Homework 6

1 Problem

Solve Problem 6a, 6b, 6d in the Davies book (p. 119)

Solution

(a) Draw a free body diagram and, from Newton's 2nd Law,

$$\sum F = -kx - b\dot{x} + F(t) = m\ddot{x}$$

The force applied is a step input $F(t) = F_0 H(t)$ so that

$$m\ddot{x} + b\dot{x} + kx = F(t) = F_0 H(t) \tag{1}$$

Taking the Laplace transform

$$\begin{align} ms^2 X(s) + bsX(s) + kX(s) &= \frac{F_0}{s} \tag{2} \\ X(s) &= \frac{F_0}{s}\frac{1}{ms^2 + bs + k} \tag{3} \\ &= \frac{(F_0/m)}{s}\frac{1}{s^2 + (b/m)s + (k/m)} \tag{4} \\ &= \frac{(F_0/m)}{s}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5} \end{align}$$

where we've made the usual substitution for damping ration and natural frequency. The poles are $p_1 = 0$ and $p_{2,3} = -\zeta\omega_n \pm \omega_d i$ (see formula sheet) where $\omega_d = \omega_n\sqrt{1-\zeta^2}$. Thus the partial fraction uses a combination of real-distinct poles (for p_1) and complex conjugate poles for p_2 and p_3. The PFE is

$$X(s) = \frac{(F_0/m)}{s}\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{A}{s} + B\frac{\omega_d}{(s+\zeta\omega_n)^2 + \omega_d^2} + C\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2 + \omega_d^2} \tag{6}$$

where $\omega_d = \omega_n = \sqrt{1-\zeta^2}$. Note that the denominators of the last two terms on the RHS are actually the same as the one on the LHS

$$\begin{align} (s+\zeta\omega_n)^2 + \omega_d^2 &= s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_d^2 \tag{7} \\ &= s^2 + 2\zeta\omega_n s + \zeta^2\omega_n^2 + \omega_n^2(1-\zeta^2) \tag{8} \\ &= s^2 + 2\zeta\omega_n s + \omega_n^2 \tag{9} \end{align}$$

For coefficient A: Multiply both sides by s and evalaute at $s = 0$ to find that

$$A = \frac{F_0}{m\omega_n^2}$$

For coefficients B and C: Multiply both sides of the PFE by $s(s^2 + 2\zeta\omega_n s + \omega_n^2)$ to obtain

$$\frac{F_0}{m} = \frac{F_0}{m\omega_n^2}(s^2 + 2\zeta\omega_n s + \omega_n^2) + B\omega_d s + C(s + \zeta\omega_n)s \tag{10}$$

$$F_0\omega_n^2 = F_0(s^2 + 2\zeta\omega_n s + \omega_n^2) + Bm\omega_d\omega_n^2 s + Cm\omega_n^2(s^2 + \zeta\omega_n s) \tag{11}$$

$$F_0\omega_n^2 = s^2(F_0 + Cm\omega_n^2) + (2F_0\zeta\omega_n + Bm\omega_d\omega_n^2 + Cm\omega_n^2\zeta\omega_n)s + F_0\omega_n^2 \tag{12}$$

Which leads to two equations for coefficients of s^2 and s. From the equation for coefficient s^2:

$$0 = F_0 + Cm\omega_n^2 \implies C = -F_0/(m\omega_n^2) \tag{13}$$

From the equation for coefficient s:

$$2F_0\zeta\omega_n + Bm\omega_d\omega_n^2 + Cm\omega_n^2\zeta\omega_n = 0$$

and with C as defined above becomes:

$$2F_0\zeta\omega_n + Bm\omega_d\omega_n^2 + (-F_0/(m\omega_n^2))m\omega_n^2\zeta\omega_n = 0 \tag{14}$$

$$2F_0\zeta\omega_n + Bm\omega_d\omega_n^2 - F_0\zeta\omega_n = 0 \tag{15}$$

$$F_0\zeta + Bm\omega_d\omega_n = 0 \tag{16}$$

$$B = \frac{F_0\zeta}{m\omega_d\omega_n} \tag{17}$$

The inverse Laplace transform (rows 2,20,21) is:

$$x(t) = A + Be^{-\zeta\omega_n}(\sin\omega_d t + C\cos\omega_d t)$$

Now using the values in the problem statement:

```
m = 13;
k = 1417;
b = 78;
F0 = 2834;

wn = sqrt(k/m)
zeta = b/(2*sqrt(k*m))
wd = wn*sqrt(1-zeta^2)
zeta*wn
A = F0/(m*wn^2)
B = F0*zeta/(m*wd*wn)
C = -F0/(m*wn^2)
```

produces the output

```
wn =
   10.4403
zeta =
    0.2873
wd =
```

```
    10
ans =
    3.0000
A =
    2.0000
B =
    0.6000
C =
   −2.0000
```

From the above values $\zeta\omega_n = 3$ and the solution is:

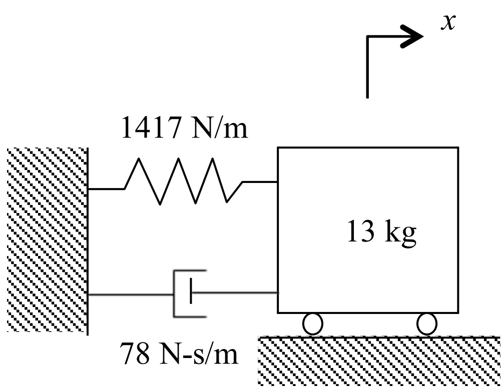
$$x(t) = 2 + 0.6e^{-3t}\sin 10t - 2e^{-3t}\cos 10t$$

For the remainder of the solution refer to the following pages (textbook solution). The approach in the solutions below is correct but has a small typo: the second line below Step 5 should read:

$$X(s) = \frac{A}{s} + \frac{Bs + c}{s^2 + 6s + 109}$$

The approach taken is discussed in the Davies book on p.44: "2.7.4 Special Case That Often Occurs with Step Inputs to Systems".

6. Consider the following model for a mechanical dynamic system consisting of a two linear springs with stiffnesses 1417 N/m and a linear damper with damping 78 N-s/m that are connected between a rigid wall and a 13 kg moving mass.



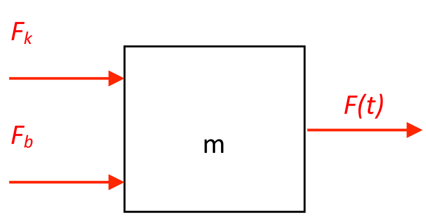
The initial position and velocity are both zero. There is a non-zero input force.

$$F(t) = 2834N \cdot 1(t)$$

(a) Find the equation of motion and solve for *x(t)* using Laplace transforms and check using the ilaplace command in MATLAB®.
(b) Calculate the natural frequency ω_n, the damping ratio ζ, the damped natural frequency and the time constant of the system.
(c) Using the symbolic manipulation capability in MATLAB® show that your answer satisfies the initial conditions and solves the original equation of motion.
(d) In MATLAB®, plot *x(t)* for a duration equal to four dominant time constants of the system.

Solution:

Part (a)
Step 1: Free body diagram (show x-direction only).



Step 2: Newton's Second Law.

$$F_k + F_b + F(t) = m\ddot{x}$$

Step 3: Write the spring force in terms of x.

$$-kx - b\dot{x} + F(t) = m\ddot{x}$$
$$m\ddot{x} + b\dot{x} + kx = F(t)$$

Step 4: Put in the values and take the Laplace transform.

$$13\ddot{x} + 78\dot{x} + 1417x = 2834 \cdot 1(t)$$
$$\ddot{x} + 6\dot{x} + 109x = 208 \cdot 1(t)$$
$$s^2 X(s) + 6sX(s) + 109X(s) = \frac{208}{s}$$
$$X(s) = \frac{208}{s\left(s^2 + 6s + 109\right)}$$

Step 5: Divide up and complete the square.

$$X(s) = \frac{208}{s\left(s^2 + 6s + 109\right)}$$
$$X(s) = \frac{A}{s} + \frac{Bs + C}{s\left(s^2 + 6s + 109\right)}$$
$$208 = A\left(s^2 + 6s + 109\right) + \left(Bs + C\right)s$$
$$208 = As^2 + 6As + 109A + Bs^2 + Cs$$
$$109A = 208 \Rightarrow A = 2$$
$$6A + C = 0 \Rightarrow C = -12$$
$$A + B = 0 \Rightarrow B = -A = -2$$

Therefore …

$$X(s) = \frac{2}{s} - \left(\frac{2s + 12}{s^2 + 6s + 109}\right)$$
$$X(s) = \frac{2}{s} - 2\left(\frac{s + 6}{s^2 + 6s + 109}\right)$$
$$X(s) = \frac{2}{s} - 2\left(\frac{s + 6}{\left(s + 3\right)^2 + 10^2}\right)$$
$$X(s) = \frac{2}{s} - 2\left(\frac{s + 3}{\left(s + 3\right)^2 + 10^2} + \frac{3}{\left(s + 3\right)^2 + 10^2}\right)$$
$$X(s) = \frac{2}{s} - 2\left(\frac{s + 3}{\left(s + 3\right)^2 + 10^2} + \frac{3}{10}\frac{10}{\left(s + 3\right)^2 + 10^2}\right)$$

Now invert with the tables.

$$x(t) = 2 \cdot 1(t) - 2e^{-3t}\left(\cos(10t) + \frac{3}{10}\sin(10t)\right)$$

Step 6: Use the *ilaplace* command in MATLAB® to invert X(s).
>> clear
>> syms X x s t
>> X=208/(s*(s^2+6*s+109));
>> x=ilaplace(X)

x =

208/109 - (208*(cos(10*t) + (3*sin(10*t))/10))/(109*exp(3*t))

When multiplied out this is the same as what we got above.

$$x(t) = \frac{208}{109} - \left(208\left(\cos(10t) + (3\sin(10t)/10)\right)\right)/\left(109e^{3t}\right)$$

$$x(t) = 2 - \frac{208}{109e^{3t}}\left(\cos(10t) + \frac{3}{10}\sin(10t)\right)$$

$$x(t) = 2 - 2e^{-3t}\left(\cos(10t) + \frac{3}{10}\sin(10t)\right)$$

Part (b)

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1417\ N/m}{13\ kg}} = 10.44\ rad/s$$

$$\zeta = \frac{b}{2\sqrt{km}} = 0.287$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 10\ rad/s$$

Part (c)
Check the answer in the original equation.

x =

208/109 - (208*(cos(10*t) + (3*sin(10*t))/10))/(109*exp(3*t))

>> x_d=diff(x)

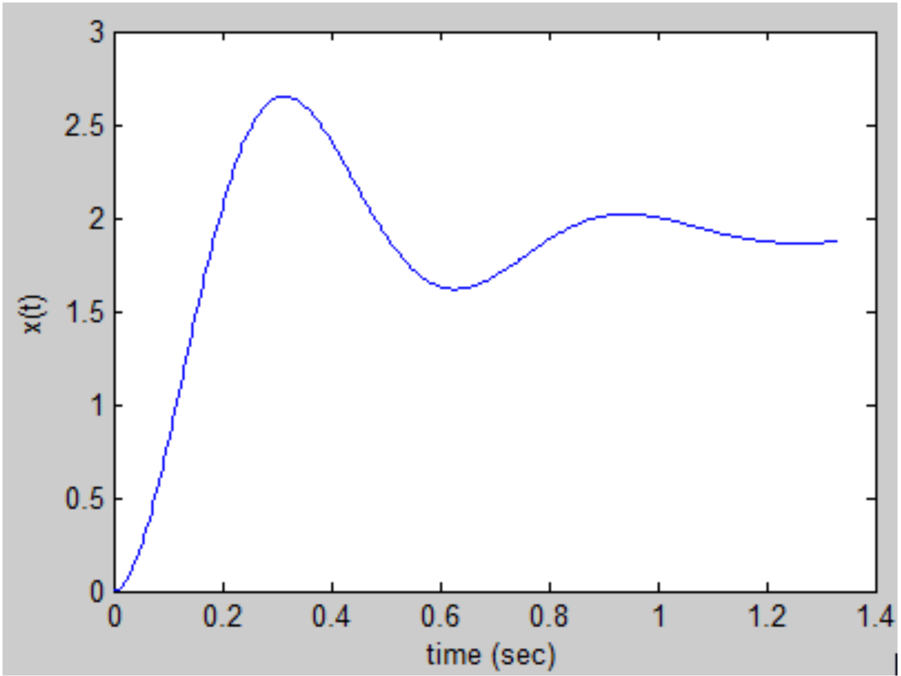
x_d =

(624*(cos(10*t) + (3*sin(10*t))/10))/(109*exp(3*t)) - (208*(3*cos(10*t) - 10*sin(10*t)))/(109*exp(3*t))

```
>> x_dd=diff(x_d)

x_dd =

(1248*(3*cos(10*t) - 10*sin(10*t)))/(109*exp(3*t)) + (208*(100*cos(10*t) +
30*sin(10*t)))/(109*exp(3*t)) - (1872*(cos(10*t) + (3*sin(10*t))/10))/(109*exp(3*t))

>> 13*x_dd+78*x_d+1417*x

ans =

(2704*(100*cos(10*t) + 30*sin(10*t)))/(109*exp(3*t)) - (270400*(cos(10*t) +
(3*sin(10*t))/10))/(109*exp(3*t)) + 2704

>> simplify(ans)

ans =

2704

>> % Oh good
```

Step 8: Plot.

```
clear
tau=1/3;  % Dominant time constant - this exponential takes the longest to die out
t=[0:0.01*tau:4*tau];
x =208/109-(208*(cos(10*t) + (3*sin(10*t))/10))./(109*exp(3*t));
figure(3)
plot(t,x);
xlabel('time (sec)');
ylabel('x(t)');
```

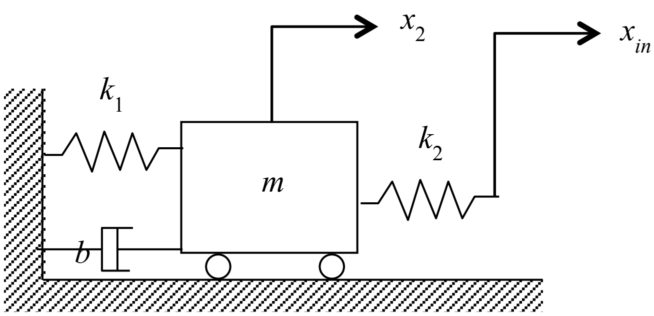

3
2.5
2
1.5
1
0.5
0
x(t)
0
0.2
0.4
0.6
0.8
1
1.2
1.4
time (sec)

2 Problem

Solve Problem 11 in the Davies book (p. 122)

Solution

11. Consider a system that can be modeled as shown. The input $x_{in}(t)$ is a prescribed motion at the right end of spring k_2. Find the system transfer function $\dfrac{X(s)}{X_{in}(s)}$.

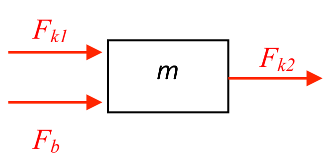


The values of the parameters are m = 30 kg, k_1 = 700 N/m k_2 = 1300 N/m, and b = 200 N-s/m. Write a MATLAB® script file that: (a) calculates the natural frequency, damping ratio, and damped natural frequency for the system; and (b) uses the impulse command to find and plot the response of the system to a unit impulse input.

Solution:

Step 1: Coordinate system – use coordinates given.

Step 2: FBD of the mass.



Step 3: Equation of motion.

$$F_{k1} + F_b + F_{k2} = 0$$

$$-k_1 x - b\dot{x} + k_2(x_{in} - x) = m\ddot{x}$$

$$m\ddot{x} + b\dot{x} + (k_1 + k_2)x = k_2 x_{in}$$

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}} = 8.816 \text{ rad/s}$$

$$\zeta = \frac{b}{2\omega_n m} = 0.408$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 7.45 \text{ rad/s}$$

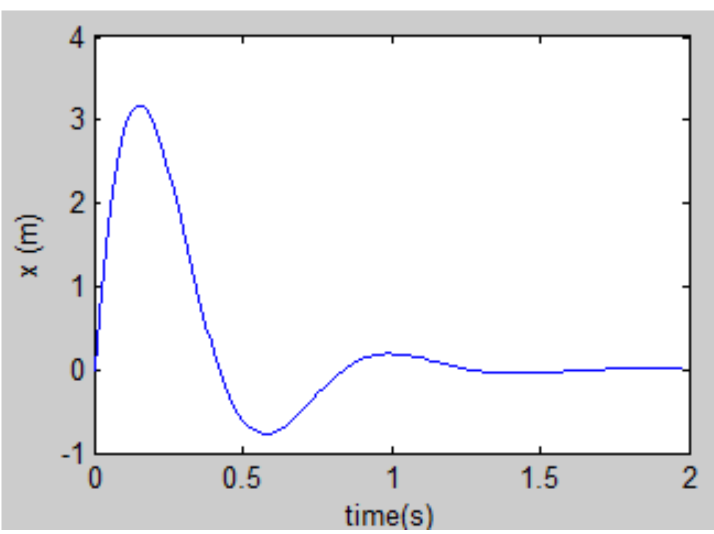
Step 4: Take the Laplace transform and find the transfer function.

$$\left(ms^2 + bs + \left(k_1 + k_2\right)\right)X(s) = k_2 X_{in}(s)$$

$$\frac{X(s)}{X_{in}(s)} = \frac{k_2}{ms^2 + bs + \left(k_1 + k_2\right)}$$

Step 5: MATLAB® solutions for an impulse input.

```
clear

% Parameters
m=30; % kg
k1=700; % N/m
k2=1300; % N/m
b=200; % N-s/m
wn=sqrt((k1+k2)/m)
zeta=b/(2*wn*m)
wd=wn*sqrt(1-zeta^2)

% Transfer Function
num=[k2];
den=[m b k1+k2];
sys=tf(num,den)

% Response
[xu,t]=impulse(sys);
plot(t,xu);
xlabel('time(s)');
ylabel('x (m)');
```



3 Problem

The rolling motion of a Mariner Class cargo ship can be approximated by the following equation:

$$(I_{44} + A_{44})\ddot{\phi} + B_{44}\dot{\phi} + C_{44}\phi = M \sin(\omega_{\text{waves}} t)$$

where ϕ is the roll angle, I_{44} is the roll inertia, A_{44} is the inertia of the added mass (i.e., the surrounding water has the effect of adding inertia to the vessel), B_{44} is the roll damping due to viscous shear forces, C_{44} is the hydrostatic restoring force. The right hand side represents sinusoidal forcing caused by waves at frequency ω_{waves}, with M being the maximum applied moment to the ship. Assume $I_{44} = 1.471 \times 10^{10}$ kg-m^2, $A_{44} = 2.1 \times 10^{10}$ kg-m^2, $C_{44} = 1.1852 \times 10^{10}$ N-m/rad, $B_{44} = 6.6018 \times 10^{9}$ N-m/(rad/s).

Determine the following parameters for the vessel by equation the coefficients of the above system to that of a damped harmonic oscillator:

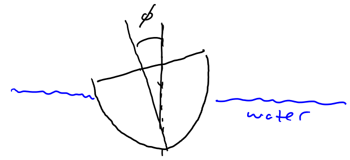
1. undamped natural frequency (in Hz)
2. damping ratio
3. damped natural frequency (in Hz)
4. period of oscillation (in seconds, assuming the damped natural frequency)
5. If the waves suddenly stopped $(M = 0)$ when the ship was at it's maximum roll angle, how long would it take the ship to settle down to 2 % of this maximum roll angle? (Hint: Use your knowledge of the time constant for the decaying oscillations.)

Side note: The system parameters above do not depend on the right hand side of the equation, which is the input or forcing term caused by the waves. However, if the frequency of the waves ω_{waves} is close to the natural frequency the ship roll angle will resonate and lead to large excursions in roll angle.

Solution

Given ODE: $(I_{44} + A_{44})\ddot{\phi} + B_{44}\dot{\phi} + C_{44}\phi = M\sin(\omega t)$

where ϕ: is the roll angle of a ship



rewrite in the form of a damped harmonic oscillator by dividing through by leading coefficient.

$$\ddot{\phi} + \frac{B_{44}}{(I_{44} + A_{44})}\dot{\phi} + \frac{C_{44}}{(I_{44} + A_{44})}\phi = \frac{M}{(I_{44} + A_{44})}\sin(\omega t)$$

equating coefficients

DHO

$$\ddot{\phi} + 2\zeta\omega_n\dot{\phi} + \omega_n^2\phi = f(t)$$

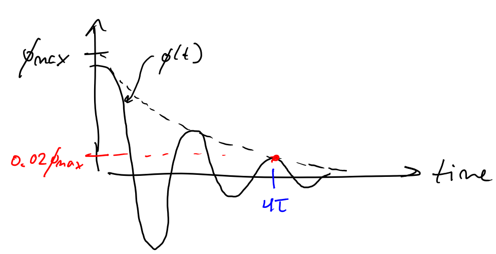
wave forcing.

$$\omega_n = \sqrt{\frac{C_{44}}{I_{44} + A_{44}}} \qquad \zeta = \left(\frac{B_{44}}{I_{44} + A_{44}}\right)\frac{1}{2\omega_n}$$

See matlab here. Note: $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Also, $\tau = \dfrac{1}{\zeta\omega_n}$ and oscillations dampen to

2% in 4 time constants (4τ)



```
clear; close all; clc;
format compact

% ship parameters
A = 2.1E10;
B = 6.6018E9;
C = 1.1852E10;
I = 1.471E10;

wn_rad = sqrt( C / (I+A) ) % natural freq. rad/s
wn_hz = wn_rad / (2 * pi) % natural freq. Hz (cycles/s)
zeta = B/(I+A)/2/wn_rad % damping ratio
wd_hz = wn_hz*sqrt(1-zeta^2) % damped natural freq. Hz
T = 1/wd_hz % damped period seconds
tau = 1/(zeta*wn_rad) % time constant, seconds
fprintf('Ship roll amplitude would damped to 2 percent after four time
 constants: %3.3f seconds\n',4*tau);
```

```
wn_rad =
    0.5761
wn_hz =
    0.0917
zeta =
    0.1605
wd_hz =
    0.0905
T =
   11.0495
tau =
   10.8183
Ship roll amplitude would damped to 2 percent after four time constants:
 43.273 seconds
```

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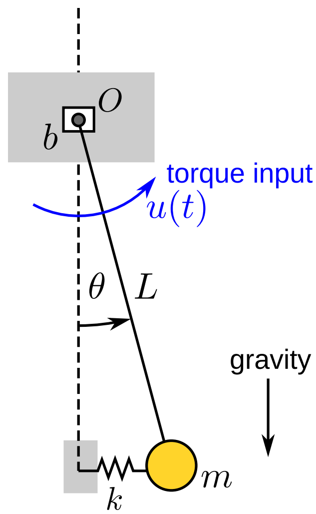
4 Problem

Consider the following mechanical system consisting of a massless rod of length L connected to a ball of mass m. The system has a rotational damper b, a spring constant k, and is subject to gravitational acceleration g. The input into the system is a *torque* $u(t)$.



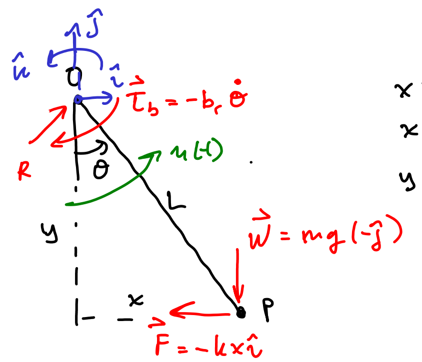
Derive and *fully simplify* the transfer function:

$$G(s) = \frac{\Theta(s)}{U(s)}$$

Make sure to expand/cancel terms such that the numerator and denominator are each a polynomial in terms of s (with coefficients in terms of the constants of the problem: k, m, b, L as needed). Use small angle approximations.

Solution

Problem



$x = L \sin\theta$

$x \approx L\theta$ (small angles)

$y = L\cos\theta$

$\approx L$

$$mL^2 \ddot{\theta} = -b_r \dot{\theta} - ykx - xmg + u(t)$$
$$= -b_r \dot{\theta} - Lk(L\theta) - (L\theta)mg + u(t)$$
$$mL^2 \ddot{\theta} + b_r \dot{\theta} + (kL^2 + Lmg)\theta = u(t)$$

L.T. $$\theta(s)\left(mL^2 s^2 + b_r s + (kL^2 + Lmg)\right) = U(s)$$

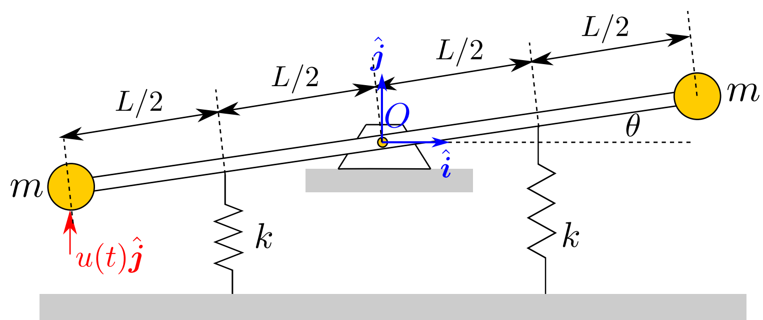
$$G(s) = \frac{\theta(s)}{U(s)} = \frac{1}{mL^2 s^2 + b_r s + (kL^2 + Lmg)}$$

5 Problem

Consider the single degree-of-freedom rotational system shown below with two masses m concentrated at each end. An upward force, $u(t)$, is applied at the leftmost point, and two springs with stiffness k are attached at distances away from the center rotation point O. The springs are unstretched when the angle of the rod is $\theta = 0$. Assume small angles and account for weight of masses.

- Derive an expression for the natural frequency in terms of the parameters given.
- Show that the transfer function for this system is

$$\frac{\Theta(s)}{U(s)} = \frac{-2}{4mLs^2 + kL}$$

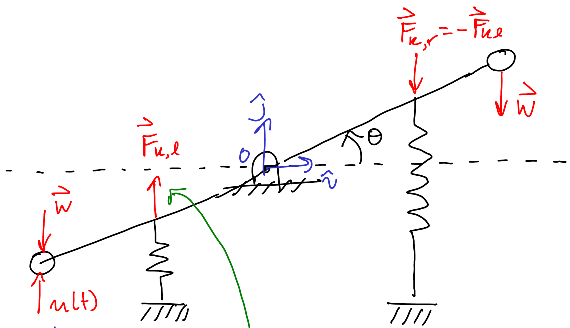


Hint: Recall that for N masses, the inertia around point O is:

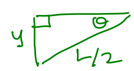
$$I_O = \sum_{i=1}^{N} m_i ||\boldsymbol{r}_{i/O}||^2$$

where $\boldsymbol{r}_{i/O}$ is the vector that points to the ith mass from point O.

Solution



The deflection of the spring is



$y = \sin\theta \frac{L}{2} \approx \frac{\theta L}{2}$ under the small angle assumption.

Moment around pt. O (Assume $\cos\theta \approx 1$)

$$\Sigma M = -L u(t) + \cancel{L m g} - \frac{L}{2}\left(\frac{k\theta L}{2}\right) - \frac{L}{2}\left(\frac{k\theta L}{2}\right) - \cancel{L(mg)}$$

$$= -L u(t) - \frac{L^2 k \theta}{2}$$

Inertia of two masses is $I_0 = 2 \cdot (mL^2)$

Newton's 2nd Law:

$$\Sigma M = I_0 \ddot{\theta}$$

$$-L u(t) - \frac{L^2 k\theta}{2} = (2mL^2)\ddot{\theta}$$

$$\Rightarrow \ddot{\theta} + \left(\frac{\cancel{L^2} k}{4m\cancel{L^2}}\right)\theta = \left(\frac{-L}{2mL^2}\right)u(t)$$

$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = 0$

Comparing to DI+O

$\omega_n^2 = \frac{k}{4m}$

$\Rightarrow \omega_n = \frac{1}{2}\sqrt{\frac{k}{m}}$

Laplace transform

$$\Theta(s)\left(s^2 + \left(\frac{k}{4m}\right)\right) = -\frac{1}{2mL} U(s)$$

$$G(s) = \frac{\Theta(s)}{U(s)} = \frac{(-1/2mL)}{s^2 + (k/4m)} = \frac{-1}{2mLs^2 + kL/2} = \frac{-2}{4mLs^2 + kL}$$