A. Wolek, Spring 2024, UNC Charlotte

MEGR 3122 Dynamic Systems II: Formula Sheet

1st Order Differential Equations:

Homogeneous $\dot{x}(t) + ax(t) = 0, \quad x(t_0) = x_0$

$$\implies x(t) = x_0 e^{-at}$$

Time Constant $\tau = 1/a$

$$x(\tau) \approx 0.368x_0, \quad x(2\tau) \approx 0.135x_0$$
$$x(3\tau) \approx 0.050x_0, \quad x(4\tau) \approx 0.018x_0$$

Inhomogeneous $\dot{x}(t) + ax(t) = u(t), \quad x(t_0) = x_0$

$$\implies x(t) = e^{-at}\int e^{at}u(t)dt + Ce^{-at}$$

2nd Order Differential Equations:

Homogeneous $\ddot{x}(t) + a\dot{x}(t) + bx(t) = 0, \quad x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0$

Characteristic Equation $\lambda^2 + a\lambda + b = 0$

Quadratic Formula $\lambda_{1,2} = (-a \pm \sqrt{a^2 - 4b})/2$

Case I: $a^2 - 4b > 0$ **(real, distinct eigenvalues)**

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

Case II: $a^2 - 4b = 0$ **(repeated eigenvalues)**

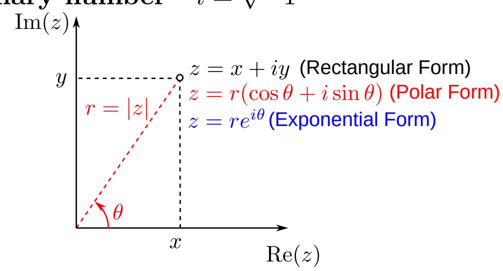
$$x(t) = C_1 e^{-at/2} + C_2 t e^{-at/2}$$

Case III: $a^2 - 4b < 0$ **(complex conjugate eigenvalues)**

$$x(t) = e^{-at/2}(A\cos\omega t + B\sin\omega t)$$

Complex Numbers:

Imaginary number $i = \sqrt{-1}$



Modulus $r = |z| = \sqrt{\text{Re}(z)^2 + \text{Im}(z)^2}$

Argument $\theta = \angle z = \arg(z) = \text{atan}(\text{Im}(z)/\text{Re}(z))$

Euler's Formula $e^{i\theta} = \cos\theta + i\sin\theta$

Properties

For $z = (x + iy)$ and $w = (u + iv)$

$$\bar{z} = x - iy$$
$$z \cdot w = (xu - yv) + i(xv + yu)$$
$$|z|^2 = z \cdot \bar{z}$$
$$\frac{z}{w} = \frac{z}{w}\left(\frac{\bar{w}}{\bar{w}}\right)$$

For $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2}$

$$\bar{z} = re^{-i\theta}$$
$$z_1 \cdot z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$
$$z_1/z_2 = r_1/r_2 e^{i(\theta_1 - \theta_2)}$$

Laplace Transform:

Definition

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}dt$$

Properties

$$\mathcal{L}[\alpha f(t)] = \alpha\mathcal{L}[f(t)] \qquad \text{(scalar multiplication)}$$
$$\mathcal{L}[f(t) + g(t)] = \mathcal{L}[f(t)] + \mathcal{L}[g(t)] \qquad \text{(addition)}$$
$$\mathcal{L}[f(t-\alpha)H(t-\alpha)] = e^{-s\alpha}F(s) \qquad \text{(translated function)}$$
$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad \text{(initial value)}$$
$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) \qquad \text{(final value)}$$
$$\mathcal{L}[\dot{f}(t)] = sF(s) - f(0) \qquad \text{(differentation)}$$
$$\mathcal{L}[\ddot{f}(t)] = s^2F(s) - sf(0) - \dot{f}(0)$$

Table of Common Laplace Transforms

Row	$f(t)$	$F(s)$
1	Unit impulse, $\delta(t)$	1
2	Unit step/Heaviside, $H(t)$	$\frac{1}{s}$
3	Ramp, t	$\frac{1}{s^2}$
4	$\frac{t^{n-1}}{(n-1)!}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{s^n}$
5	$t^n, \quad n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
6	e^{-at}	$\frac{1}{(s+a)}$
7	te^{-at}	$\frac{1}{(s+a)^2}$
8	$\frac{t^{n-1}}{(n-1)!}e^{-at}, \quad n = 1, 2, 3, \ldots$	$\frac{1}{(s+a)^n}$
9	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
10	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
11	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
12	$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
13	$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
14	$\frac{1}{a}(1 - e^{-at})$	$\frac{1}{s(s+a)}$
15	$\frac{1}{b-a}(e^{-at} - e^{-bt})$	$\frac{1}{(s+a)(s+b)}$
16	$\frac{1}{b-a}(be^{-bt} - ae^{-at})$	$\frac{s}{(s+a)(s+b)}$
17	$\frac{1}{ab}\left(1 + \frac{1}{a-b}\left(be^{-bt} - ae^{-at}\right)\right)$	$\frac{1}{s(s+a)(s+b)}$
18	$\frac{1}{a^2}(1 - e^{-at} - ate^{-at})$	$\frac{1}{s(s+a)^2}$
19	$\frac{1}{a^2}(at - 1 + e^{-at})$	$\frac{1}{s^2(s+a)}$
20	$e^{-at}\sin(\omega t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
21	$e^{-at}\cos(\omega t)$	$\frac{s+a}{(s+a)^2+\omega^2}$
22	$\frac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Inverse Laplace Transform

$$F(s) = \mathcal{L}(f(t)) \implies f(t) = \mathcal{L}^{-1}[F(s)]$$

Partial Fraction Expansion:

Polynomial form

$$F(s) = \frac{Q(s)}{R(s)} = \frac{d_m s^m + d_{m-1}s^{m-1} + \cdots + d_1 s + d_0}{c_n s^n + c_{n-1}s^{n-1} + \cdots + c_1 s + c_0}$$

Zero-pole-gain form

$$F(s) = \frac{A(s)}{B(s)} = \frac{k(s-z_1)(s-z_1)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\ ,$$

PF Form: Case I (Distinct, Real Poles)

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1}{s-p_1} + \cdots + \frac{a_k}{s-p_k} + \cdots + \frac{a_n}{s-p_n}\ ,$$

with coefficients solved by

$$a_k = \left.\frac{A(s)(s-p_k)}{B(s)}\right|_{s=p_k}$$

PF Form: Case II (Repeated Real Poles).

$$F(s) = \frac{A(s)}{(s-p)^n} = \frac{a_1}{(s-p)} + \cdots + \frac{a_k}{(s-p)^k} + \cdots + \frac{a_n}{(s-p)^n}$$

with coefficients by multiplying by $(s-p)^n$

$$F(s)(s-p)^n = a_1(s-p)^{n-1} + \cdots + a_k(s-p)^{n-k} + \cdots + a_n$$

equating coefficients, and solving the system of equations.

PF Form: Case III (Complex Poles). $p_{1,2} = -\alpha \pm i\omega$

$$F(s) = a_1\frac{\omega}{(s+\alpha)^2+\omega^2} + a_2\frac{(s+\alpha)}{(s+\alpha)^2+\omega^2}$$

with coefficients solved by writing

$$F(s) = \frac{A(s)}{B(s)} = \frac{a_1\omega + a_2(s+\alpha)}{(s+\alpha)^2+\omega^2}$$

equating coefficients of numerator above and solving the system of equations.

2nd Order System (Damped Harmonic Oscillator):

Assuming a stable mass-spring-damper model:

$$\ddot{x} + \underbrace{\left(\frac{b}{m}\right)}_{=2\zeta\omega_n}\dot{x} + \underbrace{\left(\frac{k}{m}\right)}_{=\omega_n^2}x = 0$$

$$\implies \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\implies \zeta = b/(2\sqrt{km}),\quad \omega_n = \sqrt{k/m}$$

Assuming a generic stable 2nd order system:

$$\ddot{x} + a\dot{x} + bx = 0$$

$$\implies \ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

$$\implies \zeta = a/2\omega_n,\quad \omega_n = \sqrt{b}$$

Damping ratio cases:

- Case $\zeta > 1$ overdamped.
- Case $\zeta = 1$ critically damped.
- Case $0 < \zeta < 1$ underdamped.
- Case $\zeta = 0$ undamped.

Natural frequency $\omega_n = \sqrt{k/m}$ (rad/s)

Damped natural frequency $\omega_d = \omega_n\sqrt{1-\zeta^2}$

Resonant frequency $\omega_{\text{res}} = \omega_n\sqrt{1-2\zeta^2}$

Poles $p_{1,2} = -\zeta\omega_n \pm \omega_d i$

Time constant $\tau = 1/\zeta\omega_n$

Period $T = 2\pi/\omega$ (seconds)

Frequency (Hz) $f = 1/T$ (cycles/seconds) = Hz

Transfer Functions:

General form

$$G(s) = \frac{X(s)}{U(s)} = \frac{\text{Output (s)}}{\text{Input (s)}}$$

Zero state (i.e., zero IC) response

$$x_{\text{zs}}(t) = \mathcal{L}^{-1}[G(s)U(s)] = \mathcal{L}^{-1}[X(s)]$$

First Order System

$$\dot{x} + ax = u(t) \implies G(s) = \frac{X(s)}{U(s)} = \frac{1}{s+a}$$

Second Order System

$$\ddot{m}x + b\dot{x} + kx = u(t) \implies G(s) = \frac{1}{ms^2+bs+k}$$

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = u(t) \implies G(s) = \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Mechanical Systems:

Newton's 2nd Law (translation) $\sum \boldsymbol{F} = m\ddot{x}$

Newton's 2nd Law (rotation) $\sum \boldsymbol{M} = I_O\ddot{\theta}$

Inertia of a point mass $I_O = mL^2$ where L is distance to axis of rotation

Spring force (translation) $\boldsymbol{F}_{\text{spring}} = -kx\hat{\boldsymbol{i}}$ assuming $x > 0$ is positive in $\hat{\boldsymbol{i}}$ direction

Spring force (rotation) $\boldsymbol{F}_{\text{spring}} = -k_r\theta\hat{\boldsymbol{k}}$ assuming $\theta > 0$ is positive in $\hat{\boldsymbol{k}}$ direction

Damping force (translation) $\boldsymbol{F}_{\text{damper}} = -b\dot{x}\hat{\boldsymbol{i}}$ assuming $\dot{x} > 0$ is positive in $\hat{\boldsymbol{i}}$ direction

Damping force (rotation) $\boldsymbol{F}_{\text{damper}} = -b_r\dot{\theta}\hat{\boldsymbol{k}}$ assuming $\dot{\theta} > 0$ is positive in $\hat{\boldsymbol{k}}$ direction

Small angle approximation $\sin\theta \approx \theta,\quad \cos\theta \approx 1$

Lumped model of springs in parallel: $k_{\text{eq}} = k_1 + k_2 + \cdots + k_n$

Lumped model of springs in series: $k_{\text{eq}} = \frac{k_1 k_2}{k_1+k_2}$ $k_{\text{eq}} = (\frac{1}{k_1} + \frac{1}{k_2} + \cdots \frac{1}{k_n})^{-1}$

Lumped model of dampers in parallel: $b_{\text{eq}} = b_1 + b_2 + \cdots + b_n$

Lumped model of dampers in series: $b_{\text{eq}} = \frac{b_1 b_2}{b_1+b_2}$ $k_{\text{eq}} = (\frac{1}{b_1} + \frac{1}{b_2} + \cdots \frac{1}{b_n})^{-1}$

Thermal Systems:

Governing Equation

$$\sum P = mc\dot{T}$$

Constant Input $P_{\text{constant}} = P_0 H(t)$

Convection $P_{\text{convection}} = hA(T_{\text{env}} - T)$

Conduction $P_{\text{conduction}} = \frac{kA}{L}(T_2 - T_1)$

Electrical Systems:

Impedance (Definition) $Z(s) = V(s)/I(s)$

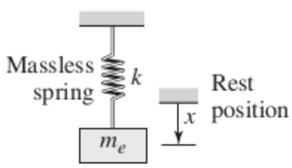
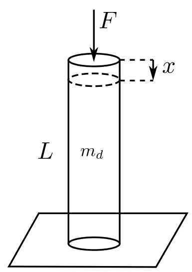
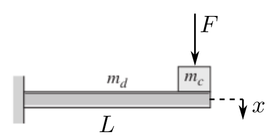
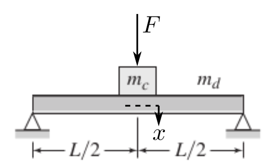
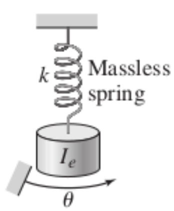
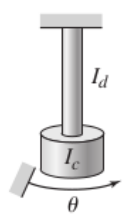
Impedance (Passive Elements)

$$Z_{\text{resistor}} = R$$
$$Z_{\text{inductor}} = Ls$$
$$Z_{\text{capacitor}} = 1/(Cs)$$

Lumped model of impedances in parallel: $Z_{\text{eq}} = \frac{Z_1 Z_2}{Z_1+Z_2}$ or $Z_{\text{eq}} = (\frac{1}{Z_1} + \frac{1}{Z_2} + \cdots \frac{1}{Z_n})^{-1}$

Lumped model of impedances in series: $Z_{\text{eq}} = Z_1 + Z_2 + \cdots Z_n$

Lumped Parameter Tables

Equivalent System 	 $m_e = m_d/3$ $k_e = \frac{EA}{L}$	 $m_e = m_c + 0.23 m_d$ $k_e = 3EI/L^3$	 $m_e = m_c + m_d/2$ $k_e = \frac{48EI}{L^3}$
Equivalent System 		$k_e = G_s J_p/L$ $I_e = I_c + I_d/3$	

Frequency Response

Sinusoidal Transfer Function: $G(i\omega)$

Gain: $|G(i\omega)|$ **Phase:** $\phi = \angle G(i\omega)$

Steady-State Sinusoidal Output: Given the sinusoidal input $u(t) = A \sin \omega t$ the steady-state output is

$$x_{ss}(t) = |G(i\omega)| A \sin(\omega t + \phi)$$

where the gain is the ratio of output-to-input amplitudes

$$|G(i\omega)| = \frac{\max(x_{ss}(t))}{\max(u(t))}$$

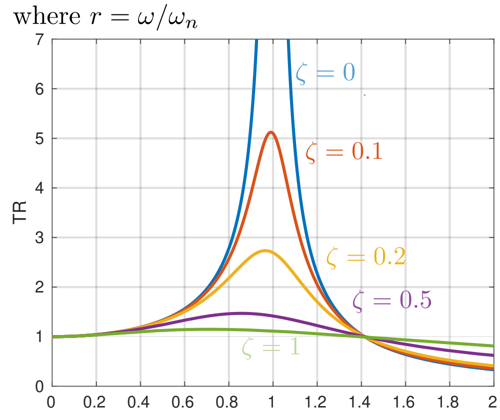
Phase (by comparing input/output peaks):

$$\phi = \left(\frac{t_{\text{input}} - t_{\text{output}}}{T} \right) 2\pi$$

Transmissibility Ratio:

$$TR(\zeta, r) = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

where $r = \omega/\omega_n$



Dynamic Vibration Absorber:

$$\omega_0 = \sqrt{k_a/m_a}$$

Decibels (dB):

$$|G|_{\mathrm{dB}} = 20\log_{10}|G|$$

$$|G| = 10^{|G|_{\mathrm{dB}}/20}$$

Vibration Mode Analysis: Assumed solutions

$$x_1(t) = A\sin\omega t$$

$$x_2(t) = B\sin\omega t$$

Transfer Function in Zero-Pole-Gain Form:

$$G(s) = \frac{K(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}$$

$$\begin{aligned}|G(i\omega)|_{\mathrm{dB}} &= 20\log_{10}|K| \\ &+ 20\log_{10}|(i\omega-z_1)| + \cdots + 20\log_{10}|i\omega-z_m| \\ &- 20\log_{10}|(i\omega-p_1)| - \cdots - 20\log_{10}|(i\omega-p_n)|\end{aligned}$$

$$\begin{aligned}\phi = \angle G(i\omega) &= \angle K + \angle(i\omega-z_1) + \cdots + \angle(i\omega-z_m) \\ &- \angle(i\omega-p_1) - \cdots - \angle(i\omega-p_n)\end{aligned}$$

Logarithm Rules

$$y = \log_b x \implies b^y = b^{\log_b x} = x$$

$$\log_b(x\cdot g) = \log_b(x) + \log_b(g)$$

$$\log_b(x^n) = n\log_b(x)$$

$$\log_b(x/g) = \log_b(x) - \log_b(g)$$

Matrix Algebra

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc$$

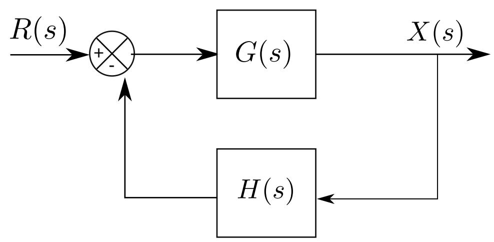
Block Diagrams

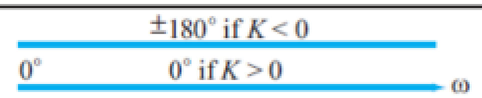
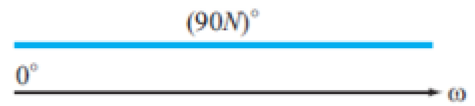
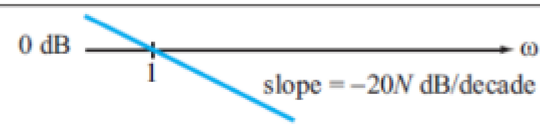
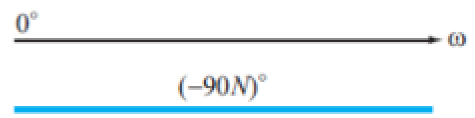
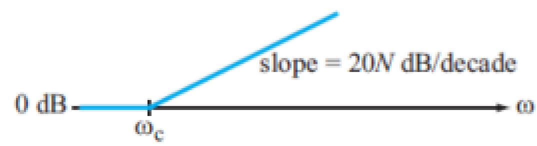
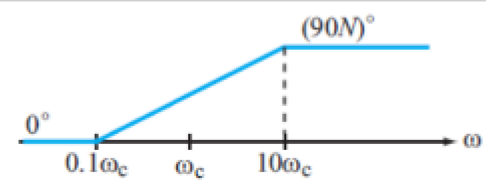
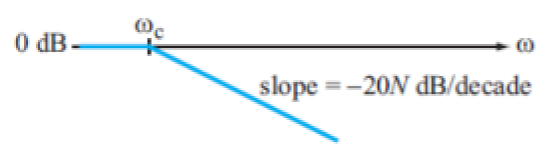
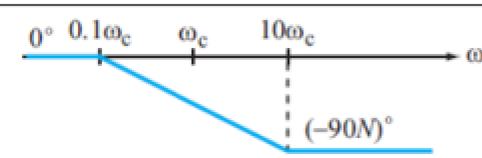
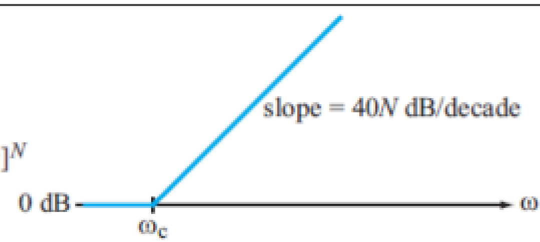
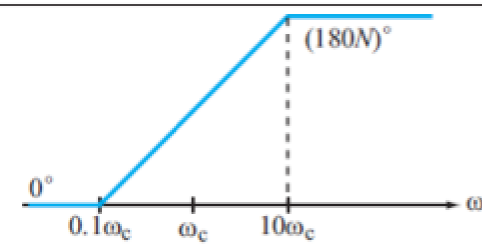
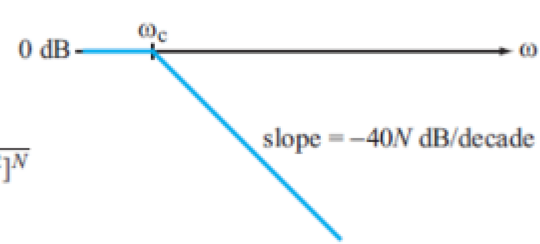
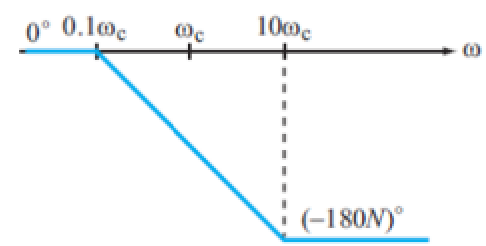
Blocks in Series: $G_{12}(s) = G_1(s)G_2(s)$

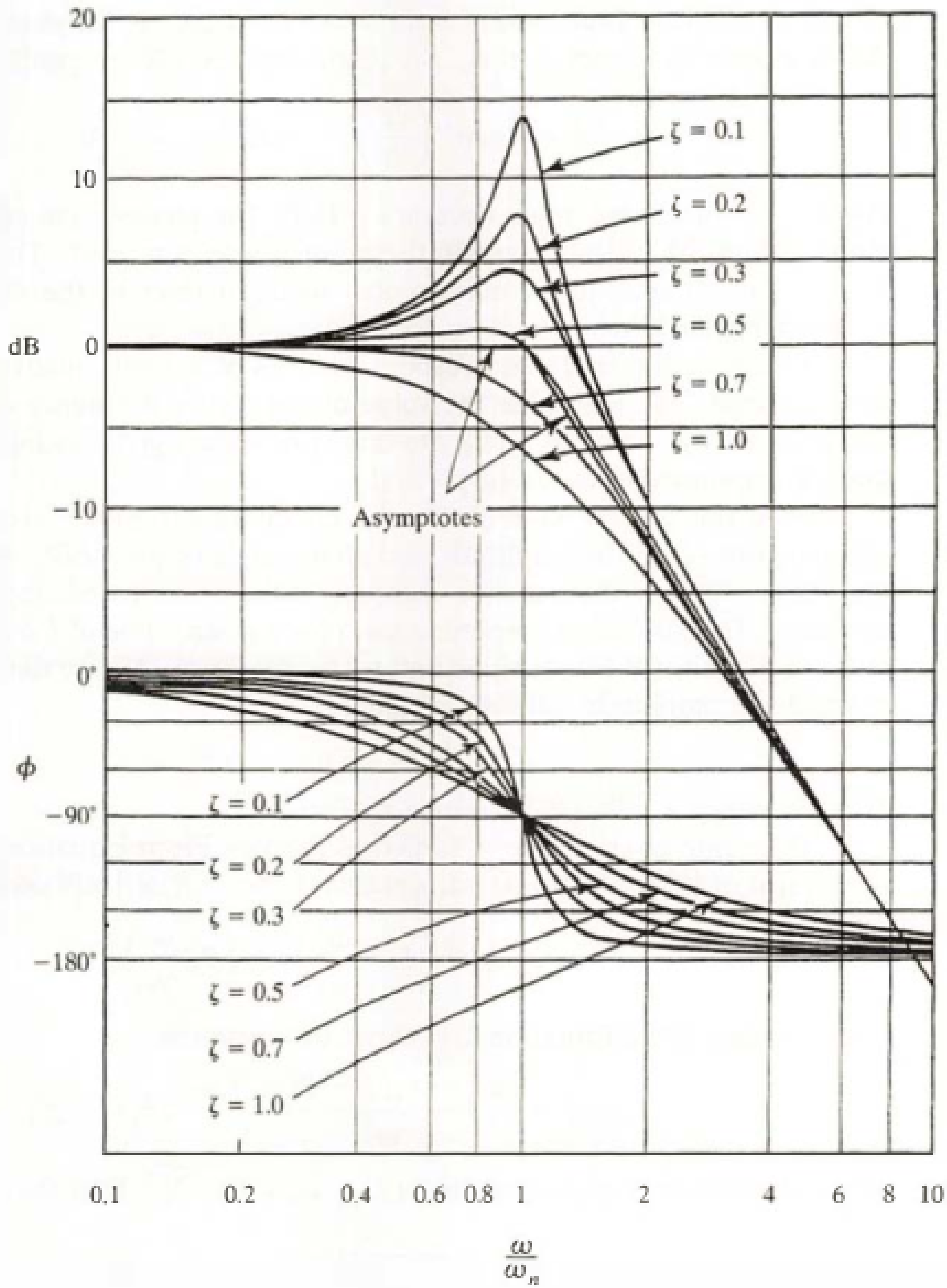
Blocks in Parallel: $G_{12}(s) = G_1(s) + G_2(s)$

Closed Loop TF:

$$G_{\mathrm{CL}}(s) = \frac{G(s)}{1+G(s)H(s)} = \frac{X(s)}{R(s)}$$



Factor	Bode Magnitude	Bode Phase
Constant K		
Zero @ Origin $(j\omega)^N$		
Pole @ Origin $(j\omega)^{-N}$		
Simple Zero $(1 + j\omega/\omega_c)^N$		
Simple Pole $\left(\frac{1}{1 + j\omega/\omega_c}\right)^N$		
Quadratic Zero $[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2]^N$		
Quadratic Pole $\frac{1}{[1 + j2\xi\omega/\omega_c + (j\omega/\omega_c)^2]^N}$		



20
10
dB
0
−10
0°
ϕ
−90°
−180°
ζ = 0.1
ζ = 0.2
ζ = 0.3
ζ = 0.5
ζ = 0.7
ζ = 1.0
Asymptotes
0.1
0.2
0.4
0.6
0.8
1
2
4
6
8
10
$\frac{\omega}{\omega_n}$