Lecture 6: The Laplace Transform

The *Laplace transform* is a mathematical tool that can be advantageous for solving LTI ODEs since it converts a differential equation in the time domain into an algebraic equation in the Laplace s-domain. Mathematically, the Laplace transform is an integral transform defined by

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

where

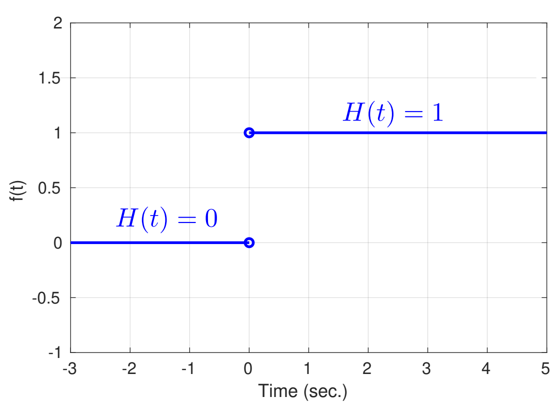
- $\mathcal{L}[\cdot]$ is the symbol for the Laplace operator. It indicates that the integral operation shown above will be applied to the function $f(t)$
- $f(t)$ is a function of time such that $f(t) = 0$ for all $t < 0$
- $s \in \mathbb{C}$ is a complex variable
- $F(s)$ is the Laplace transform of $f(t)$

The Laplace transform exists only for certain "well-behaved" functions (i.e., those for which the Laplace integral converges). We will now derive the Laplace transform for a number of common functions before introducing the Laplace transform table. The Laplace transform will eventually allow us to solve LTI differential equations with various inhomogeneous terms that we couldn't easily solve for using the methods of Lectures 3–4. Moreover, it will allow us to solve more complex LTI ODEs—beyond just first and second-order systems.

Laplace Transform of the Step Function

Define the Heaviside (unit step) function as

$$H(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases}$$



This commonly used function represens a "jump" that occurs from zero to one at the origin. For example, a suddent throttle input in a car could be represented by a jump from 0 to 100%.

Applying the integral definition of a Laplace transform:

$$\begin{aligned}
\mathcal{L}[H(t)] &= \int_0^\infty 1e^{-st}dt \\
&= \frac{1}{-s}\left[e^{-st}\right]_0^\infty \\
&= \frac{1}{-s}\left[\frac{1}{e^\infty} - e^0\right] \\
&= \frac{1}{-s}[0-1] \\
&= \frac{1}{s}
\end{aligned}$$

Warning: When a function $f(t)$ contains constants we should always interpret those constants as multiplying unit step/Heaviside functions (since we require $f(t)$ to be zero for all $t < 0$). For example, the function $f(t) = 5 + \cos t$ implies $f(t) = 5H(t) + \cos t$ when working with Laplace transforms.

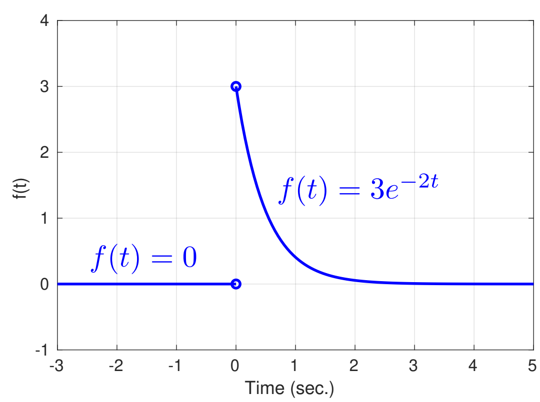
Laplace Transform of the Exponential Function

Define the exponential function as

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ Ae^{-\alpha t} & \text{for } t > 0 \end{cases}$$

where A, α are constants. As mentioned earlier, we require all of the functions to be zero for time less than zero—hence $f(t)$ above is a modified version of the exponential function you have previously encountered.

Example: Exponential function with $A = 3$ and $\alpha = 2$.



Applying the integral definition of a Laplace transform:

$$\begin{aligned}
\mathcal{L}[Ae^{-\alpha t}] &= \int_0^\infty Ae^{-\alpha t}e^{-st}dt \\
&= A\int_0^\infty e^{-(\alpha+s)t}dt \\
&= \frac{A}{-(s+\alpha)}\left[e^{-(s+\alpha)t}\right]_0^\infty \\
&= \frac{A}{-(s+\alpha)}\left[\frac{1}{e^\infty} - e^0\right] \\
&= \frac{A}{-(s+\alpha)}\left[0-1\right] \\
&= \frac{A}{(s+\alpha)}
\end{aligned}$$

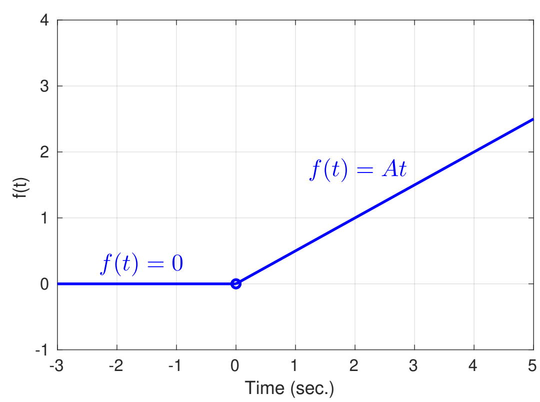
Laplace Transform of the Ramp Function

Define the ramp function as

$$f(t) = \begin{cases} 0 & \text{for } t < 0 \\ At & \text{for } t \geq 0 \end{cases}$$

where A is a constant. This function is a straight line with slope A that passes through the origin but remains zero for all time less than zero.

Example: Ramp function with $A = 1/2$.



Applying the integral definition of a Laplace transform:

$$\begin{aligned}
\mathcal{L}[f(t)] &= \int_0^\infty Ate^{-st}dt \\
&= A\int_0^\infty te^{-st}dt
\end{aligned}$$

Let $u = t$ and $dv = e^{-st}dt$. Then $v = e^{-st}/(-s)$ and $du = dt$. Using the integration by parts formula:

$$\int udv = uv\Big|_a^b - \int_a^b vdu$$

the Laplace transform becomes

$$\begin{aligned}
\mathcal{L}[f(t)] &= A\left(\left[\frac{te^{-st}}{-s}\right]_{t=0}^{t=\infty} - \int_0^\infty \frac{e^{-st}}{-s}dt\right) \\
&= \frac{A}{-s}\left(\left[\underbrace{\frac{\infty}{e^{s\infty}}}_{=0} - 0\right] - \left[\frac{e^{-st}}{-s}\right]_{t=0}^{t=\infty}\right) \\
&= \frac{A}{-s^2}\left[\underbrace{\frac{1}{e^{s\infty}}}_{=0} - 1\right] \\
\implies \mathcal{L}[f(t)] &= \frac{A}{s^2}
\end{aligned}$$

Linearity of the Laplace Transform

Since the Laplace transform is an integral operation it follows the properties of linearity

- Scalar multiplication:

$$\begin{aligned}
\mathcal{L}[Af(t)] &= \int_0^\infty Af(t)e^{-st}dt \\
&= A\underbrace{\int_0^\infty f(t)e^{-st}dt}_{\mathcal{L}[f(t)]} \\
&= A\mathcal{L}[f(t)]
\end{aligned}$$

- Addition:

$$\begin{aligned}
\mathcal{L}[f(t)+g(t)] &= \int_0^\infty A(f(t)+g(t))e^{-st}dt \\
&= \underbrace{\int_0^\infty f(t)e^{-st}dt}_{\mathcal{L}[f(t)]} + \underbrace{\int_0^\infty g(t)e^{-st}dt}_{\mathcal{L}[g(t)]} \\
&= \mathcal{L}[g(t)]
\end{aligned}$$

- *Warning (Multiplication):* The Laplace transform of two functions is not equal to the product of the two functions Laplace transforms. That is, for two functions $f(t)$ and $g(t)$ with Laplace transforms $\mathcal{L}[f(t)] = F(s)$ and $\mathcal{L}[g(t)] = G(s)$:

$$\mathcal{L}(f(t)\cdot g(t)) \neq F(s)\cdot G(s)$$

Example: Suppose the two functions are unit steps: $f(t) = H(t)$ and $g(t) = H(t)$. The Laplace transform of each of these is $F(s) = G(s) = \mathcal{L}(H(t)) = 1/s$ and the product $f(t) \cdot g(t) = H(t)$ is also a unit step. However,

$$\mathcal{L}(f(t) \cdot g(t)) \neq F(s) \cdot G(s) \tag{1}$$

$$\mathcal{L}(H(t)) \neq \frac{1}{s} \cdot \frac{1}{s} \tag{2}$$

$$\frac{1}{s} \neq \frac{1}{s^2} \tag{3}$$

This example illustrates that the Laplace transform of two functions multiplied together is not the same as the product of their individual Laplace transforms.

Laplace Transform Tables

Some common Laplace transforms are collected in the following table which will also appear on your formula sheet.

Row	$f(t)$	$F(s)$
1	Unit impulse, $\delta(t)$	1
2	Unit step/Heaviside, $H(t)$	$\frac{1}{s}$
3	Ramp, t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{(s+a)}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!}\, e^{-at}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left(1+ \frac{1}{a-b} \quad (be^{-bt} - ae^{-at})\right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}\ (1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}\ (at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2 t})$	$\frac{s+a}{(s+a)^2+\omega^2}$

Putting it all together

When finding a Laplace transform of a given function $f(t)$ we can look for the corresponding entries for each term in the Laplace transform table. This may require manipulating $f(t)$ so that it looks like it is in the appropriate form (e.g., taking advantage of the linearity of the Laplace transform or various trigonometric identities).

Example: Find the Laplace transform of

$$f(t) = e^{3t} + \cos(6t)$$

Consulting the table

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a} \qquad \text{(row 6, with } a = -3\text{)}$$
$$\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2} \qquad \text{(row 11, with } \omega = 6\text{)}$$

By linearity,

$$\begin{aligned}
\mathcal{L}[f(t)] &= \mathcal{L}[e^{3t}] + \mathcal{L}[\cos(6t)] \\
&= \frac{1}{s-3} + \frac{s}{s^2+36} \\
&= \frac{s^2+36}{(s-3)(s^2+36)} + \frac{s(s-3)}{(s-3)(s^2+36)} \\
&= \frac{2s^2-3s+36}{s^3-3s^2+36s-108}
\end{aligned}$$

Example: Find the Laplace transform of

$$f(t) = \frac{1}{2}t^4 + 5$$

Consulting the table

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \text{(row 5, with } n = 4\text{)}$$
$$\mathcal{L}[H(t)] = \frac{1}{s} \qquad \text{(row 2, recognizing that the constant 5 implies } 5H(t)\text{)}$$

By linearity,

$$\begin{aligned}\mathcal{L}[f(t)] &= \frac{1}{2}\mathcal{L}[t^4] + 5\mathcal{L}[H(t)] \\ &= \frac{1}{2}\cdot\frac{4!}{s^5} + 5\frac{1}{s} \\ &= \frac{1}{2}\cdot\frac{4\cdot 3\cdot 2\cdot 1}{s^5} + \frac{5}{s} \\ &= \frac{5s^4+12}{s^5}\end{aligned}$$

Example: Find the Laplace transform of

$$f(t) = 6e^{-5t} + 5t^3 - 9$$

Consulting the table

$$\mathcal{L}[e^{-at}] = \frac{1}{s+a} \qquad \text{(row 6, with } a = 5\text{)}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}} \qquad \text{(row 5, with } n = 3\text{)}$$

$$\mathcal{L}[H(t)] = \frac{1}{s} \qquad \text{(row 2, recognizing that the constant 9 implies } 9H(t)\text{)}$$

By linearity,

$$\begin{aligned}\mathcal{L}[f(t)] &= 6\mathcal{L}[e^{-5t}] + 5\mathcal{L}[t^3] + 9\mathcal{L}[H(t)] \\ &= 6\left(\frac{1}{s+5}\right) + 5\left(\frac{3!}{s^4}\right) + 9\frac{1}{s} \\ &= \left(\frac{6(s^4)s}{s(s+5)s^4}\right) + \left(\frac{30(s+5)s}{s(s+5)s^4}\right) + \frac{9s^4(s+5)}{s(s+5)s^4} \\ &= \frac{15s^5+45s^4+30s^2+150s}{s^6+5s^5}\end{aligned}$$

Example: Find the Laplace transform of

$$f(t) = \cos^3 t$$

This function does not appear in the table, but we can make use of the following trigonometric identities to express it in another form that does have table entries:

- Double-angle formula: $\cos^2\theta = \frac{1+\cos 2\theta}{2}$
- Product-to-sum formula: $\cos\alpha\cos\theta = \frac{1}{2}(\cos(\alpha+\theta) + \cos(\alpha-\theta))$
- Symmetry: $\cos(-\alpha) = \cos\alpha$

$$\begin{aligned}
f(t) &= \cos^3 t \\
&= \cos t \cos^2 t \\
&= \cos t \left(\frac{1 + \cos 2t}{2} \right) \\
&= \frac{1}{2} \cos t + \frac{1}{2} \cos t \cos 2t \\
&= \frac{1}{2} \cos t + \frac{1}{2} \left[\frac{1}{2} (\cos(3t) + \cos(t)) \right] \\
&= \frac{3}{4} \cos(t) + \frac{1}{4} \cos(3t)
\end{aligned}$$

Then using

$$\mathcal{L}[\cos \omega t] = \frac{s}{s^2 + \omega^2} \qquad \text{(row 11)}$$

By linearity,

$$\begin{aligned}
\mathcal{L}[f(t)] &= \frac{3}{4} \mathcal{L}[\cos(t)] + \frac{1}{4} \mathcal{L}[\cos(3t)] \\
&= \left(\frac{3}{4} \right) \frac{s}{s^2 + 1} + \left(\frac{1}{4} \right) \frac{s}{s^2 + 9} \\
&= \frac{3s(s^2 + 9) + s(s^2 + 9)}{4(s^2 + 1)(s^2 + 1)} \\
&= \frac{4s^3 + 28s}{4(s^2 + 9)(s^2 + 1)} \\
&= \frac{s^3 + 7s}{(s^2 + 9)(s^2 + 1)}
\end{aligned}$$

References and Further Reading

- Davies: Sec. 2.2, 2.5
- Ogata: Sec. 2.3