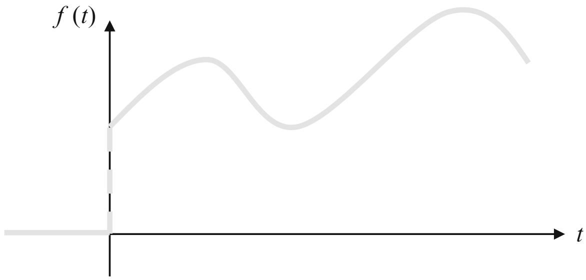


Fig. 2.8 A function, $f(t)$, that is zero before $t = 0$

2.6 Properties of the Laplace Transform

As you may have already noticed in Sect. 2.5 examples and Table 2.1, patterns appear in the Laplace transforms. In addition to Table 2.1, certain properties of the Laplace transform are important in their application.

2.6.1 Linearity

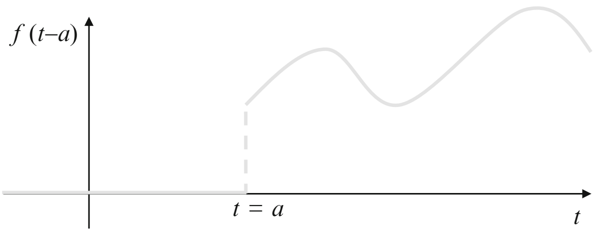
Since the Laplace transform is an integral, it follows all the principles of linearity. The most important ones are: (1) multiplication by a constant; and (2) sum of two functions. Equation (2.16) shows that the Laplace transform of a time function multiplied by a constant A yields the Laplace domain representation of the time function also multiplied by A. Additionally, we see that the Laplace transform of the sum of two time functions is the sum of the Laplace domain representation of the time functions.

$$\begin{gathered}\mathcal{L}[Af(t)] = AF(s) \\ \mathcal{L}[f(t) + g(t)] = F(s) + G(s)\end{gathered} \tag{2.16}$$

2.6.2 Laplace Transform of a Time-Delayed Function

In dynamic systems, it is common for the input to start at a delayed time (i.e., not $t = 0$). A simple example is the step input, $u(t - a)$, which could, for example, be an electric switch turned on at $t = a$ (later than $t = 0$). Alternately, a load might suddenly be applied at $t = a$. Some systems, such as the tool vibration in a machining operation, have inherent delays in the forcing function that can cause instability. Figure 2.8 shows a general function that begins at $t = 0$, while Fig. 2.9 shows the same function that is delayed until $t = a$. The goal is to determine the Laplace transform of the delayed function in terms of $F(s)$, the presumably known Laplace transform of the un-delayed function.

Fig. 2.9 A function, $f(t-a)$, that is zero before $t = a$



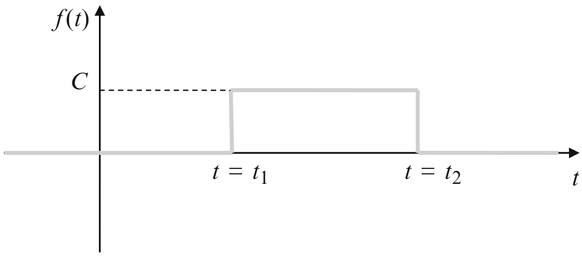


Fig. 2.10 A pulse function with amplitude C between $t = t_1$ and $t = t_2$ and zero at all other times

Applying the definition of the Laplace transform, we obtain $\mathcal{L}[f(t-a)] = \int_0^\infty f(t-a)e^{-st}dt$. Next, we substitute: $\rho = t - a$ and $d\rho = dt$ to obtain a modified version of the original integral.

$$\mathcal{L}[f(t-a)] = \int_{-a}^{\infty} f(\rho)e^{-s(\rho+a)}d\rho = e^{-sa}\int_{-a}^{\infty} f(\rho)e^{-s\rho}d\rho$$

Because $\rho = 0$ is equivalent to $t = a$, the integral limits can be redefined to be $\mathcal{L}[f(t-a)] = e^{-sa}\int_0^\infty f(\rho)e^{-s\rho}d\rho$. We see that the integral is simply $F(s)$, the Laplace transform of the un-delayed function. Substituting yields the final transform: $\mathcal{L}[f(t-a)] = e^{-sa}F(s)$.

In the Laplace domain, multiplying a function by e^{-sa} is equivalent to delaying the function by a in the time domain. For this reason, e^{-sa} is known as the delay operator and can be used to delay any function by an arbitrary time, a.

Example 2.9 Find the Laplace transform of the pulse function displayed in Fig. 2.10 that is initiated at $t = t_1$ with an amplitude of C and becomes zero again at $t = t_2$.

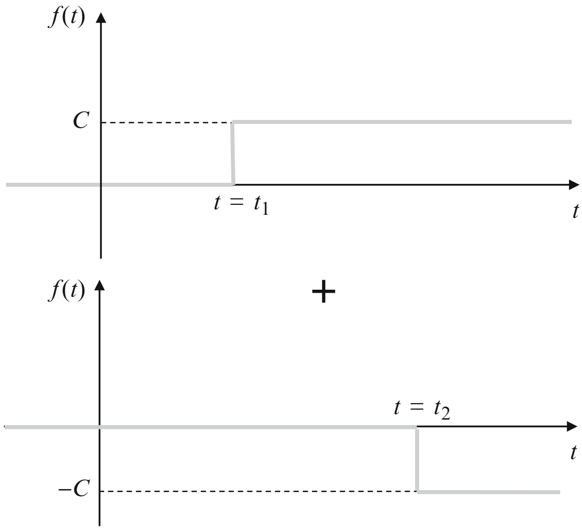


Fig. 2.11 The pulse function as a combination of two-step functions

Solution The function is the sum of the two-step functions shown in Fig. 2.11.

The straightforward solution is to sum the positive and negative steps, each the appropriate time delay, in the Laplace domain.

$$F(s) = \left(e^{-st_1}\frac{C}{s}\right) + \left(e^{-st_2}\frac{-C}{s}\right) = C\left(\frac{e^{-st_1}}{s} - \frac{e^{-st_2}}{s}\right)$$

2.6.3 Laplace Transform of a Time Derivative

In the solution of linear differential equations by Laplace transforms, it is necessary to calculate the Laplace transform of time derivatives. We use the typical "over dot" shorthand notation to represent a time derivative: $\frac{df}{dt} = \dot{f}(t)$. The Laplace transform of a time derivative proceeds as follows. Because the solution of differential equations requires knowledge of the initial conditions, we will also need to consider the inclusion of initial conditions in the Laplace transformation as we proceed.

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty \frac{df}{dt}e^{-st}dt$$

We can solve this integral using integration by parts: $\int_b^b udv = uv|_a^b - \int_b^b vdu$. We define the integration by parts variables as shown.

$$u = \mathrm{e}^{-st} dv = \frac{df}{dt} dt$$
$$du = -s\mathrm{e}^{-st} v = f(t)$$

Substitution yields the following expression.

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty \frac{df}{dt} \mathrm{e}^{-st} dt = \mathrm{e}^{-st} f(t)|_0^\infty - \int_0^\infty f(t) s\mathrm{e}^{-st} dt = f(t)\mathrm{e}^{-st}\Big|_0^\infty + s\int_0^\infty f(t)\mathrm{e}^{-st} dt$$

If *f(t)* is well-behaved, then the first term can be evaluated, while the integral in the second term is just *F(s)*.

$$\mathcal{L}\left[\dot{f}(t)\right] = \int_0^\infty \frac{df}{dt} \mathrm{e}^{-st} dt = f(t)\mathrm{e}^{-st}|_0^\infty + s\int_0^\infty f(t)\mathrm{e}^{-st} dt = (f(t)\cdot 0 - f(0)\cdot 1) + sF(s)$$

The final result is written as:

$$\mathcal{L}\left[\dot{f}(t)\right] = sF(s) - f(0), \tag{2.17}$$

where $f(0)$ is interpreted as the initial value of the function (i.e., the value at $t = 0$). To determine the Laplace transforms of higher order derivatives, we apply a recursive procedure. For example, if we wish to find the Laplace transform of a second time derivative, we make the following substitutions.

$$g(t) = \dot{f}(t)$$
$$\dot{g}(t) = \ddot{f}(t)$$

Now we take the Laplace transform of $\dot{g}(t)$ using Eq. (2.17).

$$\mathcal{L}\left[\ddot{f}(t)\right] = L[\dot{g}(t)] = sG(s) - g(0)$$

Further, we recognize that the Laplace transform of $g(t) = \dot{f}(t)$ is also given by Eq. (2.16) and combine.

$$\mathcal{L}\left[\ddot{f}(t)\right] = L[\dot{g}(t)] = s\big(sF(s) - f\big(0\big) - g(0)$$
$$\mathcal{L}\left[\ddot{f}(t)\right] = s^2F(s) - sf(0) - \dot{f}(0)$$

This procedure can be repeated to obtain the Laplace transform of any order derivative of a function. As the order of the derivative increases, the number of initial conditions required for evaluation is naturally equal to the number of initial conditions needed to solve a differential equation of that same order. For example, the second derivative of position, or acceleration, appears in the second-order equation of motion for an oscillating mass attached to ground through a spring. To determine the solution for this system, we require two initial conditions (typically an initial position and velocity).

2.6.4 Initial and Final Value Theorems

As we will see later in the text, system design and analysis can often be completed directly in the Laplace domain without requiring observation of the corresponding time domain behavior. The initial value theorem and the final value theorem are important tools for system analysis in the Laplace domain. These enable conclusions about the time domain behavior to be drawn without explicitly determining the time domain response of the system. We state them without proof here and then provide examples that illustrate how to practically apply these theorems.

The *initial value theorem* enables $f(0)$ to be determined from $F(s)$ only.

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} (sF(s)) \tag{2.18}$$

The limit is taken as t approaches 0 from the positive side ($t > 0$) because we assume that our functions are zero until $t = 0$. This is also implicit in the selection of the Laplace transform lower limit.

The *final value theorem* enables the limiting value of $f(t)$ as t becomes very large (infinite) to be determined.

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} (sF(s)) \tag{2.19}$$

Notice that in these two theorems, as t becomes large, s becomes small and vice versa. Let us explore this for the initial value theorem. Consider Eq. (2.17), which is reproduced here.

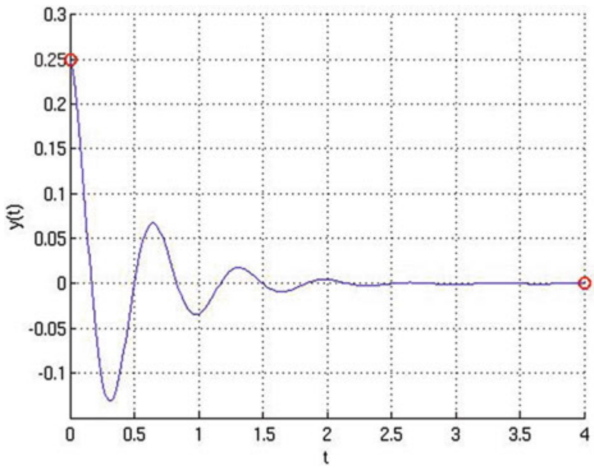
$$\mathcal{L}\left[\dot{f}(t)\right] = \mathcal{L}\left[\frac{d}{dt}f(t)\right] = \int_0^\infty \left[\frac{d}{dt}f(t)\right] \mathrm{e}^{-st} dt = sF(s) - f(0)$$

Now apply the limit to both sides of the equation as $s \to \infty$.

$$\lim_{s \to \infty} \int_0^\infty \left[\frac{d}{dt}f(t)\right] \mathrm{e}^{-st} dt = 0 = \lim_{s \to \infty} [sF(s) - f(0)]$$

The left-hand side becomes zero as $s \to \infty$ due to the exponential term. Rearranging, we obtain $\lim_{s \to \infty} sF(s) = f(0)$.

Example 2.10 Consider the following function which represents the displacement of a vibrating structure: $y(t) = 0.25e^{-2t}\cos(3\pi t)$. The time domain response was plotted in MATLAB®. We observe that the starting value is 0.25 and the ending value approaches zero. Verify this result using the initial and final value theorems.



Solution To demonstrate the initial and final value theorems, we use entry 21 of Table 2.1 to find the function's Laplace transform.

$$Y(s) = 0.25\frac{s+2}{(s+2)^2 + (3\pi)^2}$$

We apply Eq. (2.18) to determine the initial value.

$$\lim_{t\to 0} y(t) = \lim_{s\to\infty} sY(s) = \lim_{s\to\infty}\left(s\frac{0.25(s+2)}{(s+2)^2 + (3\pi)^2}\right)$$

We expand the numerator and denominator and divide through by the highest power in s for both. We then set all terms with s in the denominator equal to zero (as $s \to \infty$) in order to obtain the result.

$$\lim_{s\to\infty} sY(s) = \lim_{s\to\infty}\left(0.25\frac{(s^2+2s)}{s^2+4s+4+(3\pi)^2}\right) = \lim_{s\to\infty}\left(0.25\frac{(1+\frac{2}{s})}{1+\frac{4}{s}+\frac{4+(3\pi)^2}{s^2}}\right)$$

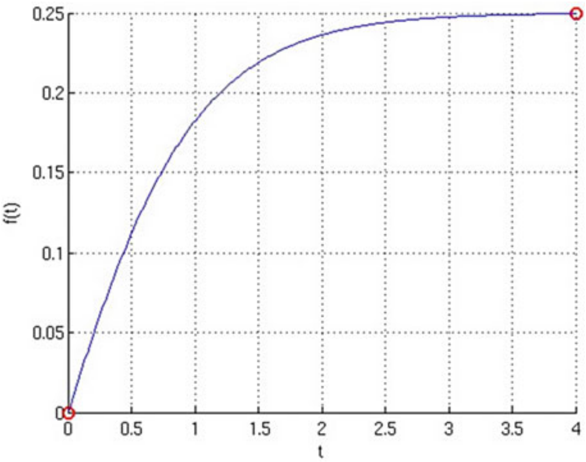
$$= 0.25\frac{1}{1} = 0.25$$

This matches the time domain graphical result. Next, we apply the final value theorem. In this case, we do not divide by the highest power in s. Rather, we simply set all the s terms equal to zero (as $s \to 0$).

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0}\left(0.25\frac{(s^2+2s)}{s^2+2s+4+(3\pi)^2}\right) = 0.25\frac{0}{4+(3\pi)^2} = 0$$

This also agrees with the time domain behavior of the function.

Example 2.11 Consider the function $f(t) = \frac{1}{4}(1 - \mathrm{e}^{-2t} - 2t\mathrm{e}^{-2t})$. The MATLAB® time domain response is shown, where we see that the function is zero at $t = 0$ and approaches 0.25 as time gets large.



Suppose this was the solution for the model of a dynamic system. Sometimes it is important to know, in a control system for example, the initial and final values of the solution. However, we may only have the solution in the Laplace domain.

Solution We find the Laplace transform of the function using entry 18 of Table 2.1.

$$F(s) = \frac{1}{s(s+2)^2}$$

Applying Eq. (2.18), we find the initial value.

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{1}{(s+2)^2} = \lim_{s\to\infty}\frac{1}{s^2+4s+4} = \lim_{s\to\infty}\frac{\frac{1}{s^2}}{1+\frac{4}{s}+\frac{4}{s^2}} = \frac{0}{1} = 0$$

Applying Eq. (2.19), we find the final value.

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} \frac{1}{(s+2)^2} = \lim_{s\to 0} \frac{1}{s^2+4s+4} = \frac{1}{4}$$

These agree with the time domain plot of the function.

2.7 Inverting Laplace Transforms

In order to solve[7] a differential equation (that models a dynamic system) using Laplace transforms, we need to first understand how to apply the *forward Laplace transform* (Sect. 2.5, Table 2.1). We can then perform simple algebraic manipulations in the Laplace domain and finally apply the *inverse Laplace transform* to obtain the solution in the time domain. We represent the inverse Laplace transform as shown in Eq. (2.20).

$$\mathcal{L}^{-1}[F(s)] = f(t) \tag{2.20}$$

The inverse Laplace transform is an integral, so it also obeys the linearity properties listed in Eq. (2.16).

$$\begin{aligned} &\mathcal{L}^{-1}[AF(s)] = A\mathcal{L}^{-1}[F(s)] = Af(t) \\ &\mathcal{L}^{-1}[F(s)+G(s)] = \mathcal{L}^{-1}[F(s)] + \mathcal{L}^{-1}[G(s)] = f(t) + g(t) \end{aligned} \tag{2.21}$$

Typically, in system dynamics, the solution for a selected model will involve determining the inverse Laplace transform of a function with the form $F(s) = \frac{B(s)}{A(s)}$. The numerator and denominator of $F(s)$ are typically polynomials in s. That is, $A(s)$ and $B(s)$ can be described as sums of terms with descending powers of s.

$$\begin{aligned} A(s) &= c_n s^n + c_{n-1}s^{n-1} + \ldots + c_1 s + c_0 \\ B(s) &= d_m s^m + d_{m-1}s^{m-1} + \ldots + d_1 s + d_0 \end{aligned}$$

We say that $A(s)$ is a polynomial of order n and $B(s)$ is a polynomial of order m because these are the highest powers of s. The system can also be written in *pole-zero* form by factoring the numerator and denominator.

[7] By solve, we mean that we wish to determine the time domain solution of the differential equation.

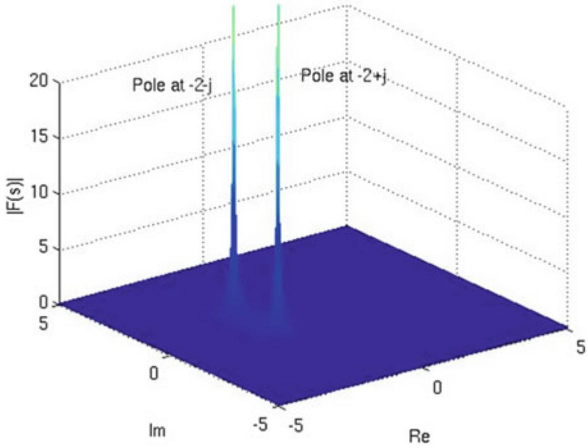


Fig. 2.12 The magnitude of $F(s)$ plotted as a function of s. The axes represent the real and imaginary parts of the complex variable s and the surface represents the magnitude of $F(s)$

$$F(s) = \frac{K(s - z_1)(s - z_2)(s - z_3)\ldots(s - z_m)}{(s - p_1)(s - p_2)(s - p_3)\ldots(s - p_n)}$$

$F(s)$ is zero when $s = z_1$, $s = z_2$, etc., so these values of s are called the *zeroes* of $F(s)$. $F(s)$ approaches infinity when $s = p_1$, $s = p_2$, etc. These are called the *poles* of $F(s)$. Note that both poles and zeroes can be real or complex (i.e., they include an imaginary part). Later we will see that the function $F(s)$ will represent a transfer function of a system. In that case the poles provide direct system properties.

We can visualize the effect of poles graphically if we plot the magnitude of $F(s)$ for a particular example.

$$F(s) = \frac{1}{s^2 + 4s + 5} = \frac{1}{(s - (-2 + j))(s - (-2 - j))}$$

For each complex value of s, the magnitude of $F(s)$ has a corresponding value. That value approaches infinity when s is equal to a pole of $F(s)$ as shown in Fig. 2.12, where the axes represent the real and imaginary parts of the Laplace variable s. The poles are $s = -2 + j$ and $s = -2 - j$.

The primary procedure we will apply to determine the inverse Laplace transform for engineering system functions is *partial fraction expansion*. Partial fractions are used to convert a function that is not in a form we can find in our Laplace transform table into a form that is available in the table. We will demonstrate this by examples. In these examples, it is assumed that the numerator and denominator polynomials have already been factored and like terms have been canceled from the two.

There are three primary types of partial fraction expansions; these provide "shortcuts" for inverting a particular Laplace transform. These types are based on the $F(s)$ poles (or roots of the denominator). A fourth case occurs frequently when a system is subjected to a step input.

1. Distinct real poles (or roots).
2. Complex poles.
3. Repeated real poles.
4. Special case that often occurs with step inputs to systems.

There is no "magic" to partial fractions. The central question to answer is this: Given a certain form of $F(s)$ that is not easily invertible (i.e., does not appear in our Laplace transform table), is it possible to write $F(s)$ in another form that is more easily invertible? There are typically multiple ways to invert a Laplace transform and even if the different results do not "look" the same, they must give the same answer. The Laplace transform and its inverse are unique, which means there cannot be two different functions in the time domain that represent the inverse of the same function in the Laplace domain.

2.7.1 Distinct Real Poles

In this case, when the denominator is factored, the poles are all *distinct real numbers*. This is illustrated by the following example. In this case, a useful partial fractions expansion is the sum of a series of fractions with the individual poles appearing in the denominators.

$$\begin{aligned} F(s) &= \frac{K(s-z_1)(s-z_2)(s-z_3)\ldots(s-z_m)}{(s-p_1)(s-p_2)(s-p_3)\ldots(s-p_n)} \\ &= \frac{a_1}{s-p_1}+\frac{a_2}{s-p_2}+\ldots+\frac{a_n}{s-p_n} \end{aligned} \tag{2.22}$$

Here the values $a_1, a_2 \ldots a_n$ are constants. We apply the linearity principle of the Laplace transform and its inverse to obtain Eq. (2.23).

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{a_n}{s+p_n}\right]+\ldots+\mathcal{L}^{-1}\left[\frac{a_2}{s+p_2}\right]+\mathcal{L}^{-1}\left[\frac{a_1}{s+p_1}\right] \\ &= a_n\mathrm{e}^{-p_n t}+\ldots+a_2\mathrm{e}^{-p_2 t}+a_1\mathrm{e}^{-p_1 t} \end{aligned} \tag{2.23}$$

We illustrate this case by an example.

Example 2.12 Find the inverse Laplace transform of the Laplace domain function $F(s)=\frac{s-3}{s^2+6s+5}$.

Solution First, we factor the denominator.

$$F(s) = \frac{s-3}{(s+1)(s+5)}$$

If there were common terms between the numerator and denominator, we would cancel them at this stage. The method of partial fractions says that $F(s)$ can be expanded using Eq. (2.22).

$$F(s) = \frac{s-3}{(s+1)(s+5)} = \frac{a_1}{s+1} + \frac{a_2}{s+5}$$

To determine a_1 and a_2, we multiply both sides of the equation by the left-hand side denominator. We then group terms in powers of s.

$$\begin{aligned}
&s-3 = \frac{a_1(s+1)(s+5)}{s+1} + \frac{a_2(s+1)(s+5)}{s+5}\\
&s-3 = a_1(s+5) + a_2(s+1)\\
&s(a_1+a_2-1) + (5a_1+a_2+3) = 0
\end{aligned}$$

Because s is a variable and can take on any real or complex value, the only way for this expression to always be zero is for both the multiplier of s (in the first parenthesis) and the constant term (in the second parenthesis) to be equal to zero. To determine a_1 and a_2, we solve the two simultaneous linear equations.

$$a_1 + a_2 - 1 = 0$$

$$5a_1 + a_2 + 3 = 0$$

The only pair of values that satisfies both equations is $a_1 = -1$ and $a_2 = 2$.

There is an alternate way to find a_1 and a_2 that takes advantage of s being a variable that can take on any value, so that the expansion must be true no matter the value of s. To find a_1, we begin with $F(s) = \frac{s-3}{(s+1)(s+5)} = \frac{a_1}{s+1} + \frac{a_2}{s+5}$ and multiply both sides by $s+1$. We then let s approach the pole, $s=-1$, in the limit.

$$\begin{aligned}
&\lim_{s\to -1}\left[\frac{s-3}{(s+5)}\right] = \lim_{s\to -1}\left[a_1 + \frac{a_2(s+1)}{s+5}\right]\\
&\frac{-1-3}{(-1+5)} = a_1 + \frac{a_2(-1+1)}{-1+5}\\
&\frac{-4}{4} = a_1 + \frac{a_2(0)}{4}
\end{aligned}$$

This yields $a_1 = -1$ as before. We use a limit because we know that the original function becomes infinite when s is equal to -1.

To find a_2, we multiply both sides by $(s+5)$ and repeat the procedure.

$$\lim_{s\to -5}\left[\frac{s-3}{(s+1)}\right] = \lim_{s\to -5}\left[\frac{a_1(s+5)}{s+1} + a_2\right]$$
$$\frac{-5-3}{(-5+1)} = \frac{a_1(-5+5)}{-5+1} + a_2$$
$$\frac{-8}{-4} = \frac{a_1(0)}{-4} + a_2$$

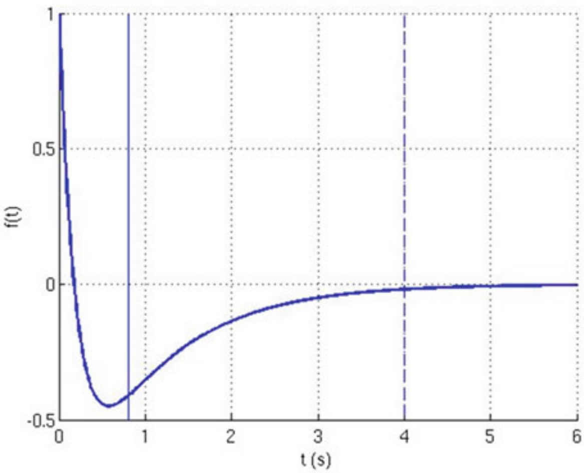
We see that $a_2 = 2$ as before. Substituting gives our final partial fraction expansion, where we subdivided our original function into two partial fractions.

$$F(s) = \frac{-1}{s+1} + \frac{2}{s+5}$$

calculating the inverse Laplace transform and making use of linearity, we find that the time domain solution for the selected Laplace function with real poles is a sum of exponential functions; this is the general case.

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{-1}{s+1}\right] + \mathcal{L}^{-1}\left[\frac{2}{s+5}\right] = -\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + 2\mathcal{L}^{-1}\left[\frac{1}{s+5}\right]$$
$$f(t) = -\mathrm{e}^{-t} + 2\mathrm{e}^{-5t}$$

A plot of this function is provided.



Notice the effect of the two exponentials. The first exponential has a time constant of 1 s and is relatively slow, while the second is faster with a shorter time constant of 0.2 s (1/5). Therefore, the second exponential changes rapidly over about four time constants, 4(0.2) = 0.8 s (solid vertical line), while the first exponential changes more slowly over an interval of about $4(1)=4$ s (dashed vertical line). Notice also that the time constants are known as soon as we have found the poles (roots of the denominator); they are simply the inverse of the absolute value of p_i for the pole form $(s-p_i)$.

2.7.2 Complex Poles

In this case the denominator can still be factored to produce distinct poles/roots, but the poles/roots are complex numbers. More specifically, they appear as pairs of complex conjugates. In this situation, the most convenient approach is not the partial fractions expansion solution we demonstrated for real and distinct roots, although this can be done as long as you are familiar with manipulating complex numbers and adept at using Euler's formula. As an alternative, for a second-order system, we *complete the square* in the denominator and write the function $F(s)$ in a form that represents exponentially decaying sine and cosine terms (in the time domain) that are found in Table 2.1. This is illustrated by the following example.

Example 2.13 Determine the inverse Laplace transform of the function $F(s)=\frac{3}{s^2+4s+20}$.

Solution Using the quadratic equation, we find the roots to the denominator to be:

$$s_{1,2}=\frac{-4\pm\sqrt{4^2-4(1)20}}{2(1)}=-2\pm\frac{\sqrt{-64}}{2}=-2\pm j4.$$

Rewriting gives:

$$F(s)=\frac{3}{s^2+4s+20}=\frac{3}{(s+2-j4)(s+2+j4)}.$$

To complete the square in the denominator, we take half of the constant multiplying s, square it, and then "borrow" that value from the constant term. In this case, the multiplier on s for the polynomial representation of the denominator is 4. Half of 4 squared is again 4. Borrowing 4 from 20 leaves 16.

$$F(s)=\frac{3}{s^2+4s+4+16}$$

Because s^2+4s+4 is a perfect square, we obtain the following.

$$F(s)=\frac{3}{(s+2)^2+16}=\frac{3}{(s+2)^2+4^2}$$

This can now be compared to entry 20 in Table 2.1 where $a = 2$ and $\omega = 4$. However, because use of this entry also requires ω (or 4) in the numerator we can manipulate the expression as follows.

$$F(s) = \frac{3}{4}\left(\frac{4}{(s+2)^2 + 4^2}\right)$$

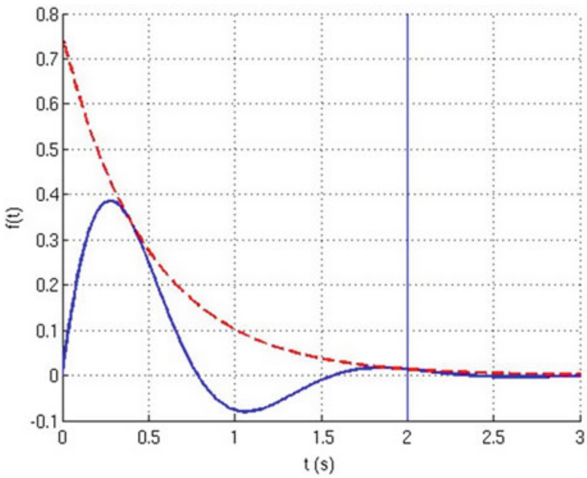
Now the linearity property is applied:

$$f(t) = \mathcal{L}^{-1}\left[\frac{3}{4}\left(\frac{4}{(s+2)^2 + 4^2}\right)\right] = \frac{3}{4}\mathcal{L}^{-1}\left[\frac{4}{(s+2)^2 + 4^2}\right]$$

and we obtain the final time domain function.

$$f(t) = \frac{3}{4}e^{-2t}\sin 4t$$

A plot of this function is provided (solid line), while the *exponential envelope*, $\frac{3}{4}e^{-2t}$, is shown as a dashed line. The function has decayed appreciably when the time reaches four time constants of the exponential envelope, or $4(0.5) = 2$ s (solid vertical line).



2.7.3 Repeated Real Poles

Sometimes, the poles are real but they are repeated. For example, the denominator might contain the factor $(s - 3)^2$ in which case the “strength” of the pole/root has been increased. Since true repeated roots require exact values for some of the physical constants in the system to make a perfect square, the most common/practical cause of repeated roots is a ramp input (Table 2.1, entry 3). In this case, the term s^2 appears in the denominator of the Laplace transform and there are two poles (roots in the denominator) at $s = 0$. For repeated roots, it is necessary to modify the partial fractions expansion for real roots. In particular, a single repeated root is treated using the partial fraction expansion in Eq. (2.24), where the root p_1 is repeated.

$$\begin{aligned} F(s) &= \frac{K(s-z_1)(s-z_2)(s-z_3)\ldots(s-z_m)}{(s-p_1)^2(s-p_3)\ldots(s-p_n)} \\ &= \frac{a_1}{s-p_1} + \frac{a_2}{(s-p_1)^2} + \frac{a_3}{s-p_3} + \ldots + \frac{a_n}{s-p_n} \end{aligned} \tag{2.24}$$

The inverse Laplace transform for this expansion is provided in Eq. (2.25).

$$f(t) = a_1 \mathrm{e}^{-p_1 t} + a_2 t \mathrm{e}^{-p_1 t} + a_3 \mathrm{e}^{-p_3 t} + \ldots + a_n \mathrm{e}^{-p_n t} \tag{2.25}$$

Example 2.14 Determine the inverse Laplace transform of the function $F(s) = \frac{2s+10}{(s+1)^2(s+4)}$ with the repeated pole $s = –1$.

Solution This function has two roots at –1 and one root at –4. According to Eq. (2.24), we expand as follows.

$$F(s) = \frac{2s+10}{(s+1)^2(s+4)} = \frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \frac{a_3}{(s+4)}$$

To identify the constants, we multiply both sides by the left-hand side denominator and group terms by powers of s.

$$\begin{aligned} &2s + 10 = a_1(s+1)(s+4) + a_2(s+4) + a_3(s+1)^2 \\ &0 = (a_1 + a_3)s^2 + (5a_1 + a_2 + 2a_3 - 2)s + (4a_1 + 4a_2 + a_3 - 10) \end{aligned}$$

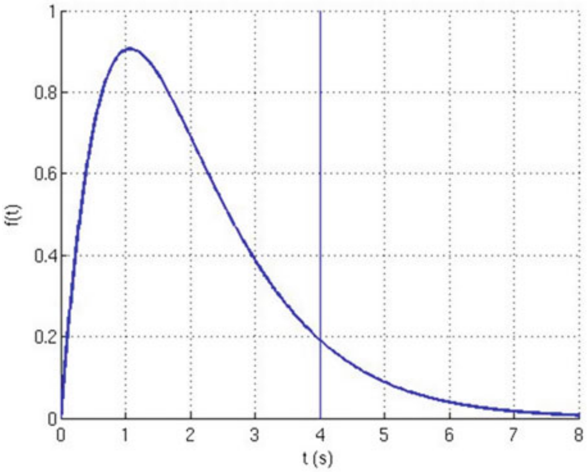
This gives us three simultaneous equations:

$$\begin{aligned} &a_1 + a_3 = 0 \\ &5a_1 + a_2 + 2a_3 - 2 = 0 \\ &4a_1 + 4a_2 + a_3 - 10 = 0, \end{aligned}$$

which are then solved for the constants $a_1 = -\frac{2}{9}$, $a_2 = \frac{8}{3}$, and $a_3 = \frac{2}{9}$. The inverse Laplace transform is then applied to determine the time domain solution.

$$f(t) = -\frac{2}{9}e^{-t} + \frac{8}{3}te^{-t} + \frac{2}{9}e^{-4t}$$

A plot of this function is provided.



Notice that, even though the two time constants for the exponentials are 0.25 and 1, the function still has a significant value at a time of 4(1) = 4 s (solid vertical line). This is the effect of the repeated root, which results in the term te^{-t}. For this term the multiplicative factor t reduces the rate of decay; therefore, our time constant estimation is not strictly applicable.

2.7.4 Special Case That Often Occurs with Step Inputs to Systems

Another common situation is Laplace domain functions with a third order in s; this occurs when a system is driven by a step input. In this case, the denominator can first be subdivided by the partial fraction expansion shown in Eq. (2.26). Here, $p(s)$ is a polynomial of order two or less.

$$F(s) = \frac{p(s)}{s(s^2 + bs + c)} = \frac{a_1}{s} + \frac{a_2 s + a_3}{s^2 + bs + c} \tag{2.26}$$

Example 2.15 Determine the inverse Laplace transform of the function $F(s) = \frac{s+3}{s(s^2+4s+20)}$.

Solution We begin by attempting the expansion suggested by Eq. (2.26).

$$\frac{s+3}{s(s^2+4s+20)}=\frac{a_1}{s}+\frac{a_2 s+a_3}{s^2+4s+20}$$

We multiply both sides by the left-hand side denominator, expand, and separate powers of s.

$$s+3=a_1(s^2+4s+20)+(a_2 s+a_3)s$$
$$(a_1+a_2)s^2+(4a_1+a_3-1)s+(20a_1-3)=0$$

Setting the coefficients of the different powers of s equal to zero and solving the three simultaneous equations results in $a_1=\frac{3}{20}$, $a_2=-\frac{3}{20}$, and $a_3=\frac{2}{5}$. Substituting these coefficients into the expansion results in the following expression.

$$F(s)=\frac{3}{20}\left(\frac{1}{s}-\frac{s-\frac{8}{3}}{s^2+4s+20}\right)$$

Next, we recognize that the second term has two complex conjugate poles (see Example 2.13). Completing the square gives the following expression, where we have divided the numerator into two parts. In the final step, we extract the constant $\frac{7}{6}$ to leave 4 in the numerator for the third term. These modifications result in expressions that appear in Table 2.1.

$$F(s)=\frac{3}{20}\left(\frac{1}{s}-\frac{s+2-\frac{14}{3}}{(s+2)^2+4^2}\right)=\frac{3}{20}\left(\frac{1}{s}-\frac{s+2}{(s+2)^2+4^2}+\frac{\frac{14}{3}}{(s+2)^2+4^2}\right)$$
$$=\frac{3}{20}\left(\frac{1}{s}-\frac{s+2}{(s+2)^2+4^2}+\frac{7}{6}\frac{4}{(s+2)^2+4^2}\right)$$

Now we can invert the expression using Table 2.1 and determine the time domain result

$$f(t)=\frac{3}{20}\left(u(t)-\mathrm{e}^{-2t}\cos(4t)+\frac{7}{6}\mathrm{e}^{-2t}\sin(4t)\right)$$

Notice that even after the two exponential terms have decayed to zero, $f(t)$ retains a constant value of $\frac{3}{20}$. We confirm this result by applying the final value theorem.

$$\lim_{t\to\infty}f(t)=\lim_{s\to 0}(sF(s))=\lim_{s\to 0}\left(\frac{3}{20}\left(\frac{s}{s}-\frac{s(s+2)}{(s+2)^2+4^2}+\frac{7}{6}\frac{4s}{(s+2)^2+4^2}\right)\right)$$
$$=\frac{3}{20}\left(1-\frac{0(0+2)}{(0+2)^2+4^2}+\frac{7}{6}\frac{4\cdot 0}{(0+2)^2+4^2}\right)=\frac{3}{20}$$

2.8 Using MATLAB® to Find Laplace and Inverse Laplace Transforms

MATLAB® can also be used to obtain the Laplace and inverse Laplace transform of a function using the commands `laplace` and `ilaplace`. When we apply these commands, we first must tell MATLAB® that it is working with algebraic symbols and not numeric quantities. The command `syms` is used to do this. The quantities that follow the commands `syms` are treated as algebraic by MATLAB®. In the following example, the command `diff` is used to differentiate $x(t)$, the solution to Example 2.13, to obtain $\dot{x}(t)$ by applying the product rule.

```
>>syms x t
>>x = 3/4*exp(-2*t)*sin(4*t);
>>dx_dt = diff(x)
dx_dt =
3*cos(4*t)*exp(-2*t) - (3*sin(4*t)*exp(-2*t))/2
```

Next, we use the command `laplace` to find the Laplace transform of $x(t)$ and we see that this does indeed match with $F(s)$ from Example 2.13 after we completed the square.

```
>>syms x t X
>>x = 3/4*exp(-2*t)*sin(4*t);
>>X = laplace(x)
X =
3/((s + 2)^2 + 16)
```

Now consider Example 2.14. The inverse Laplace transform is easily determined using MATLAB®.

```
>>syms F s f t
>>F = (2*s+10)/((s+1)^2*(s+4));
>>f = ilaplace(F)
f =
(2*exp(-4*t))/9 - (2*exp(-t))/9 + (8*t*exp(-t))/3
```

We will now demonstrate how to use the symbolic capability of MATLAB® to obtain the solution of linear differential equations using Laplace transforms.

2.9 Solving Differential Equations Using Laplace Transforms

As we stated previously, many dynamic systems may be modeled using linear differential equations. Therefore, our primary motivation for learning Laplace transforms is to analyze the behavior of these models. In this section, we demonstrate the solution of linear differential equations using Laplace transforms.

Example 2.16 Solve the differential equation $3\ddot{x} + 12\dot{x} + 60x = 0$ assuming the initial conditions $x(0) = 0$ and $\dot{x}(0) = 3$.

Solution To solve, we convert from the time to Laplace domain using the Laplace transform.

$$3\left(s^2X(s) - sx(0) - \dot{x}(0)\right) + 12(sX(s) - x(0)) + 60X(s) = 0$$

Substituting the initial conditions and rearranging enables us to solve for $X(s)$.

$$\begin{aligned} &X(s)(3s^2 - 3s \cdot 0 + 12s + 60) - 3(3) - 12(0) = 0 \\ &X(s) = \frac{9}{3s^2 + 12s + 60} = \frac{3}{s^2 + 4s + 20} \end{aligned}$$

Converting back to the time domain yields the same result as Example 2.13.

$$x(t) = \frac{3}{4}e^{-2t}\sin 4t$$

Example 2.17 Assuming $x(0) = 0$ and $\dot{x}(0) = 3$, solve the differential equation $3\ddot{x} + 18\dot{x} + 15x = 0$ for $x(t)$.

Solution We begin by eliminating the common factor of 3, converting to the Laplace domain using the Laplace transform, and applying the initial conditions.

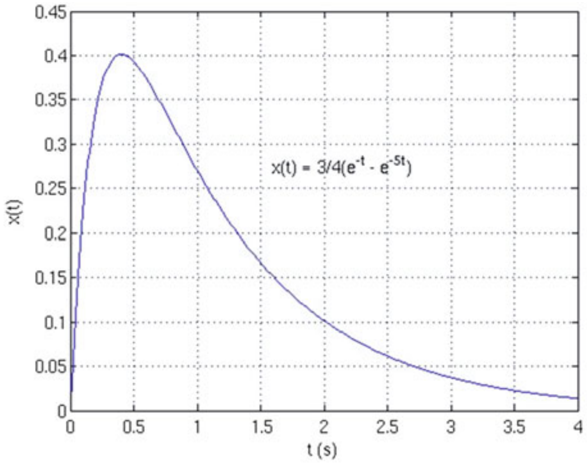
$$\begin{aligned} &s^2X(s) - sx(0) - \dot{x}(0) + 6(sX(s) - x(0)) + 5X(s) = 0 \\ &s^2X(s) - s \cdot 0 - 3 + 6sX(s) - 6 \cdot 0 + 5X(s) = 0 \end{aligned}$$

Finally, we solve for $X(s)$.

$$X(s) = \frac{3}{s^2 + 6s + 5}$$

```
>>syms X s x t
>>X = 3/(s^2+6*s+5);
>>x = ilaplace(X)
x =
(3*exp(-t))/4 - (3*exp(-5*t))/4
>>tau = 1;
>>t = [0:tau/100:4*tau];
>>plot(t, eval(x));
>>xlabel('t(s)')
>>ylabel('x(t)')
```

The symbolic result was: `x = (3*exp(-t))/4 - (3*exp(-5*t))/4`. In the `plot` command, we used `eval` to insert the value of $x(t)$ over a time span equal to four times the dominant (larger) time constant (`tau` = `1`). We selected the time step to be `tau/100`.



Example 2.18 Solve the following differential equation $10\ddot{x} + 90\dot{x} + 200x = -400$ $u(t)$ with zero initial conditions. Note the right-hand side input is a negative step with a magnitude of 400.

Solution First, we compute the Laplace transform and solve for $X(s)$. Recall that the Laplace transform of a unit step is $\frac{1}{s}$.

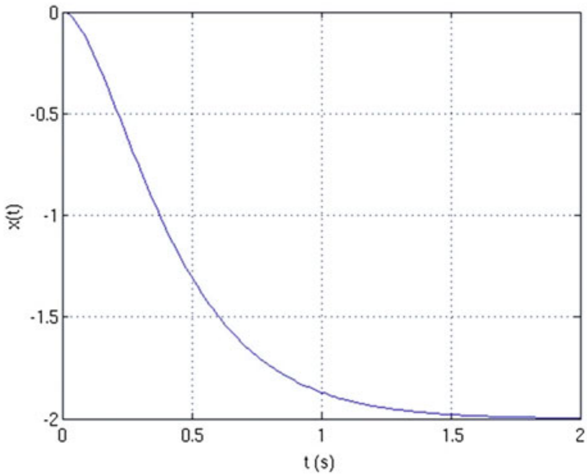
$$X(s) = \frac{-400}{s(10s^2 + 90s + 200)}$$

```
>>clear
>>syms X t s x
>>X = -400/(s*(10*s^2+90*s+200));
>>x = ilaplace(X)
x =
10/exp(4*t) - 8/exp(5*t) - 2
```

We can rearrange to obtain a form with which we are more familiar.

$$x(t) = 10e^{-4t} - 8e^{-5t} - 2$$

A plot of this solution is provided. Note that the final value approaches –2.



Problems

1. Solve the following equations and plot your answers in the complex plane with the real part along the horizontal axis and the imaginary part along the vertical axis.
   (a) $3s^2 + 21s + 36 = 0$
   (b) $s^2 + 4s + 29 = 0$
   (c) $3s^2 + 75 = 0$
   (d) $s^3 + 7s^2 + 32s + 60 = 0$

2. Solve the following equations and plot your answers in the complex plane with the real part along the horizontal axis and the imaginary part along the vertical axis.
   (a) $s^2 + 5s + 4 = 0$
   (b) $s^2 + 6s + 13 = 0$
   (c) $s^2 + 20 = 0$
   (d) $s^3 + 7s^2 + 24s + 18 = 0$

3. Type the following lines at the MATLAB® command prompt.

```
>> % f(t) = e^-(5t)*cos(20*t)
>> a = 5;
>> tau = 1/a;
>> w = 20*pi;
>> t = [0:tau/100:8*tau];
>> f = exp(-a*t).*cos(w*t);
>> plot(t, f)
>> xlabel('t(s)')
>> ylabel('f(t)')
```

Suppose the plot depicts position, x, versus time, t, of a mechanical system. The response in the plot represents a *damped oscillation*. What is the *frequency* in Hertz (cycles/second) of the oscillations you see in the plot? How long does it take the oscillations to decay to within approximately 2% of 0 (the final value)?

4. Use integration by parts to determine the Laplace transform of the following function.

$$f(t) = \begin{cases} 0 & t < 0 \\ t^2 e^{-4t} & t \geq 0 \end{cases}$$

5. Use integration by parts to determine the Laplace transform of the following function. Check your answer against the Laplace transform table.

$$f(t) = \begin{cases} 0 & t < 0 \\ \sin(2t) & t \geq 0 \end{cases}$$

6. Calculate the Laplace transform of the following functions using the Laplace transform table.
   (a) e^{-10t}
   (b) $e^{-10t}\cos(4t)$
   (c) $e^{-5t}e^{4t}$
   (d) $e^{-10t}\sin(2t)$

7. Calculate the Laplace transform of the following functions using the Laplace transform table.
   (a) e^{-2t}
   (b) $e^{-2t}\cos(16t)$
   (c) te^{-2t}
   (d) $e^{-2t}\sin(16t) + e^{-2t}\cos(16t)$

8. Calculate the inverse Laplace transforms of the following functions both analytically and using the MATLAB® command `ilaplace`.
   (a) $F_1(s) = \frac{s+1}{(s+3)(s+4)}$
   (b) $F_2(s) = \frac{s}{s^2+8s+52}$
   (c) $F_3(s) = \frac{52}{s(s^2+8s+52)}$
   (d) $F_4(s) = \frac{6}{(s+2)(s+1)^2}$

9. Solve the following differential equations using Laplace transforms. Substitute your answers into the original equations to verify them.
   (a) $\ddot{x} + 36x = 0,\quad x(0) = 2,\ \dot{x}(0) = 0$
   (b) $\ddot{x} + 10\dot{x} + 41x = \mathrm{u}(t)\quad x(0) = 0;\ \dot{x}(0) = 1$
   (c) $\dot{y} + 10y = t,\quad y(0) = 0$
   (d) $\ddot{y} + 4\dot{y} + 10y = 0,\quad y(0) = 2,\ \dot{y}(0) = 0$