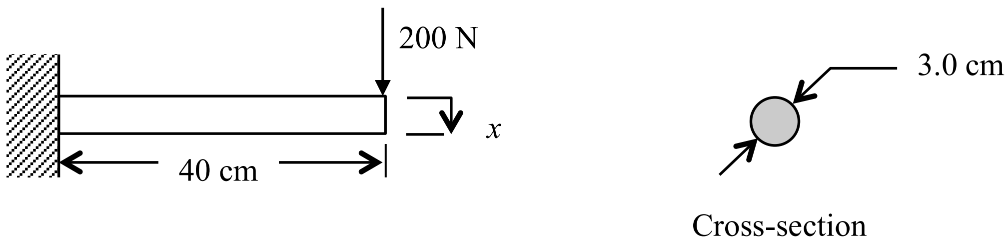
Homework 7

1 Problem

Solve Problem 7 in the Davies book (p. 74). Note that the area moment of inertia of circular cross-section is $J = \pi R^4/4$. Assume aluminimum has an elastic modulus of $E = 70 \times 10^9$ N/m^2

Solution

7. A solid aluminum cantilever beam is 40 cm in length and has a circular cross-section with a diameter of 3.0 cm. Calculate the lateral stiffness of the beam, k, in N/mm if subjected to a downward load as shown. Determine the displacement of the tip in mm when the load is 200 N.



Solution:

Step 1: Calculate the area moment of inertia.

$$J_A = \frac{\pi R^4}{4} = \frac{\pi\left(3.0\times 10^{-2} m\right)^4}{4} = 3.976\times 10^{-8} m^4$$

Step 2: Calculate the stiffness.

$$k = \frac{3EJ_A}{L^3} = \frac{3\left(70\times 10^9 \, N/m^2\right)\left(3.976\times 10^{-8} m^4\right)}{\left(40\times 10^{-2} m\right)^3} = 1.3046\times 10^5 N/m$$

$$k = 130.5 N/mm$$

Step 3: Calculate the deflection due to a 200 N load.

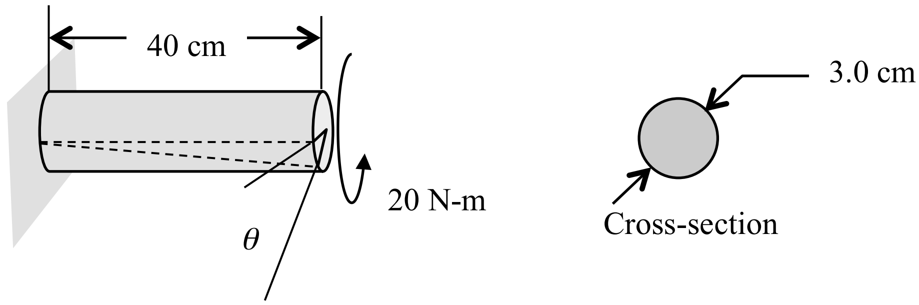
$$x = \frac{F}{k} = \frac{200N}{130.5N/mm} = 1.53mm$$

2 Problem

Solve Problem 8 in the Davies book (p. 75). Note that the polar moment of inertia of circular cross-section is $J = \pi R^4/2$. Assume a shear modulus of $G = 25.5 \times 10^9$ N/m^2

Solution

8. A solid aluminum cantilever beam is 40 cm in length and has a circular cross-section with a diameter of 3.0 cm. Calculate the torsional stiffness of the beam, k_r, in N-m/rad and the angular displacement at the end in deg when it is subjected to a moment of 20 N-m.



Solution:

Step 1: Calculate the area moment of inertia.

$$J_P = \frac{\pi R^4}{2} = \frac{\pi\left(3.0\times10^{-2}m\right)^4}{4} = 7.9522\times10^{-8}m^4$$

Step 2: Calculate the torsional stiffness.

$$k_r = \frac{GJ_P}{L} = \frac{\left(25.5\times10^9\,N/m^2\right)\left(7.9522\times10^{-8}m^4\right)}{40\times10^{-2}m} = 1.3046\times10^5 N/m$$

$$k_r = 5069.5N-m/rad$$

Step 3: Calculate the angular deflection due to a 20 N-m moment.

$$\theta = \frac{M}{k_r} = \frac{20N-m}{5069.5N-m/rad} = 3.9\times10^{-3}rad$$

$$\theta = \left(3.9\times10^{-3}rad\right)\left(\frac{180°}{\pi}\right) = 0.224°$$

3 Problem

An 80 kg boy jumps from rest on the ground onto the middle of slackline that is tied across two trees that are about 9 meters apart with an initial tension of $T = 1,000$ N.

- Using the effective spring table from Lecture 15, develop an equivalent mass-spring model of this system. That is, draw a free body diagram of the equivalent system and write down a differential equation that describes the dynamics of $y(t)$. Assume that $u(t)$ is a step force input equal to the weight of the boy and ignore any damping effects.
- Write down the solution the solution to the differential equation $y(t)$ assuming that $y(0) = \dot{y}(0) = 0$.
- After jumping on the slackline the boy bounces up and down. How many seconds does each up-and-down cycle last?

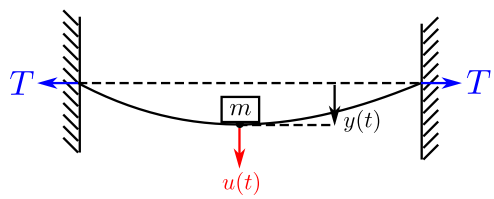


Figure 1: Left: Image Source: [Link], Right: Model

Solution

From the table in Lecture 15, the effective spring stiffness for a taut string is $k_{\text{eff}} = 4T/L$. There are two forces acting on the mass: the step input weight force $u(t) = mgH(t)$, and the spring force $-ky$ (where y is positive downard). Then, applying N2L,

$$\sum F = mgH(t) - k_{\text{eff}}y = m\ddot{y} \tag{1}$$
$$\ddot{y} + (k_{\text{eff}}/m)y = gH(t) \tag{2}$$

which is in the standard form of a damped harmonic oscillator with an inhomogeneous term (input on the RHS). Comparing to the DHO equation we see that $\zeta = 0$ and $\omega_n^2 = \frac{k_{\text{eff}}}{m}$. Thus, we may immediately conclude that each cycle will have a period of

$$T_{\text{period}} = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{k_{\text{eff}}/m}} = \frac{2\pi}{\sqrt{4T/mL}} = 2.6657 \text{ sec.}$$

Take the Laplace transform of the differential equation describing the dynamics:

$$s^2Y(s) + (k_{\text{eff}}/m)Y(s) = g\frac{1}{s} \tag{3}$$
$$Y(s) = g\frac{1}{s(s^2 + (k_{\text{eff}}/m))} \tag{4}$$

The denominator has a combination of real and complex poles, $p_1 = 0$ and $p_{2,3} = \pm\sqrt{k_{\text{eff}}/m)}i = \pm\omega i$ thus we consider the PFE of the form

$$Y(s) = g\frac{1}{s(s^2+\omega^2)} = \frac{c_1}{s} + c_2\frac{s}{s^2+\omega^2} + c_3\frac{\omega}{s^2+\omega^2} \tag{5}$$

The first coefficient is found by multiplying by s and evaluating both sides at $s = 0$ so that $c_1 = \frac{g}{\omega^2}$. Then

$$\begin{aligned} g\frac{1}{s(s^2+\omega^2)} &= \frac{g}{s\omega^2} + c_2\frac{s}{s^2+\omega^2} + c_3\frac{\omega}{s^2+\omega^2} && (6)\\ g\omega^2 &= g(s^2+\omega^2) + c_2 s^2\omega^2 + c_3 s\omega^3 && (7)\\ &= s^2(g + c_2\omega^2) + s(c_3\omega^3) + g\omega^2 && (8) \end{aligned}$$

which implies that $c_2 = -g/\omega^2$ and $c_3 = 0$. Thus, the PFE is

$$Y(s) = \frac{g}{w^2}\left[\frac{1}{s} - \frac{s}{s^2+\omega^2}\right] \tag{9}$$

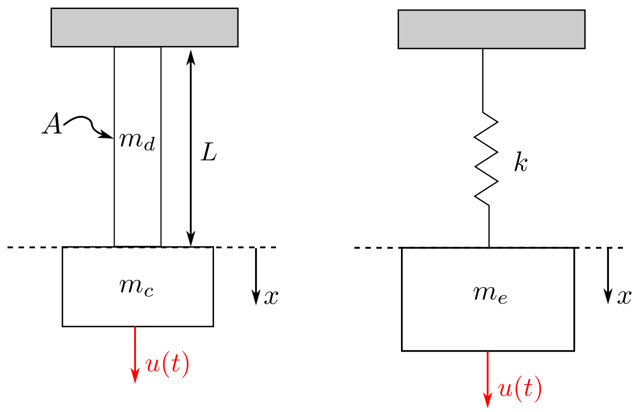
and taking the inverse Laplace

$$\begin{aligned} y(t) &= \frac{g}{w^2}[H(t) - \cos\omega t] && (10)\\ &= \frac{g}{(k_{\text{eff}}/m)^2}[1 - \cos\{(k_{\text{eff}}/m)t\}] && (11)\\ &= \frac{g}{[4T/(mL)]^2}[1 - \cos\{[4T/(mL)]t\}] && (12)\\ &= 2.3570[1 - \cos\{1.7658 \cdot t\}] && (13) \end{aligned}$$

Note: the model indicates that the maximum deflection would be about 4.7 meters. In practice, the slackline would be mounted lower to the ground and hit the ground. More tension would reduce the amplitude of the oscillation.

4 Problem

Consider an aluminum rod of length $L = 2$ m and cross-sectional area $A = 0.01$ m^2 (shown on the left). Attached to the rod tip is a concentrated mass $m_c = 10$ kg, and a force $u(t)$ is applied which causes a small amount of extension/compression of the rod. An equivalent lumped parameter model of the system is shown on the right. Assume that the density of aluminum is $\rho = 2710$ kg/m^3 and Young's modulus is $E = 68 \times 10^9$ Pa.



Using the lumped parameter tables from Lecture 15 to find:

- the equivalent lumped mass m_e
- the equivalent stiffness k
- using the above expressions determine the transfer function $G(s) = X(s)/U(s)$.

Solution

Problem : Answers may vary depending on ρ, E.

Let ρ be the density of aluminum.

Then $m_d = \rho \cdot \underbrace{A \cdot L}_{\text{volume}}$ is the distributed mass.

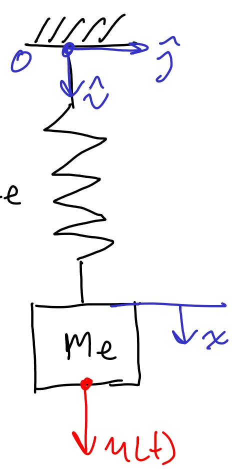
From equivalent mass table:

$$m_e = m_c + \frac{m_d}{3}$$

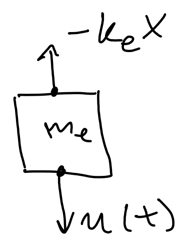
From equivalent stiffness table:

$$k_e = \frac{EA}{L}$$

The equivalent system is thus:



Free-body diagram



Newton's 2nd Law

$$u(t) - k_e x = m_e \ddot{x}$$

$$m_e \ddot{x} + k_e x = u(t).$$

Laplace transform

$$X(s)\left(s^2 + \frac{k_e}{m_e}\right) = (1/m_e)U(s)$$

Transfer function

$$\frac{X(s)}{U(s)} = \frac{1/m_e}{s^2 + \frac{k_e}{m_e}} = \frac{1}{m_e s^2 + k_e}$$

```
clear; close all; clc;

rho = 2710; % kg/m^3, density of aluminum
E = 68*1E9; % Pa, Young's modulus of aluminum
L = 2; % m, length
A = 0.01; %m^2, area
mc = 10; %kg, concentrated mass


% calculations
md = rho*A*L;
me = mc + md/3 % kg, equivalent mass
ke = E*A/L % N/m, equivalent spring stiffness

num = [1];
den = [me 0 ke];
G = tf(num,den)
```

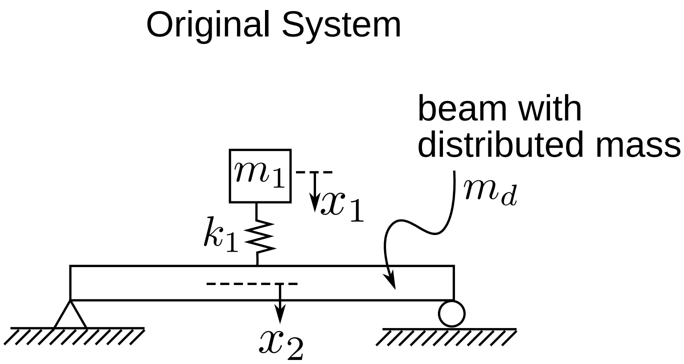
```
me =

   28.0667


ke =

   340000000


G =

          1
  ------------------
  28.07 s^2 + 3.4e08

Continuous-time transfer function.
```

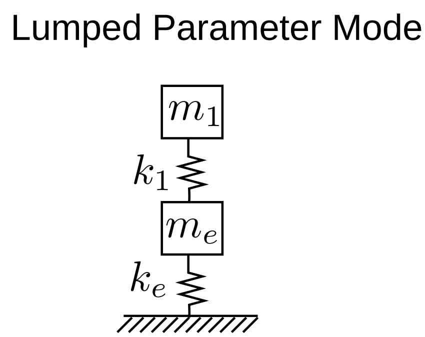
Published with MATLAB® R2021b

5 Problem

A vibrating machine of mass m_1 is mounted on a flexure with stiffness k_1 to a simply supported beam with elastic modulus E, inertia I and length L and modeled by the original system below. The beam has a distributed mass of m_d. The gravitational force with constant g acts on the element of this system. Using the lumped parameter tables from Lecture 15 to find:

- The equivalent lumped mass m_e of the beam (note: the machine with mass m_1 is connected by a spring to the beam and cannot be lumped together with m_d as a concentrated mass)
- The equivalent stiffness k_e
- Then, derive two ODEs (one for $\ddot{x}_1$ and one for $\ddot{x}_2$) that model the motion of the system. The equations should *only* be in terms of the original system variables: $m_1, k_1, x_1, m_d, x_2, g, E, I, L$.





Solution

From the tables in Lecture 15, the effective mass is $m_e = m_c + m_d/3 = m_d/3$ (since $m_c = 0$) and the effective spring constant is $k_e = 48EI/L^3$. There are only two forces acting on the first mass, the k_1 spring force and gravity:

$$\sum F_1 = k_1(x_2 - x_1) + m_1 g = m_1 \ddot{x}_1$$

and there are three forces acting on the second mass (both spring forces and gravity)

$$\sum F_2 = -k_1(x_2 - x_1) + k_e x_2 + m_e g = m_e \ddot{x}_2$$

or, in terms of the original problem variables,

$$\sum F_2 = -k_1(x_2 - x_1) + (48EI/L^3)x_2 + (m_d/3)g = (m_d/3)\ddot{x}_2$$