Homework 2

1 Problem

For each of the following ODEs determine if the eigenvalues are (a) real and distinct, (b) repeated, (c) complex conjugate pairs:

1. $\ddot{x} + 2\dot{x} + 3x = 0$
2. $\ddot{x} + 4\dot{x} + x = 0$
3. $\ddot{x} + 4\dot{x} + 4x = 0$
4. $\ddot{x} + 3x = 0$

Solution

The solution is found by evaluating $\sqrt{b^2 - 4ac}$ in each case.

1. $\sqrt{b^2 - 4ac} = 2.82i \implies$ Complex conjugate pairs
2. $\sqrt{b^2 - 4ac} = 3.4641 \implies$ Real
3. $\sqrt{b^2 - 4ac} = 0 \implies$ Repeated
4. $\sqrt{b^2 - 4ac} = 3.4641i \implies$ Complex conjugate pairs

2 Problem

For each of the following linear, time-invariant, 2nd order homogeneous ODEs solve for the particular solution that satisfies the initial values given.

1. $\ddot{x} - 4\dot{x} + 4x = 0$, Initial Values: $x(0) = 12$, $\dot{x}(0) = -3$
2. $\ddot{x} + 3\dot{x} - 10x = 0$, Initial Values: $x(0) = 4$, $\dot{x}(0) = -2$
3. $\ddot{x} - 8\dot{x} + 17x = 0$, Initial Values: $x(0) = -4$, $\dot{x}(0) = -1$

Solution

1. $\ddot{x} - 4\dot{x} + 4x = 0$, Initial Values: $x(0) = 12$, $\dot{x}(0) = -3$

Example 1 Solve the following IVP.

$$y'' - 4y' + 4y = 0 \quad y(0) = 12 \quad y'(0) = -3$$

Hide Solution ▾

The characteristic equation and its roots are.

$$r^2 - 4r + 4 = (r-2)^2 = 0 \quad r_{1,2} = 2$$

The general solution and its derivative are

$$y(t) = c_1 \mathbf{e}^{2t} + c_2 t \mathbf{e}^{2t}$$
$$y'(t) = 2c_1 \mathbf{e}^{2t} + c_2 \mathbf{e}^{2t} + 2c_2 t \mathbf{e}^{2t}$$

Don't forget to product rule the second term! Plugging in the initial conditions gives the following system.

$$12 = y(0) = c_1$$
$$-3 = y'(0) = 2c_1 + c_2$$

This system is easily solved to get $c_1 = 12$ and $c_2 = -27$. The actual solution to the IVP is then.

$$y(t) = 12\mathbf{e}^{2t} - 27t\mathbf{e}^{2t}$$

2. $\ddot{x} + 3\dot{x} - 10x = 0$, Initial Values: $x(0) = 4$, $\dot{x}(0) = -2$

Example 2 Solve the following IVP

$$y'' + 3y' - 10y = 0 \quad y(0) = 4 \quad y'(0) = -2$$

Hide Solution ▾

The characteristic equation is

$$r^2 + 3r - 10 = 0$$
$$(r+5)(r-2) = 0$$

Its roots are $r_1 = -5$ and $r_2 = 2$ and so the general solution and its derivative is.

$$y(t) = c_1 \mathbf{e}^{-5t} + c_2 \mathbf{e}^{2t}$$
$$y'(t) = -5c_1 \mathbf{e}^{-5t} + 2c_2 \mathbf{e}^{2t}$$

Now, plug in the initial conditions to get the following system of equations.

$$4 = y(0) = c_1 + c_2$$
$$-2 = y'(0) = -5c_1 + 2c_2$$

Solving this system gives $c_1 = \frac{10}{7}$ and $c_2 = \frac{18}{7}$. The actual solution to the differential equation is then

$$y(t) = \frac{10}{7}\mathbf{e}^{-5t} + \frac{18}{7}\mathbf{e}^{2t}$$

3. $\ddot{x} - 8\dot{x} + 17x = 0$, Initial Values: $x(0) = -4$, $\dot{x}(0) = -1$

Example 2 Solve the following IVP.

$$y'' - 8y' + 17y = 0 \quad y(0) = -4 \quad y'(0) = -1$$

Hide Solution ▾

The characteristic equation this time is.

$$r^2 - 8r + 17 = 0$$

The roots of this are $r_{1,2} = 4 \pm i$. The general solution as well as its derivative is

$$\begin{aligned} y(t) &= c_1 \mathbf{e}^{4t} \cos(t) + c_2 \mathbf{e}^{4t} \sin(t) \\ y'(t) &= 4c_1 \mathbf{e}^{4t} \cos(t) - c_1 \mathbf{e}^{4t} \sin(t) + 4c_2 \mathbf{e}^{4t} \sin(t) + c_2 \mathbf{e}^{4t} \cos(t) \end{aligned}$$

Notice that this time we will need the derivative from the start as we won't be having one of the terms drop out. Applying the initial conditions gives the following system.

$$\begin{aligned} -4 &= y(0) = c_1 \\ -1 &= y'(0) = 4c_1 + c_2 \end{aligned}$$

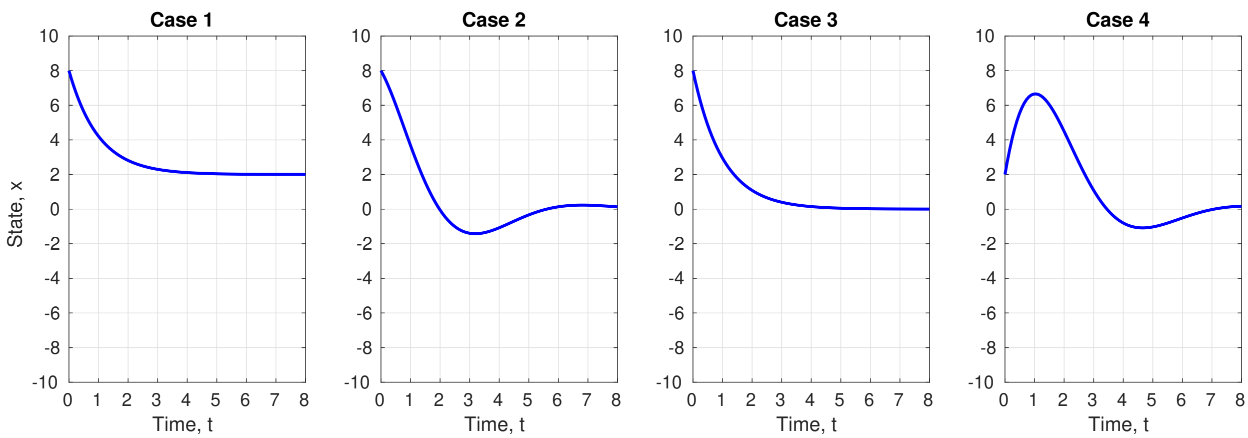
Solving this system gives $c_1 = -4$ and $c_2 = 15$. The actual solution to the IVP is then.

$$y(t) = -4\mathbf{e}^{4t} \cos(t) + 15\mathbf{e}^{4t} \sin(t)$$

3 Problem

Match each one of the responses shown below (Cases 1-4) with one of the following IVPs. That is, for each case pick just one response (a) through (j) that is the best match.

Hint: This problem can be solved "by inspection" (i.e., no need to workout full solutions) although it may require some short calculations. The initial conditions (ICs) give a clue and can eliminate some possibilities. Checking some basic properties such as whether the 1st order system is stable (decaying) or unstable (diverging) is another clue. Perhaps the time-constant or whether the system is homogeneous or in-homogeneous. Note also that first order systems can only exhibit 3 types of behaviors, while 2nd order systems can be one of three cases, including cases where the direction of changes or oscillates with decreasing amplitude.



(a) $\ddot{x} + \dot{x} + x = 0, x(0) = 8, \dot{x}(0) = 0$

(b) $\dot{x} - x = 2,\ x(0) = 8$

(c) $\dot{x} + x = 2,\ x(0) = 8$

(d) $\ddot{x} + \dot{x} + x = 2, x(0) = 0, \dot{x}(0) = -8$

(e) $\ddot{x} + \dot{x} + x = 2, x(0) = 2, \dot{x}(0) = 0$

(f) $\dot{x} + x = 8,\ x(0) = 2$

(g) $\dot{x} + x = 0,\ x(0) = 8$

(h) $\ddot{x} + \dot{x} + x = 0, x(0) = 8, \dot{x}(0) = -3$

(i) $\ddot{x} + \dot{x} + x = 0, x(0) = 2, \dot{x}(0) = 10$

(j) $\dot{x} + 8x = 0,\ x(0) = 0$

Solution

- By inspection, Case 1 is a first-order ODE with initial condition $x(0) = 8$. Since the solution decays to $x = 2$ it must have $a > 0$ and be inhomogeneous. Answer: (c)
- By inspection, Case 2 is a second-order ODE with initial condition $x(0) = 8$ and $\dot{x}(0) < 0$. Answer: (h)
- By inspection, Case 3 is a first-order ODE with initial condition $x(0) = 8$. Since the solution decays to zero it must have $a > 0$ and be homogeneous. Answer: (g)
- By inspection, Case 4 is a second-order ODE with initial condition $x(0) = 2$ and $\dot{x}(0) > 0$. Answer: (i)

4 Problem

Let $z = x + iy$. Determine the value of x and y in each of the following cases 1-3 below. Show all of your work for full credit. You can check your answer in MATLAB by typing in the expression with `1i` representing the imaginary number (e.g., `(3+1i)*(1+3i)`). For cases 4-5, answer the question directly.

1. $z = (3 + i)(1 + 3i)$
2. $z = i^4 - 1$
3. $z = \frac{3+i}{1+3i}$
4. If $w = 1 + 2i$, then what is $|w|$ and $\theta = \arg(w)$?
5. If $z_1 = -i$ and $z_2 = e^{i\pi/2}$, then what is the sum $z_1 + z_2$?

Solution

Let $z = x + iy$. Determine the value of x and y in each of the following cases:

1. $x = 0$ and $y = 10$

$$\begin{aligned} z &= (3 + i)(1 + 3i) \\ &= 3 + 9i + i + 3i^2 \\ &= 3 + 9i + i - 3 \\ &= 10i \end{aligned}$$

2. $x = 0$ and $y = 0$

$$\begin{aligned} z &= i^4 - 1 \\ &= (i^2)(i^2) - 1 \\ &= (-1)(-1) - 1 \\ &= 0 \end{aligned}$$

3. $x = 3/5$ and $y = -4/5$

$$\begin{aligned}
z &= \frac{3+i}{1+3i} \\
&= \frac{3+i}{1+3i}\left(\frac{1-3i}{1-3i}\right) \\
&= \frac{3-9i+i-3i^2}{1-3i+3i-9i^2} \\
&= \frac{3-8i+3}{1+9} \\
&= \frac{3-8i+3}{1+9} \\
&= \frac{6-8i}{10} \\
&= \frac{3-4i}{5}
\end{aligned}$$

4. $|w| = \sqrt{1+2^2} = \sqrt{5}$ and $\theta = \arg(w) = \text{atan}(2) \approx 63.435$ deg.

5. Using Euler's Formula (or by inspection) $z_2 = i$ so that $z_1 + z_2 = -i + i = 0$.

5 MATLAB Problem

The McGuire Nuclear Station in Huntersville, NC is testing a new isotope of radioactive material called *nineridium*. Engineers have determined that the material exhibits exponential decay according to the ODE:

$$\dot{N} = -kN$$

where $N(t)$ is the number of parent atoms, time t has units of years, and $k = 1$ (1/year) is the decay rate. If the reactor starts with a chunk of nineridium that consists of $N(t_0) = 1000$ atoms at time $t_0 = 0$, then:

- Part A (5 pts). What is the expression that gives the number of atoms, $N(t)$, for any future time $t \geq t_0$?
- Part B (5 pts). What is the time constant describing the decay?
- Part C (5 pts). Using MATLAB, plot the function $N(t)$ out to 5 time constants. Label your axes and include appropriate units. Your submission should include both your code and the resulting graph. MATLAB Hints:
  - If you don't have MATLAB installed you can download it from `software.uncc.edu`. If you are feeling rusty, please review the MATLAB help files provided (you may wish to complete the "On Ramp" tutorial).
  - to plot a solid line of width 2 with circular markers in MATLAB use the the command `plot(time, x, 'ro-','linewidth',2)` where `time` is a vector of increasing time values, `x` is a vector of data points to be plotted with time. You can change the color of the line by replacing the `r` in `ro-` with other letters corresponding to colors e.g., blue `b`, magenta `m`, green `g`.
- Part D (5 pts). Based on your plot, what is the approximate half-life of nineridium (i.e., how many years does it take the material to decay to 50% of the initial amount)?

Solution

See following page.

MATLAB Problem

Part A. Since this is a first order ODE we can write the solution directly from the problem statement as:

$$N(t) = N_0 e^{-kt}$$

$$N(t) = 1000 e^{-t}$$

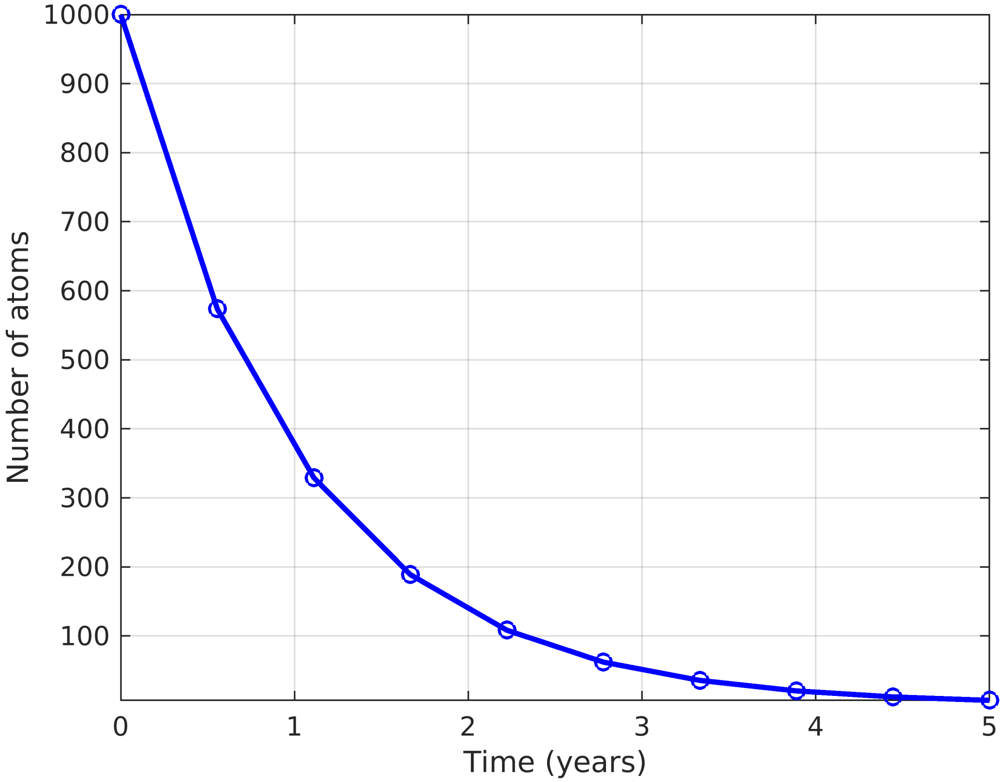
Part B. The time constant describing the decay is:

$\tau = \frac{1}{k} = 1$ year

Part C.

```
clear; close all; clc;
k = 1; % decay rate
N0 = 1000; % initial number of atoms
tau = 1/k; % time constants
t = linspace(0,5*tau,10);  % vector of time values at which to evaluate N(t)
N = N0*exp(-k*t); % from Part A

figure;
plot(t,N,'bo-','linewidth',2)
grid on;
xlabel('Time (years)')
ylabel('Number of atoms')
set(gca,'FontSize',10)
axis tight;
xticks([0:1:5])
```



Part D.

From the graph above it seems the half life (500 atoms) occurs after about 0.7 years. To precisely calculate this number we can use the expression from Part A. The exact value is 0.6931 years.

```
log(500/1000)/-k
```

```
ans = 0.6931
```