Lecture 13: Transfer Functions for Rotational Motion

Rotational systems are analogous to the translational systems we discussed in the previous lecture. The approach for obtaining the transfer function for simple (one degree-of-freedom) rotational systems is as follows:

1. **Derive the equations of motion** by drawing a free-body diagram and applying Newton's 2nd Law. In the case of rotational systems, the system's output/response is an angle $\theta(t)$ and we apply the rotational form of Newton's 2nd Law:

$$\sum \boldsymbol{M} = I_O \ddot{\theta}$$

where $\sum \boldsymbol{M}$ is the sum of moments around the rotation axis O, I_O is the inertia of the system around O, and $\ddot{\theta}$ is the angular acceleration. Recall that the inertia of a mass rotating around point O at a distance L is $I_O = mL^2$. For rotational systems we often encounter torsion springs which produce a restoring moment

$$\boldsymbol{M}_{\text{spring}} = -k_r(\theta - \theta_{\text{nominal}})$$

where θ_{nominal} is the nominal (unstretched) angle of the torsion spring (often taken to be zero) and rotational dampers that produce a moment proportional to the angular rate

$$\boldsymbol{M}_{\text{damper}} = -b_r \dot{\theta} \ .$$

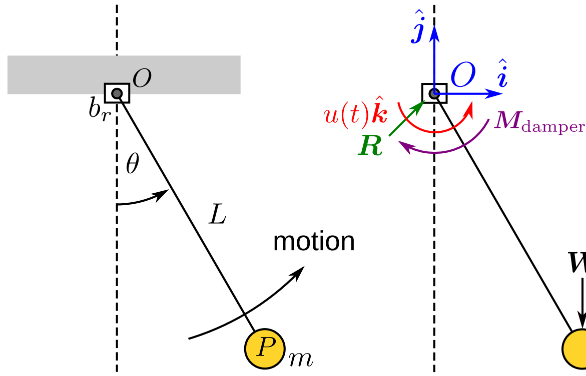
2. **Simplify using small angle assumptions**. After deriving an expression in terms of θ we often assume that the system oscillates with small angles around an equilibrium. This assumption implies that:

$$\sin\theta \approx \theta \qquad \text{and} \qquad \cos\theta \approx 1 \ .$$

With these substitutions we eliminate the nonlinear trigonometric terms so the system can be analyzed as a LTI ODE. These approximates are a result of a truncated Taylor series around $\theta = 0$ (see Lecture 5, page 7 for more details).

3. **Obtain the transfer function** by computing the Laplace transform assuming zero initial conditions $\theta(0) = \dot{\theta}(0) = 0$, keeping the system input unknown (i.e., transforming $u(t)$ as $U(s)$) and rearranging for $G(s) = \Theta(s)/U(s)$.

Example: Consider the torque-actuated damped pendulum shown below. The pendulum rotates around point O and has mass m concentrated at point P, a distance L away from the rotation point. The external torque applied is $u(t)$ and the damping coefficient is b_r.



The moment of inertia of the particle P about point O is

$$I_0 = mL^2 \tag{1}$$

From the free-body diagram, it is clear that the reaction force $\boldsymbol{R}$ creates no moment on the pendulum since its line of action passes through point O. There are three moments acting on the pendulum: the applied torque $u(t)\hat{\boldsymbol{k}}$, the moment due to gravity

$$\begin{aligned}
\boldsymbol{M}_{\text{gravity}} &= \boldsymbol{r}_{P/O} \times \boldsymbol{W} &(2)\\
&= (L\sin\theta\hat{\boldsymbol{i}} - L\cos\theta\hat{\boldsymbol{j}}) \times (mg(-\hat{\boldsymbol{j}})) &(3)\\
&= -L\sin\theta mg\hat{\boldsymbol{k}} &(4)
\end{aligned}$$

where $\boldsymbol{W}$ is the weight force and $\boldsymbol{r}_{P/O}$ is the position of point P with respect to point O, and the damping moment

$$\boldsymbol{M}_{\text{damper}} = -b_r\dot{\theta}\hat{\boldsymbol{k}}\ .$$

The rotational dynamics are found by applying the rotational form of Newton's 2nd Law for a pinned body

$$\sum \boldsymbol{M} = (u(t) - b_r\dot{\theta} - Lmg\sin\theta)\hat{\boldsymbol{k}} = I_O\ddot{\theta}\hat{\boldsymbol{k}} \tag{5}$$

Substiuting (1) the scalar form of this equation (in the $\hat{\boldsymbol{k}}$ direction) can then be re-written as

$$mL^2\ddot{\theta} + b_r\dot{\theta} + Lmg\sin\theta = u(t) \tag{6}$$

Assume small angles, $\sin\theta \approx \theta$, and divide by mL^2 so the system becomes

$$\ddot{\theta} + \left(\frac{b_r}{mL^2}\right)\dot{\theta} + \left(\frac{g}{L}\right)\theta = \frac{u(t)}{mL^2} \tag{7}$$

with initial conditions $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$. Notice that this equation is analogous to a damped harmonic oscillator with the coordinate $x = \theta$:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 = \frac{u(t)}{mL^2} \tag{8}$$

Comparing the two equations, it is clear that the pendulum's natural frequency is

$$\omega_n = \sqrt{\frac{g}{L}} \, .$$

This implies that the frequency of the oscillation depends only on the length of the pendulum (regardless of the mass). Also, we find that the damping ratio is

$$\frac{b_r}{mL^2} = 2\zeta\omega_n \implies \zeta = \frac{b_r}{2mL^2\omega_n} \tag{9}$$

and we can write

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{u(t)}{mL^2} \, . \tag{10}$$

Taking the Laplace transform (with zero initial conditions)

$$\Theta(s)(s^2 + 2\zeta\omega_n s + \omega_n^2) = (1/mL^2)U(s) \tag{11}$$

and rearranging, the transfer function is found to be

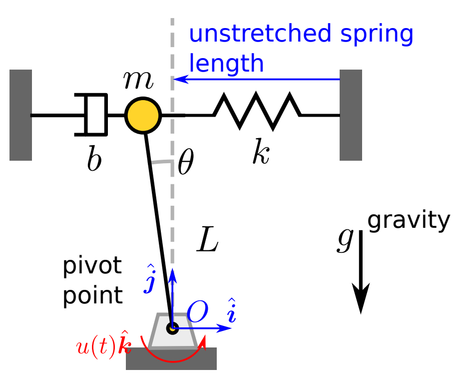
$$\implies G(s) = \frac{\Theta(s)}{U(s)} = \frac{(1/mL^2)}{(s^2 + 2\zeta\omega_n s + \omega_n^2)} \, . \tag{12}$$

Example: Consider an inverted torque-driven pendulum consisting of a mass m connected to a massless rigid rod of length L that rotates around point O (the mass moment of inertia is $I_0 = mL^2$). The mass is also connected to a damper and spring as shown. Gravity acts on the mass and the rod makes an angle θ with the vertical (θ is positive CCW, as shown below). A torque $u(t)$ is applied at the axis of rotation.

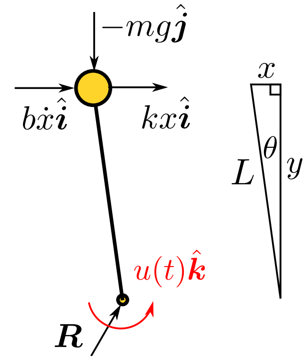
Find the second order differential equation describing the motion of the system using small angle approximations. Write your answer in standard form:

$$\ddot{\theta} + c_1\dot{\theta} + c_2\theta = g(t)$$

where c_1 and c_2 are constants. Include a free body diagram.



Begin by drawing a free body diagram. Three forces act on the mass-rod system: weight $\boldsymbol{W} = -mg\hat{\boldsymbol{j}}$, the spring force $\boldsymbol{F}_k = kx\hat{\boldsymbol{\imath}}$, and the damping force $\boldsymbol{F}_b = b\dot{x}\hat{\boldsymbol{\imath}}$, and the reaction force $\boldsymbol{R}$ at the pivot point.



Although this problem involves both rotation and translation, there is only one degree-of-freedom since the rotation and translation are directly related (e.g., we cannot rotate the arm without translating the mass). From the free-body diagram we sum the moments around point O (positive CCW):

$$\sum \boldsymbol{M} = (x(mg) - y(b\dot{x}) - y(kx) + u(t))\hat{\boldsymbol{k}}$$

Newton's 2nd Law (rotational form) is

$$\sum \boldsymbol{M} = I_0\ddot{\theta}\hat{\boldsymbol{k}}$$

and we can express the variables $x, \dot{x}, y$ that appear in the moment terms as functions of θ using the geometry:

$$x = L\sin\theta \qquad \text{and} \qquad y = L\cos\theta$$

and thus

$$\dot{x} = L\cos\theta\dot{\theta}\ .$$

Using small angle approximations $\sin\theta \approx \theta$ and $\cos\theta \approx 1$ it follows that

$$\begin{aligned} x &\approx L\theta \\ y &\approx L \\ \dot{x} &\approx L\dot{\theta} \end{aligned}$$

and then

$$\sum \boldsymbol{M} = (Lmg\theta - L^2b\dot{\theta} - L^2k\theta + u(t))\hat{\boldsymbol{k}} = I_0\ddot{\theta}\hat{\boldsymbol{k}}$$

Writing this is a scalar equation in the $\hat{\boldsymbol{k}}$ direction and rearranging:

$$\begin{aligned} mL^2\ddot{\theta} &= Lmg\theta - L^2b\dot{\theta} - L^2k\theta + u(t) \\ \implies \ddot{\theta} + \left(\frac{b}{m}\right)\dot{\theta} + \left(\frac{k}{m} - \frac{g}{L}\right)\theta &= \left(\frac{1}{mL^2}\right)u(t) \end{aligned}$$

The above form is in standard form so that the coefficients of the LHS can be used to compute parameters such as damping ratio and natural frequency. Bonus: We can also compute the transfer function. Taking the Laplace transform

$$s^2\Theta(s) + \left(\frac{b}{m}\right) s\Theta(s) + \left(\frac{k}{m} - \frac{g}{L}\right)\Theta(s) = \left(\frac{1}{mL^2}\right)U(s)$$
$$\Theta(s)\left[s^2 + \left(\frac{b}{m}\right) s + \left(\frac{k}{m} - \frac{g}{L}\right)\right] = \left(\frac{1}{mL^2}\right)U(s)$$

The transfer function is thus

$$\implies G(s) = \frac{\Theta(s)}{U(s)} = \frac{1}{mL^2\left[s^2 + \left(\frac{b}{m}\right)s + \left(\frac{k}{m} - \frac{g}{L}\right)\right]}$$
$$= \frac{1}{mLs^2 + L^2bs + (kL^2 - mgL)}$$

Rotational Drive Systems

Rotational drive systems have an applied moment or rotational motion with various elements (e.g., inertia, springs, dampers, etc.) distributed along the axis of rotation. The torques produced are all coincident along the same axis but the method of analysis is the same as above. Consider the following example.

Example (Davies, p. 146-147): Consider the following position drive system.

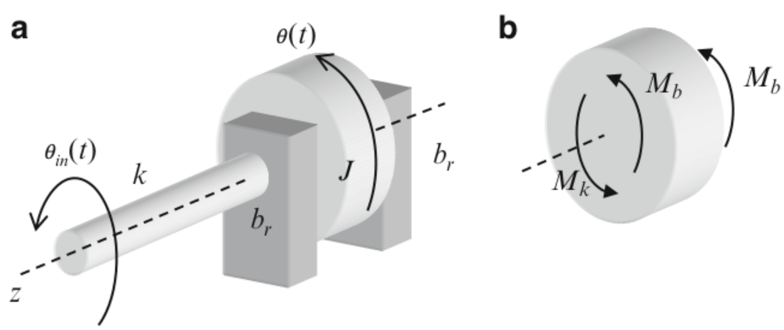


Fig. 5.6 (**a**) The inertia, J, is driven though a flexible input shaft with stiffness, k. The shaft is supported by two bearings, each with a rotational damping constant b_r; and (**b**) free-body diagram of J. The input angle is $\theta_{in}(t)$ and the angle of rotation for the inertia is $\theta(t)$

The system input is the angle of the outward facing tip of the flexible shaft $\theta_{\text{in}}(t)$. The shaft itself is flexible and modeled as a torsional spring with stiffness k_r. The angle $\theta(t)$ models the position of the rotor, and the rotor has inertia J (another common symbol used for inertia along with I). We assume that both angles are measured from the same reference point such that when $\theta_{\text{in}} = \theta$ the shaft is not twisted. Let the unit vector $\hat{\boldsymbol{k}}$ be aligned with the positive

z axis shown above. The moment from the spring is positive when $\theta_{\text{in}} > \theta$, so that

$$\boldsymbol{M}_{\text{spring}} = k_r(\theta_{\text{in}} - \theta)\hat{\boldsymbol{k}} \ .$$

The moment from the each damper depends only on the angular velocity of the rotor

$$\boldsymbol{M}_{\text{damper}} = -b_r(\dot{\theta})\hat{\boldsymbol{k}} \ .$$

Summing the moments and applying Newton's 2nd Law:

$$\begin{aligned} \sum \boldsymbol{M} = \boldsymbol{M}_{\text{spring}} - 2\boldsymbol{M}_{\text{damper}} &= J\ddot{\theta}\hat{\boldsymbol{k}} \\ k_r(\theta_{\text{in}} - \theta)\hat{\boldsymbol{k}} - 2b_r(\dot{\theta})\hat{\boldsymbol{k}} &= J\ddot{\theta}\hat{\boldsymbol{k}} \end{aligned}$$

Proceeding with the scalar form of the equation and rearranging, the equations of motion are

$$J\ddot{\theta} = 2b_r\dot{\theta} + k_r(\theta_{\text{in}} - \theta) \tag{13}$$

$$\ddot{\theta} + \left(\frac{2b_r}{J}\right)\dot{\theta} + \left(\frac{k_r}{J}\right)\theta = \frac{k_r}{J}\theta_{\text{in}} \tag{14}$$

Can write this system using the notation for a damped harmonic oscillator

$$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \frac{k_r}{J}\theta_{\text{in}} \tag{15}$$

where $\omega_n = \sqrt{k_r/J}$ and $2\zeta\omega_n = 2_b r/J$ implies that $\zeta = b_r/(J\omega_n)$. The transfer function is

$$\frac{\Theta(s)}{\Theta_{\text{in}}(s)} = \frac{k_r/J}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{16}$$

References

- Davies and Schmitz, Chapter 5