Lecture 16: Thermal Systems

Thermal systems involve the transfer of heat between objects or fluids. Examples of thermal systems include:

- home water boilers, refrigerators, and air conditioning units

- engine cooling systems, radiators

- heat exchangers in steam power plants

- furnaces for metallurgy, industrial/chemical processes

The dynamics of a thermal system are governed by temperature differences—heat flows from elements with higher temperature to lower temperature. To analyze thermal systems we will make two assumptions about the elements in the system:

1. The temperature of each element is uniform.

2. Each element is characterized either as a *thermal damper/resistor* or a *thermal mass/capacitor*

The above assumptions are reasonable for the purposes of many engineering problems and permit using ordinary differential equations to develop a system model. However, in reality, objects experience non-uniform temperature that varies throughout each object leading to thermal gradients (i.e., a *distributed parameter model* governed by a partial differential equation). The second bullet above is the lumped-parameter assumption. If the object is a thermal resistor it has negligible heat capacitance and vice versa. Engineering systems that require a more careful analysis (without the above assumptions) can be treated using methods discussed in a course on heat transfer/thermodynamics.
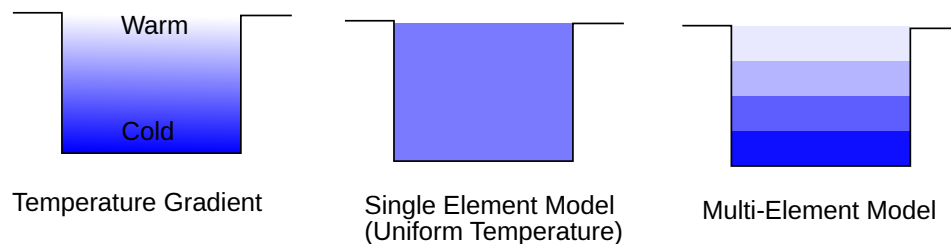

Figure 1: Several thermal models of swimming pol

An everyday example is the heating of an outdoor swimming pool. On a warm sunny day the temperature at the surface of the pool will be greater than the temperature at the bottom of the pool and there exists a smooth variation—a gradient—at intermediate depths as shown on the left of Fig. 1. To simplify our analysis (e.g., suppose only the average temperature was of interest) the water in the pool can be assumed to be uniform and act as a single thermal element (Fig. 1 middle). Another assumption might be to assume the water in the pool is stratified (no mixing) and treat it as several uniform layers, each with their own temperature (Fig. 1 right).

Thermal Energy and Capacitance (Thermal Mass). The amount of *thermal energy* stored in an object is, of course, related to its temperature. Intuitively, the thermal energy denoted E should also depend on the mass m (in kg) of the object and its material properites, specifically, the

object's specific heat capacity c with units K/(kg·°C) as shown in the Table below for common substances.

Substance	Specific heat capacity, c, J/(kg · °C)
Air	1005
Aluminum	897
Brass	375
Brick	840
Concrete	880
Steel	490
Water	4182

Table 1: Specific heat capacity of common substances

The thermal energy is:

$$E = mc(T - T_{\text{ref}}) \tag{1}$$

where T_{ref} is a reference temperature. The actual reference temperature is irrelevant to describing the dynamics of heat transfer here—much like gravitational potential energy can be referenced to any point—since it will be the *change* in heat energy that is of interest. The *thermal capacitance* or *thermal mass* is the product

$$C = mc$$

with SI units of J/°C and is the amount of thermal energy needed to raise the temperature by one degree.

Heat transfer.   Heat, denoted Q, is the amount of thermal energy transferred from one object to another and is measured in joules (J). Other units commonly used include the kilocalorie (kcal) and Btu (British thermal unit) and they are related by:

$$1 \text{ J} = 2.389 \times 10^{-4} \text{ kcal} = 9.840 \times 10^{-4} \text{ Btu}$$

It is important to emphasize that heat, Q, is the *change* in energy flowing into or out of an object, whereas the energy before or after is E. Thus the change in thermal energy (i.e., heat) is:

$$Q = mc\Delta T = \Delta E \tag{2}$$

Equation (2) describes a static relationship between heat Q and temperature change ΔT. In differential form,

$$\dot{Q} = mc\dot{T} \tag{3}$$

The quantity $\dot{Q}$ is sometimes referred to as *thermal power* or heat-flow rate and is denoted $P(t) = \dot{Q}$ with units of J/s or Watts. Thus, we may write:

$$\dot{Q} = P = mc\dot{T} \tag{4}$$

$$\implies \dot{T} = \frac{1}{mc}P(t) \tag{5}$$

This equation is similar to Newton's 2nd Law where $P(t)$ is analogous to a force, mc is analogous to the mass, and $\dot{T}$ is analgous to the acceleration.

$$\sum \underbrace{P}_{\text{"Sum of thermal powers"}} = \underbrace{(mc)}_{\text{"thermal mass"}} \cdot \underbrace{(\dot{T})}_{\text{"temperature-rate"}} \tag{6}$$

Heat can be exchanged via direct heat input, conduction, convection, or radiation. We will focus on the first three—radiation is only significant if the difference between the heat emitter and receiver is extremely large. For conduction or convection heat transfer, the heat flow rate is proportional to the temperature difference.

Heat Exchange with a Constant Thermal Power Input

The simplest model of heat flow is for a perfectly insulated mass—with no convection/conduction/radiation at the boundaries—and a single source of thermal power $P(t)$. Suppose that at the intial time $t_0 = 0$ the thermal mass has temperature $T(t_0) = T_0$.
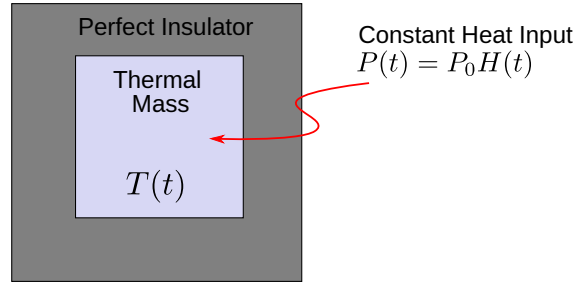

Figure 2: Heat transfer for a constant heat input

Assume that the only input is a constant power (described in units of Watts)

$$P_{\text{constant}}(t) = P_0 H(t)$$

If $P_0 > 0$ is positive then heat is flowing into the object and the input is heating it so that the temperature $T(t)$ will rise. If $P_0 < 0$ is negative then heat is flowing out of the object and the input is cooling it so $T(t)$ will decrease. In either case (5) becomes:

$$\dot{T} = \frac{1}{mc} P_0 H(t) \tag{7}$$

Taking the Laplace transform,

$$sT(s) - T_0 = \frac{P_0}{mc} \frac{1}{s} \tag{8}$$

rearranging for

$$T(s) = \frac{P_0}{mc} \left[\frac{1}{s^2}\right] + T_0 \left[\frac{1}{s}\right] , \tag{9}$$

and using the inverse Laplace transform

$$T(t) = \frac{P_0}{mc} \mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + T_0 \mathcal{L}^{-1}\left[\frac{1}{s}\right] \tag{10}$$

$$\implies T(t) = \frac{P_0}{mc} t + T_0 \tag{11}$$

From (11) it is clear that the thermal mass will increase or decrease temperature linearly, starting from T_0. The slope of the change depends inversely on the thermal mass and is proportional to the thermal power applied. If the heat source is no longer applied then, in this ideal model, the perfect insulator maintains the temperature at a constant (i.e., if $P_0 = 0$ in (7) then $\dot{T} = 0$).

Heat Exchange with an Ambient Environment (Convection)

Another model for heat flow involves the circulation/movement of a fluid or gas. This may occur when warmer fluid rises and is replaced by cooler fluid, or when the fluid is blowing past the plate as in a fan/blower arrangement. In such cases the heat exchange is governed by *Newton's law of cooling*:

$$P_{\text{convection}}(t) = hA(T_{\text{env}} - T) \tag{12}$$

where

- h is a convection or heat transfer coefficient, Watts/(m^2·°C)

- A is the area normal to the heat flow, m^2

- T_{env} is the temperature of the ambient environment and

- T is the temperature of the object

If (12) is viewed through the lens of a mechanical system, then the "force" $P_{\text{convection}}$ is related to the difference in "velocities" $(T_{\text{env}} - T)$ and depends on the "damping coefficient" hA. The heat transfer coefficient is determined by factors such as whether the air is being forced (e.g., a blower) or unforced/natural convection, the temperatuere difference, fluid viscosity, heat capacity of the fluid, and geometry of fluid/object interaction. In this class we will make the simplifying assumption that h is a constant. Assuming only convection with the environment (5) with (12)
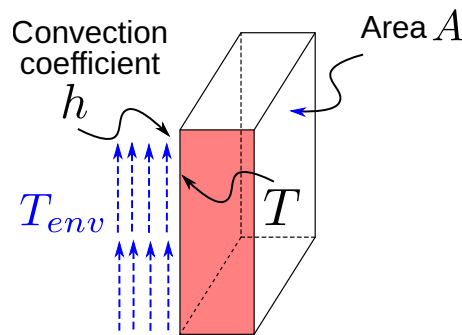

Figure 3: Heat transfer via convection

becomes:

$$mc\dot{T} = P_{\text{convection}} \tag{13}$$
$$mc\dot{T} = [hA(T_{\text{env}} - T)] \tag{14}$$

Suppose the environment is at a constant temperature $T_{\text{env}}(t) = T_{\text{env}}$ and take the Laplace transform

$$\mathcal{L}[mc\dot{T}] = \mathcal{L}[hAT_{\text{env}} - haT] \tag{15}$$

$$mc(sT(s) - T_0) = \frac{hAT_{\text{env}}}{s} - hAT(s) \tag{16}$$

$$mcs^2 T(s) + hAsT(s) = hAT_{\text{env}} + mcT_0 s \tag{17}$$

$$T(s)(mcs^2 + hAs) = hAT_{\text{env}} + mcT_0 s \tag{18}$$

$$T(s) = \frac{hAT_{\text{env}}}{mcs^2 + hAs} + \frac{(mcT_0)s}{s(mcs + hA)} \tag{19}$$

$$T(s) = \frac{(hA/mc)T_{\text{env}}}{s^2 + (hA/mc)s} + \frac{T_0}{s + (hA/mc)} \tag{20}$$

Define the constant $a = (hA/mc)$ then the above system becomes

$$T(s) = aT_{\text{env}} \frac{1}{s(s + a)} + T_0 \frac{1}{s + a} \tag{21}$$

and taking the inverse Laplace transform

$$T(t) = aT_{\text{env}} \mathcal{L}^{-1} \left[\frac{1}{s(s + a)} \right] + T_0 \mathcal{L}^{-1} \left[\frac{1}{s + a} \right] \tag{22}$$

$$\implies T(t) = T_{\text{env}}(1 - e^{-at}) + T_0 e^{-at} \tag{23}$$

The above equation suggests that $\tau = 1/a = mc/(hA)$ is a thermal time constant. At times $t >> 4\tau$ the exponential terms decay to near zero and $T(t) \to T_{\text{env}}$. That is, the temperature of the plate becomes equal to that of the environment.

Heat Exchange via Conduction.

When heat is transferred by diffusion through a substance it is referred to as conduction. Consider a plate-shaped object that has an exposed face with area A and thickness L. The object is exposed to a temperature T_2 on one face and temperature T_1 on the other face. According to *Fourier's law of heat conduction*, the heat transfer rate (thermal power) is

$$P_{\text{conduction}}(t) = \frac{kA}{L}(T_2 - T_1) \tag{24}$$

where

- k is a thermal conductivity coefficient, Watts/(m·°C)
- A is the plate area, m²
- L is the thickness of the plate, m
- $(T_2 - T_1)$ is the temperature difference, °C

In this idealized model we ignore any movement of air/fluid around the edges of the plate. To model the heat exchange into a plate with two different temperatures on each face we can use a lumped parameter assumption. In reality there will be a gradient of temperature across the

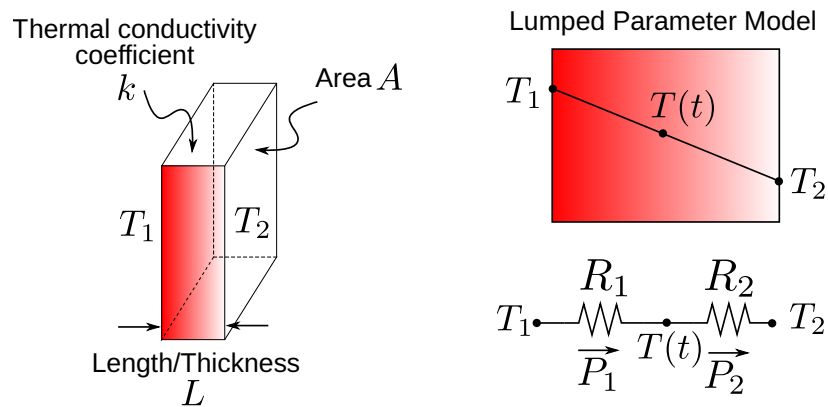| Substance	Thermal conductivity coefficient, k, Watts/(m·°C)
Air	0.0262
Aluminum	237
Brass	111
Brick	0.80
Concrete	0.50
Steel	14.4
Water	0.606

Table 2: Thermal conductivity coefficient

Figure 4: Heat transfer via conduction

plate; however, assuming a linear gradient, we can design our model for the average temperature T at the center of the plate. The quantities $R_1 = R_2 = (L/2)/(kA)$ are analogous to thermal resistors (similar to dampers) that slow down the heat exchange. Suppose that the left side of the plate is hotter than the right side, $T_1 > T_2$ and the initial temperature of the plate midpoint is $T_1 > T_0 > T_2$. Then the left side adds positive thermal power:

$$P_1 = \frac{kA}{L/2}(T_1 - T) > 0$$

and the right side subtracts thermal power

$$P_2 = \frac{kA}{L/2}(T_2 - T) < 0$$

By conservation of heat energy the total conduction thermal power is

$$P_{\text{conduction}}(t) = P_1 + P_2$$

and using (12) the temperature dynamics are:

$$\dot{T} = \frac{1}{mc}(P_1 + P_2) \tag{25}$$

$$= \frac{1}{mc}\left(\frac{kA}{L/2}(T_1 - T) + \frac{kA}{L/2}(T_2 - T)\right) \tag{26}$$

$$= \frac{kA}{mcL/2}(T_1 - T + T_2 - T) \tag{27}$$

$$= \frac{kA}{mcL/2}(-2T + T_1 + T_2) \tag{28}$$

Taking the Laplace transform,

$$sT(s) - T_0 = -\frac{4kA}{mcL}T(s) + \frac{2kA(T_1 + T_2)}{mcL}\frac{1}{s} \tag{29}$$

$$(mcL)s^2 T(s) + 4kAsT(s) = (T_0 mcL)s + 2kA(T_1 + T_2) \tag{30}$$

$$T(s) = \frac{(T_0 mcL)s + 2kA(T_1 + T_2)}{(mcL)s^2 + 4kAs} \tag{31}$$

$$T(s) = \frac{T_0 s + [2kA(T_1 + T_2)]/(mcL)}{s^2 + [4kA/(mcL)]s} \tag{32}$$

Let $a = 4kA/(mcL)$, then

$$T(s) = \frac{T_0 s + \frac{1}{2}a(T_1 + T_2)}{s^2 + as} \tag{33}$$

the denominator has roots at $s = 0$ and $s = -a$, thus we use a PFE of the form:

$$T(s) = \frac{T_0 s + \frac{1}{2}a(T_1 + T_2)]}{s(s + a)} = \frac{c_1}{s} + \frac{c_2}{s + a} \tag{34}$$

and the coefficients are

$$c_1 = \left.\frac{T_0 s + \frac{1}{2}a(T_1 + T_2)}{(s + a)}\right|_{s=0} = \frac{\frac{1}{2}a(T_1 + T_2)}{a} = \frac{1}{2}(T_1 + T_2) \tag{35}$$

and

$$c_2 = \left.\frac{T_0 s + [\frac{1}{2}(T_1 + T_2)]a}{s}\right|_{s=-a} = \frac{-T_0 a + \frac{1}{2}(T_1 + T_2)a}{-a} = T_0 - \frac{1}{2}(T_1 + T_2) \tag{36}$$

so that

$$T(s) = \frac{1}{2}(T_1 + T_2)\frac{c_1}{s} + (T_0 - \frac{1}{2}(T_1 + T_2))\frac{1}{s + a}$$

and taking the inverse Laplace transform

$$T(t) = \frac{1}{2}(T_1 + T_2) + (T_0 - \frac{1}{2}(T_1 + T_2))e^{-at}$$

At the initial time $T(t_0) = T_0$ is satisfied by the above equation and as $t \to \infty$ the temperature $T(t) \to \frac{1}{2}(T_1 + T_2)$. The result is consistent with our common sense: as time goes on the plate achieves a temperature at its midpoint that is the average between T_1 and T_2, regardless of what temperature T_0 it started with.

Heat Exchange with a Constant Thermal Power and Convection

In the three previous examples we considered heat exchange with a single source of thermal power: a constant input, convection, and conduction. However, just like in Newton's 2nd Law there may be several forces that are summed to produce a net force, similarly in thermal systems there may be multiple heat exchange mechanisms (thermal power sources) that drive the temperature dynamics.

Suppose that a thermal mass is subject to both a direct and constant thermal input as well as convection. A heated swimming pool can be modeled this way—the heating element provides a constant heat flow into the pool, whereas the convection cools the pool (assuming the ambient temperature is less than that of the pool). Again, using (5)

$$mc\dot{T} = \sum P \tag{37}$$

$$mc\dot{T} = P_{\text{constant}} + P_{\text{convection}} \tag{38}$$

$$mc\dot{T} = P_0 + hA(T_{\text{env}} - T) \tag{39}$$

$$mc\dot{T} = P_0 H(t) + hAT_{\text{env}}H(t) - hAT \tag{40}$$

where, as usual, we have added the Heaviside function for any constants that appear before taking the Laplace transform. Taking the Laplace transform gives

$$mcsT(s) - T_0 = P_0\frac{1}{s} + hAT_{\text{env}}\frac{1}{s} - hAT(s) \tag{41}$$

$$s(mcs + hA)T(s) = [P_0 + hAT_{\text{env}}] + T_0 s \tag{42}$$

$$T(s) = \frac{[P_0 + hAT_{\text{env}}] + T_0 s}{s(mcs + hA)} \tag{43}$$

$$T(s) = \frac{[P_0/(mc) + aT_{\text{env}}] + T_0/(mc)s}{s(s + a)} \tag{44}$$

where we have introduced $a = (hA)/(mc)$. The roots are $s = 0$ and $s = -a$. As before, we can use partial fraction expansion and he inverse Laplace transform. However, if we simply expand the right-hand-side into the sum of three factions we find that they are all in the Laplace transform table:

$$mcsT(s) - mcT_0 = P_0\frac{1}{s} + hAT_{\text{env}}\frac{1}{s} - hAT(s) \tag{45}$$

$$s(mcs + hA)T(s) = [P_0 + hAT_{\text{env}}] + mcT_0 s \tag{46}$$

$$T(s) = \frac{[P_0 + hAT_{\text{env}}] + mcT_0 s}{s(mcs + hA)} \tag{47}$$

$$T(s) = \frac{[P_0/(mc) + aT_{\text{env}}]}{s(s + a)} + \frac{T_0}{(s + a)} \tag{48}$$

which then implies (using rows 6 and 14) that:

$$\implies T(t) = \left[\frac{P_0}{mc} + aT_{\text{env}}\right]\frac{1}{a}(1 - e^{-at}) + T_0 e^{-at} \tag{49}$$

From the above expression we see that at $t_0 = 0$ the temperature is $T(t_0) = T_0$ and as $t \to \infty$ the

temperature

$$T(t) \rightarrow \left[\frac{P_0}{mc} + aT_{\text{env}} \right] \frac{1}{a} \tag{50}$$

$$= \frac{P_0}{hA} + T_{\text{env}} \tag{51}$$

Thus, the swimming pool will eventually reach a temperature that is above the surrounding ambient temperature by an amount P_0/hA.

References

- Davies and Schmitz: Chapter 9
- Ogata: Chapter 7.6
- Yang and Abramova: Chapter 5
- Cochin and Cadwallender: Chapter 4