## Example 3.3.1

A rotor (motor–propeller assembly) of a quadcopter is shown in Figure 3.3.10, where the propeller is driven by a torque $\tau_m$ of the electric motor, and it is subject to a drag torque due to aerodynamic effect. The motor and the propeller are connected by a flexible shaft, which can be modeled as a torsional spring of coefficient $k_T$ and the motor shaft is supported by a bearing of coefficient $b$. The mass moment of inertia of the rotating parts of the motor is $J_m$ and the mass moment of inertia of the propeller is $J_p$ . Let the rotation angles of the motor and propeller be $\theta_m$ and $\theta_p$, respectively. Derive the equations of motion for the rotor system.

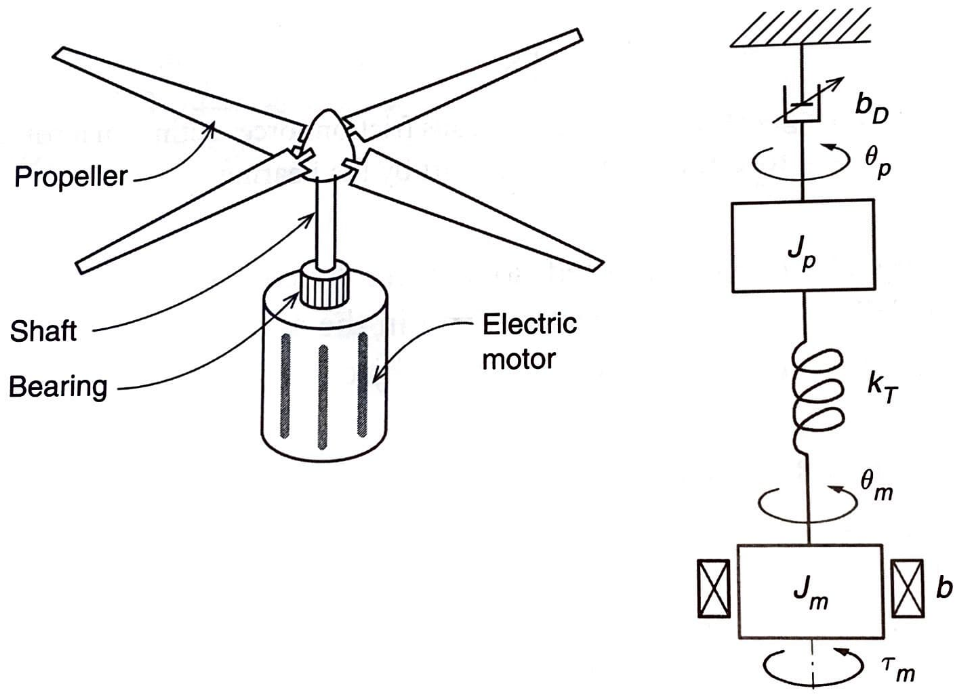


**Figure 3.3.10** Simplified model of a motor–propeller assembly

**Solution**

The free-body diagrams of the motor and propeller and the auxiliary plot of the spring (flexible shaft) are drawn in Figure 3.3.11, where $\tau_s$, $\tau_b$, and $\tau_D$ are the spring torque, bearing torque, and drag torque, respectively. The application of Newton's second law, Eq. (3.3.1), to the two mass elements gives

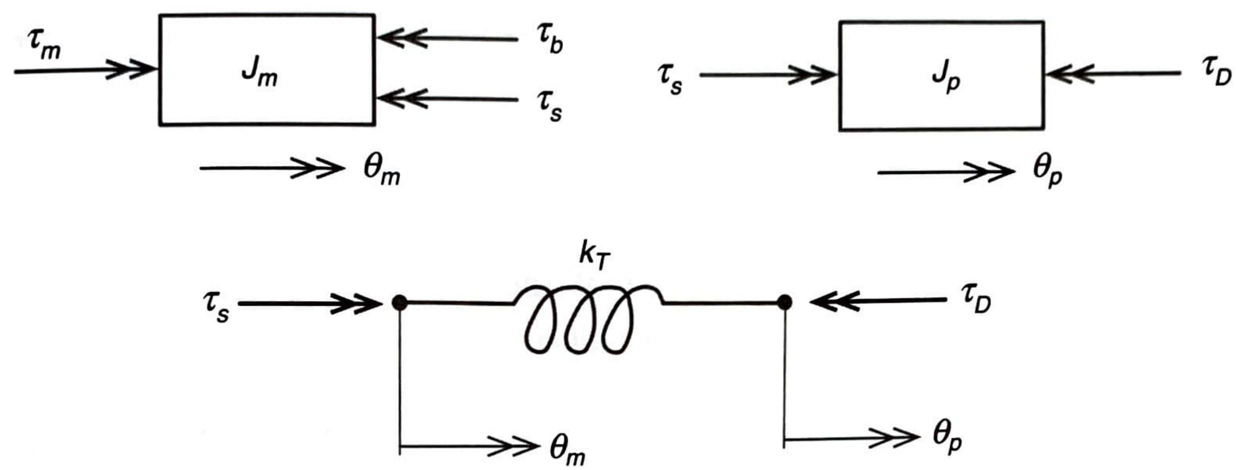


**Figure 3.3.11** Free-body diagrams and auxiliary plot of the motor–propeller assembly in Example 3.3.1

$$\text{Motor:} \quad J_m\ddot{\theta}_m = \tau_m - \tau_b - \tau_s \tag{a}$$

$$\text{Propeller:} \quad J_p\ddot{\theta}_p = \tau_s - \tau_D \tag{b}$$

Also, the torques of the spring and damping elements are given by

$$\begin{gathered} \tau_s = k_T\Delta\theta = k_T(\theta_m - \theta_p) \\ \tau_b = b\dot{\theta}_m, \; \tau_D = b_D\dot{\theta}_p^2 \operatorname{sgn}(\dot{\theta}_p) \end{gathered} \tag{c}$$

where, according to Figure 3.3.5(b), the relative rotation of the spring is $\Delta\theta = \theta_m - \theta_p$. Substituting Eq. (c) into Eqs. (a) and (b) yields the equations of motion of the motor–propeller assembly as follows

$$\begin{gathered} J_m\ddot{\theta}_m + b\dot{\theta}_m + k_T\theta_m - k_T\theta_p = \tau_m \\ J_p\ddot{\theta}_p + b_D\dot{\theta}_p^2 \operatorname{sgn}(\dot{\theta}_p) + k_T\theta_p - k_T\theta_m = 0 \end{gathered} \tag{d}$$

## Example 3.3.3

The pulley–mass system in Figure 3.3.18(a) has an extendible cable, which can be modeled as a spring of coefficient $k$. Derive the equations of motion for the pulley-mass system.

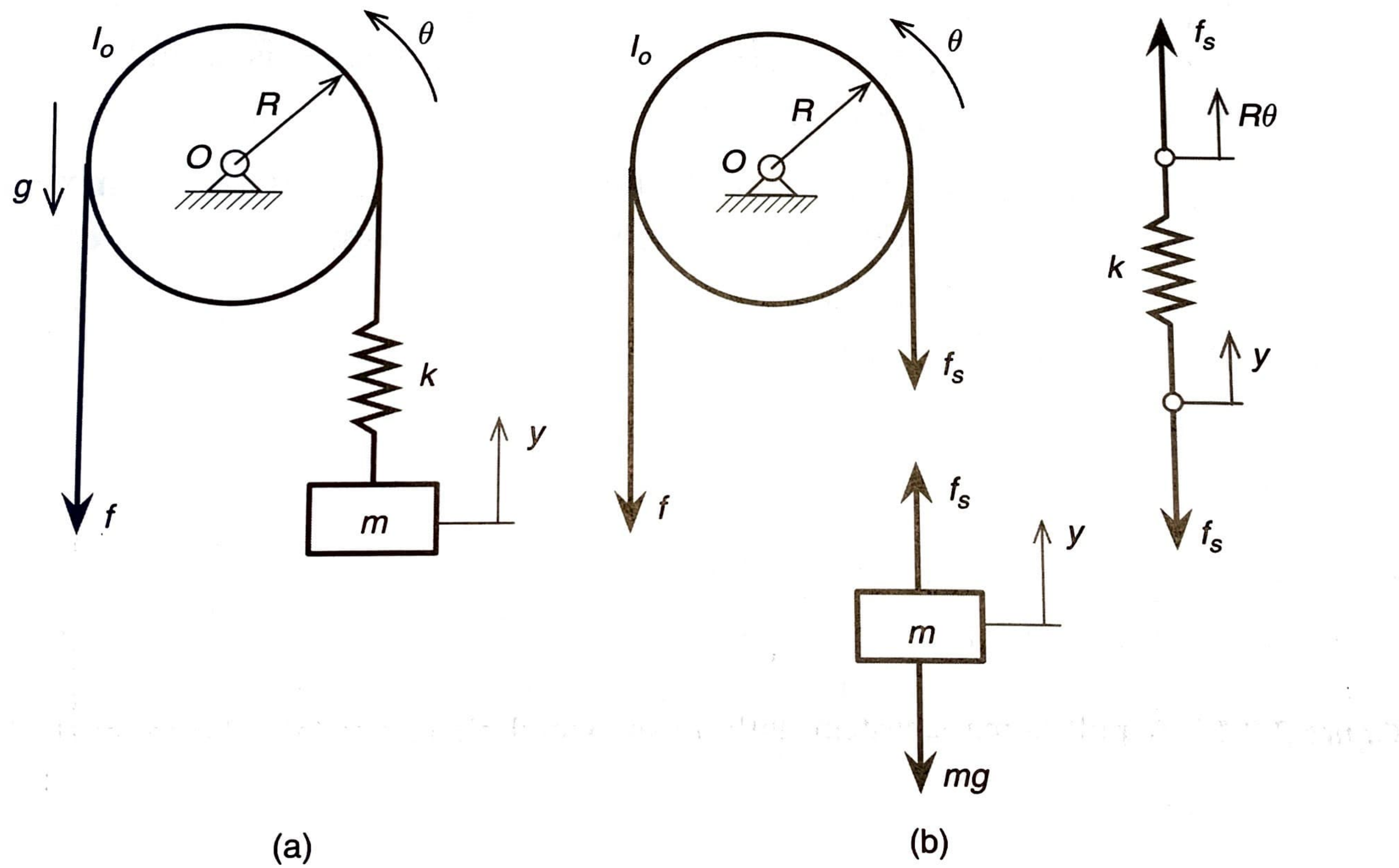


**Figure 3.3.18** A pulley–mass system with an extendible cable: (a) schematic; (b) free-body diagrams and auxiliary plot

**Solution**

Because the cable is extendible, the rotation angle $\theta$ and displacement $y$ are independent of each other. In other words, $y \neq R\theta$. Thus, these displacement parameters are all required to fully describe the motion of the pulley–mass system. Figure 3.3.18(b) shows the free-body diagrams of the pulley and mass and the auxiliary plot of the spring (cable). Thus, by Newton's second law,

$$
\begin{aligned}
I_O\ddot{\theta} &= fR - f_s R \\
m\ddot{y} &= f_s - mg
\end{aligned}
$$

With the spring force being $f_s = k(R\theta - y)$, the equations of motion of the system are obtained as follows

$$
\begin{aligned}
I_O\ddot{\theta} + kR^2\theta - kRy &= fR \\
m\ddot{y} + ky - kR\theta &= -mg
\end{aligned}
$$