Exam 2 – Solutions   Spring 2022

Multiple choice

1) If $\omega_n = 2$ rad/s and $\zeta = 0.5$, then

D

$$\ddot{x} + \underbrace{2\zeta\omega_n}_{=2(0.5)2 \,=\, 2}\dot{x} + \underbrace{\omega_n^2}_{=4}x = 0$$

$$\Rightarrow \boxed{\ddot{x} + 2\dot{x} + 4x = 0}$$

2) If $k = 1000$, $m = 10$, $b = 10$ then

A

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

$$2\zeta\omega_n = b \Rightarrow \zeta = \frac{b}{2\omega_n} = \frac{10}{2\cdot 10} = \frac{1}{2}$$

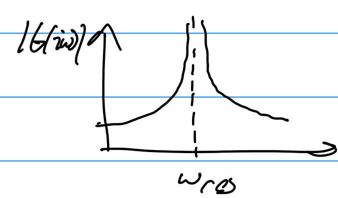
$$\omega_{res} = \omega_n\sqrt{1-2\zeta^2} = 10\sqrt{1-2\left(\tfrac{1}{2}\right)^2} = 10\sqrt{1-\tfrac{1}{2}}$$

$$= 10\sqrt{1-0.5}$$

$$\boxed{\omega_{res} = 10/\sqrt{2}}$$

D 3) $|G(i\omega)| = \dfrac{20}{10} = 2 \Rightarrow$ occurs at 400 rad/s

C 4) $|G(i\omega)| \to \infty$ as $\omega \to \omega_{res}$.



A 5) $G(i\omega) = \dfrac{-1}{(i\omega)^2} = \dfrac{-1}{-\omega^2} = \dfrac{1}{\omega^2} \Rightarrow \phi = 0°$

C 6) $\ddot{x} + \left(\frac{b}{m}\right)\dot{x} + \left(\frac{k}{m}\right)x = 0$

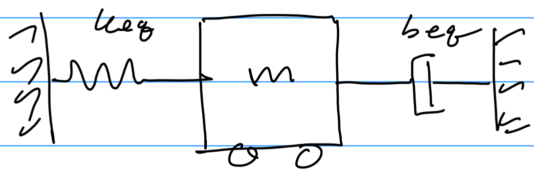
$$\omega_n = 2 \text{ rad/s}$$

$$2\zeta\omega_n = 4 \Rightarrow \zeta = 1$$

$$x(t) = e^{-\zeta\omega_n t}(\text{———— '' -————})$$

$$\tau = \frac{1}{\zeta\omega_n} = \frac{1}{2} = 0.5 \text{ sec} \Rightarrow \boxed{4\tau = 2\text{sec.}}$$

B 7)



$$b_{eq} = \frac{b_1 b_2}{b_1 + b_2} = 0.5 \qquad k_{eq} = k_1 + k_2 = 2$$

$$m\ddot{x} + b\dot{x} + kx = 0$$

$$\boxed{\ddot{x} + 0.5\dot{x} + 2x = 0}$$

C 8) KE equivalence

A 9) $\ddot{x} + 12\dot{x} + 4x = 0$

$$\omega_n = 2$$

$$2\zeta\omega_n = 12 \Rightarrow \zeta = 3$$

B 10)

$$\begin{array}{lccc} & s^2 & s^1 & s^0 \\ \text{num} = & [5 & 2 & 0] \end{array}$$

$$\begin{array}{lcccccc} \text{den} = & [3 & 3 & 0 & 1 & 2 & 1] \\ & s^5 & s^4 & s^3 & s^2 & s^1 & s^0 \end{array}$$

Recall: Diff'th

$$\mathcal{L}[\dot{x}] = sX(s) - x_0$$

$$\mathcal{L}[\ddot{x}] = s^2 X(s) - s x_0 - \dot{x}_0$$

Workout Problem 1

$$\ddot{x} + 6\dot{x} + 34x = 0 \quad , \quad x(0) = 3, \quad \dot{x}(0) = -11$$

$$(s^2 X(s) - s x_0 + 11) + 6[sX(s) - x_0] + 34 X(s) = 0$$

$$(s^2 X(s) - 3s + 11) + 6[sX(s) - 3] + 34 X(s) = 0$$

$$X(s)(s^2 + 6s + 34) = 3s + 6(3) - 11$$

$$= 3s + 7$$

$$X(s) = \frac{(3s + 7)}{(s^2 + 6s + 34)}$$

Poles: $p_{1,2} = \dfrac{-6 \pm \sqrt{36 - 4(34)}}{2} = -3 \pm \sqrt{36 - 136}/2$

$$= -3 \pm \frac{10i}{2}$$

$$p_{1,2} = -3 \pm 5i$$

P.F.E. $\quad X(s) = \dfrac{(3s+7)}{(s^2 + 6s + 34)} = C_1 \dfrac{5}{(s+3)^2 + 5^2} + C_2 \dfrac{(s+3)}{(s+3)^2 + 5^2}$

$$3s + 7 = 5C_1 + C_2(s+3)$$

$$= C_2 s + (5C_1 + 3C_2)$$

$$\Rightarrow C_2 = 3 \qquad 7 = 5C_1 + 9$$

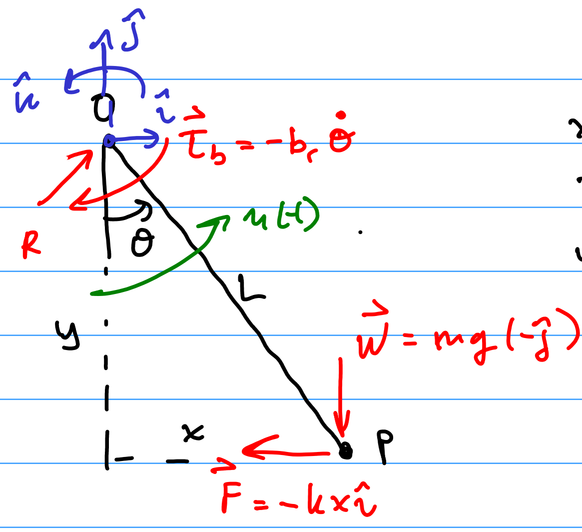
$$\Rightarrow C_1 = -2/5$$

$$\Rightarrow x(s) = 3\left(\frac{5}{(s+3)^2 + 5^2}\right) + \frac{-2}{5}\left(\frac{(s+3)}{(s+3)^2 + 5^2}\right)$$

I.L.T

$$\Rightarrow x(t) = 3e^{-3t}\cos 5t - \frac{2}{5} e^{-3t} \sin 5t$$

Problem



$x = L\sin\theta$

$x \approx L\theta$ (small angles)

$y = L\cos\theta$

$\approx L$

$$mL^2\ddot{\theta} = -b_r\dot{\theta} - ykx - xmg + u(t)$$

$$= -b_r\dot{\theta} - Lk(L\theta) - (L\theta)mg + u(t)$$

$$mL^2\ddot{\theta} + b_r\dot{\theta} + (kL^2 + Lmg)\theta = u(t)$$

L.T. $\quad \theta(s)\left(mL^2s^2 + b_rs + (kL^2 + Lmg)\right) = U(s)$

$$G(s) = \frac{\theta(s)}{U(s)} = \frac{1}{mL^2s^2 + b_rs + (kL^2 + Lmg)}$$

S.T.F

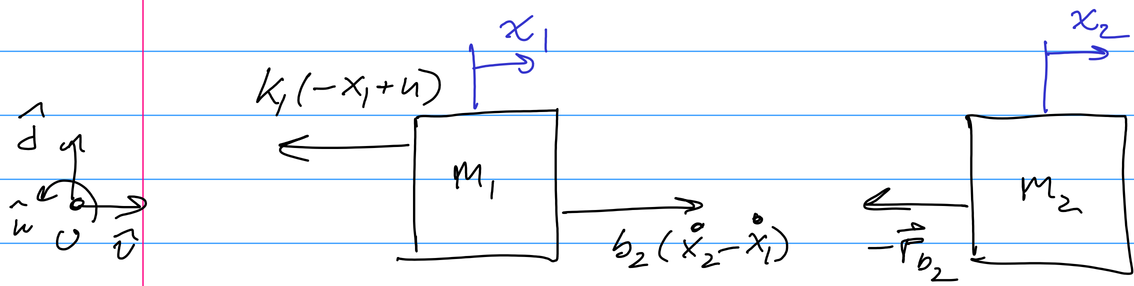
$$G(i\omega) = \frac{1}{-mL^2\omega^2 + ib\omega + (kL^2 + Lmg)}$$

$$|G(i\omega)| = \frac{1}{\sqrt{b^2\omega^2 + [kL^2 + Lmg - mL^2\omega^2]}}$$

$$G(i\omega) = \frac{[kL^2 + Lmg - mL^2\omega] - ib\omega}{\sqrt{\quad - \text{"} - \quad}}$$

$$\phi = \operatorname{atan}\left(\frac{-b\omega}{(kL^2 + Lmg - mL^2\omega)}\right)$$

Problem



$$m_1\ddot{x}_1 = k_1(-x_1+u) + b_2(\dot{x}_2-\dot{x}_1) \quad (1) \qquad m_2\ddot{x}_2 = -b_2(\dot{x}_2-\dot{x}_1) \quad (2)$$

From (1)

$$X_1(s)(m_1 s^2) = X_1(s)(-k_1 - b_2 s) + U(s)(k_1) + X_2(s)(s b_2) \quad (3)$$

From (2)

$$X_2(s)(m_2 s^2 + b_2 s) = X_1(s)(b_2 s) \quad (4)$$

Sub. (4) → (3)

$$X_1(s)(m_1 s^2 + b_2 s + k_1) = U(s)\,k_1 + \frac{(b_2 s)^2}{(m_2 s^2 + b_2 s)} X_1(s)$$

$$X_1(s)\left[(m_1 s^2 + b_2 s + k_1)(m_2 s^2 + b_2 s) - (b_2 s)^2\right] = U(s)\,k_1$$

$$X_1(s)\left[m_1 m_2 s^4 + m_1 b_2 s^3 + m_2 b_2 s^3 + \cancel{(b_2 s)^2} + m_2 k_1 s^2 + k_1 b_2 s - \cancel{(b_2 s)^2}\right] = U(s)\,k_1$$

$$\frac{X_1(s)}{U(s)} = \frac{k_1}{(m_1 m_2)s^4 + (m_1+m_2)b_2 s^3 + m_2 k_1 s^2 + k_1 b_2 s}$$