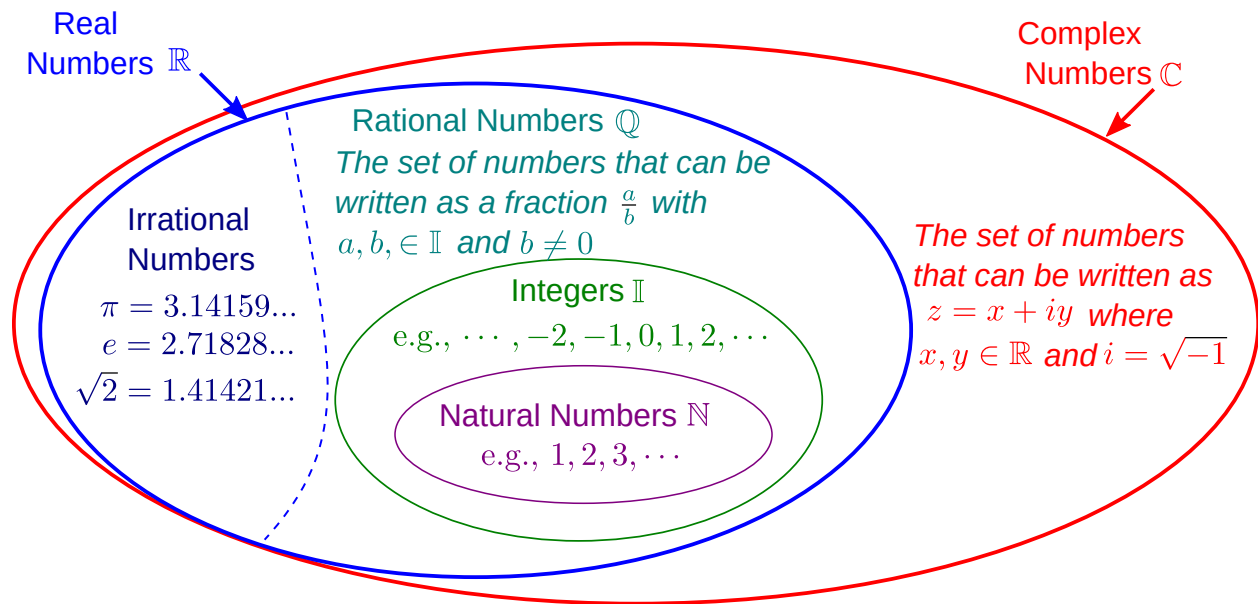


## Lecture 5: Complex Arithmetic

### Types of Numbers

Numbers can be categorized into particular types that depend on their value, as illustrate by the Venn diagram below. For example, the natural numbers are a subset of the integers, the set of real numbers consists of all rational numbers and irrational number, and complex numbers extend the real numbers to include the *imaginary number*. The imaginary number is usually denoted with the symbol  $i$  or  $j$  and is equal to  $i = \sqrt{-1}$ . From this definition it immediately follows that  $i^2 = \sqrt{-1}\sqrt{-1} = -1$ . To denote the type of a number we use the symbol " $\in$ " which means "is an element of the set" or simply "in the set". For example if  $z = 3 + 1i$  is a complex number, we write  $z \in \mathbb{C}$  meaning "z is in the set of complex numbers  $\mathbb{C}$ ".



*Remark:* Equations that contain all real constants or coefficients may have complex-valued solutions. Consider the quadratic equation

$$x^2 - 4x + 5 = 0.$$

This equation represents a parabola and has all integer coefficients. Yet the solution is a complex number,  $x = 2 \pm i$ , and indicates that the parabola does not intercept the  $x$  axis.

### Rectangular and Polar Forms

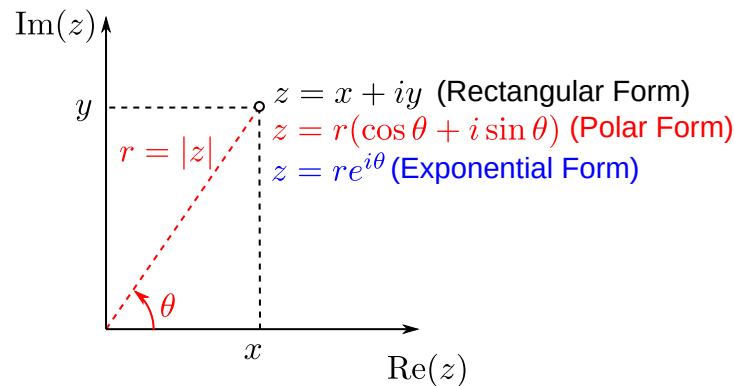
A complex number  $z$  can be represented either in rectangular or polar form. In *rectangular form*,  $z$  is written as:

$$z = \underbrace{x}_{\text{real part}} + i \underbrace{y}_{\text{imaginary part}}.$$

To denote the real and imaginary parts of  $z$  we use the notation

$$\operatorname{Re}(z) = x \quad \text{and} \quad \operatorname{Im}(z) = y$$

Complex numbers can be sketched as a point (or a ray extending from the origin) in the complex plane. The real part of a complex number gives the abscissa coordinate (the horizontal axis) in the complex plane, and the imaginary part gives the ordinate (vertical axis), as shown below.



The *modulus* (also called the absolute value or magnitude) of a complex number is its distance from the origin

$$|z| = \sqrt{x^2 + y^2}$$

and the *argument* is the angle that the ray joining the origin to  $z$  makes with the Re (real) axis

$$\theta = \arg(z) = \operatorname{atan}\left(\frac{y}{x}\right).$$

*Remark:* The function  $\operatorname{atan}(y/x)$  uses the ratio  $y/x$  to return an angle that is always between  $-\pi/2$  and  $\pi/2$ . However, this angle is not unique, and it does distinguish between  $(y = 1, x = 1)$  and  $(y = -1, x = -1)$  since they both have the same ratio of  $y/x = 1$ . Instead, the four-quadrant arctangent function  $\operatorname{atan2}(y, x)$  can be used as both a mathematical function (and a MATLAB function) to correctly distinguish between the two cases (the first one giving 45 deg. and the latter one giving 225 deg.).

From the geometry, it is clear that

$$x = |z| \cos \theta$$

$$y = |z| \sin \theta$$

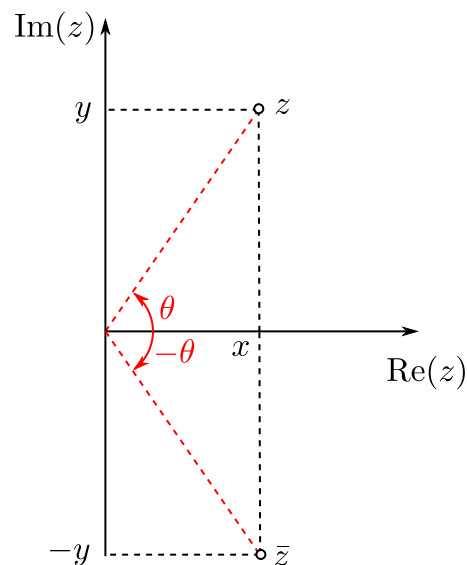
so that we may re-write the complex number in *polar form* as:

$$z = r(\cos \theta + i \sin \theta)$$

where it is understood that  $r = |z|$ . The polar form is sometimes abbreviated as  $(r, \theta)$ .

## Complex Conjugate

The complex conjugate of  $z$  is the reflection of  $z$  about the Re axis as shown below.



If  $z = x + iy$ , the complex conjugate is denoted by an overbar and defined as  $\bar{z} = x - iy$ . In polar form, the complex number  $(r, \theta)$  has conjugate  $(r, -\theta)$ . Thus, using trigonometric identities,

$$\begin{aligned}\bar{z} &= r(\cos(-\theta) + i \sin(-\theta)) \\ &= r(\cos \theta - i \sin \theta)\end{aligned}$$

## Complex Arithmetic (Rectangular Form)

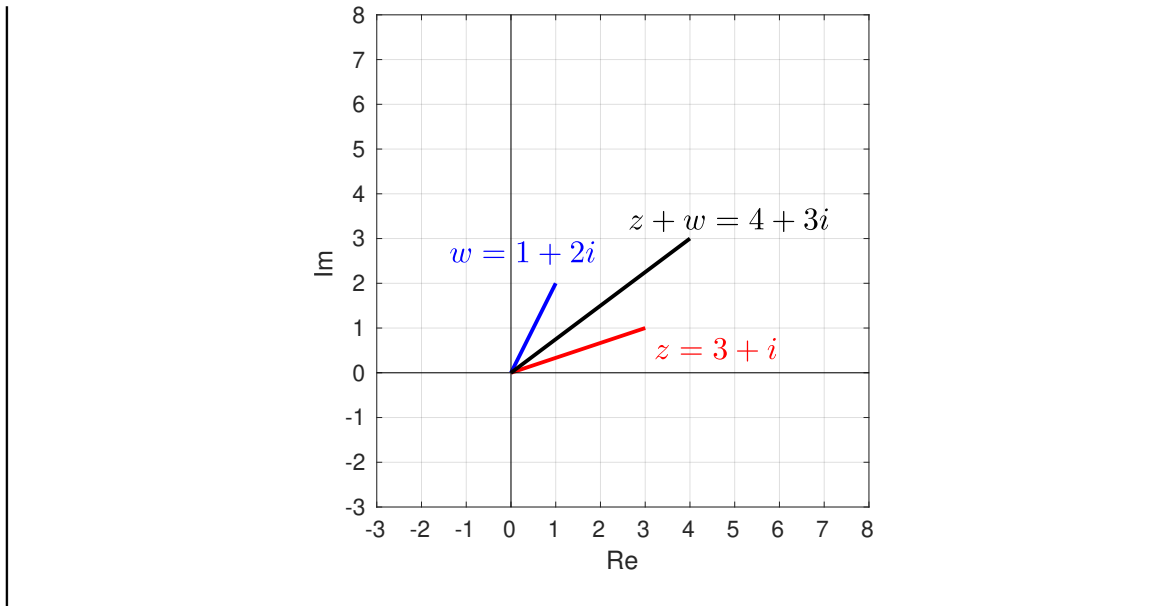
Consider the two complex numbers in rectangular form:  $z = x + iy$  and  $w = u + iv$ . In the following, we develop expressions and show examples for the addition, subtraction, multiplication, and division of these complex numbers.

- Addition: The real and imaginary components are simply added together, similar to vector addition:

$$\begin{aligned}z + w &= (x + iy) + (u + iv) \\ &= (x + u) + i(y + v)\end{aligned}$$

*Example:* Let  $z = 3 + i$  and  $w = 1 + 2i$ , then

$$z + w = (3 + 1) + (1 + 2)i = 4 + 3i$$

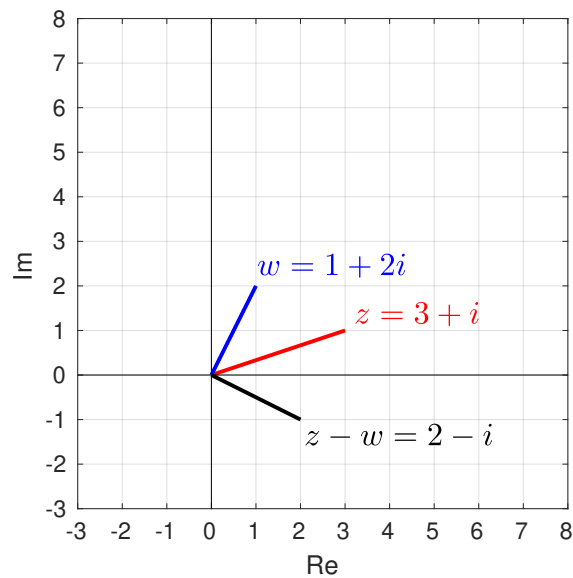


- Subtraction: The real and imaginary components are subtracted from one another, similar to vector subtraction:

$$\begin{aligned} z - w &= (x + iy) - (u + iv) \\ &= (x - u) + i(y - v) \end{aligned}$$

*Example:* Let  $z = 3 + i$  and  $w = 1 + 2i$ , then

$$z - w = (3 - 1) + (1 - 2)i = 2 - i$$



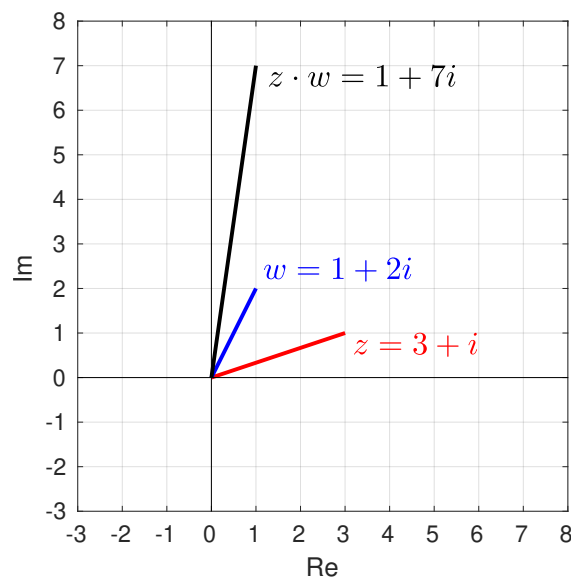
- Multiplication: We expand the two terms multiplied and simplify using the fact that  $i^2 = -1$

$$\begin{aligned}
 z \cdot w &= (x + iy)(u + iv) \\
 &= xu + ixv + iyu + \underbrace{i^2}_{=-1} yv \\
 &= (xu - yv) + i(xv + yu)
 \end{aligned}$$

*Example:* Let  $z = 3 + i$  and  $w = 1 + 2i$ , then

$$z \cdot w = (3 \cdot 1 - 1 \cdot 2) + (3 \cdot 2 + 1 \cdot 1)i = 1 + 7i$$

Notice that multiplication of  $w$  by  $z$  both scales and rotates the vector  $w$ .



- Multiplication by a complex conjugate gives the modulus squared

$$\begin{aligned}
 z \cdot \bar{z} &= (x + iy)(x - iy) \\
 &= x^2 - \underbrace{ixy + iyx}_{=0} - i^2 y^2 \\
 &= x^2 + y^2 = |z|^2
 \end{aligned}$$

*Example:* Let  $z = 3 + i$ , then  $\bar{z} = 3 - i$  and

$$z \cdot \bar{z} = (3 + i)(3 - i) = 9 - 3i + 3i - i^2 = 10$$

which is equal to the modulus since

$$|z|^2 = 3^2 + 1^2 = 10$$

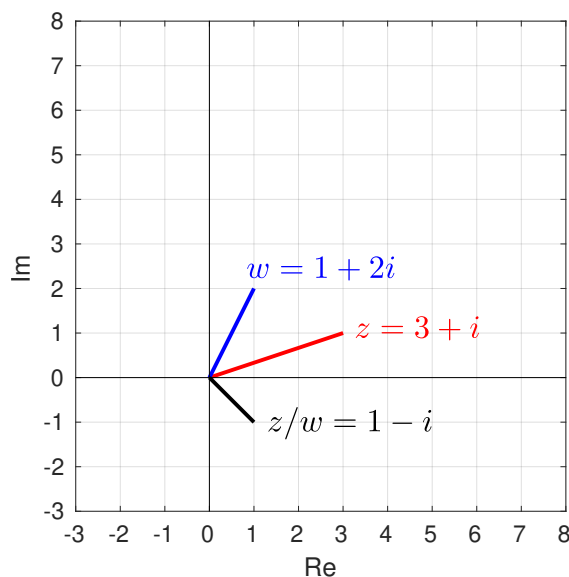
- Division: We first multiply the top and bottom by a complex conjugate of the denominator ( $\bar{w} = u - iv$ ), then simplify using the fact that  $w \cdot \bar{w} = (u + iv)(u - iv) = |w|^2 = u^2 + v^2$

$$\begin{aligned} \frac{z}{w} &= \frac{z}{w} \left( \frac{\bar{w}}{\bar{w}} \right) \\ &= \frac{(x + iy)(u - iv)}{(u + iv)(u - iv)} \\ &= \frac{xu - ixv + iyu - i^2 yv}{u^2 + v^2} \\ &= \frac{(xu + yv) + i(yu - xv)}{u^2 + v^2} \end{aligned}$$

Example: Let  $z = 3 + i$  and  $w = 1 + 2i$ . Then,  $\bar{w} = 1 - 2i$  and

$$\frac{z}{w} = \frac{(3 + i)(1 - 2i)}{(1 + 2i)(1 - 2i)} = \frac{3 - 6i + i - 2i^2}{1^2 + 2^2} = \frac{5 - 5i}{5} = 1 - i$$

Notice, again, that division of  $z$  by  $w$  both scales and rotates the vector  $z$ .



## Euler's Formula

To obtain expressions for the arithmetic operations described above with  $z$  and  $w$  expressed in polar form we could simply substitute their polar equivalents. But another representation using

exponentials is quite useful and relies on *Euler's formula*:

$$e^{i\theta} = \cos \theta + i \sin \theta .$$

By using Euler's formula, the complex number  $z = r(\cos \theta + i \sin \theta)$  is converted into *exponential form* as  $z = re^{i\theta}$ . Evaluating Euler's formula at  $\theta = \pi$  we find that

$$\begin{aligned} e^{i\pi} &= \cos \pi + i \sin \pi \\ &= -1 + i(0) \end{aligned}$$

which can be rearranged to give *Euler's identity*:

$$e^{i\pi} + 1 = 0$$

For an explanation of how Euler's formula arises refer to the remark below.

*Remark:* At first glance, Euler's formula looks a bit unusual—what does cos and sin have to do with the exponential  $e^{i\theta}$ ? To gain some insight, we can look at the Taylor series representations of these functions. Recall that a Taylor series can represent a function  $f(x)$  as an infinite summation (a power series) of the form:

$$\begin{aligned} \sum_{n=0}^{\infty} a_n(x-b)^n &= a_0 \underbrace{(x-b)^0}_{=1} + a_1(x-b)^1 + a_2(x-b)^2 + \dots \\ &= a_0 + a_1(x-b) + a_2(x-b)^2 + \dots \end{aligned}$$

where the coefficients  $a_0, a_1, \dots, a_n$  are determined by evaluating the function  $f(x)$  and its derivatives at a point  $x = b$  where the approximation is centered:

$$a_0 = f(b) \quad a_1 = f'(b) \quad a_2 = f''(b) \quad \dots \quad a_i f^{(i)}(x) \Big|_{x=b}$$

Using this method, the Taylor series approximation of the exponential, sine, and cosine functions around  $x = 0$  are

$$\begin{aligned} e^x &= 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \end{aligned}$$

Now, returning to our problem, substitute in  $x = i\theta$  into the Taylor series for the expo-

nential function above and replace  $i^2 = -1$  everywhere it appears

$$\begin{aligned}
 e^{i\theta} &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \dots \\
 &= 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^2i\theta^3}{3!} + \frac{i^2i^2\theta^4}{4!} + \frac{ii^2i^2\theta^5}{5!} + \dots \\
 &= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \frac{i\theta^5}{5!} + \dots \\
 &= \underbrace{\left[1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \dots\right]}_{\cos \theta} + i \underbrace{\left[\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} + \dots\right]}_{\sin \theta}
 \end{aligned}$$

The simplified expressions can be rearranged into the Taylors series for sine and cosine that is related through Euler's formula.

### Complex Arithmetic (Exponential Form)

Recall that a complex number in rectangular form is written as  $z = x + iy$ . Let  $r = |z|$  be the distance of the complex-valued point from the origin (i.e., the modulus) and let  $\theta$  be the angle with the horizontal, each of the rectangular components can then be written as:

$$\begin{aligned}
 x &= r \cos \theta \\
 y &= r \sin \theta .
 \end{aligned}$$

So that, using Euler's formula,  $z$  becomes

$$\begin{aligned}
 z &= (r \cos \theta) + i(r \sin \theta) \\
 &= r(\cos \theta + i \sin \theta) \\
 &= re^{i\theta} .
 \end{aligned}$$

Now, consider two complex numbers represented in exponential form

$$\begin{aligned}
 z_1 &= r_1 e^{i\theta_1} \\
 z_2 &= r_2 e^{i\theta_2} ,
 \end{aligned}$$

and the corresponding arithmetic operations. Addition and subtraction is not particularly simplified by using the exponential form:

- Addition:

$$z_1 + z_2 = r_1 e^{i\theta_1} + r_2 e^{i\theta_2}$$

- Subtraction:

$$z_1 - z_2 = r_1 e^{i\theta_1} - r_2 e^{i\theta_2}$$

But using the exponential form for multiplication and division takes advantage of the power and quotient rule for exponents.



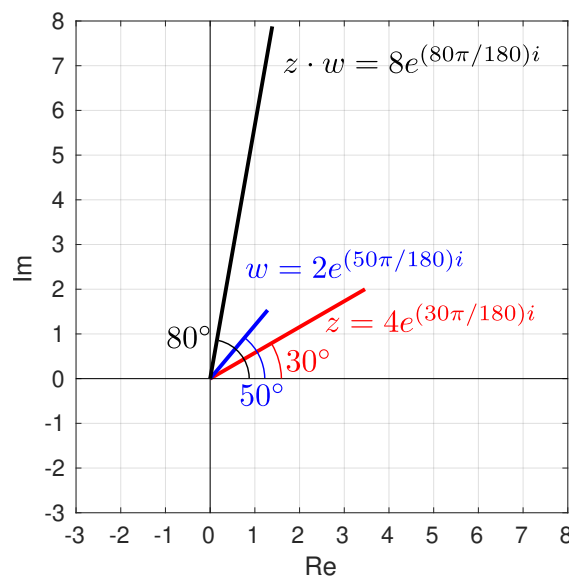
- Multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} \\ &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

Now it is much easier to interpret multiplying two complex numbers. If  $z_1$  has length  $r_1$  and  $z_2$  has length  $r_2$ , the  $z_1 \cdot z_2$  has length  $r_1 r_2$ . Moreover, the angle of the resulting vector  $z_1 \cdot z_2$  is the addition of the two angles for  $z_1$  and  $z_2$ .

*Example:* Let  $z = 4e^{(30\pi/180)i}$  and  $w = 2e^{(50\pi/180)i}$ . Then,

$$z \cdot w = (4e^{(30\pi/180)i})(2e^{(50\pi/180)i}) = (4 \cdot 2)e^{[(30+50)\pi/180]i} = 8e^{(80\pi/180)i}$$



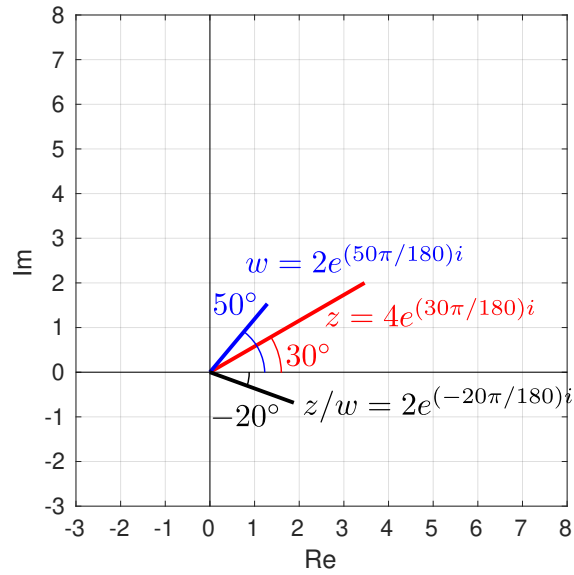
- Division:

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1 - \theta_2)}$$

As with multiplication, it is now much easier to interpret division of two complex numbers when using exponential form. If  $z_1$  has length  $r_1$  and  $z_2$  has length  $r_2$ , the  $z_1/z_2$  has length  $r_1/r_2$ . Moreover, the angle of the resulting vector  $z_1/z_2$  is the difference of the two angles for  $z_1$  and  $z_2$ .

*Example:* Let  $z = 4e^{(30\pi/180)i}$  and  $w = 2e^{(50\pi/180)i}$ . Then,

$$z/w = (4e^{(30\pi/180)i})(2e^{(50\pi/180)i}) = (4/2)e^{[(30-50)\pi/180]i} = 2e^{(-20\pi/180)i}$$



*Example. Part A: Consider the two complex numbers  $z_1 = 5e^{i\pi/2}$  and  $z_2 = 1 + i$ . Convert  $z_2$  into exponential form and compute  $z_1/z_2$ .*

The modulus of  $z_2$  is  $|z_2| = \sqrt{2}$  and the polar angle is  $\theta = \text{atan2}(1,1) = \pi/4$ . Thus,  $z_2 = \sqrt{2}e^{i\pi/4}$  and

$$\frac{z_1}{z_2} = \frac{5e^{i\pi/2}}{\sqrt{2}e^{i\pi/4}} = \frac{5}{\sqrt{2}}e^{i\pi/4}. \quad (1)$$

*Part B: Convert  $z_1$  into rectangular form and compute  $z_1/z_2$ .*

In rectangular coordinates,

$$z_1 = (5 \cos(\pi/2)) + i(5 \sin(\pi/2)) = 5i \quad (2)$$

and

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{5i}{1+i} \frac{1-i}{1-i} \\ &= \frac{5i - 5i^2}{1+i+1-i} \\ &= \frac{5+5i}{2} \end{aligned}$$

*Part C: Show that the two expressions are equivalent. Using Euler's formula:*

$$\begin{aligned} \frac{5}{\sqrt{2}}e^{i\pi/4} &= \frac{5}{\sqrt{2}}(\cos(\pi/4) + i\sin(\pi/4)) \\ &= \frac{5}{\sqrt{2}}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \\ &= \frac{5+5i}{2} \end{aligned}$$

## References and Further Reading

- Davies: Sec. 2.3, 2.4
- Ogata: Sec. 2.2