1 \cos \omega t + C_2 \sin \omega t$$

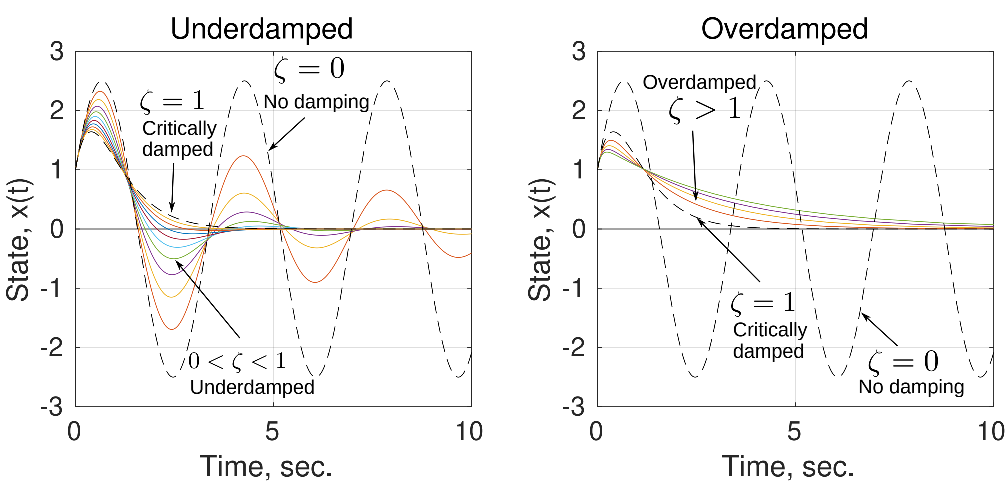
with $C_1 = x_0$ and $C_2 = v_0/\omega$. Using trignometric identities the solution above can be re-written as: $x(t) = A\cos(\omega t + \phi)$

The response of the system to generic initial conditions is very similar to our first example with $x_0 = 0$. The effect of $x_0 \neq 0$ is to change the starting location of the solution, and the effect of v_0 is to change the slope of the solution at the initial condition (and overall amplitude of the oscillation). This is illustrated in the following examples

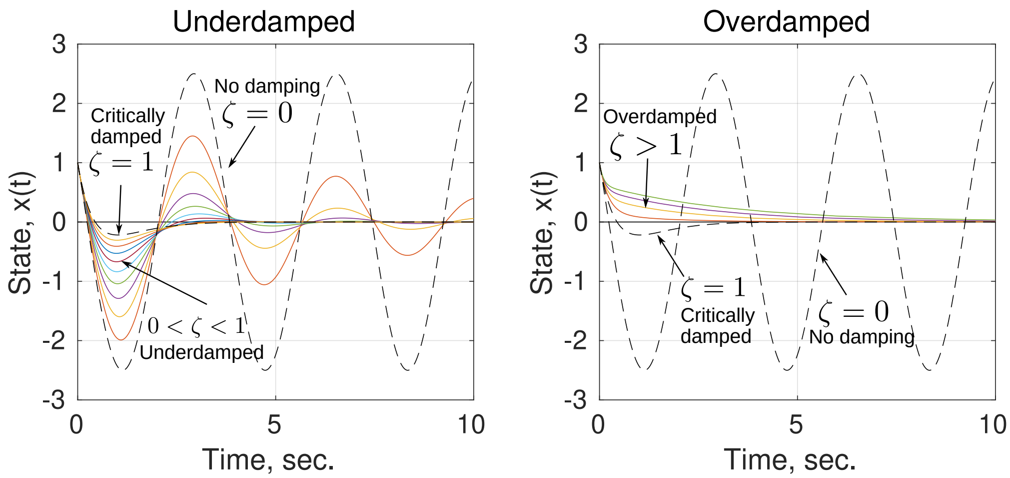
- **No initial velocity (released from rest).** Initial conditions: $x_0 \neq 0$ and $v_0 = 0$.



- **Positive initial velocity.** Initial conditions: $x_0 \neq 0$ and $v_0 > 0$.



- **Negative initial velocity.** Initial conditions: $x_0 \neq 0$ and $v_0 < 0$.



References and Further Reading

- Davies: Sec. 4.2-4.3
- Ogata: Sec. 3.3
- `https://beltoforion.de/en/harmonic_oscillator/`