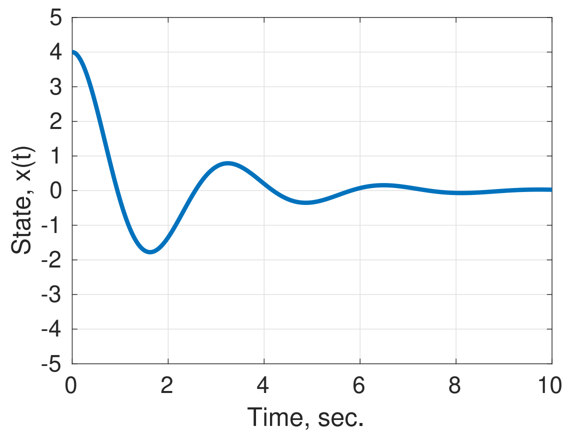
Name: ______________________________

MEGR 3122 Dynamics Systems II: Exam 2, Spring 2023

Directions: Circle the best answer. Show your work and explain your reasoning on all problems to receive full credit (unless otherwise specified).

1. Consider the response of a homogeneous (unforced) second-order system from the initial condition $x(0) = 4$ and $\dot{x}(0) = 0$.



Which ODE best matches the response shown?
A. $2\ddot{x} + 1\dot{x} - 0.5x = 0$
B. $4\ddot{x} + 4\dot{x} + 8x = 0$
C. $\ddot{x} + 2\dot{x} - 4x = 0$
D. $\ddot{x} + 8\dot{x} + 4x = 0$
E. $\ddot{x} + 2x = 0$

Solution (B). The response is underdamped so it cannot be E which has no damping. The response is also stable so it cannot be A or C which have negative spring coefficients. System D is already in standard form and has a natural frequency of $\omega_n = 2$ and a damping ratio of

$$2\zeta\omega_n = 8 \implies \zeta = 8/(2\omega_n) = 2$$

Since this is an overdamped response one can conclude the solution must be B. This can be confirmed by normalizing the ODE to

$$\ddot{x} + \dot{x} + 4x = 0$$

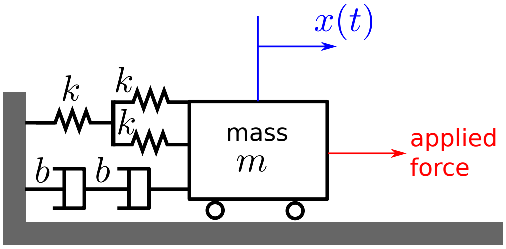
which has a natural frequency of 2 rad/s. The damping ratio is

$$2\zeta\omega_n = 1 \implies \zeta = 1/(2\omega_n) = 0.25$$

which is underdamped as expected.

2. What is the damping ratio of the system shown below?
A. $\dfrac{b}{\sqrt{(2/3)km}}$

B. $\sqrt{3k/2m}$



C. $\sqrt{1-b^2}\sqrt{3k/m}$

D. $\dfrac{b}{4\sqrt{(2/3)km}}$

E. $\dfrac{3b^2}{2\sqrt{k/m}}$

Solution (D). The equivalent spring is

$$k_e = \frac{k(2k)}{k+2k} = \frac{2k^2}{3k} = \frac{2}{3}k$$

and the equivalent damper is

$$b_e = \frac{b^2}{2b} = \frac{1}{2}b$$

The natural frequency is

$$\omega_n = \sqrt{\frac{k_e}{m}} = \sqrt{\frac{2k}{3m}}$$

and the damping ratio is

$$\implies \zeta = \frac{b_e}{2\sqrt{k_e m}} = \frac{(1/2)b}{2\sqrt{(2/3)km}} = \frac{b}{4\sqrt{(2/3)km}}$$

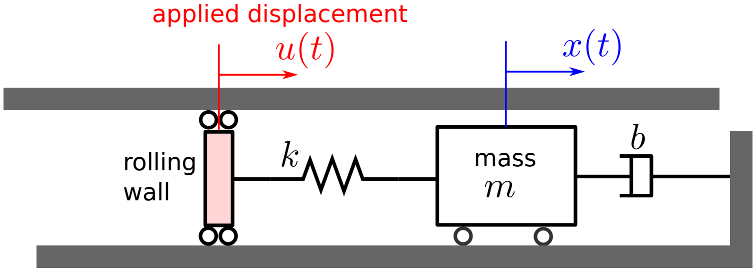
and

$$\zeta^2 = \frac{b^2}{16(2/3)km} = \frac{3b^2}{32km}$$

The damped natural frequency is then

$$\omega_d = \omega_n(1-\zeta^2) = \sqrt{\frac{2k}{3m}}\left(1 - \frac{3b^2}{32km}\right)$$

3. Find the transfer function $X(s)/U(s)$ for the mechanical system shown below



A. $\frac{k}{ms^2+bs+k}$

B. $\frac{1}{ms^2-ks-b}$

C. $\frac{k}{ms^2+ks+b}$

D. $\frac{b}{ms^2+bs-k}$

E. $\frac{b}{ms^2+bs+k}$

Solution (A). Let $\boldsymbol{i}_1$ point to the right. The free body diagram on the mass has two forces (for the spring and damper) so that N2L is:

$$m\ddot{x} = -k(x-u) - b\dot{x}$$

Taking the Laplace transform

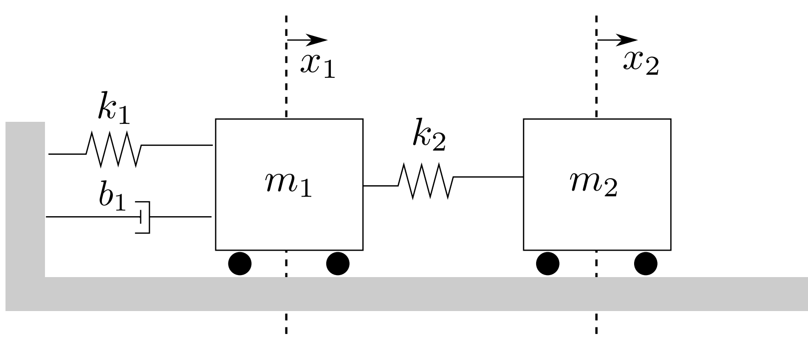
$$ms^2X(s) = -k(X(s) - U(s)) - bsX(s)$$

$$ms^2X(s) + kX(s) + bsX(s) = kU(s)$$

and rearranging

$$\frac{X(s)}{U(s)} = \frac{k}{ms^2 + bs + k}$$

4. Suppose that $k_1 = 4$, $k_2 = 10$, and $b_1 = 1$. Both masses are equal $m_1 = m_2 = 1$. The first mass is at position $x_1 = 0.5$ with velocity $\dot{x}_1 = -1$ and the second mass is at position $x_2 = 0.5$ with velocity $\dot{x}_2 = 1$. What is the total force that acts on the first mass at this instant? Assume that the positive directions for x_1 and x_2 are as indicated by the arrows below. Neglect gravity/normal forces in the vertical direction (consider horizontal forces only).

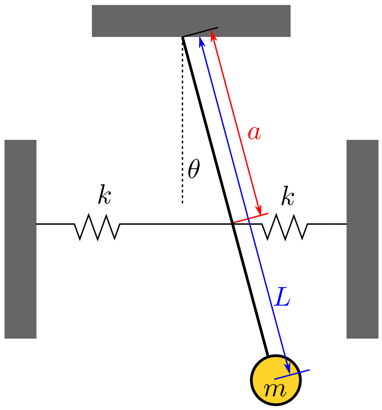


A. The net force has magnitude 2 and points to the right
B. The net force has magnitude 1 and points to the right
C. The net force has magnitude 0
D. The net force has magnitude 1 and points to the left
E. The net force has magnitude 2 and points to the left

Solution (D). Since the two masses are displaced by the same amount there is no spring force due to spring k_2. The forces on mass one are

$$\sum F = -k_1 x_1 - b_1 \dot{x}_1 = -4(0.5) - 1(-1) = -2 + 1 = -1$$

5. The system below consists of a pendulum with a masless rod and bob of mass m. Two spring forces and gravity act on the system. What is the correct equation of motion? Assume small angles.



A. $\ddot{\theta} + \left(\frac{g}{L} + \frac{2ka^2}{mL^2}\right)\theta = 0$

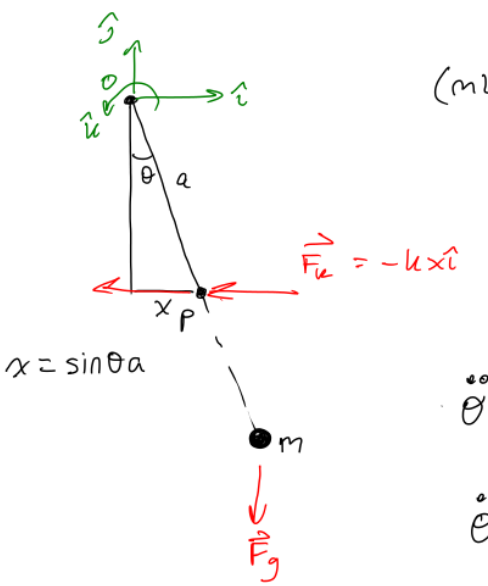
B. $\ddot{\theta} + \left(-\frac{g}{L} + \frac{2ka^2}{mL^2}\right)\theta = 0$

C. $\ddot{\theta} + \frac{g}{L}\dot{\theta} + \left(\frac{2ka}{mL^2}\right)\theta = 0$

D. $\ddot{\theta} + \frac{2ka}{mL^2}\dot{\theta} + \left(\frac{g}{L}\right)\theta = 0$

E. $\ddot{\theta} + \left(-\frac{g}{L} + \frac{2ka}{mL^2}\right)\theta = 0$

Solution (A). See below.



$$(mL^2)\ddot{\theta} = \vec{r}_{P/O} \times (2\vec{F}_k) + \vec{r}_{m/O} \times \vec{F}_g$$

$$= -2(\cos\theta)a \cdot kx - mg\sin\theta L$$

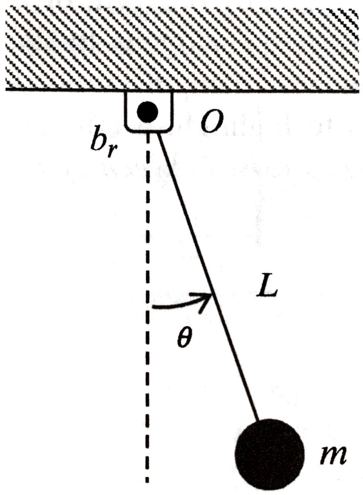
$$= -2(\cos\theta a)\cdot k(\sin\theta a) - mg\sin\theta L$$

$$= -2ka^2\cos\theta\sin\theta - mg\sin\theta L$$

$$\approx -2ka^2\theta - mgL\theta$$

$$\ddot{\theta} + \frac{2ka^2\theta}{mL^2} + \frac{mgL\theta}{mL^2} = 0$$

$$\ddot{\theta} + \left(\frac{2ka^2}{mL^2} + \frac{g}{L}\right)\theta = 0$$



6. Suppose that the highly underdamped system below has mass $m = 3$ and length $L = 2.5$. It is released from rest and begins to oscillate (with very gradually decreasing amplitude due to the small damping $b_r = 1$). What is the period of each oscillation in seconds? Assume small angles and $g \approx 10$.
   A. About two seconds
   B. About half a second
   C. About three seconds
   D. About one second
   E. About one tenth of a second

   Solution (C). The equation of motion can be derived as

   $$\sum M = -mgL\sin\theta - b_r\dot{\theta} = mL^2\ddot{\theta}$$

   which is re-written as

   $$\ddot{\theta} + \frac{b_r}{mL^2}\dot{\theta} + \frac{mgL}{mL^2}\sin\theta = 0$$

   Using small angles $\sin\theta \approx \theta$ and simplifying

   $$\ddot{\theta} + \frac{b_r}{mL^2}\dot{\theta} + \frac{g}{L}\theta = 0$$

   The natural frequency is

   $$\omega_n = \sqrt{\frac{g}{L}} = \sqrt{10/2.5} = 2 \text{ rad/s}$$

   One period corresponds to 2π radians and thus

   $$T = 2\pi/\omega_n = 3.146$$

   The damping of the system is negligible so the damped natural frequency is very close to the natural frequency.

7. Consider the multi-degree-of-freedom system of coupled ODEs with zero initial conditions:

   $$\begin{aligned} \ddot{x}_1 + kx_1 - kx_2 &= 0 \\ \ddot{x}_2 + kx_2 &= f(t) \end{aligned}$$

where $f(t)$ is an input into the system. What is the transfer function $G_1(s) = X_1(s)/F(s)$?

A. $\frac{2k}{(s^2+k)^2}$
B. $\frac{k^2}{(s^2+k)}$
C. $\frac{k}{(s^2+k^2)^2}$
D. $\frac{k}{(s^2+k)^2}$
E. $\frac{3k}{(s^2+2ks+k)}$

Solution (D). Take the Laplace transform of each system

$$X_1(s)(s^2 + k) - kX_2(s) = 0 \quad (1)$$
$$X_2(s)(s^2 + k) = F(s) \quad (2)$$

Solving the first equation for $X_2(s)$

$$X_2(s) = \frac{s^2 + k}{k} X_1(s)$$

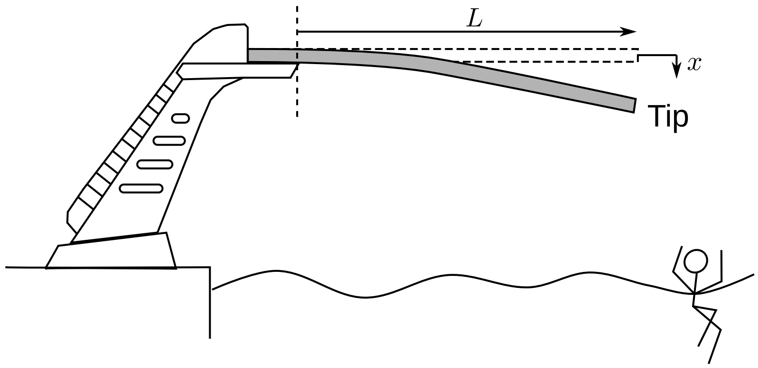
and plugging into the second equation

$$\frac{s^2 + k}{k} X_1(s)(s^2 + k) = F(s)$$

then rearranging

$$\frac{X_1(s)}{F(s)} = \frac{k}{(s^2 + k)^2}$$

8. A diving board has Young's modulus of E, width a, thickness b, area moment of inertia I , length of L, and (volumetric) density of ρ. Use a lumped-parameter model to model the displacement of the tip of the diving board. What is the natural frequency of the tip (in rad/s)?



A. $\frac{3}{4}\sqrt{\frac{0.23EI}{\rho abL^3}}$

B. $\frac{3.61}{L^2}\sqrt{\frac{EI}{\rho ab}}$

C. $\sqrt{\frac{0.23E(ab^2)}{4\rho L^3}}$

D. $\frac{0.4796}{L^2}\sqrt{\frac{EI}{\rho ab}}$

E. $\frac{L^2 b}{0.277}\sqrt{\frac{\rho a}{EI}}$

Solution (B). The diving board is approximately a cantilever beam as shown in the lumped parameter tables. The total distributed mass of the beam is

$$m_d = \rho abL$$

thus the effective mass is

$$m_e = 0.23m_d = 0.23\rho abL$$

The effective spring constant is

$$k_e = \frac{3EI}{L^3}$$

The model of the equivalent system is

$$m_e\ddot{x} + k_e x = 0$$

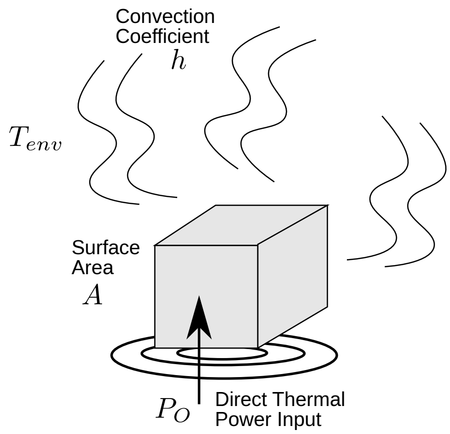
and the natural frequency is

$$\omega_n = \sqrt{\frac{k_e}{m_e}} = \sqrt{\frac{3EI/L^3}{0.23\rho abL}} = \frac{3.61}{L^2}\sqrt{\frac{EI}{\rho ab}}$$

9. Suppose that an object initially at temperature $T_0 = 65$ deg C has thermal mass of $mc = 3$ J/(deg C) and is being simulatenously heated with a constant thermal power input of $P_0 = 100$ Watts and cooled via convection with the environment at $T_{\text{env}} = 15$ deg C. The exposed surface area for convection is $A = 0.8$ m^2 and the convection coefficient is $h = 2.5$ Watts/(m$^2\cdot$ deg C). What will happen to the temperature of the object after a long period of time?

A. The temperature will decay to a value $T_{\text{env}} + P_0/(mc)$

B. The temperature will approach an asymptote at $(T_{\text{env}} + P_0/(mc))/2$
C. The temperature will stay constant at T_0
D. The temperature will decay to T_{env} after approximately four time constants
E. The temperature will oscillate between T_{env} and $T_0 + P_0/(mc)$



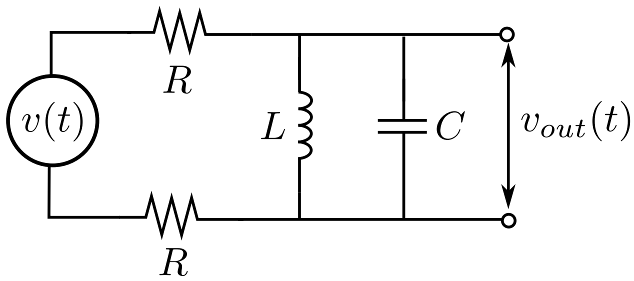
Solution (C). If we sum the thermal powers

$$\sum P = P_0 + hA(T_{\text{env}} - T) \tag{3}$$

$$= 100 + 0.8 \cdot 2.5(15 - 65) = 0 \tag{4}$$

thus the temperature will remain constant

10. For the following circuit, what is the transfer function $G(s) = V_{\text{out}}(s)/V(s)$?



A. $\frac{CLs^2+1}{CLs^2+2RLs+1}$

B. $\frac{Ls}{2RLCs^2+Ls+2R}$

C. $\frac{1}{RLCs^2+2RLs+LC}$

D. $\frac{(1/RL)}{s^2+2Cs+1}$

E. $\frac{RCs^2+Ls+R}{CLs^2+LRs+1}$

Solution (B). The impedance of the inductor and capacitor in parallel is:

$$Z_{LC} = \left[\frac{1}{Ls} + \frac{1}{1/Cs}\right]^{-1} = \left[\frac{1}{Ls} + Cs\right]^{-1} = \left[\frac{CLs^2+1}{Ls}\right]^{-1} = \frac{Ls}{CLs^2+1}$$

The impedance of the entire circuit is

$$Z_{\text{circuit}} = R + \frac{Ls}{CLs^2+1} + R = \frac{Ls + 2R(CLs^2+1)}{CLs^2+1} = \frac{2RLCs^2 + Ls + 2R}{CLs^2+1}$$

The desired transfer function is then

$$G(s) = \frac{Z_{LC}}{Z_{\text{circuit}}} = \frac{Ls}{2RLCs^2 + Ls + 2R}$$