The mass moment of inertia for a hollow cylinder is:

$$J_P = \frac{1}{2} m \left(R_o^2 - R_i^2 \right),$$

which gives $2.4 \cdot 10^{-3}$ kg-m^2 for this shaft. Thus, the mass moment of inertia of the wheel is approximately 260 times greater than that for the shaft. The error from considering all of the inertia to be concentrated in the wheel is negligible and a lumped parameter approach is justified.

5.4.2 Nonzero Input Angle ($\theta_{in} \neq 0$)

While the previous section provided the basic vibrational characteristics for the shaft–wheel–bearing combination, a more common situation is for the inertia to be the rotor of a spindle or a robot arm that is commanded to move to a certain angle, $\theta_{in}(t)$. In that case, if we assume zero initial conditions, we can calculate the Laplace transform of Eq. (5.27) to obtain the transfer function for the system; see Eq. (5.31). This transfer function can be used to examine the system behavior for arbitrary inputs (recall the MATLAB® functions `step`, `impulse`, and `lsim`).

$$\frac{\Theta(s)}{\Theta_{\text{in}}(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{5.31}$$

Example 5.9 Suppose a counter-balanced section of a robot arm, modeled as shown in Fig. 5.7, has a rotational inertia of 1.0 kg-m^2 and a damping ratio from all velocity-dependent sources of 10%. It is driven through a shaft and associated drive components with a combined rotational stiffness of 5 kN-m/rad. Determine the total damping in the system to achieve this damping ratio, the natural frequency of the system, and the damped natural frequency of the system. Next, find the step response for a commanded step angular rotation of 5° and determine the maximum angular overshoot (beyond the commanded 5°) in degrees using the `max` command in MATLAB®. Assume zero initial conditions.

Solution For this case all of the damping is combined into one velocity-dependent moment; there are not two separate dampers for this model. Therefore, the equation of motion becomes:

$$\ddot{\theta} + \frac{b_r}{J}\dot{\theta} + \frac{k_r}{J}\theta = \frac{k_r}{J}\theta_{in} \tag{5.32}$$

and the natural frequency and damping ratio are given by Eq. (5.33). These are analogous to Eqs. (4.7) and (4.22).

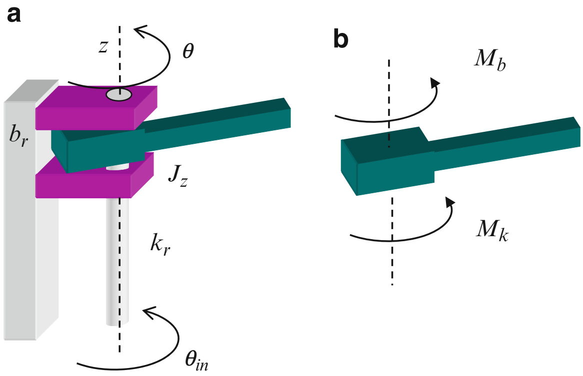


Fig. 5.7 (**a**) A robot arm with a (rotational) mass moment of inertia, J_z, is driven about the z-axis. The arm is balanced about the z-axis so the weight does not cause a moment on the bearings. It is driven though a flexible input shaft with stiffness, k_r, and is supported by two bearings. The system has a total rotational damping constant, b_r, from all sources. The input is $\theta_{in}(t)$ and the angle of rotation of the inertia is $\theta(t)$; and (**b**) free body diagram of the arm

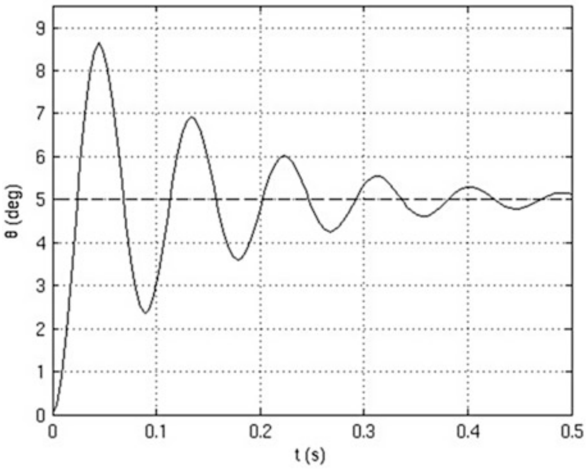
$$\begin{aligned} \omega_n &= \sqrt{\frac{k_r}{J}} \\ \zeta &= \frac{b_r}{2J\omega_n} = \frac{b_r}{2\sqrt{k_r J}} \end{aligned} \tag{5.33}$$

We use Eq. (5.31) to define the transfer function in MATLAB® and write the code to simulate the step response of the system. The MATLAB® code follows.

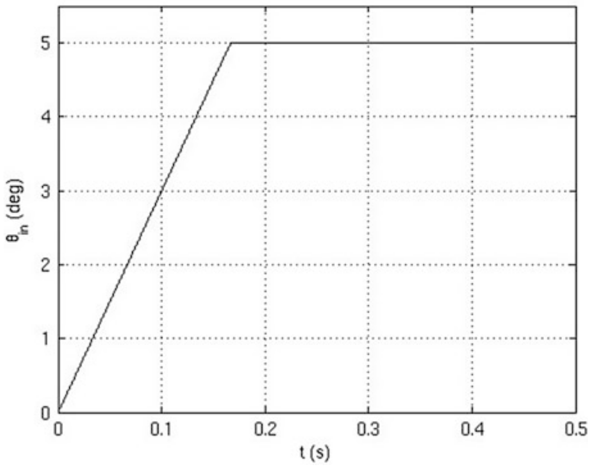
```
clear all
clc
close all

% Parameters
kr = 5e3;            % N-m/rad
J = 1.0;             % kg-m^2
wn = sqrt(kr/J);     % rad/s
zeta = 0.1;
br = 2*zeta*wn*J;    % N-m-s
q_max = 5;           % deg

% Define the system transfer function
num = [wn^2];
den = [1 2*zeta*wn wn^2];
sys = tf(num,den);

% Find the unit step response
[qu, t] = step(sys);

% Scale the unit step response by the input size
q = q_max*qu;
```

```
% Define a step input vector for comparison to the response
qin = q_max*ones(1, length(t));

% Plot the results
figure(1)
plot(t, qin, 'k-', t, q, 'k-')
grid
xlabel('t (s)');
ylabel('\theta (deg)');
axis([0 0.5 0 1.1*max(q)])

% Find the maximum overshoot in deg
q_os = max(q)-q_max;
display(q_os)
```

The corresponding plot is provided. We observe that the maximum overshoot of 3.65° (i.e., the difference between the maximum value of 8.65° and the commanded angle of 5°) occurs at 0.044 s. This could certainly be a concern depending on the required accuracy of the operation being performed and could cause damage to the components either immediately or due to repeated shock loads.



Example 5.10 One reason for the large overshoot in the previous example is that a step input produces a significant shock load on the system. In reality, most input drives would not be able to realize a true step input even if commanded to do so. Therefore, to produce a more controlled motion, a more typical input is a truncated ramp as shown.



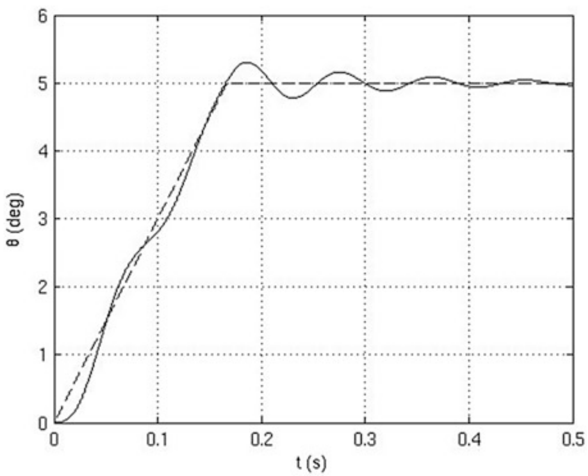
This θ_{in} input profile commands the arm to move at 30°/s to a final angle of 5° and then stops. Use the MATLAB® command `lsim` to find and plot the response of the robot arm detailed in Example 5.9. Find the maximum overshoot for this case and compare it to Example 5.9.

Solution The MATLAB® code first generates the input profile and then uses that profile to drive the system and find the response. To generate the input profile, the `find` command is again useful. The MATLAB® code is provided.

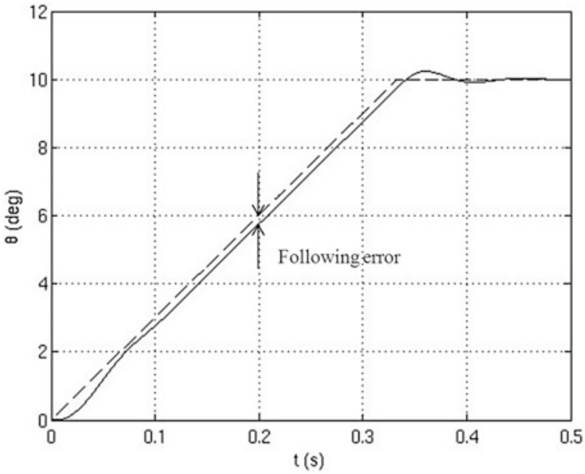
```
clear all
clc
close all

% Parameters
kr = 5e3;            % N-m/rad
J = 1.0;             % kg-m^2
wn = sqrt(kr/J);     % rad/s
zeta = 0.1;
br = 2*zeta*wn*J;    % N-m-s
q_max = 5;           % step, deg
q_rate = 30;         % ramp rate, deg/s

% Define the system transfer function
num = [wn^2];
den = [1 2*zeta*wn wn^2];
sys = tf(num, den);

% Define the truncated ramp input
t = [0:0.0001:0.5];
qin = q_rate*t;
index = find(qin > q_max);
qin(index) = q_max;
```

```
figure(1)
plot(t, qin, 'k-')
set(gca,'FontSize', 14)
xlabel('t (s)')
ylabel('\theta_{in} (deg)')
grid
axis([0 max(t) 0 1.1*q_max])

% Use the lisim command to find and plot the system response
figure(2)
[q, t] = lsim(sys, qin, t);
set(gca,'FontSize', 14)
plot(t, qin, 'k-', t, q, 'k-')
grid
xlabel('t (s)');
ylabel('\theta (deg)');

% Maximum overshoot
q_os = max(q)-q_max;
display(q_os)
```

The resulting output plot is shown.



Comparing this figure to the result from Example 5.9, we observe that the maximum overshoot is reduced from the large 3.6° to a more reasonable 0.31°. As an exercise, verify that by decreasing the ramp rate from 30 to 10°/s, the maximum overshoot is again reduced to 0.12°. Also, notice that due to the low damping, the angle does not exactly track the ramp. The transients persist throughout the ramp time. However, if the ramp is longer and/or the damping is larger, the transient damps out and the system does track the ramp, but with a lag. For example, if the truncated ramp ends at 10° and the damping ratio is increased to 30%, the response changes.



The vertical offset, or lag, in following the ramp is known as the *following error*. It is very important in the design of equipment, such as machine tools and robots, that must accurately follow a prescribed profile. In machine tools, the following error can lead to an error in the shape of a machined component for multi-axis motions if the following error is not identical for all axes. In robots, a following error could lead to a collision between the robot and some other component in the workspace.

5.5 Multiple Degree of Freedom Rotational Systems

Suppose that we now have two rotational inertias driven by flexible shafts as shown in Fig. 5.8. Bearings support the inertias and supply damping. Free body diagrams for the two inertias are provided in Fig. 5.9 with the positive sign convention chosen to be in the direction of positive angle (counter-clockwise about the z-axis as seen from the left). Notice that if the spring moment, M_{k_2}, is assumed to act in the positive direction on inertia J_1, it must act with an equal magnitude, but opposite direction on inertia J_2 if the spring element is assume to have zero inertia. Applying Newton's second law to each inertia, we obtain the equations of motion.

$$\begin{aligned} M_{k1} + M_{b1} + M_{k2} &= J_1\ddot{\theta}_1 \\ M_{b2} - M_{k2} &= J_2\ddot{\theta}_2 \end{aligned} \tag{5.34}$$