Name: ____________________________________

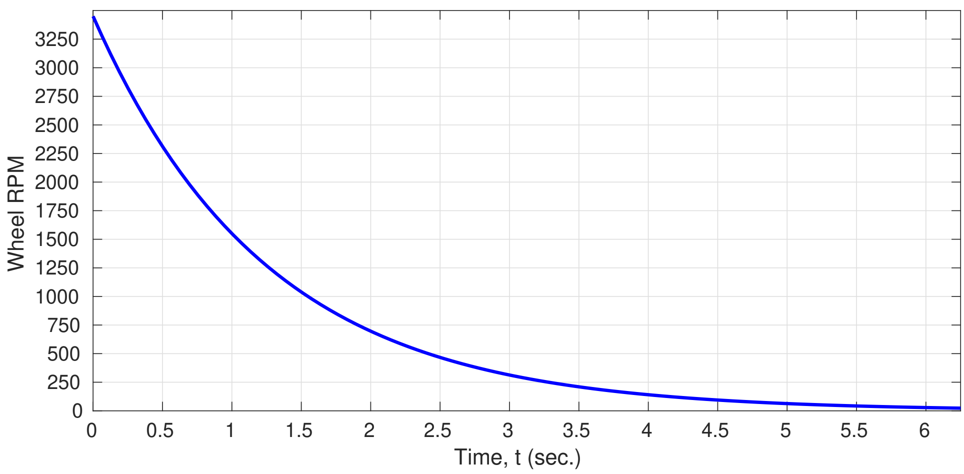
MEGR 3122 Dynamics Systems II: Exam 1, Spring 2023

Directions: Circle the best answer. Show your work to receive full credit.

1. (2 points) A bench grinder is spinning at a rate of $\omega_0 = 3{,}450$ RPM when it is turned off and the wheel coasts to a stop. The wheel velocity is modeled according to:

$$\dot{\omega} + b\omega = 0,\ \omega(t_0) = \omega_0,\ t_0 = 0$$

where b is a damping coefficient. A plot of the angular velocity measured with a tachometer is shown below. What is a reasonable estimate for the value of b?



A. 1/3
B. 0.8
C. 1.4
D. 3.1
E. 6.3

Solution (B). Reading the plot we see that the RPM decays to $0.368\omega_0 \approx 1,269$ at around 1.2 seconds which implies that $\tau = 1.2$ and thus $a = 1/1.2 \approx 0.8$.

2. (4 points) What is the imaginary part of the quantity below?

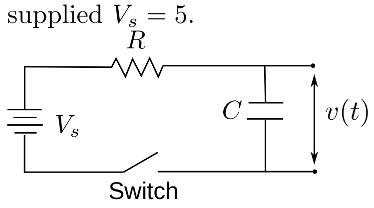
$$z = \frac{i-4}{2i-3} \cdot e^{i\pi/2}$$

A. 14/13
B. 5/13
C. 5/14
D. $5/(2\pi)$
E. $-5/13$

Solution (A). Note that $e^{i\pi/2} = i$

$$\begin{aligned} z &= \frac{i-4}{2i-3}\frac{2i+3}{2i+3} \cdot i && (1)\\ &= \frac{i(2i+3)-4(2i+3)}{-4-9} \cdot i && (2)\\ &= \frac{-2+3i-8i-12}{-13} \cdot i && (3)\\ &= \frac{-14i+5}{-13} && (4)\\ &= -\frac{5}{13} + \frac{14}{13}i && (5) \end{aligned}$$

3. (4 points) The RC circuit shown below has a resistor $R = 0.5$ and a capacitor $C = 2$ and voltage supplied $V_s = 5$.



The equation modeling the system is

$$RC\frac{dv(t)}{dt} + v(t) = V_s$$

where $v(t)$ is the voltage measured at the output across the capacitor and V_s is a constant voltage supplied by a battery. What is the value of $v(t)$ at one second after the switch is closed? Assume the initial output voltage is $v(t_0) = 0$. (Hint: re-write the above equation in more familair notation.)
A. 0.47 V
B. 1.66 V
C. 2.30 V
D. 3.16 V
E. 5.00 V

Solution (D). First, we write this equation in more familiar notation. Divide through by RC and let $a = 1/(RC)$ and $u = V_s/RC$:

$$\dot{v} + av = u$$

For a constant u the general solution is:

$$\begin{aligned} x(t) &= ue^{-at}\int e^{at}dt + Ce^{-at} && (6)\\ &= ue^{-at}[(1/a)e^{at}] + Ce^{-at} && (7)\\ &= (u/a) + Ce^{-at} && (8)\end{aligned}$$

and with the initial condition

$$x(0) = 0 = (u/a) + C$$

implies that

$$\begin{aligned} x(t) &= (u/a)(1 - e^{-at}) && (9)\\ &= ([V_s/RC]/[1/RC])(1 - e^{-at}) && (10)\\ &= V_s(1 - e^{-(1/(RC))t}) && (11)\end{aligned}$$

Evaluating with the values provided gives $v(t) = 3.16$ V at $t = 1$.

4. (4 points) The general solution of a second order ODE with initial conditions $x(0) = 1$ and $\dot{x}(0) = 3$ is found to be $x(t) = e^{-t}(c_1 \cos 2t + c_2 \sin 2t)$. What is the particular solution?

A. $x(t) = e^{-t}(6\cos 2t + 4\sin 2t)$
B. $x(t) = e^{-t}(\cos 2t + 4\sin 2t)$
C. $x(t) = e^{-t}(\cos 2t + 2\sin 2t)$
D. $x(t) = 2e^{-t}(\cos 2t + \sin 2t)$
E. $x(t) = 6e^{-t}\sin 2t$

Solution (C). Using the first initial condition

$$\begin{aligned} x(0) = 1 &= e^0(c_1 \cos 0 + c_2 \sin 0) && (12)\\ \implies c_1 &= 1 && (13)\end{aligned}$$

Differentiating the general solution,

$$\dot{x}(t) = -e^{-t}(c_1 \cos 2t + c_2 \sin 2t) + e^{-t}(c_1[-2\sin 2t] + c_2[2\cos 2t]) \quad (14)$$

and applying the second initial condition

$$\begin{aligned} \dot{x}(0) = 3 &= -c_1 + 2c_2 && (15)\\ \implies c_2 &= 2 && (16)\end{aligned}$$

Thus,

$$x(t) = e^{-t}(\cos 2t + 2\sin 2t)$$

5. (4 points) What is the partial fraction expansion of the Laplace transform of $\ddot{x} + 4\dot{x} + 5x = 0$ with $x(0) = 1$ and $\dot{x}(0) = -1$?

   A. $X(s) = \frac{(s+2)}{(s+2)^2+1}$

   B. $X(s) = \frac{s+3}{s^2+4s+6}$

   C. $X(s) = \frac{1}{(s+2)^2+1} - \frac{2(s+1)}{(s+1)^2+2}$

   D. $X(s) = \frac{1}{(s+2)^2+1} - \frac{(s+2)}{(s+2)^2+1}$

   E. $X(s) = \frac{1}{(s+2)^2+1} + \frac{(s+2)}{(s+2)^2+1}$

Solution (E). Taking the Laplace transform

$$\mathcal{L}[\ddot{x}] + 4\mathcal{L}[\dot{x}] + 5\mathcal{L}[x] = 0 \tag{17}$$

$$[s^2X(s) - sx(0) - \dot{x}(0)] + 4[sX(s) - x(0)] + 5X(s) = 0 \tag{18}$$

$$[s^2X(s) - s + 1] + 4[sX(s) - 1] + 5X(s) = 0 \tag{19}$$

$$s^2X(s) - s + 1 + 4sX(s) - 4 + 5X(s) = 0 \tag{20}$$

$$s^2X(s) + 4sX(s) + 5X(s) = s + 3 \tag{21}$$

$$X(s) = \frac{s+3}{s^2+4s+5} \tag{22}$$

The poles are $p_{1,2} = (-4 \pm \sqrt{16-20}/2 = -2 \pm i$ and therefore the PFE is of the form

$$X(s) = c_1\frac{1}{(s+2)^2+1} + c_2\frac{(s+2)}{(s+2)^2+1} \tag{23}$$

Equating numerators,

$$s + 3 = c_1 + c_2(s+2) \tag{24}$$

$$= c_2 s + (c_1 + 2c_2) \tag{25}$$

which implies that $c_1 = 1$ and $c_2 = 1$ so

$$X(s) = \frac{1}{(s+2)^2+1} + \frac{(s+2)}{(s+2)^2+1} \tag{26}$$

6. (4 points) What is the initial value of $x(t)$ if the Laplace transform of $x(t)$ is the following?

$$X(s) = \frac{5s^2 + 2s + 7}{4s^3 + 3s^2 + 2s}$$

   A. $x(0) = 7/3$
   B. $x(0) = 5/4$
   C. $x(0) = 2/3$
   D. $x(0) = 7/9$
   E. $x(0) = 4/5$

Solution (B). Using the initial value theorem

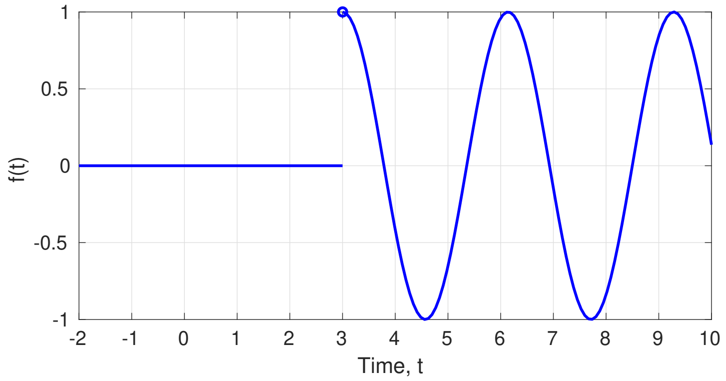
$$\lim_{t \to 0} x(t) = \lim_{s \to \infty} s \left(\frac{5s^2 + 2s + 7}{4s^3 + 3s^2 + 2s} \right) \tag{27}$$

$$= \lim_{s \to \infty} s \left(\frac{5s^2 + 2s + 7}{4s^2 + 3s + 2} \right) \tag{28}$$

$$= \lim_{s \to \infty} s \left(\frac{5 + 2/s + 7/s^2}{4 + 3/s + 2/s^2} \right) \tag{29}$$

$$= 5/4 = 1.25 \tag{30}$$

7. (4 points) The input signal $f(t)$ below is applied to a dynamic system starting at $t = 3$ seconds. The signal has a frequency $\omega = 2$ rad/s. What is the Laplace transform $F(s)$?



A. $e^{-s} \frac{(s+3)}{(s+3)^2+4}$

B. $e^{-(s-3)} \frac{(s-3)}{(s-3)^2+4}$

C. $e^{-3s} \frac{s}{s^2+4}$

D. $\frac{1}{s^2+4}$

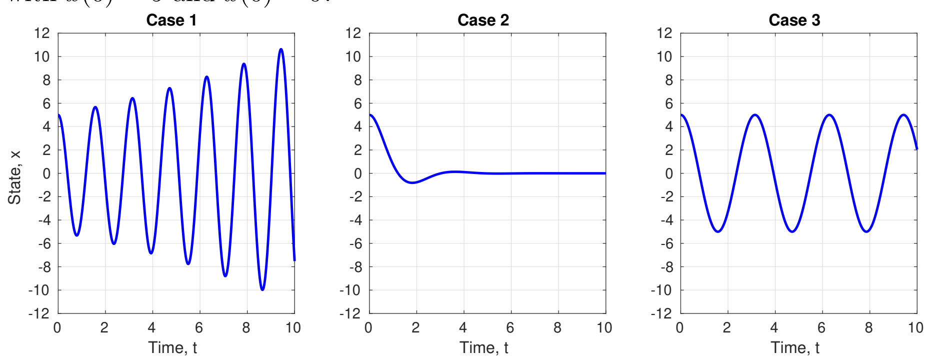
E. $\frac{s}{s^2+4}$

Solution (C). The unshifted/nominal signal is $f(t) = \cos \omega t$ whereas the shifted signal is $f(t - 3)H(t - 3)$. Applying the Laplace transform for shifted signals:

$$\begin{aligned} \mathcal{L}[f(t-3)H(t-3)] &= e^{-3s}F(s) \\ &= e^{-3s}\frac{s}{s^2+4} \end{aligned}$$

where we've used the LT table entry for $\cos(\omega t)$ with $\omega = 2$.

8. (2 points) Which case below could plausibly represent the response of a system $\ddot{x} - 0.16\dot{x} + 16x = 0$ with $x(0) = 5$ and $\dot{x}(0) = 0$?



A. Case 1
B. Case 2
C. Case 3
D. None of the above
E. All of the above

Solution (A). All of the cases satisfy the initial condition. The ODE is second order with $a < 0$, hence the response should be unstable as shown in Case 1.

9. (4 points) The inverse Laplace transform of

$$X(s) = \frac{7}{(s+3)(s+5)}$$

is which of the following?

A. $x(t) = \frac{5}{2}e^{3t} - \frac{5}{2}e^{5t}$

B. $x(t) = \frac{7}{2}e^{5t} - \frac{5}{2}e^{3t}$

C. $x(t) = \frac{7}{2}e^{-3t} - \frac{7}{2}te^{-5t}$

D. $x(t) = \frac{7}{2}e^{-3t} - \frac{7}{2}e^{-5t}$

E. $x(t) = \frac{5}{3}\cos 3t - \frac{5}{3}\sin 5t$

Solution(D). There are two real-distinct pole $p_1 = -3$ and $p_2 = -5$. Thus, the PFE has the form

$$X(s) = \frac{a_1}{(s+3)} + \frac{a_2}{(s+5)}$$

We can solve for a_1 as

$$a_1 = \left.\frac{7}{(s+5)}\right|_{s=-3} = 7/2$$

We can solve for a_2 as

$$a_2 = \left.\frac{7}{(s+3)}\right|_{s=-5} = -7/2$$

Then the solution is:

$$x(t) = \frac{7}{2}e^{-3t} - \frac{7}{2}e^{-5t} \tag{31}$$

10. (6 points) What is the solution to the dynamic system below?

$$\ddot{x} + 16x = \cos 3t,\ x(0) = 0,\ \dot{x}(0) = 0$$

A. $x(t) = \sin 4t + \sin 3t$

B. $x(t) = \frac{1}{7}\cos 3t - \frac{1}{7}\cos 4t$

C. $x(t) = 4\cos 3t - \cos 4t$

D. $x(t) = \frac{9}{16}\cos 3t - \frac{9}{16}\cos 4t$

E. $x(t) = \frac{9}{16}\sin 3t + \frac{1}{7}\cos 4t$

Solution (B). Take the Laplace transform

$$X(s)(s^2+16) = \frac{s}{s^2+9} \tag{32}$$

$$X(s) = \frac{s}{(s^2+16)(s^2+9)} \tag{33}$$

The poles are $p_{1,2} = \pm 3i$ and $p_{3,4} = \pm 4i$ so the partial fraction expansion is

$$X(s) = \frac{s}{(s^2+16)(s^2+9)} = \frac{a_1}{s^2+9} + \frac{a_2 s}{s^2+9} + \frac{a_3}{s^2+16} + \frac{a_4 s}{s^2+16} \tag{34}$$

Multiplying to eliminate the denominators

$$\begin{aligned} s &= a_1(s^2+16) + a_2 s(s^2+16) + a_3(s^2+9) + a_4 s(s^2+9) && (35)\\ &= a_1 s^2 + 16a_1 + c_2 s^3 + 16c_2 s + c_3 s^2 + 9c_3 + c_4 s^3 + 9c_4 s && (36)\\ &= (c_2+c_4)s^3 + (c_1+c_3)s^2 + (16c_2+9c_4)s + 16c_1 + 9c_3 && (37)\end{aligned}$$

Equating coefficients:

$$\begin{aligned} s^3:&\quad c_2 + c_4 = 0 \implies c_2 = -c_4 && (38)\\ s^2:&\quad c_1 + c_3 = 0 \implies c_1 = -c_3 && (39)\\ s^1:&\quad 16c_2 + 9c_4 = 1 \implies -7c_4 + 9c_4 = 1 && (40)\\ s^0:&\quad 16c_1 + 9c_3 = 0 \implies -16c_2 + 9c_3 = 0 && (41)\\ & && (42)\end{aligned}$$

so that $c_3 = c_1 = 0$, $c_2 = 1/7$ and $c_4 = -1/7$. The solution is then

$$\begin{aligned} x(t) &= \frac{1}{7}\mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right] - \frac{1}{7}\mathcal{L}^{-1}\left[\frac{s}{s^2+16}\right] && (43)\\ &= \frac{1}{7}\cos 3t - \frac{1}{7}\cos 4t && (44)\end{aligned}$$