Homework 3

1 Problem

Find the Laplace transform $F(s) = \mathcal{L}[f(t)]$ for each of the following functions. For full credit, show all of your work by expanding each Laplace transform into one or more Laplace transforms of the common functions listed in the provided Laplace Transform Table. Indicate which rows of the table you used to obtain your solution and fully simplify the result to a single term.

1. $f(t) = e^{at+b}$ where a, b are constant
2. $f(t) = \sin(\omega t - \phi)$ where ω, ϕ are constant
3. $f(t) = t^3 - 1/2$
4. $f(t) = (e^{-t}/4)[2 + t^2 + \cos(3t)]$. (Note: For this problem there is no need to simplify to a common denominator.)
5. $f(t) = (t-2)H(t-2)$, where $H(\cdot)$ is the unit step or Heaviside function

Solution

1. $f(t) = e^{at+b}$ where a, b are constant
   Using Laplace transform table (rows 2 and 6):

$$\begin{aligned}\mathcal{L}[e^{at+b}] &= \mathcal{L}[e^{at}e^{b}] \\ &= e^{b}\mathcal{L}[e^{at}] \\ &= e^{b}\left(\frac{1}{s-a}\right)\end{aligned}$$

2. $f(t) = \sin(\omega t - \phi)$ where ω, ϕ are constant
   Using Laplace transform table (rows 10 and 11) and the sine angle difference identity

$$\begin{aligned}\mathcal{L}[\sin(\omega t - \phi)] &= \mathcal{L}[\sin(\omega t)\cos(\phi) - \cos(\omega t)\sin(\phi)] \\ &= \mathcal{L}[\sin(\omega t)\cos(\phi)] - \mathcal{L}[\cos(\omega t)\sin(\phi)] \\ &= \cos(\phi)\mathcal{L}[\sin(\omega t)] - \sin(\phi)\mathcal{L}[\cos(\omega t)] \\ &= \cos(\phi)\left(\frac{\omega}{s^2+\omega^2}\right) - \sin(\phi)\left(\frac{s}{s^2+\omega^2}\right) \\ &= \left(\frac{\cos(\phi)\omega - \sin(\phi)s}{s^2+\omega^2}\right)\end{aligned}$$

3. $f(t) = t^3 - 1/2$

Using Laplace transform table (rows 2 and 5):

$$\begin{aligned}
\mathcal{L}[t^3 - 1/2] &= \mathcal{L}[t^3] - \mathcal{L}[1/2H(t)] \\
&= \frac{3!}{s^4} - \frac{1}{2}\frac{1}{s} \\
&= \frac{6}{s^4} - \frac{1}{2s} \\
&= \frac{-s^3 + 12}{2s^4}
\end{aligned}$$

4. $f(t) = \frac{e^{-t}}{4}[2 + t^2 + \cos(3t)]$
   First expand the function

$$f(t) = \frac{1}{2}e^{-t} + \frac{1}{4}e^{-t}t^2 + \frac{1}{4}e^{-t}\cos 3t$$

   then look in the Laplace transform table to find similar forms on row 6, row 8 (with $n = 3$) and on row 21 (with $a = 1$ and $\omega = 3$). The Laplace transform is then

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{1}{2}\mathcal{L}[e^{-t}] + \frac{1}{2}\mathcal{L}[\frac{t^2e^{-t}}{2}] + \frac{1}{4}\mathcal{L}[e^{-t}\cos 3t] \\
&= \frac{1}{2}\left(\frac{1}{s+1}\right) + \frac{1}{2}\left(\frac{1}{(s+1)^3}\right) + \frac{1}{4}\left(\frac{s+1}{(s+1)^2+9}\right) \\
&= \frac{1}{4}\left(\frac{2[(s+1)^2+9](s+1)^2 + 2[(s+1)^2+9] + (s+1)^3}{(s+1)((s+1)^2+9)}\right) \\
&= \frac{1}{4}\left(\frac{2[(s+1)^2+9](s+1)^2 + 2[(s+1)^2+9] + (s+1)^3}{(s+1)(s^2+2s+10)}\right)
\end{aligned}$$

5. Notice that the function $(t-2)$ is just a ramp (t) shifted by $\alpha = 2$ units. The Laplace transform of a ramp is $\mathcal{L}[t] = 1/s^2$. Then using the LT transform theorem for translated functions:

$$F(s) = e^{-\alpha s}\mathcal{L}[t] = \frac{e^{-2s}}{s^2}$$

2 Problem

What is the final value of $x(t)$ as $t \to \infty$ if the Laplace transform of $x(t)$ is the following?

$$X(s) = \frac{6}{s(s+2)}$$

Solution

From the initial value theorem,

$$\begin{align}
\lim_{t\to\infty} x(t) &= \lim_{s\to 0} sX(s) \tag{1}\\
&= \lim_{s\to 0} s\frac{6}{s(s+2)} \tag{2}\\
&= \lim_{s\to 0} \frac{6}{(s+2)} \tag{3}\\
&= \frac{6}{2} = 3 \tag{4}
\end{align}$$

3 Problem

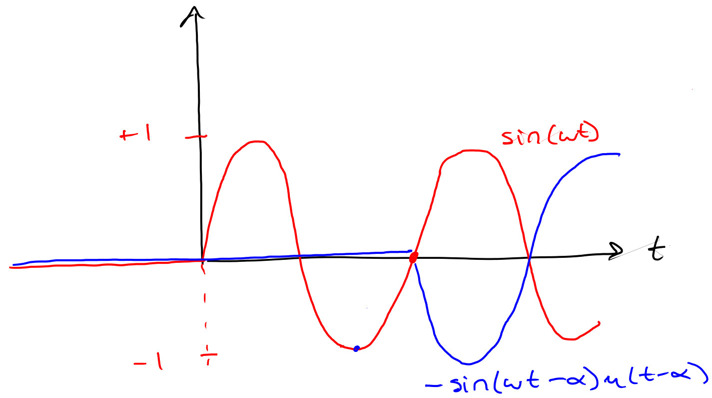
Sketch (by hand) the following function on the interval $t \in [-2\pi, 4\pi]$ and find its Laplace transform:

$$f(t) = \sin(t) - \sin(t-\alpha)H(t-\alpha)$$

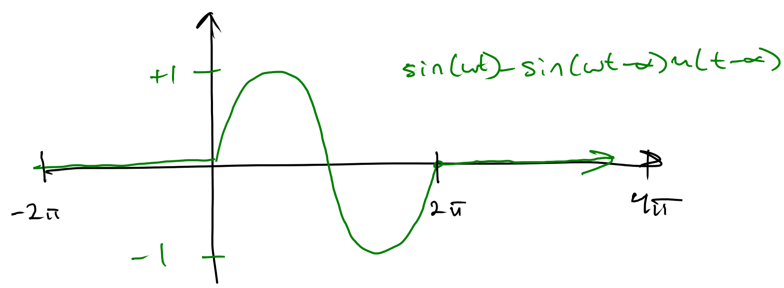
where $H(t-\alpha)$ is the unit step function delayed to start at time $\alpha = 2\pi$.

Hint: Use the Laplace transform for a translated function.

Solution



Adding together the two functions:



$$\mathcal{L}[f(t)] = \mathcal{L}[\sin(\omega t)] - \mathcal{L}[\sin(\omega(t-\alpha))u(t-\alpha)]$$

$\downarrow$ row 10 of L.T. table $\qquad \downarrow$ see topic 10

$$= \frac{\omega}{s^2+\omega^2} - e^{-\alpha s}\mathcal{L}^{-1}\Big[\underbrace{\sin(\omega t - 2\pi)}_{=\sin(\omega t)}\Big] \qquad \alpha = 2\pi$$

$$= \frac{\omega}{s^2+\omega^2} - e^{-2\pi s}\frac{\omega}{s^2+\omega^2}$$

$$= \frac{\omega}{s^2+\omega^2}\left(1 - e^{-2\pi s}\right)$$

4 Problem

Find the Laplace transform fraction for the following function and rearrange it such that $X(s)/F(s)$ is the only term on the left-hand-side:

$$\dddot{x}(t) + 2\zeta\omega\ddot{x}(t) + \omega^2\dot{x}(t) = f(t)$$

Assume the initial conditions are all zero, $x(t_0) = \dot{x}(t_0) = \ddot{x}(t_0) = \dddot{x}(t_0) = 0$ with initial time $t_0 = 0$. Hint: Use the differentiation theorem.

Solution

Since

$$\mathcal{L}[\dddot{x}] = s^3X(s) - \underbrace{s^2\ddot{x}(0) - s\dot{x}(0) - x(0)}_{=0}$$

$$\mathcal{L}[\ddot{x}] = s^2X(s) - \underbrace{s\dot{x}(0) - x(0)}_{=0}$$

By initial conditions

$$\mathcal{L}[\dot{x}] = sX(s) - \underbrace{x(0)}_{=0}$$

$$\mathcal{L}[f(t)] = F(s)$$

Then

$$\mathcal{L}[\dddot{x} + 2\zeta\omega_n\ddot{x} + \omega^2\dot{x}] = \mathcal{L}[f(t)]$$

By linearity.

$$\mathcal{L}[\dddot{x}] + 2\zeta\omega_n\mathcal{L}[\ddot{x}] + \omega^2\mathcal{L}[\dot{x}] = F(s)$$

Plug in from above.

$$s^3X(s) + 2\zeta\omega_n s^2X(s) + \omega^2 sX(s) = F(s)$$

Rearrange

$$X(s)\left(s^3 + 2\zeta\omega_n s^2 + \omega^2 s\right) = F(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{s(s^2 + 2\zeta\omega_n s + \omega^2)}$$

5 Problem

Compute the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of the following function:

$$F(s) = \frac{1}{s+\sigma}(e^{-as} - e^{-bs})$$

where a, b, and σ are constants.

Hint: Recall that the following property holds for translated functions $\mathcal{L}[f(t-\alpha)H(t-\alpha)] = e^{-s\alpha}F(s)$, which implies that

$$f(t) = \mathcal{L}^{-1}[e^{-s\alpha}F(s)] = f(t-\alpha)H(t-\alpha)$$

The expression above can be written as a sum of two functions of this form.

Solution

Expand $F(s)$:

$$F(s) = \frac{1}{s+\sigma}(e^{-as} - e^{-bs}) \tag{5}$$

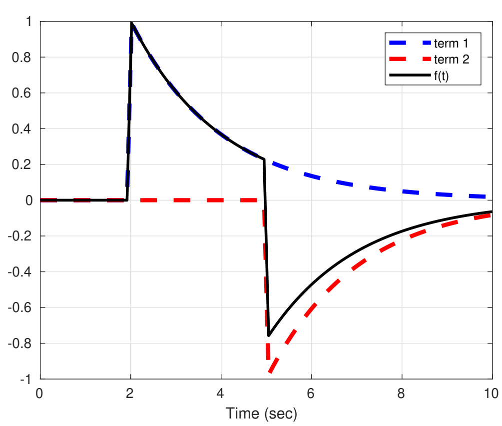
$$= e^{-as}\frac{1}{s+\sigma} + e^{-as}\frac{1}{s+\sigma} \tag{6}$$

The above expression is in the form of a translated function. The first term delays the signal $\frac{1}{s+\alpha}$ until $t = a$ and the second term delays the signals $\frac{1}{s+\sigma}$ until $t = b$. Recall that $\mathcal{L}^{-1}[\frac{1}{s+\alpha}] = e^{-\sigma t}$ then

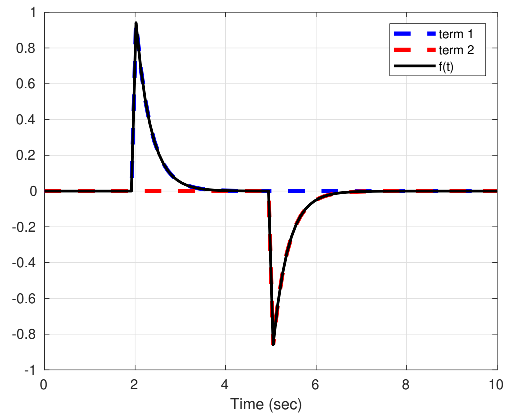
$$f(t) = \mathcal{L}^{-1}\left[e^{-as}\frac{1}{s+\sigma}\right] + \mathcal{L}^{-1}\left[e^{-as}\frac{1}{s+\sigma}\right] \tag{7}$$

$$= H(t-a)e^{-\sigma(t-a)} - H(t-b)e^{-\sigma(t-b)} \tag{8}$$

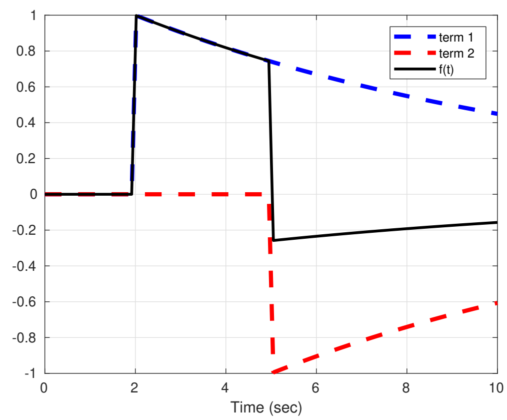
The above solution indicates that $f(t)$ is the sum of two delayed exponential functions. For example, with $a = 2$, $b = 5$ and $\sigma = 0.5$ we obtain the black curve $f(t)$ shown below as the sum of the two terms. The first term "turns on" at $t = 2$ and the second term turns on at $t = 5$ seconds.



If instead $\sigma = 3$, then the decay is faster:



Or... if $\sigma = 0.1$, then the decay is slower:



6 Problem

Compute the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of the following function:

$$F(s) = \frac{s+1}{s^2+6s+9}$$

Hint: Use a partial fraction expansion for repeated poles. Show that the solution is:

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left[\frac{1}{(s+3)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right] && (9) \\ &= e^{-3t} - 2te^{-3t} && (10) \end{aligned}$$

Solution

The denominator can be factored as $(s+3)^2$ which indicates that there are two repeated poles $p_{1,2} = -3$

$$F(s) = \frac{s+1}{s^2+6s+9} = \frac{s+1}{(s+3)^2}$$

The partial fraction expansion is thus:

$$F(s) = \frac{c_1}{s+3} + \frac{c_1}{(s+3)^2} \tag{11}$$

Multiplying both sides by $(s+3)^2$:

$$F(s)(s+3)^2 = c_1(s+3) + c_2 \tag{12}$$

$$(s+1) = c_1 s + (3c_1 + c_2) \tag{13}$$

Equating coefficiens we see immediately that $c_1 = 1$ and can solve for c_2 as:

$$1 = 3c_1 + c_2 \tag{14}$$

$$1 = 3(1) + c_2 \tag{15}$$

$$\implies c_2 = -2 \tag{16}$$

Thus the partial fracton expansion is:

$$F(s) = \frac{1}{(s+3)} - 2\frac{1}{(s+3)^2} \tag{17}$$

and taking the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+3)}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right] \tag{18}$$

$$= -2te^{-3t} \tag{19}$$

7 Problem

Compute the inverse Laplace transform of

$$F(s) = \frac{s+1}{s(s+2)}$$

Hint: Use a partial fraction expansion for real distinct poles. Show that the solution is:

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}$$

Solution

From the denominator it is clear that the poles are $p_1 = 0$ and $p_2 = -2$ hence real and distinct and thus we wish to expand $F(s)$ as

$$F(s) = \frac{a_1}{s} + \frac{a_2}{(s+2)}$$

The coefficients are found from

$$a_1 = \left[\frac{(s+1)s}{s(s+2)}\right]_{s=0} = \left[\frac{(s+1)}{(s+2)}\right]_{s=0} = \frac{1}{2} \tag{20}$$

$$a_2 = \left[\frac{(s+1)(s+2)}{s(s+2)}\right]_{s=-2} = \left[\frac{(s+1)}{s}\right]_{s=-2} = \frac{1}{2} \tag{21}$$

Thus, $F(s)$ is equivalent to

$$F(s) = \frac{1}{2}\frac{1}{s} + \frac{1}{2}\frac{1}{(s+2)}$$

which is in a form for which we can easily compute the inverse Laplace by table lookup. Using rows 2 and 6 in the Laplace transform table:

$$\begin{aligned} f(t) = \mathcal{L}^{-1}[F(s)] &= \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{s}\right] + \frac{1}{2}\mathcal{L}^{-1}\left[\frac{1}{(s+2)}\right] \\ &= \frac{1}{2}H(t) + \frac{1}{2}e^{-2t} \end{aligned}$$

which, for $t \geq 0$ is equivalent to

$$f(t) = \frac{1}{2} + \frac{1}{2}e^{-2t}$$