Lecture 15: Electrical Systems

Electrical systems, just like mechanical and thermal systems, can be modeled as LTI differential equations and analyzed using Laplace transforms. A commonly used analogy compares electrical circuits to fluid circuits, as shown below. Imagine a pump that does work (adds energy into the

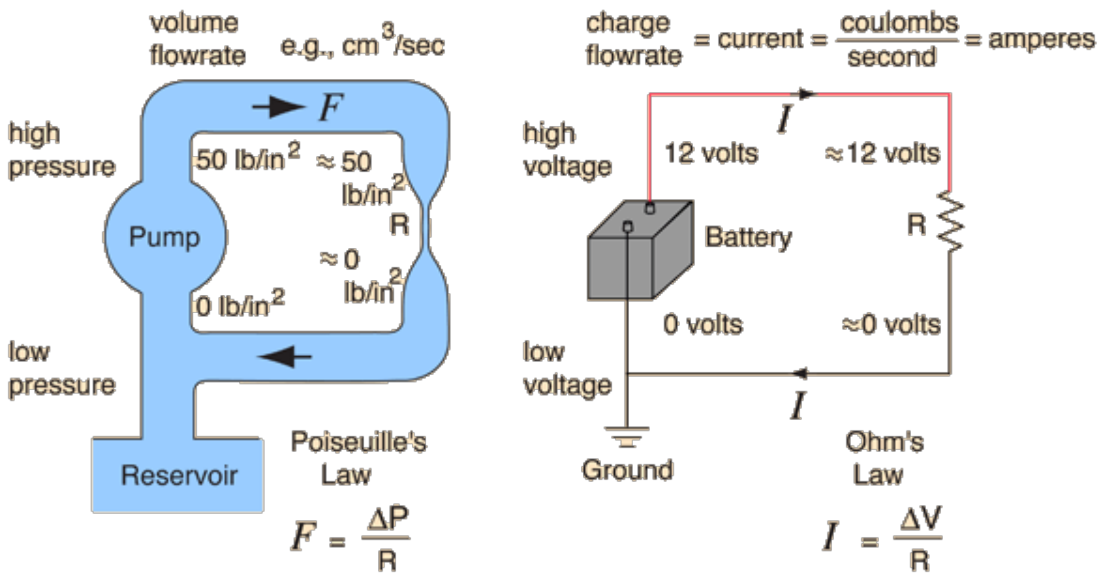


Figure 1: Image source: Carl Nave, Hyperphiscs, Georgia State

system) to increase the pressure of a fluid and push it through a pipe. The diameter of the pipe determines how much water can flow and the restriction in the diameter creates *impedance* which allows the pressure to build. At the other side of the restriction the pressure equalizes to zero and the flow exits the pipe to enter a large reservoir. Instead of a pump we could also imagine that the fluid is flowing from an elevated point, with large potential energy, like the top of a mountain or a water tower. Similarly, in electrical circuits the battery is a source of *electric potential energy* (in units of Volts) and positive charge (or *current*) is "flowing" instead of water. An electrical component can cause impedance that causes the voltage to drop. As the charged flow completes the electric circuit it ends up with the same zero potential energy as the ground.

Aside: Engineers like to visualize the flow of charge from the positive terminal of the battery (+) to the negative terminal (-) and ground. That is, we draw arrows indicating motion of *positive charge flow*. In reality, the only charges moving in a circuit are actually the electrons. The electrons are small particles that move from one atom to the next while the protons stay stationary. Thus, the actual electron flow is opposite in direction to the positive charge flow indicated by arrows in circuit diagrams. For more details see here.

In the above description, voltage is used to describe the potential energy per unit charge and is defined according to:

$$1 \text{ Volt} = \frac{1 \text{ Joule}}{\text{Coulomb}} \tag{1}$$

One coulomb is equal to the charge carried by a very large number of protons (to be precise, 6.241509 $\times 10^{18}$ protons). As an example, a standard AA battery holds about 5000 Coulombs of charge. Let $i(t)$ denote the time-varying current in a circuit and $q(t)$ denote the amount of charge in a circuit. The current is defined as the rate of charge flow, that is,

$$i(t) = \frac{d}{dt}q(t) \tag{2}$$

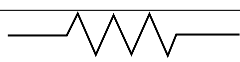
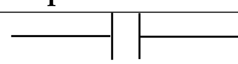
$$\text{Units: Amps} = \frac{\text{Coulombs}}{\text{second}} \tag{3}$$

Taking the Laplace transform above we have the relationship:

$$I(s) = sQ(s) \tag{4}$$

Passive Electrical Elements

Passive circuit elements do not add external energy into the system. The three common passive electrical elements in an electric circuit are the resistor, capacitor, and the inductor.

	Resistor	**Capacitor**	**Inductor**
Diagram			
Symbol	R	C	L
Units	Ω, Ohm (Volts/Amp)	F, Farad (Amp-second/Volt)	H , Henries (Volt-second/Amp)
Impedance	$Z_{\text{resistor}} = R$	$Z_{\text{inductor}} = Ls$	$Z_{\text{capacitor}} = 1/(Cs)$

Impedance When analyzing electric circuits we will use the concept of *impedence* which is the ratio of voltage $V(s)$ to current $I(s)$. In the Laplace domain

$$Z(s) = \frac{V(s)}{I(s)} \tag{5}$$

Each passive element has a different impedance that we will describe below. Note that some textbooks use $e(t)$ for voltage in the time domain.

Resistor A resistor impedes the flow of current when a voltage is applied across it and dissipates energy as heat. A simple resistor is any conductive material. Wires themselves have resistance but it is often negligible. The value of resistance, measured in Ohms, is determined by:

$$R = \frac{\rho_c L}{A} \tag{6}$$

where ρ_c is the charge density, L is the length, and A is the cross-sectional area of the conductor.

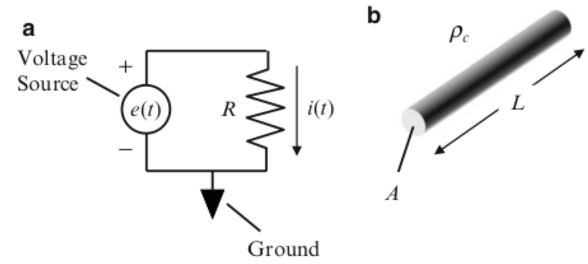


Figure 2: Image Source: Davies and Schmitz, "System Dynamics for Mechanical Engineering", Springer, 2015.

Resistors are commonly used to reduce the current voltage supplied to other circuit elements so that they work properly. Current that flows through a resistor is proportional to the voltage applied across it according to *Ohm's Law*:

$$v(t) = i(t)R \tag{7}$$

Taking the Laplace transform

$$V(s) = I(s)R \tag{8}$$

and thus the impedance for a resistor is

$$Z_{\text{resistor}} = \frac{V(s)}{I(s)} = R \tag{9}$$

Inductor An inductor is a device that stores energy in the form of a magnetic field. A simple example is a coil of conductive wire wrapped around a ferromagnetic core. When a current runs through the coil a magnetic field develops (this basic idea is used in electromagnets and solenoids). If the current through the coil changes, then the magnetic field must change also.

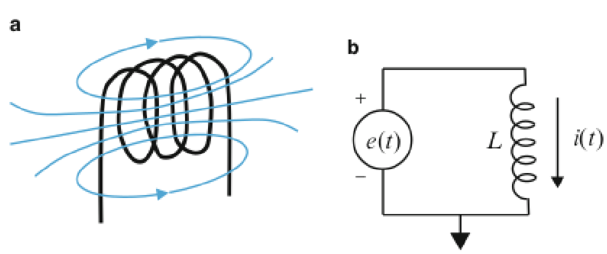


Figure 3: Image Source: Davies and Schmitz, "System Dynamics for Mechanical Engineering", Springer, 2015.

However, this changing magnetic field introduces a *back voltage* that resists the change. *Faraday's Law* relates the voltage across an inductor to the change in current flow

$$v(t) = L\frac{di}{dt} \tag{10}$$

The unit of inductance is the Henry and we see from the above equation that L must have units of voltage per charge-rate, i.e.,

$$\text{Henry} = \frac{\text{Volt}}{\text{Amps/sec}} = \frac{\text{Volt-seconds}}{\text{Amp}} \tag{11}$$

Taking the Laplace transform

$$V(s) = LsI(s) \tag{12}$$

Rearranging gives the impedance of an inductor

$$Z_{\text{inductor}} = \frac{V(s)}{I(s)} = Ls \tag{13}$$

Capacitor A capacitor stores energy (i.e., a charge) in the form of an electric field. When a voltage is applied across two plates that are a distance d apart the charge of $+q$ develops on one plate and a charge of $-q$ develops on the other plate. The capacitance is determined by

$$C = \frac{\epsilon A}{d} \tag{14}$$

where A is the area of the plate and ϵ is the electrical permittivity. The amount of charge on the

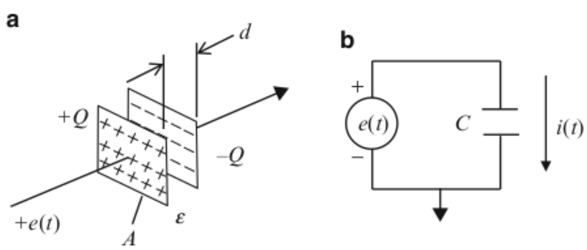


Figure 4: Image Source: Davies and Schmitz, "System Dynamics for Mechanical Engineering", Springer, 2015.

capacitor plates $q(t)$ is directly proportional to the voltage $v(t)$:

$$q(t) = Cv(t) \tag{15}$$

The unit of inductance is the Farad and we see from the above equation that C must have units of charge per volt, i.e.,

$$\text{Farad} = \frac{\text{Coloumb}}{\text{Volt}} = \frac{\text{Amp-seconds}}{\text{Volt}} \tag{16}$$

where we've made use of the definition of an Ampere (3) above. Taking the derivative we have

$$\frac{d}{dt}q(t) = C\frac{d}{dt}v(t) \tag{17}$$

$$i(t) = C\frac{dv(t)}{dt} \tag{18}$$

Then the Laplace transform is

$$I(s) = CsV(s) \tag{19}$$

Rearranging, the impedance for a capacitor is

$$Z_{\text{capacitor}} = \frac{V(s)}{I(s)} = \frac{1}{C \cdot s} \tag{20}$$

An uncharged capacitor has no impedance but as it charges the impedance builds and it becomes harder to add more charged particles to the plate.

Mass-spring-damper analogy. From the above discussion we see that an electrical system is in many ways like a mass-spring-damper mechanical system:

- a mass and spring store kinetic and potential energy, whereas in an electrical circuit the analogous components are a capacitor (stores energy in electric field) and inductor (stores energy in a magnetic field)
- a damper dissipates mechanical energy as heat, whereas a resistor dissipates electrical energy as heat
- a mass resists sudden changes in velocity, while an inductor resists sudden changes in electric current (acting as an "electrical inertia")
- when charge builds up on a capacitor it resists further charge build-up, similar to how a spring force increases in proportion to its displacement

Impedance in Parallel and Series

Similar to how springs and dampers can be lumped in series and parallel, the impedances of various passive components (resistors, capacitors, inductors) can be lumped together as well. For impedances Z_1 and Z_2 arranged in parallel, the equivalent impedance is:

$$Z_{\text{eq}} = \frac{Z_1 Z_2}{Z_1 + Z_2} = \frac{1}{Z_1} + \frac{1}{Z_2} \tag{21}$$

or for N components in parallel:

$$Z_{\text{eq}} = \left[\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N}\right]^{-1} \tag{22}$$

For impedances Z_1 and Z_2 arranged in series, the equivalent impedance is:

$$Z_{\text{eq}} = Z_1 + Z_2 \tag{23}$$

or for N components in series:

$$Z_{\text{eq}} = Z_1 + Z_2 + \cdots + Z_N \tag{24}$$

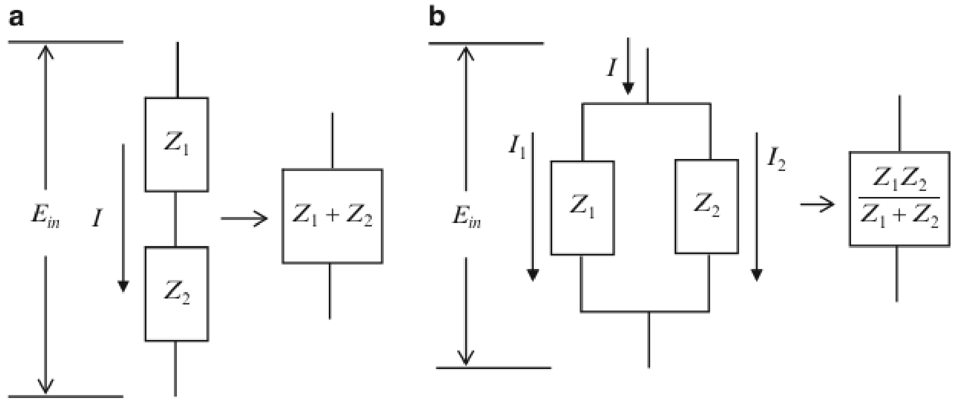


Figure 5: Image Source: Davies and Schmitz, "System Dynamics for Mechanical Engineering", Springer, 2015.

Kirchoff's Laws

Kirchoff's voltage law (KVL) and Kirchoff's current law (KCL) are fundamental equations in circuit modeling. If we consider a *node* in a circuit (i.e., any junction where two or more wires flow into or out of the node) the KCL states that the total sum of the currents must remain constant. In other words, the current in must equal the current out. Suppose that for a particular node there are k incoming currents $\{i_1, i_2, \ldots i_k\}$ and n outgoing currents $\{i_{k+1}, i_{k+2}, \ldots, i_{k+n}\}$. Kirchoff's Current Law is then

$$\sum_{j=1}^{n} i_j = \sum_{j=k+1}^{k+n} i_j \tag{25}$$

If we use a sign convention to denote incoming and outgoing current then KCL can be restated as

$$\sum_{j=1}^{k+n} i_j = 0 \tag{26}$$

KVL states that the sum of voltages across the impedance elements of a closed circuit are zero. For a simple circuit with just one battery, the battery is assumed to have a positive voltage V_{in} and each impedance element has a negative voltage drop $\{-V_1, \ldots, -V_m\}$. KVL implies that the voltage drops the entire circuit must equal the supply voltage

$$V_{in} = V_1 + V_2 + \cdots + V_m \tag{27}$$

Impedance Method: Total Circuit Current

Let $Z_{\text{circuit}}(s)$ denote the total equivalent impedance obtained after simplifying circuit elements using the parallel/series laws described above. From the definition of impedance (5):

$$Z_{\text{circuit}}(s) = \frac{V_{\text{in}}(s)}{I(s)} \tag{28}$$

and the transfer function from input voltage to the circuit current can be obtained according to:

$$G(s) = \frac{1}{Z_{\text{circuit}}(s)} = \frac{I(s)}{V_{\text{in}}(s)} = \frac{\text{Output (current)}}{\text{Input (Voltage)}} \tag{29}$$

The reciprocal of impedance above is called the *admittance*. With this transfer function the current that develops in a circuit can be simulated in response to an applied voltage. Remember that transfer functions assume zero initial conditions, so the response to an input voltage illustrates the transient as the circuit turns on. If the input voltage V_O is suddenly applied as a step input then $v_{in}(t) = V_0 H(t)$ and the Laplace transform is $V_{\text{in}}(s) = V_0/s$. The current response can be found as:

$$i(t) = \mathcal{L}^{-1}[G(s)V_{\text{in}}(s)]$$

Impedance Method: Component Voltage

Suppose that a circuit can be reduced to a single loop with several (possibly lumped) impedance elements $Z_1(s), Z_2(s), \ldots, Z_n(s)$ along the loop. From KCL, the current through each impedance element is equal. Thus, again using the definition (5):

$$V_1(s) = Z_1(s)I(s) \tag{30}$$

$$V_2(s) = Z_2(s)I(s) \tag{31}$$

$$\vdots \tag{32}$$

$$V_n(s) = Z_n(s)I(s) \tag{33}$$

If the total circuit impedance from (28) has been computed then

$$V_{\text{in}}(s) = Z_{\text{circuit}}(s)I(s) \tag{34}$$

A transfer function can be found that describes the voltage across a particular component i as a function of the input voltage $V_{\text{in}}(s)$ by simply dividing the two expressions to cancel the current $I(s)$:

$$G(s) = \frac{V_i(s)}{V_{\text{in}}(s)} = \frac{Z_i(s)I(s)}{Z_{\text{circuit}}(s)I(s)} = \frac{Z_i(s)}{Z_{\text{circuit}}(s)} \tag{35}$$

Examples

Several examples were covered in class.

References

7.1–7.4 then Chapter 7.6: Impedance Methods