Homework 1

1 Problem

For each of the ODEs in the 1st column, indicate whether it is:

1. linear time-invariant (LTI), linear time-varying (LTV), or nonlinear
2. 1st order, 2nd order, or higher order
3. homogeneous or inhomogeneous (only for the case of linear systems—for nonlinear systems you can leave the last column blank)

by marking the appropriate column. The unknown function $x(t)$ represents the state of some mechanical system and t represents time. Hint: An ODE is considered nonlinear only if the nonlinearity involves the unknown function $x(t)$. Hint: if an ODE is nonlinear it doesn't make sense to classify it as homogeneous or non-homogeneous — leave the columns blank.

	Linearity?			Order?			Homogeneity?	
System ODE	LTI	LTV	NL	1st	2nd	Higher	Homog.	Inhomog.
$\ddot{x} + 3tx = 0$								
$(\dot{x} - x)^2 + 1 = 0$								
$t^2 x + bx + c\dot{x} = 0$								
$\dddot{x} = 0$								
$\ddot{x} + \dot{x} + x - 2 = 0$								
$\ddot{x} + \sin(x) = 0$								
$e^t x + \dot{x} = \sin t$								
$\dot{x} + x = 0$								
$\dot{x}x + a + bt = 0$								
$\ddot{x} - b\dot{x}^2 = 0$								

Solution

Note to graders: For systems that are nonlinear the last column regarding homogeneity can be ignored.

2 Problem

Consider the following IVP:

$$\dot{x} + 2x = 0$$

with initial condition $x(t_0) = -10$ and $t_0 = 0$.

1. What is the particular solution, $x(t)$?

	Linearity?			Order?			Homogeneity?	
System ODE	LTI	LTV	NL	1st	2nd	Higher	Homog.	Inhomog.
$\ddot{x} + 3tx = 0$		x			x		x	
$(\dot{x} - x)^2 + 1 = 0$			x	x			–	–
$t^2 x + bx + c\dot{x} = 0$		x		x			x	
$\dddot{x} = 0$	x					x	x	
$\ddot{x} + \dot{x} + x - 2 = 0$	x				x			x
$\ddot{x} + \sin(x) = 0$			x		x		–	–
$e^t x + \dot{x} = \sin t$		x		x				x
$\dot{x} + x = 0$	x			x			x	
$\dot{x}x + a + bt = 0$			x	x			–	–
$\ddot{x} - b\dot{x}^2 = 0$			x		x		–	–

2. What is the value of x as time $t \to \infty$?

A. $x \to -\infty$
B. $x \to -10$
C. $x \to 0$
D. $x \to +10$
E. $x \to +\infty$

Solution

The solution to the IVP for this homogeneous first-order ODE is

$$x(t) = -10e^{-2t} \ .$$

As $t \to \infty$ grows large the term e^{-2t} approaches zero. Thus, the solution is C.

3 Problem

Consider the ODE $\dot{x} + 2x = e^{-2t}$ with initial condition $x(t_0) = 10$ and $t_0 = 0$. What is the particular solution, $x(t)$?

Solution

The ODE is first-order and inhomogeneous. Thus, the general solution is:

$$x(t) = \underbrace{e^{-at} \int e^{at} g dt}_{\text{inhomogeneous component}} + \underbrace{Ce^{-at}}_{\text{homogeneous component}} \ .$$

For the problem given, $a = 2$ and $g = e^{-2t}$ and

$$\begin{aligned} x(t) &= e^{-2t} \int e^{2t} e^{-2t} dt + Ce^{-2t} \\ &= e^{-2t} \int dt + Ce^{-2t} \\ &= e^{-2t} t + Ce^{-2t} \,. \end{aligned}$$

To solve for the particular solution, evaluate the above expression at the initial condition $x(0) = 10$.

$$x(0) = 10 = e^{-2\cdot 0} 0 + Ce^{-0\cdot t} = C$$

Thus, the particular solution is

$$\implies x(t) = e^{-2t} t + 10 e^{-2t}$$

4 Problem

Consider the ODE $\dot{x} = 2 - 3x$ with initial condition $x(t_0) = 1$ and $t_0 = 0$. What is the value of the state x when $t = 2$? Estimate when the system will decay to a constant value using the time constant of the system (i.e., after 4 time constants)?

Solution

The ODE is first-order and inhomogeneous and can be re-written in standard form as :

$$\dot{x} + 3x = 2$$

The general solution is:

$$x(t) = \underbrace{e^{-at} \int e^{at} g dt}_{\text{inhomogeneous component}} + \underbrace{Ce^{-at}}_{\text{homogeneous component}} \,.$$

For the problem given, $a = 3$ and $g = 2$ and

$$\begin{aligned} x(t) &= e^{-3t} \int e^{3t} 2 dt + Ce^{-3t} \\ &= 2e^{-3t} \int e^{3t} dt + Ce^{-2t} \\ &= 2e^{-3t} \left[\frac{1}{3} e^{3t} \right] + Ce^{-3t} \\ &= \frac{2}{3} + Ce^{-3t} \end{aligned}$$

To solve for the particular solution, evaluate the above expression at the initial condition $x(0) = 1$.

$$x(0) = 1 = \frac{2}{3} + C$$

which implies that $C = 1/3$. Thus, the particular solution is

$$x(t) = \frac{2}{3} + \frac{1}{3}e^{-3t}$$

The above implies that as t grows large the second term decays to zero and the state asymptotically approaches the value of $2/3$. You can also confirm this equilibrium by evaluating the ODE with $\dot{x} = 0$. That is, if $\dot{x} = 0$, then

$$3x = 2 \implies x(t) = 2/3$$

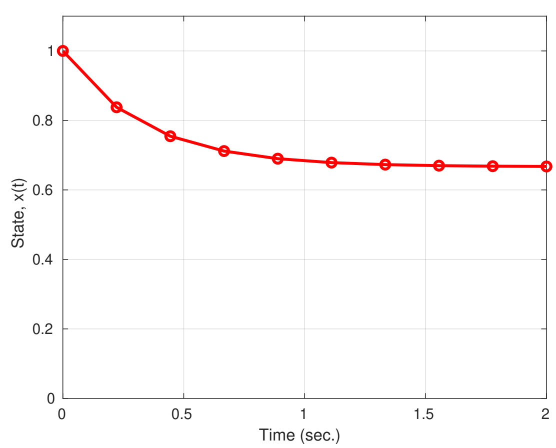
The problem asks to evaluate the solution at $t = 2$ seconds:

$$\implies x(2) = \frac{2}{3} + \frac{1}{3}e^{-3(2)} = 0.6675$$

The time constant of the system is $\tau = 1/a = 1/3$. So the solution will decay to the steady value of $2/3$ after approximately

$$4\tau = 4/3 = 1.33 \text{ sec}$$

Looking at the solution plotted below you'll agree that after 1.33 seconds the state has reached a steady value.



5 Problem

Consider the ODE $\dot{x} - 3x = 0$ with initial condition $x(1) = 40$ and $t_0 = 0$. (Note that the initial condition is not given at $t = 0$!)

1. What is the particular solution, $x(t)$?
2. What is the value of x as time $t \to \infty$?

A. $x \to -\infty$
B. $x \to -10$
C. $x \to 0$
D. $x \to +10$
E. $x \to +\infty$

Solution

This is a 1st order homogeneous ODE with $a = -3$. The general solution is

$$x(t) = Ce^{3t}$$

We can use any point on the solution curve to find the unknown constant C. We are given the point $x(1) = 40$. Thus,

$$x(1) = Ce^3 = 40 \implies C = 40/e^3 = 1.9915 \approx 2$$

The particular solution is

$$x(t) = 2e^{3t}$$

As time grows larger the solution diverges and $x(t) \to \infty$.