On my honor, I submit that I have neither given or received assistance on this exam or consulted any prohibited materials (beyond the one page crib sheet allowed for the exam).

Name: ______________________________ Date: February 10, 2022

MEGR 3122 Dynamics Systems II: Exam 1, Spring 2022

Multiple Choice Problems (Total 20 Points)

Directions: Circle the best answer. Each question is worth 2 points.

1. In this course, system dynamics are modeled as:
   A. partial differential equations
   B. ordinary differential equations
   C. hyperbolic equations
   D. asymptotic equations

2. If $x(t)$ is a function of time (the state of a mechanical system), and a and b are constants, then $\ddot{x} + a\dot{x} + \sqrt{b}t = 1$ is which of the following?
   A. linear, time-varying, second-order, homogeneous
   B. linear, time-invariant, second-order, homogeneous
   C. linear, time-invariant, second-order, inhomogeneous
   D. nonlinear

   $\underbrace{\ddot{x} + a\ddot{x}}_{\text{time invariant}} = g(t)$
   $= \underbrace{1 - \sqrt{b}t}_{\neq 0 \text{ inhomog.}}$

3. Which case in Fig. 1 (below) could plausibly represent the response of a system $\dot{x} - 2x = 0$? (Note: the initial condition is not necessary to answer this question.)
   A. Case 1
   B. Case 2
   C. Case 3
   D. Both Case 1 and Case 2

   if $a < 0$ then $x(t)$ diverges to $+$ or $-\infty$

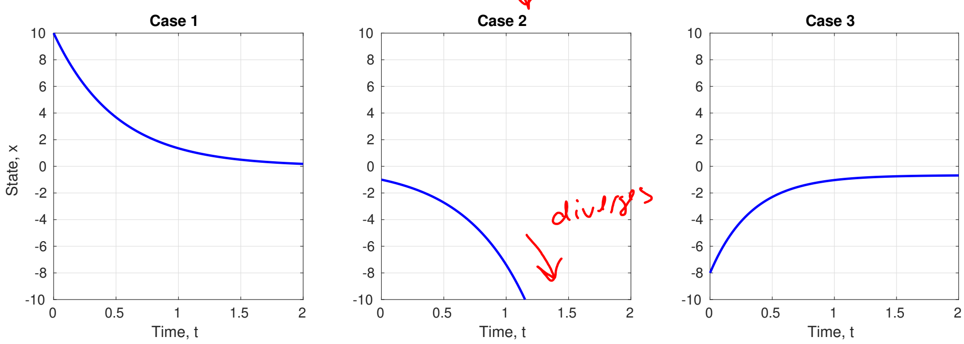

Figure 1: System response

4. Consider the following initial value problem:

$$\dot{x} + 4x = 0\ , \qquad x(0) = 5$$

   The solution is:

   A. $x(t) = 5e^{-4t}$ B. $x(t) = 4e^{-5t}$ C. $x(t) = e^{-4t}\cos 5t$ D. $x(t) = e^{4t}\sin t$

5. If $z_1 = -i$ and $z_2 = e^{i\pi/2}$, then what is the sum $z_1 + z_2$?
   A. i
   B. 1
   C. -1
   D. 0

   $z_2 = i \Rightarrow z_1 + z_2 = 0$
   $z_1 = -i$

6. Consider the following system

$$\dot{x} + ax = 0$$

with initial condition of $x(0) = 100$. The state $x(t)$ decays to a value of 36.8 after 3 seconds. That is, $x(3) = 36.8$. What is the value of a?

A. $a = 100/3$
B. $a = 1/3$
C. $a = 3$
D. $a = (100 - 36.8)/3$

36.8% of 100 implies 1 time constant
$\tau = 3$, $\Rightarrow a = \frac{1}{\tau} = \frac{1}{3}$

7. What is the Laplace transform of the function $x(t) = (t-2)H(t-2)$, where $H(\cdot)$ is the unit step or Heaviside function? $\mathcal{L}[t] = 1/s^2$ ramp shifted by $\alpha = 2$

using translated LT theorem

A. $\frac{1}{s^2}$ B. $\frac{1}{(s+1)^2}$ C. $\frac{e^{-2s}}{s^2}$ D. $\frac{2!}{(s-2)^2}$

8. Consider the second order system $6\ddot{x} + 3x = 0$ and define a new set of variables $z_1 = x$ and $z_2 = \dot{x}$. Which of the following first-order systems of two equations (in z_1 an z_2) is equivalent to the second order system (in x)?

$\dot{z}_1 = \dot{x} = z_1$
$\dot{z}_2 = \ddot{x} = -\frac{3}{6}\dot{x} = -\frac{1}{2} z_1$

A.
$$z_1 = \dot{z}_2$$
$$z_2 = 6\dot{z}_1 + 3\dot{z}_2$$

B.
$$\dot{z}_1 = 3z_2$$
$$\dot{z}_2 = 6z_1$$

C.
$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = 6z_2 + 3z_1$$

D.
$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = -(1/2)z_1$$

9. Which case in Fig. 2 (below) could plausibly represent the response of a system $\ddot{x} + \dot{x} + x = 0$ with $x(0) = 4$ and $\dot{x}(0) = -10$?

A. Case 1
B. Case 2
C. Case 3
D. None of the above

slope should be downward

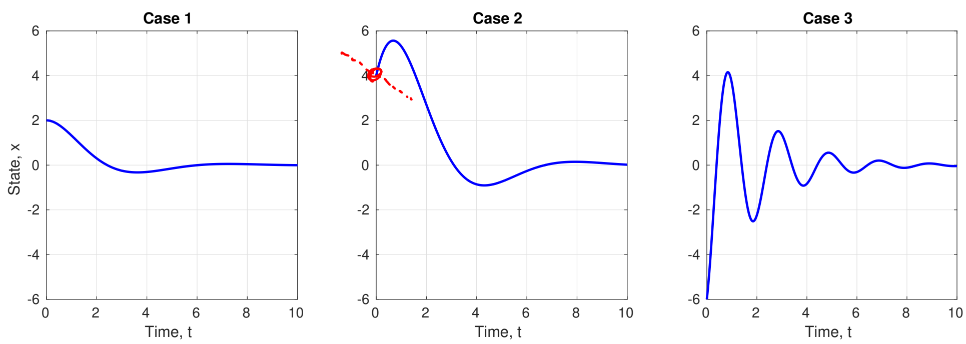

Figure 2: System response

10. What is the final value of $x(t)$ as $t \to 0$ if the Laplace transform of $x(t)$ is the following?

$$X(s) = \frac{s + 10}{5s^2 + 2s + 1}$$

A. $x(t) \to 10$
B. $x(t) \to 5$
C. $x(t) \to 2$
D. $x(t) \to 1$

skipped

Workout Problem Instructions

To receive full credit on the workout problems show all of your work.

Workout Problem 1 (5 pts)

Consider the ODE

$$\ddot{x} + 4\dot{x} + 8x = 0$$

with initial conditions $x(0) = 0$ and $\dot{x}(0) = 10$.

- Find the eigenvalues of the system
- State the general solution of the ODE
- Determine the particular solution $x(t)$ that satisfies the initial conditions

Eigenvalues

$$\lambda^2 + 4\lambda + 8 = 0$$

$$(\lambda+2)^2 = \lambda^2 + 4\lambda + 4 = -4$$

$$\lambda_{1,2} = -2 \pm 2i$$

General sol'n:

$$x(t) = e^{-2t}(A\cos(2t) + B\sin(2t))$$

I.Cs

$$x(0) = 0 = A \cdot \underbrace{\cos(0)}_{=1} \Rightarrow A = 0$$

$\Rightarrow$

$$\dot{x}(0) = 10 = -2e^{-2t} B \underbrace{\sin(2t)}_{=0} + e^{-2t}(B \cdot 2 \cdot \cos(2t))$$

$$10 = 2B \Rightarrow B = 5$$

$$\boxed{x(t) = 5e^{-2t}\sin(2t)}$$

Workout Problem 2 (5 pts)

Compute the Laplace transform $F(s) = \mathcal{L}[f(t)]$ of the following function:

$$f(t) = \frac{e^{-t}}{4}[2 + t^2 + \cos(3t)]$$

$$f(t) = \frac{e^{-t}}{4}\left(2 + t^2 + \cos(3t)\right)$$

$$= \frac{2}{4} e^{-t} + \frac{1}{2}\left(\frac{t}{2} e^{-t}\right) + \frac{1}{4} e^{-t} \cos 3t$$

$$F(s) = \underbrace{\frac{1}{2} \mathcal{L}[e^{-t}]}_{\text{row } 6} + \underbrace{\frac{1}{2} \mathcal{L}\left[\frac{t^2 e^{-t}}{2}\right]}_{\substack{\text{row } 8 \\ n=3}} + \underbrace{\frac{1}{4} \mathcal{L}[e^{-t} \cos(3t)]}_{\text{row } 21} \qquad \begin{matrix} a=1 \\ \omega=3 \end{matrix}$$

$$= \frac{1}{2}\left[\frac{1}{s+1}\right] + \frac{1}{2}\left[\frac{1}{(s+1)^3}\right] + \frac{1}{4}\left[\frac{s+1}{(s+1)^2+9}\right]$$

Workout Problem 3 (5 pts)

Compute the inverse Laplace transform $f(t) = \mathcal{L}^{-1}[F(s)]$ of the following function:

$$F(s) = \frac{s+1}{s^2 + 6s + 9}$$

$$F(s) = \frac{(s+1)}{(s^2+6s+9)} = \frac{(s+1)}{\underbrace{(s+3)^2}_{\text{repeated roots}}} = \frac{c_1}{(s+3)} + \frac{c_2}{(s+3)^2}$$

$$(s+1) = c_1(s+3) + c_2$$

$$s + 1 = c_1 s + (3c_1 + c_2)$$

$$\Rightarrow \boxed{c_1 = 1}$$

$$1 = 3 + c_2 \Rightarrow \boxed{c_2 = -2}$$

$$F(s) = \frac{1}{(s+3)} - \frac{2}{(s+3)^2}$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] - 2\mathcal{L}^{-1}\left[\frac{1}{(s+3)^2}\right]$$

$$= e^{-3t} - 2te^{-3t}$$

Table 2.1 Laplace transforms [2]

	$f(t)$	$F(s)$
1	Unit impulse $\delta(t)$	1
2	Unit step $u(t)$	$\frac{1}{s}$
3	t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n=1,\ 2,\ 3,\ \ldots$	$\frac{1}{s^n}$
5	$t^n, \quad n=1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{s+a}$
7	$t\mathrm{e}^{-at}$	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!}\mathrm{e}^{-at}, \quad n=1,\ 2,\ 3,\ \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n\mathrm{e}^{-at}, \quad n=1, 2, 3, \ldots$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1-\mathrm{e}^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(\mathrm{e}^{-at}-\mathrm{e}^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(b\mathrm{e}^{-bt}-a\mathrm{e}^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left(1+\frac{1}{a-b}(b\mathrm{e}^{-at}-a\mathrm{e}^{-bt})\right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}(1-\mathrm{e}^{-at}-at\mathrm{e}^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at-1+\mathrm{e}^{-at})$	$\frac{1}{s^2(s+a)}$
20	$\mathrm{e}^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$\mathrm{e}^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}\mathrm{e}^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t\right)$	$\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

(continued)

Table 2.1 (continued)

	$f(t)$	$F(s)$
23	$-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{s}{s^2+2\zeta\omega_n s+\omega_n^2}$
24	$1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}t+\phi\right)$ $\phi=\tan^{-1}\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)$	$\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$
25	$1-\cos(\omega t)$	$\frac{\omega^2}{s(s^2+\omega^2)}$
26	$\omega t-\sin(\omega t)$	$\frac{\omega^3}{s^2(s^2+\omega^2)}$
27	$\sin(\omega t)-\omega t\cos(\omega t)$	$\frac{2\omega^3}{(s^2+\omega^2)^2}$
28	$\frac{1}{2\omega}t\sin(\omega t)$	$\frac{s}{(s^2+\omega^2)^2}$
29	$t\cos(\omega t)$	$\frac{s^2-\omega^2}{(s^2+\omega^2)^2}$
30	$\frac{1}{\omega_2^2-\omega_1^2}(\cos(\omega_1 t)-\cos(\omega_2 t)),\ \omega_1^2\neq\omega_2^2$	$\frac{s}{\left(s^2+\omega_1^2\right)\left(s^2+\omega_2^2\right)}$
31	$\frac{1}{2\omega}(\sin(\omega t)+\omega t\cos(\omega t))$	$\frac{s^2}{(s^2+\omega^2)^2}$