Comparison of the coefficients of the s power terms on the two sides of the previous equation yields

$$\begin{aligned} s^2 &: 1 = r + a \\ s^1 &: 2 = 2r + b \\ s^0 &: 1 = 2r \end{aligned}$$

The solution of the above coupled algebraic equations gives

$$r = \frac{1}{2}, \ a = \frac{1}{2}, \ b = 1$$

Hence,

$$\begin{aligned} X(s) &= \frac{1}{2s} + \left(\frac{1}{2}\right)\left(\frac{s+2}{s^2+2s+2}\right) = \frac{1}{2s} + \left(\frac{1}{2}\right)\left(\frac{s+2}{(s+1)^2+1}\right) \\ &= \frac{1}{2s} + \left(\frac{1}{2}\right)\left(\frac{s+1}{(s+1)^2+1}\right) + \left(\frac{1}{2}\right)\left(\frac{1}{(s+1)^2+1}\right) \end{aligned}$$

where the expression for $X(s)$ is properly scaled and arranged for utility of Table 2.4.1. It follows from the Laplace transform table that the solution of the differential equation is

$$x(t) = \frac{1}{2} + \frac{1}{2}e^{-t}(\cos t + \sin t)$$

which is the same as obtained previously.

Examples 2.4.7 and 2.4.13 indicate a method for solution of linear differential equations with constant coefficients, which will be further explored later on in Section 2.6.

2.4.6 Piecewise-Continuous Functions in the Laplace Transformation

In many engineering applications, dynamic systems are subject to piecewise-continuous inputs such as, for example, pulses shown in Figure 2.4.2. These pulses physically represent external excitations persisting for the finite time duration.

The Laplace transform of this type of function can be performed through the application of the principle of superposition as described by Eq. (2.4.12) and the formula for time-shifted functions as given in Eq. (2.4.15). In general, a piecewise-continuous function can be decomposed as

$$f(t) = f_0(t)\, u(t) + f_1(t - T_1)\, u(t - T_1) + f_2(t - T_2)\, u(t - T_2) + \ldots \tag{2.4.44}$$

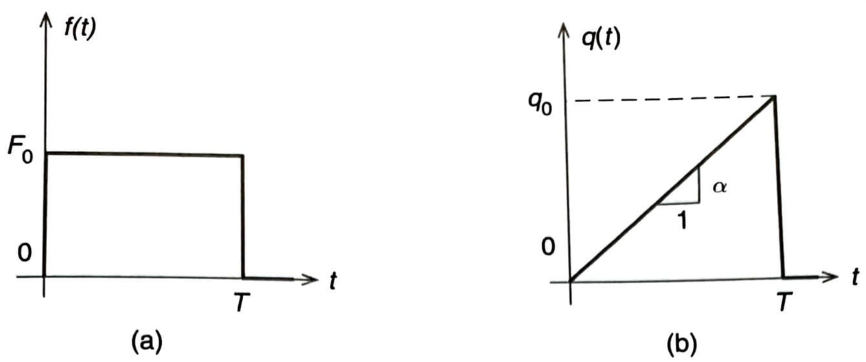

Figure 2.4.2 Piecewise-continuous functions: (a) rectangular pulse $f(t)$, (b) triangular pulse $q(t)$

where $f_0(t),\ f_1(t),\ f_2(t),\ \ldots$ are basic functions, as those in Table 2.4.1, $f_1(t-T_1)$, $f_2(t-T_2),\ \ldots$ are time-shifted functions, and $T_1,\ T_2,\ \ldots$ are positive time-delay parameters. The Laplace transform of $f(t)$ then has the form

$$F(s) = F_0(s) + F_1(s)e^{-T_1 s} + F_2(s)e^{-T_2 s} + \ldots \tag{2.4.45}$$

where $F_0(s),\ F_1(s),\ F_2(s), \ldots$ are the Laplace transforms of $f_0(t), f_1(t), f_2(t), \ldots$, respectively.

Example 2.4.14 Deriving the Laplace transform of a rectangular pulse

Consider the rectangular pulse $f(t)$ in Figure 2.4.2(a), which mathematically is given by

$$f(t) = \begin{cases} F_0, & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

It is easy to show that $f(t)$ can be expressed in a single expression

$$f(t) = F_0\, u(t) - F_0\, u(t-T)$$

where $u(t)$ is the unit step function defined in Eq. (2.4.5), $u(t-T)$ is the time-shifted unit step function given in Example 2.4.4, and T is the time-delay parameter. Graphically, $f(t)$ is illustrated in Figure 2.4.3(a). The Laplace transform of the rectangular pulse is

$$F(s) = \mathcal{L}\{F_0\, u(t) - F_0\, u(t-T)\} = \frac{F_0}{s}(1 - e^{-Ts})$$

where Eq. (2.4.15) has been used.

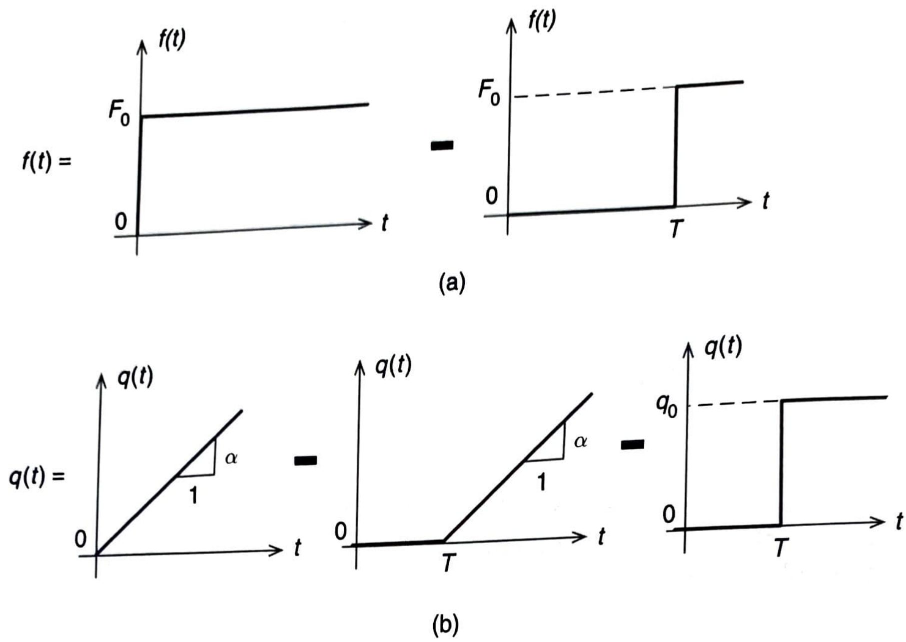

Figure 2.4.3 Graphical derivation of piecewise-continuous functions described in Figure 2.4.2: (a) rectangular pulse, (b) triangular pulse

Example 2.4.15 Deriving the Laplace transform of a triangular pulse

Consider the triangular pulse $q(t)$ in Figure 2.4.2(b), which is defined by

$$q(t) = \begin{cases} \alpha t, & 0 \le t \le T \\ 0, & t > T \end{cases}$$

with the slope $\alpha = q_0/T$. The pulse can be written as

$$q(t) = \alpha\, t\, u(t) - \alpha(t - T)\, u(t - T) - q_0\, u(t - T)$$

which is a linear combination of a ramp function ($\alpha\, t$), a delayed ramp function ($\alpha\,(t - T)$), and a delayed step function ($q_0\, u(t - T)$), as shown in Figure 2.4.3(b). The Laplace transform of this triangular pulse is

$$Q(s) = \frac{\alpha}{s^2}\left(1 - e^{-Ts}\right) - \frac{q_0}{s} e^{-Ts}.$$

$$= \lim_{a \to 0} \frac{\quad}{\dfrac{d}{da}(a)} \qquad \lim_{a \to 0} \quad 1$$

$$= -s + 2s = s$$

Problem A–2–5

Obtain the Laplace transform of the function $f(t)$ shown in Figure 2–10.

Solution The given function $f(t)$ can be defined as follows:

$$\begin{aligned} f(t) &= 0 && t \le 0 \\ &= \frac{b}{a}t && 0 < t \le a \\ &= 0 && a < t \end{aligned}$$

Notice that $f(t)$ can be considered a sum of the three functions $f_1(t)$, $f_2(t)$, and $f_3(t)$ shown in Figure 2–11. Hence, $f(t)$ can be written as

$$\begin{aligned} f(t) &= f_1(t) + f_2(t) + f_3(t) \\ &= \frac{b}{a}t \cdot 1(t) - \frac{b}{a}(t-a) \cdot 1(t-a) - b \cdot 1(t-a) \end{aligned}$$

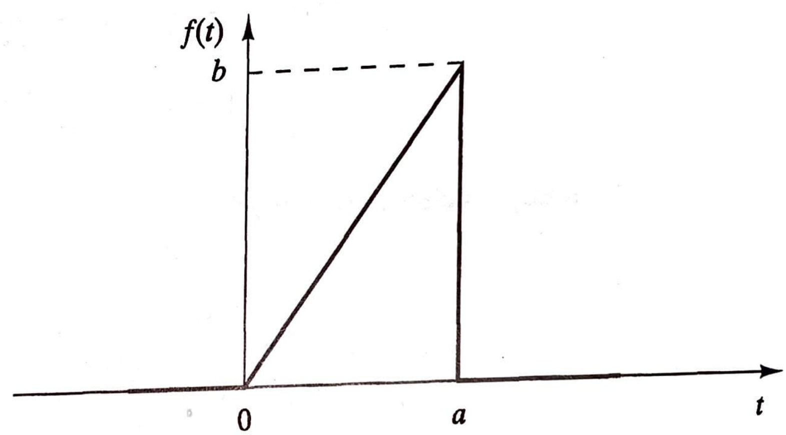

Figure 2–10 Function $f(t)$.

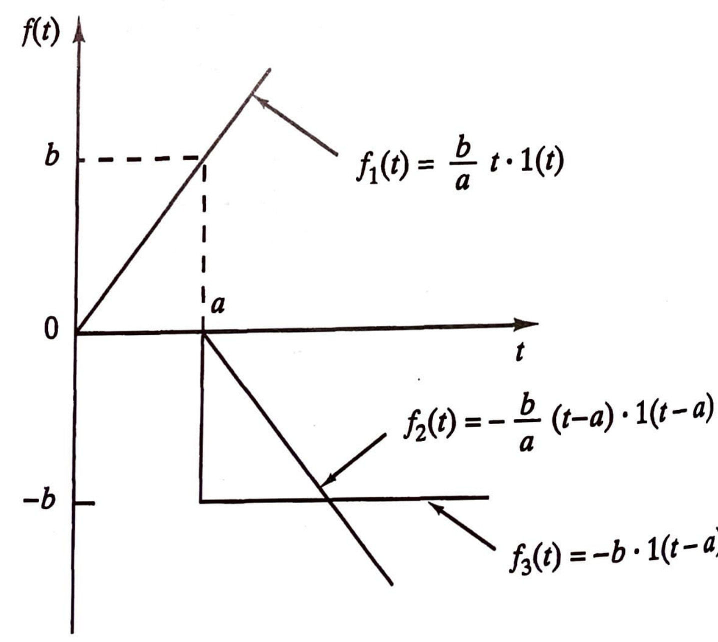

Figure 2–11 Functions $f_1(t)$, $f_2(t)$, and $f_3(t)$.

Then the Laplace transform of $f(t)$ becomes

$$
\begin{aligned}
F(s) &= \frac{b}{a}\frac{1}{s^2} - \frac{b}{a}\frac{1}{s^2}e^{-as} - b\frac{1}{s}e^{-as} \\
&= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}
\end{aligned}
$$

The same $F(s)$ can, of course, be obtained by performing the following Laplace integration:

$$
\begin{aligned}
\mathscr{L}[f(t)] &= \int_0^a \frac{b}{a}te^{-st}\,dt + \int_a^\infty 0\,e^{-st}\,dt \\
&= \frac{b}{a}t\frac{e^{-st}}{-s}\bigg|_0^a - \int_0^a \frac{b}{a}\frac{e^{-st}}{-s}\,dt \\
&= b\frac{e^{-as}}{-s} + \frac{b}{as}\frac{e^{-st}}{-s}\bigg|_0^a \\
&= b\frac{e^{-as}}{-s} - \frac{b}{as^2}(e^{-as} - 1) \\
&= \frac{b}{as^2}(1 - e^{-as}) - \frac{b}{s}e^{-as}
\end{aligned}
$$

2.6.2 Laplace Transform of a Time-Delayed Function

In dynamic systems, it is common for the input to start at a delayed time (i.e., not $t = 0$). A simple example is the step input, $u(t - a)$, which could, for example, be an electric switch turned on at $t = a$ (later than $t = 0$). Alternately, a load might suddenly be applied at $t = a$. Some systems, such as the tool vibration in a machining operation, have inherent delays in the forcing function that can cause instability. Figure 2.8 shows a general function that begins at $t = 0$, while Fig. 2.9 shows the same function that is delayed until $t = a$. The goal is to determine the Laplace transform of the delayed function in terms of $F(s)$, the presumably known Laplace transform of the un-delayed function.

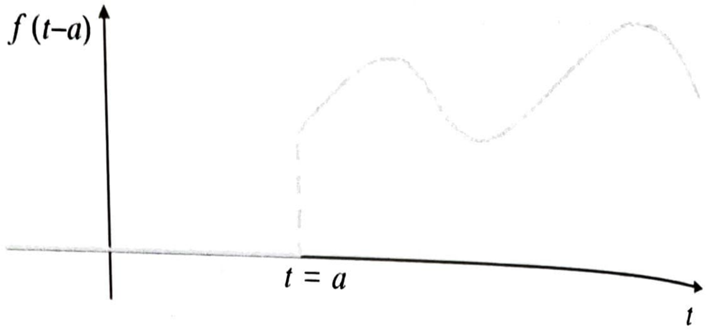

Fig. 2.9 A function, $f(t-a)$, that is zero before $t=a$

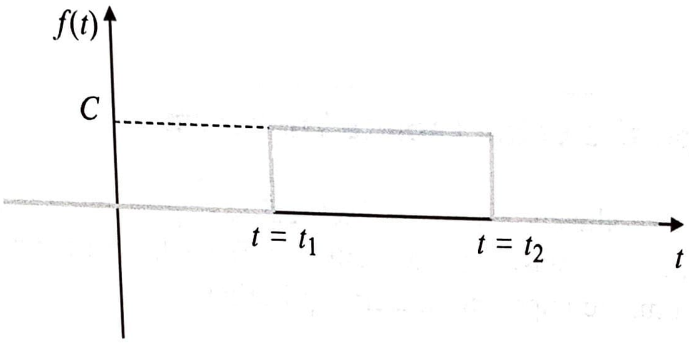

Fig. 2.10 A pulse function with amplitude C between $t=t_1$ and $t=t_2$ and zero at all other times

Applying the definition of the Laplace transform, we obtain $\mathcal{L}[f(t-a)] = \int_0^\infty f(t-a)e^{-st}dt$. Next, we substitute: $\rho = t-a$ and $d\rho = dt$ to obtain a modified version of the original integral.

$$\mathcal{L}[f(t-a)] = \int_{-a}^{\infty} f(\rho)e^{-s(\rho+a)}d\rho = e^{-sa}\int_{-a}^{\infty} f(\rho)e^{-s\rho}d\rho$$

Because $\rho = 0$ is equivalent to $t=a$, the integral limits can be redefined to be $\mathcal{L}[f(t-a)] = e^{-sa}\int_0^\infty f(\rho)e^{-s\rho}d\rho$. We see that the integral is simply $F(s)$, the Laplace transform of the un-delayed function. Substituting yields the final transform: $\mathcal{L}[f(t-a)] = e^{-sa}F(s)$.

In the Laplace domain, multiplying a function by e^{-sa} is equivalent to delaying the function by a in the time domain. For this reason, e^{-sa} is known as the delay operator and can be used to delay any function by an arbitrary time, a.

Example 2.9 Find the Laplace transform of the pulse function displayed in Fig. 2.10 that is initiated at $t=t_1$ with an amplitude of C and becomes zero again at $t=t_2$.

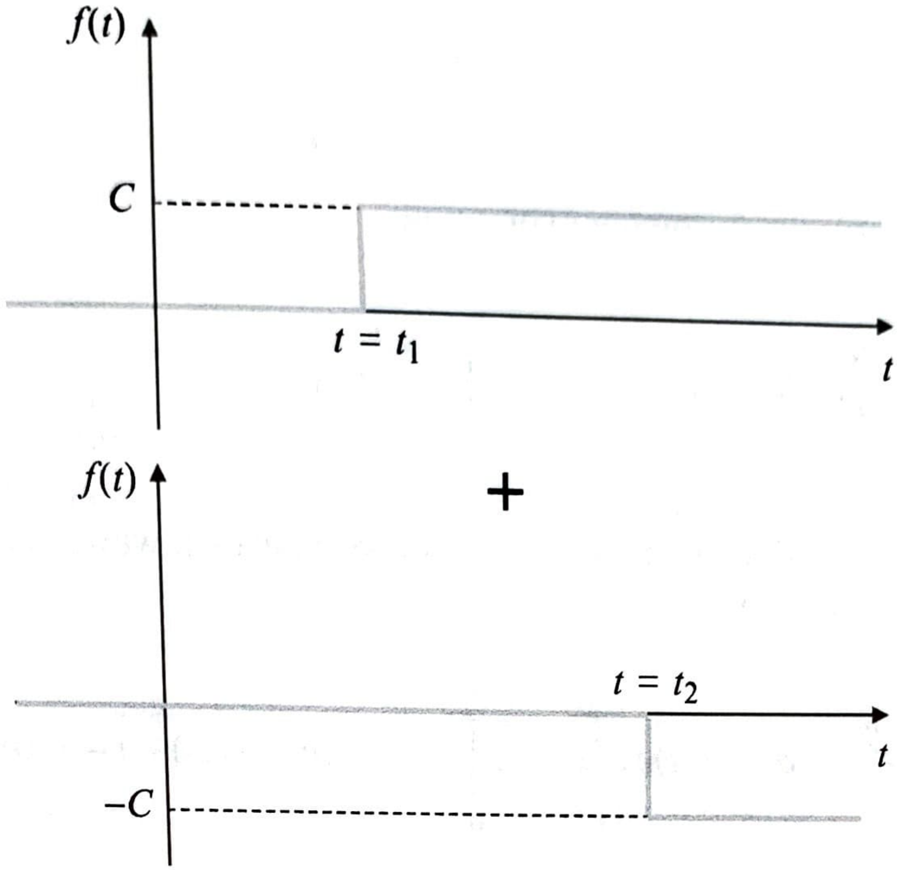

Fig. 2.11 The pulse function as a combination of two-step functions

Solution The function is the sum of the two-step functions shown in Fig. 2.11.

The straightforward solution is to sum the positive and negative steps, each the appropriate time delay, in the Laplace domain.

$$F(s) = \left(e^{-st_1}\frac{C}{s}\right) + \left(e^{-st_2}\frac{-C}{s}\right) = C\left(\frac{e^{-st_1}}{s} - \frac{e^{-st_2}}{s}\right)$$