MEGR 3122 Dynamic Systems II: Formula Sheet

1st Order Differential Equations:

Homogeneous $\dot{x}(t) + ax(t) = 0, \quad x(t_0) = x_0$

$$\implies x(t) = x_0 e^{-at}$$

Time Constant $\tau = 1/a$

$$x(\tau) \approx 0.368x_0, \quad x(2\tau) \approx 0.135x_0$$
$$x(3\tau) \approx 0.050x_0, \quad x(4\tau) \approx 0.018x_0$$

Inhomogeneous $\dot{x}(t) + ax(t) = u(t), \quad x(t_0) = x_0$

$$\implies x(t) = e^{-at}\int e^{at}u(t)dt + Ce^{-at}$$

2nd Order Differential Equations:

Homogeneous $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0$

Characteristic Equation $\lambda^2 + a\lambda + b = 0$

Quadratic Formula $\lambda_{1,2} = (-a \pm \sqrt{a^2 - 4b})/2$

Case I: $a^2 - 4b > 0$ **(real, distinct eigenvalues)**

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Case II: $a^2 - 4b = 0$ **(repeated eigenvalues)**

$$x(t) = C_1 e^{-at/2} + C_2 t e^{-at/2}$$

Case III: $a^2 - 4b < 0$ **(complex conjugate eigenvalues)**

$$x(t) = e^{-at/2}(A\cos\omega t + B\sin\omega t)$$

where ω is the frequency:

$$\omega = \frac{\sqrt{|a^2 - 4b|}}{2}$$

Laplace Transform:

Definition

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Properties

$$\mathcal{L}[\alpha f(t)] = \alpha\mathcal{L}[f(t)] \qquad \text{(scalar multiplication)}$$
$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] \qquad \text{(addition)}$$
$$\mathcal{L}[f(t-\alpha)H(t-\alpha)] = e^{-s\alpha}F(s) \qquad \text{(translated function)}$$
$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{(initial value)}$$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{(final value)}$$
$$\mathcal{L}[\dot{f}(t)] = sF(s) - f(0) \qquad \text{(differentation)}$$
$$\mathcal{L}[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$$

Table of Common Laplace Transforms

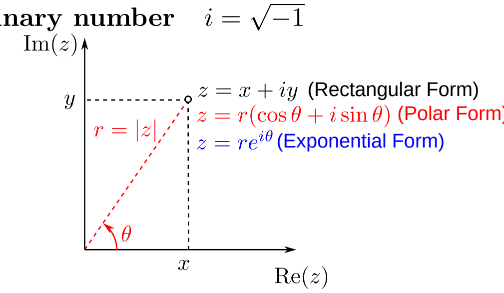
Row	$f(t)$	$F(s)$
1	Unit impulse, $\delta(t)$	1
2	Unit step/Heaviside, $H(t)$	$\frac{1}{s}$
3	Ramp, t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{(s+a)}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left(1 + \frac{1}{a-b}(be^{-bt} - ae^{-at})\right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse Laplace Transform

$$F(s) = \mathcal{L}(f(t)) \implies f(t) = \mathcal{L}^{-1}[F(s)]$$

Complex Numbers:

Imaginary number $i = \sqrt{-1}$



Modulus $r = |z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$
Argument $\theta = \angle z = \arg(z) = \text{atan}(\text{Im}(z)/\text{Re}(z))$
Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$

Properties

For $z = (x + iy)$ and $w = (u + iv)$

$$\begin{aligned} \bar{z} &= x - iy \\ z \cdot w &= (xu - yv) + i(xv + yu) \\ |z|^2 &= z \cdot \bar{z} \\ \frac{z}{w} &= \frac{z}{w}\left(\frac{\bar{w}}{\bar{w}}\right) \end{aligned}$$

For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$\begin{aligned} \bar{z} &= re^{-i\theta} \\ z_1 \cdot z_2 &= r_1 r_2 e^{i(\theta_1+\theta_2)} \\ z_1/z_2 &= r_1/r_2 e^{i(\theta_1-\theta_2)} \end{aligned}$$

Partial Fraction Expansion:

Polynomial form

$$F(s) = \frac{Q(s)}{R(s)} = \frac{d_m s^m + d_{m-1}s^{m-1} + \cdots + d_1 s + d_0}{c_n s^n + c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}$$

Zero-pole-gain form

$$F(s) = \frac{A(s)}{B(s)} = \frac{k(s - z_1)(s - z_1)\cdots(s - z_m)}{(s - p_1)(s - p_2)\cdots(s - p_n)} \ ,$$

PF Form: Case I (Distinct, Real Poles)

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1}{s - p_1} + \cdots + \frac{a_k}{s - p_k} + \cdots + \frac{a_n}{s - p_n} \ ,$$

with coefficients solved by

$$a_k = \left.\frac{A(s)(s - p_k)}{B(s)}\right|_{s=p_k}$$

PF Form: Case II (Repeated Real Poles).

$$F(s) = \frac{A(s)}{(s-p)^n} = \frac{a_1}{(s-p)} + \cdots + \frac{a_k}{(s-p)^k} + \cdots + \frac{a_n}{(s-p)^n}$$

with coefficients by multiplying by $(s - p)^n$

$$F(s)(s-p)^n = a_1(s-p)^{n-1} + \cdots + a_k(s-p)^{n-k} + \cdots + a_n$$

equating coefficients, and solving the system of equations.

PF Form: Case III (Complex Poles). $p_{1,2} = -\alpha \pm i\omega$

$$F(s) = a_1 \frac{\omega}{(s+\alpha)^2 + \omega^2} + a_2 \frac{(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

with coefficients solved by writing

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1\omega + a_2(s+\alpha)}{(s+\alpha)^2 + \omega^2}$$

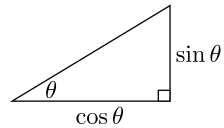
equating coefficients of numerator above and solving the system of equations.

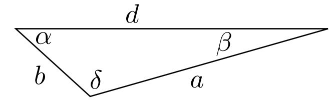
Trigonometry:

Identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\quad \sin^2\theta + \cos^2\theta = 1$

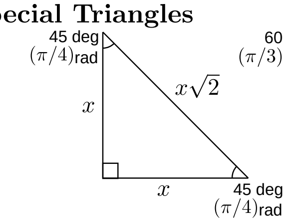
Sine Law $\sin\alpha/a = \sin\beta/b = \sin\delta/d$

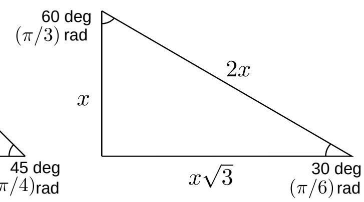
Cosine Law $d^2 = a^2 + b^2 - 2ab\cos\delta$





Special Triangles





Product to sum

$$\begin{aligned} \sin x \sin y &= \frac{1}{2}\big[\cos(x-y) - \cos(x+y)\big] \\ \cos x \cos y &= \frac{1}{2}\big[\cos(x-y) + \cos(x+y)\big] \\ \sin x \cos y &= \frac{1}{2}\big[\sin(x+y) + \sin(x-y)\big] \end{aligned}$$

Half angles

$$\begin{aligned} \sin\frac{x}{2} &= \pm\sqrt{\frac{1-\cos x}{2}} \\ \cos\frac{x}{2} &= \pm\sqrt{\frac{1+\cos x}{2}} \\ \tan\frac{x}{2} &= \frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x} \end{aligned}$$

Sum to product

$$\begin{aligned} \sin x + \sin y &= 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \sin x - \sin y &= 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \\ \cos x + \cos y &= 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) \\ \cos x - \cos y &= -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right) \end{aligned}$$

Sum and difference

$$\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x \sin y \\ \sin(x-y) &= \sin x\cos y - \cos x \sin y \\ \cos(x+y) &= \cos x\cos y - \sin x \sin y \\ \cos(x-y) &= \cos x\cos y + \sin x \sin y \\ \tan(x+y) &= \frac{\tan x + \tan y}{1 - \tan x \tan y} \\ \tan(x-y) &= \frac{\tan x - \tan y}{1 + \tan x \tan y} \end{aligned}$$