Homework 4

1 Problem

Compute the inverse Laplace transform of

$$F(s) = \frac{s+2}{(s+1)(s+4)^2}$$

Hint: Use a combination of partial fraction expansions for real distinct and repeated poles.

Solution

There are three poles of the system: $p_1 = -1$ and $p_{2,3} = -4$ which is a distinct pole and pair of repeated poles. Thus the partial fraction expansion we seek is of the form

$$F(s) = \frac{c_1}{s+1} + \frac{c_2}{s+4} + \frac{c_3}{(s+4)^2}s \tag{1}$$

$$\tag{2}$$

To find c_1 we can use the approach for distinct poles:

$$c_1 = \left[\frac{(s+2)(s+1)}{(s+1)(s+4)^2}\right]_{s=-1} \tag{3}$$

$$= \left[\frac{(s+2)}{(s+4)^2}\right]_{s=-1} = \frac{1}{9} \tag{4}$$

To find c_2 and c_3 we multiply both sides of (2) by $(s+1)(s+4)^2$

$$F(s)(s+4)^2(s+1) = \frac{1}{9}(s+4)^2 + c_2(s+4)(s+1) + c_3(s+1) \tag{5}$$

$$(s+2) = \frac{1}{9}(s^2+8s+16) + c_2(s^2+5s+4) + c_3(s+1) \tag{6}$$

$$(s+2) = s^2(\frac{1}{9}+c_2) + s(\frac{8}{9}+5c_2+c_3) + (\frac{16}{9}+4c_2+c_3) \tag{7}$$

Then, equating coefficients on the LHS and RHS:

$$s^2: \qquad 0 = \frac{1}{9} + c_2 \tag{8}$$

$$\implies c_2 = -\frac{1}{9} \tag{9}$$

$$s: \qquad 1 = \frac{8}{9} + 5(\frac{-1}{9}) + c_3 \tag{10}$$

$$\implies c_3 = \frac{9-8+5}{9} = \frac{2}{3} \tag{11}$$

The partial fraction expnasion is then:

$$F(s) = \frac{1}{9}\frac{1}{s+1} - \frac{1}{9}\frac{1}{s+4} + \frac{2}{3}\frac{1}{(s+4)^2}s \tag{12}$$

Taking the inverse Laplace transform (with rows 6 and 7):

$$
\begin{aligned}
f(t) &= \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s+4}\right] + \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{(s+4)^2}s\right] && (13)\\
&= \frac{1}{9}e^{-t} - \frac{1}{9}e^{-4t} + \frac{2}{3}te^{-4t} && (14)
\end{aligned}
$$

2 Problem

Compute the inverse Laplace transform of

$$F(s) = \frac{3s+9}{s^2+4s+5}$$

Solution

Consider the example:

$$Y(s) = \frac{3s+9}{s^2+4s+5}$$

The roots of the denominator are -2+/-i. We can complete the square for the denominator. We have

$$s^2+4s+5 = s^2+4s+4+1 = (s+2)^2+1$$

Hence, we have

$$Y(s) = \frac{3s+9}{(s+2)^2+1}$$

Note the denominator (s+2)^2+1 is similar to that for Laplace transforms of exp(-2t)cos(t) and exp(-2t)sin(t). We need to manipulate the numerator. Note that in the formula in the table, we have a=-2 and b=1. We look for a decomposition of the form

$$\frac{3s+9}{(s+2)^2+1} = \frac{A(s+2)+B}{(s+2)^2+1}$$

If we can find A and B, then:

$$\frac{3s+9}{(s+2)^2+1} = A\frac{s+2}{(s+2)^2+1} + B\frac{1}{(s+2)^2+1}$$

The inverse transform is

$$y(t) = L^{-1}[Y(s)](t) = Ae^{-2t}\cos t + Be^{-2t}\sin t$$

We can determine A and B by equating numerators in the expression

$$\frac{3s+9}{(s+2)^2+1} = \frac{A(s+2)+B}{(s+2)^2+1} = \frac{As+2A+B}{(s+2)^2+1}$$

Comparing coefficients of s in the numerator we conclude 3=A. Comparing the constant terms we conclude 2A+B=9. Hence A=3 and B=3.

3 Problem

For each of the following differential equations, use Laplace transforms to find the solution to the IVP.

1. $3\ddot{x} + 12\dot{x} + 60x = \delta(t);\ x(0) = 0;\ \dot{x}(0) = 0$ where $\delta(t)$ is the impulse or dirac delta function (row 1 in the Laplace transform table).
2. $\ddot{x} + 10\dot{x} + 25x = 0;\ x(0) = 1;\ \dot{x}(0) = 0$

3. $\ddot{x} + 5\dot{x} + 6x = 2e^{-t}$; $x(0) = 1$; $\dot{x}(0) = 0$
4. $\ddot{x} + 2\dot{x} = 8t$; $x(0) = 0$; $\dot{x}(0) = 0$

Show all your work/intermediate steps. Other solution methods besides Laplace transforms will not receive any credit.

Solution

Problem 1.1

Take the Laplace transform of the system (note $\mathcal{L}[\delta(t)] = 1$):

$$3s^2X(s) + 12sX(s) + 60X(s) = 1$$

and rearrange

$$X(s) = \frac{(1/3)}{s^2 + 4s + 20}$$

There are two complex-valued poles:

$$p_{1,2} = \frac{-4 \pm \sqrt{16 - 4(20)}}{2} = \frac{-4 \pm \sqrt{-64}}{2} - 2 \pm 4i$$

Since the poles are complex in the form $p_{1,2} = -\alpha \pm i\omega$, expand as damped sinusoid and cosines (PFE):

$$X(s) = \frac{(1/3)}{s^2 + 4s + 20} = \frac{c_1(s + \alpha)}{(s + \alpha)^2 + \omega^2} + \frac{c_2\omega}{(s + \alpha)^2 + \omega^2} \tag{15}$$

with $\alpha = 2$ and $\omega = 4$. Equating the numerators gives two equations (one for coefficient of s and one for the constant):

$$s^2 : \qquad 0 = c_1 s \tag{16}$$

$$(\text{constant}) : \qquad (1/3) = c_1\alpha + c_2\omega \tag{17}$$

The first equation implies that $c_1 = 0$ and the second equation implies that

$$(1/3) = c_2(4) \implies c_2 = 1/12$$

So the partial fraction expansion is:

$$X(s) = \frac{1}{12}\left(\frac{4}{(s+2)^2 + 16}\right)$$

and taking the inverse Laplace transform

$$\begin{aligned} x(t) &= \frac{1}{12}\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2 + 16}\right] \\ &= \frac{1}{12}e^{-2t}\sin 4t \end{aligned}$$

1.2 $\quad \ddot{x} + 10\dot{x} + 25x = 0 \qquad x(0) = 1 \qquad \dot{x}(0) = 0$

Laplace Transform: $\left[s^2 X(s) - s\underbrace{x(0)}_{=1} - \underbrace{\cancel{\dot{x}(0)}}_{=0}\right] + 10\left[sX(s) - \underbrace{x(0)}_{=1}\right] + 25X(s) = 0$

$$s^2X(s) - s + 10sX(s) - 10 + 25X(s) = 0$$

$$X(s) = \frac{s+10}{\underbrace{s^2+10s+25}}$$

poles: $p_{1,2} = \dfrac{-10 \pm \sqrt{100-4(25)}}{2}$

$= -5$

Thus, $X(s) = \dfrac{s+10}{(s+5)^2}$

Since poles are repeated, we expand as powers of the denom.

$$X(s) = \frac{(s+10)}{(s+5)^2} = \frac{c_1}{(s+5)} + \frac{c_2}{(s+5)^2}$$

Multiply both sides by highest power denom $(s+5)^2$

$$(s+10) = c_1(s+5) + c_2$$
$$= c_1 s + (5c_1 + c_2)$$

Equate coefficients: $\boxed{1 = c_1}$

$10 = 5(1) + c_2 \Rightarrow \boxed{c_2 = 5}$

Thus, $X(s) = \dfrac{1}{(s+5)} + \dfrac{5}{(s+5)^2}$

Inverse L.T. $\quad x(t) = \mathcal{L}^{-1}\left[\dfrac{1}{s+5}\right] + 5\,\mathcal{L}^{-1}\left[\dfrac{1}{(s+5)^2}\right]$

row 6 $\qquad$ row 7

$$\boxed{x(t) = e^{-5t} + 5te^{-5t}}$$

1.3

$$\ddot{x} + 5\dot{x} + 6x = 2e^{-t} \qquad x(0) = 1 \quad \dot{x}(0) = 0$$

Take L.T. $\left[s^2X(s) - s\underbrace{x(0)}_{=1} - \cancel{\dot{x}(0)}_{=0}\right] + 5\left[sX(s) - \underbrace{x(0)}_{=1}\right] + 6X(s) = 2\left(\frac{1}{s+1}\right)$ row 6

$$s^2X(s) - s + 5sX(s) - 5 + 6X(s) = 2\left(\frac{1}{s+1}\right)$$

$$X(s)(s^2+5s+6) = 2\left(\frac{1}{s+1}\right) + (s+5)$$

$$X(s) = \frac{2}{(s+1)(s^2+5s+6)} + \frac{(s+5)}{(s^2+5s+6)}$$

make common denom.

$$= \frac{2 + (s+5)(s+1)}{(s+1)(s^2+5s+6)}$$

$$= \frac{2 + s^2 + 6s + 5}{(s+1)(s^2+5s+6)}$$

poles: $p_1 = -1$ $\qquad p_{2,3} = \frac{(-5 \pm \sqrt{25-4(6)})}{2}$

$$= \frac{-5 \pm 1}{2}$$

$p_2 = -2$

$p_3 = -3$

Thus, $X(s) = \dfrac{s^2+6s+7}{(s+1)(s+2)(s+3)}$

Since all poles are distinct, expand as partial fractions with corresponding distinct denominators

$$X(s) = \frac{s^2+6s+7}{(s+1)(s+2)(s+3)} = \frac{C_1}{(s+1)} + \frac{C_2}{(s+2)} + \frac{C_3}{(s+3)}$$

To solve for coefficients we multiply both sides by denominator and set s equal to the pole. (see topic 11)

$$C_1 = \left.\frac{(s^2+6s+7)}{(s+2)(s+3)}\right|_{s=-1} = \frac{1-6+7}{1 \cdot 2} = 1$$

$$C_2 = \left.\frac{(s^2+6s+7)}{(s+1)(s+3)}\right|_{s=-2} = \frac{4-12+7}{-1\cdot 1} = 1$$

$$C_3 = \left.\frac{(s^2+6s+7)}{(s+1)(s+2)}\right|_{s=-3} = \frac{9-18+7}{-2\cdot -1} = -1$$

Taking I.L.T. $x(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{1}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+3}\right]$

↓ row 6

$$\boxed{x(t) = e^{-t} + e^{-2t} - e^{-3t}}$$

1.4 $\ddot{x} + 2\dot{x} = 8t$ $\dot{x}(0) = 0$ $x(0) = 0$

row 3

L.T. $s^2X(s) + 2sX(s) = 8\frac{1}{s^2}$

$$X(s) = 8\frac{1}{s^2}\frac{1}{(s^2+2s)}$$

$$= \frac{8}{s^3(s+2)}$$

$p_4 = -2$ (distinct)

$p_{1,2,3} = 0$ (repeated poles)

This case involves mixed poles, hence we expand using both methods (for distinct and for real)

$$X(s) = \frac{8}{s^3(s+2)} = \underbrace{\frac{C_1}{s} + \frac{C_2}{s^2} + \frac{C_3}{s^3}}_{\text{method for repeated}} + \underbrace{\frac{C_4}{(s+2)}}_{\text{method for distinct}}$$

Multiply both sides by $s^3(s+2)$:

$$8 = C_1s^2(s+2) + C_2s(s+2) + C_3(s+2) + C_4s^3$$

$$= C_1s^3 + 2C_1s^2 + C_2s^2 + 2C_2s + C_3s + 2C_3 + C_4s^3$$

$$= (C_1 + C_4)s^3 + (2C_1 + C_2)s^2 + (2C_2 + C_3)s + 2C_3$$

Equating coefficients const. : $8 = 2C_3 \Rightarrow \boxed{C_3 = 4}$

s : $0 = 2C_2 + C_3 \Rightarrow C_2 = -\frac{C_3}{2}$

$\boxed{C_2 = -2}$

s^2: $0 = 2C_1 + C_2 \Rightarrow \boxed{C_1 = 1}$

s^3: $0 = C_1 + C_4 \Rightarrow \boxed{C_4 = -1}$

I.L.T: $x(t) = \mathcal{L}^{-1}\left[\frac{1}{s}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] + 4\mathcal{L}^{-1}\left[\frac{1}{s^3}\right] - \mathcal{L}^{-1}\left[\frac{1}{s+2}\right]$

row 2 3 4 6

$$\boxed{\begin{array}{l} x(t) = u(t) - 2t + 4\frac{t^2}{2} - e^{-2t} \\ \text{or} \\ x(t) = 1 - 2t + 2t^2 - e^{-2t} \end{array}}$$

4 Problem

Use the MATLAB function `dsolve` to verify your answer for Problem 3.4. Generate a plot of the solution over the time interval $t \in [0, 3]$ seconds. Submit your code.

Solution

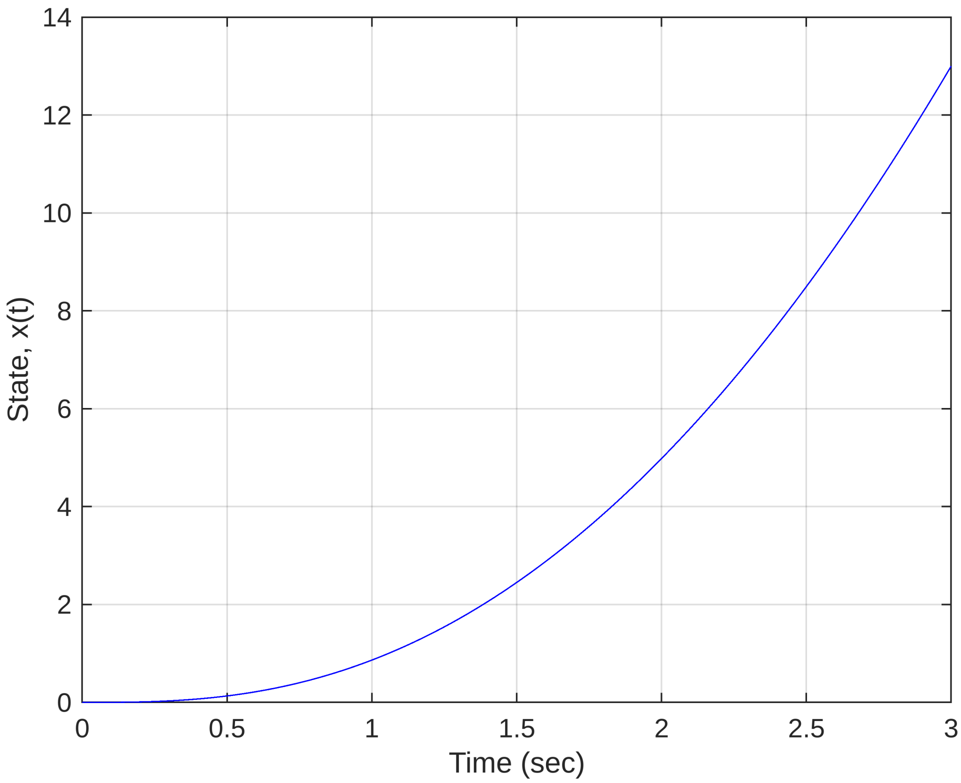
Using the symbolic toolbox with dsolve to solve Problem 1.4

```
syms x(t);
xdot = diff(x,t);
xddot = diff(x,t,2);
assume(t>=0)
eqn = xddot == -2*xdot + 8*t;
cond = [x(0)==0, xdot(0)==0];
xsol = dsolve(eqn,cond)
```

xsol = $2\,t^2 - e^{-2\,t} - 2\,t + 1$

Now to plot the solution

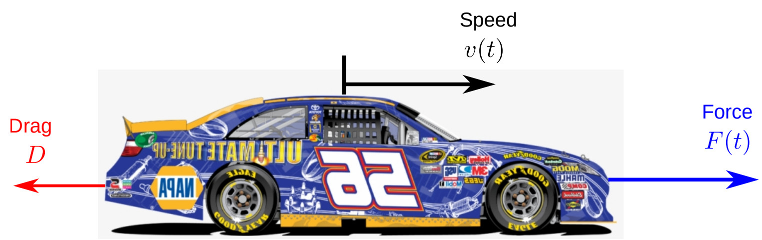
```
tvals = linspace(0,3);
xvals = double(subs(xsol,tvals));
figure;
plot(tvals,xvals,'b-')
grid on;
xlabel('Time (sec)')
ylabel('State, x(t)');
```

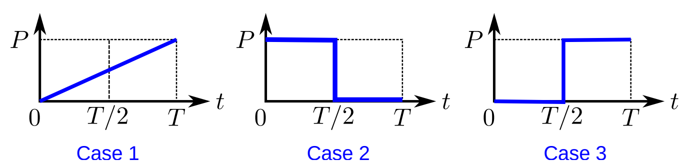


5 Problem

Suppose the racecar below has a mass of $m = 750$ kg and is moving down a track with an initial speed of $v(t_0) = 45$ m/s at time $t_0 = 0$ sec. The drag on the car is modeled as a linear function of velocity: $D = bv$, where $b = 20$ N/(m/s).

- Using the free-body diagram below, where $F(t)$ is an applied force, apply Newton's 2nd Law to find the equations of motion. Since $a(t) = \dot{v}(t)$ you can write this equation as a first-order ODE in speed (i.e., $\sum F = m\dot{v}$).





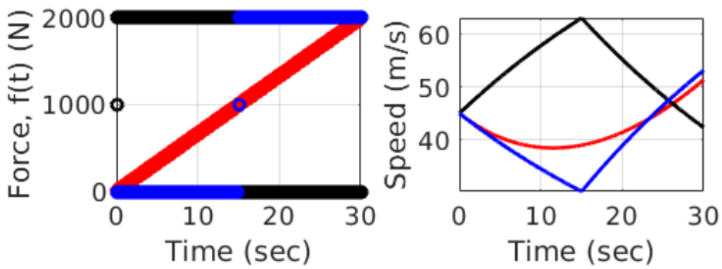
Suppose that over the next $T = 30$ seconds the driver can choose from the three possible force profiles, $F(t)$, shown above, where $P = 2000$ N is the same maximum force reached during each profile.

- Write down an expression for each of the force profiles $F_1(t)$, $F_2(t)$, $F_3(t)$ as a function of the magnitude P and time. You can construct the force profiles from a combination of Heaviside functions and ramps (straight lines) with appropriate slope. Reviewing the doublet example (Lecture 7 PDF, p.2) may be helpful.

Interestingly, each profile has the same impulse (area under the force-time curve) but results in a different final displacement and velocity. Determine the velocity profile $v(t)$ that results from each case by following these steps:

- Solve for the velocity profile in each of the three cases using MATLAB (following the methods of Lecture 10 e.g., using `dsolve`). and plot the three solutions on the same axes. Which case results in the largest final speed? Label your axes, add a legend for each line, and use a thick line type for clarity.

Note that MATLAB defines the step function as: `heaviside(t)`.Your solution should look similar to the one below:



Bonus: Which case results in the furthest distance traveled at time T? Justify your answer with a plot of distance traveled in MATLAB.

Solution

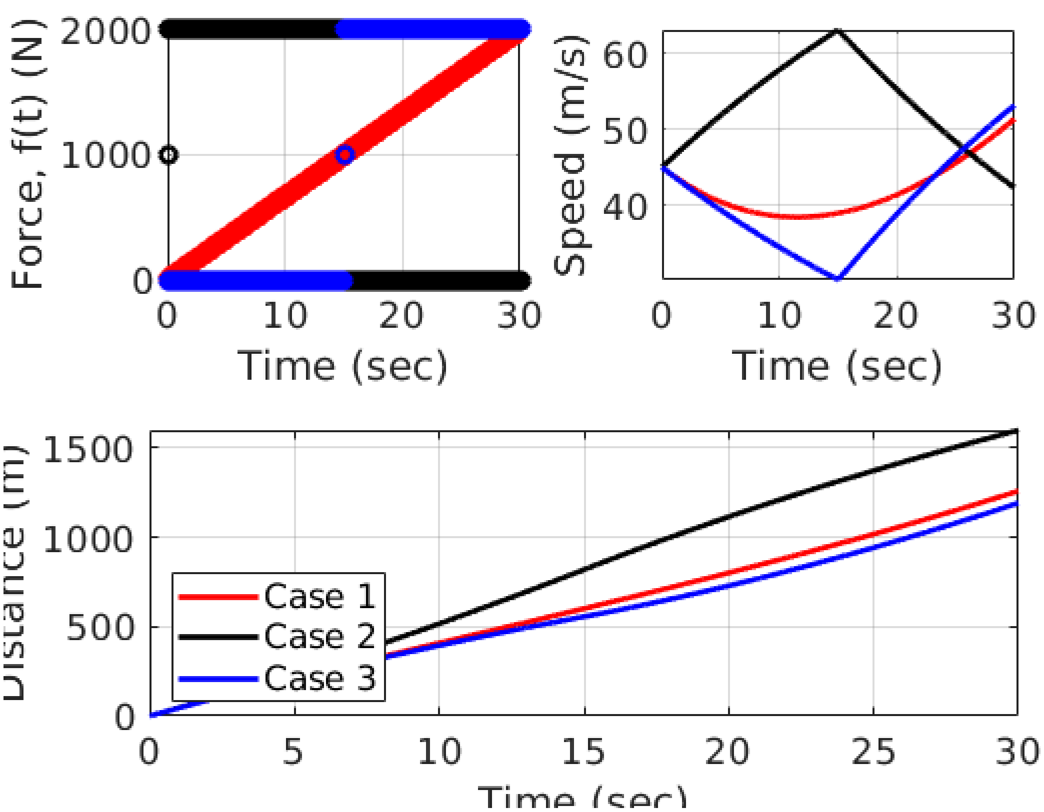
Answers may vary. Solution below uses `laplace`, `solve`, `ilaplace`, `eval`. Another approach may use `dsolve`.

```
clear; close all; clc; % prepare workspace

% constants
b = 20; % N/(m/s)
m = 750; % kg
v0 = 45; % x(0) IC, initial position
P = 2000;
T = 30;
tvals = [0:1/50:T]; % 50 frames per second

% v' + b/m*v = f

% case 1
fprintf('---------------\nCase 1:\n');
syms V s v t f; % define symbolic variables
f = t*P/T;
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist1 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist1 = eval(subs(int(v,0,t),tvals));
fhist1 = eval(subs(f,tvals));

% case 2
fprintf('---------------\nCase 2:\n');
f = P*heaviside(t) -P*heaviside(t-T/2);
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist2 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist2 = eval(subs(int(v,0,t),tvals));
fhist2 = eval(subs(f,tvals));

% case 3
fprintf('---------------\nCase 3:\n');
f = P*heaviside(t-T/2);
F = laplace(f,t,s); % take laplace transform
V1 = s*V - v0; % laplace transform of x-dot
Vsol = solve(V1 + b/m*V == F/m, V); % solve for X(s)
v = ilaplace(Vsol); % take inverse in MATLAB for x(t)
pretty(v)
vhist3 = eval(subs(v,tvals)); % evaluate x(t) for tvals given
dhist3 = eval(subs(int(v,0,t),tvals));
fhist3 = eval(subs(f,tvals));

figure;
subplot(2,2,1)
plot(tvals,fhist1,'ro','linewidth',2); hold on;
```

```
plot(tvals,fhist2,'ko','linewidth',2)
plot(tvals,fhist3,'bo','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Force, f(t) (N)')
grid on;
hold on;
axis tight;

subplot(2,2,2)
plot(tvals,vhist1,'r','linewidth',2); hold on;
plot(tvals,vhist2,'k','linewidth',2)
plot(tvals,vhist3,'b','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Speed (m/s)')
grid on;
hold on;
axis tight;

subplot(2,2,[3:4])
plot(tvals,dhist1,'r','linewidth',2); hold on;
plot(tvals,dhist2,'k','linewidth',2)
plot(tvals,dhist3,'b','linewidth',2)
set(gca,'FontSize',14);
xlabel('Time (sec)')
ylabel('Distance (m)')
legend('Case 1','Case 2','Case 3','location','southwest');
grid on;
hold on;
axis tight;

fprintf('*******************************************************\n')
fprintf('Case 3 leads to the greatest speed at t = 30 sec\n')
fprintf('Case 2 leads to the greatest distance at t = 30 sec\n')
fprintf('Case 1 has intermediate performance compared to Case 2 and
 3\n')
fprintf('*******************************************************\n')
```

```
---------------
Case 1:
10 t      /   2 t \
---- + exp| - --- | 170 - 125
  3       \    75 /

---------------
Case 2:
                  /    / 2   2 t \         \
                  | exp| - - --- | 75      |
                  |    \ 5    75 /      75 |
heaviside(t - 15) | ----------------- - -- | 8
                  \         2          2 /        /   2 t \
---------------------------------------------- - exp| - --- | 55 + 100
                      3                             \    75 /
```

```
---------------
Case 3:
                                    /    / 2   2 t \        \
                                    | exp| - - --- | 75      |
                                    |    \ 5   75 /      75 |
                  heaviside(t - 15) | ---------------- - -- | 8
   /   2 t \                        \        2            2 /
exp| - --- | 45 - ---------------------------------------------
   \   75 /                              3

******************************************************
Case 3 leads to the greatest speed at t = 30 sec
Case 2 leads to the greatest distance at t = 30 sec
Case 1 has intermediate performance compared to Case 2 and 3
******************************************************
```



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