Homework 9

1 Problem

For circuit A in the appendix:

- Determine the current profile $i(t)$ in response to a step-input of V_0 volts using the impedance method. (Hint: find the impedance of the total circuit Z_{circuit}, solve for $I(s)$ assuming $V(s)$ is the desired step input, and use the inverse Laplace transform.)
- Determine the voltage profile $v_{\text{cap}}(t)$ across the capacitor in response to a step-input of V_0 volts using the impedance method. (Hint: use the ratio of Z_{circuit} to Z_{cap} to find the desired transfer function, as described in the notes.)
- If $R = 10$ ohm and $C = 15\mu$F how long does it take the capacitor's voltage to reach ≈ 2 % of V_0? (Hint: determine the time constant first)

Solution

Since the resistor and capacitor are in series the total impedance of the circuit is given by

$$Z_{\text{circuit}} = \frac{V(s)}{I(s)} = R + \frac{1}{Cs} = \frac{RCs+1}{Cs} \tag{1}$$

The transfer function from voltage to current (i.e., the admittance) is then

$$G_{V\to I}(s) = \frac{I(s)}{V(s)} = \frac{1}{Z_{\text{circuit}}} = \frac{Cs}{RCs+1} \tag{2}$$

In the time domain, a step-input of V_0 is $v(t) = V_0 H(t)$ which has Laplace transform $V(s) = V_0/s$. Then,

$$I(s) = G_{V\to I}(s)V(s) = \frac{Cs}{(RCs+1)}\frac{V_0}{s} = \frac{V_0 C}{RCs+1} \tag{3}$$

$$= \frac{(V_0/R)}{s+(1/RC)} \tag{4}$$

The above transfer function is in the form of a standard first-order system and the inverse Laplace transform is

$$i(t) = (V_0/R)e^{-\frac{1}{RC}t} \tag{5}$$

At the initial time, $t_0 = 0$, the current is $i(0) = V_0/R$ and as time increases the current decays to zero. The electrical time constant is $\tau = 1/(RC)$. The impedance across the capacitor is

$$Z_{\text{capacitor}} = \frac{1}{Cs} = \frac{V_{\text{cap}}(s)}{I(s)} \tag{6}$$

We can obtain a transfer from from V to V_{cap} by dividing the total circuit impedance and the capacitor impedance:

$$G(s)_{V\to V_{\text{cap}}} = \frac{Z_{\text{capacitor}}}{Z_{\text{circuit}}} = \frac{V_{\text{cap}}/I(s)}{V(s)/I(s)} \tag{7}$$

$$= \frac{1/(Cs)}{(RCs+1)/(Cs)} \tag{8}$$

$$= \left[\frac{1}{RC}\right]\frac{1}{s+(1/RC)} \tag{9}$$

The response to the step input is

$$V_{\text{cap}} = G(s)_{V\to V_{\text{cap}}} V(s) \tag{10}$$

$$= \left[\frac{1}{RC}\right]\frac{1}{s+(1/RC)}\frac{V_0}{s} \tag{11}$$

$$= \left[\frac{V_0}{RC}\right]\frac{1}{s(s+(1/RC))} \tag{12}$$

Taking the inverse Laplace transform

$$v_{\text{cap}}(t) = \left[\frac{V_0}{RC}\right] RC(1-e^{-(1/RC)t}) \tag{13}$$

$$= V_0(1-e^{-(1/RC)t}) \tag{14}$$

Again, the time constant is clearly $\tau = RC$. For the choice of parameters

$$\tau = (10 \text{ ohm})(15\times 10^{-6} \text{ Farads}) = 150\times 10^{-6} \text{ sec.} = 150\ \mu s$$

The steady-state voltage is achieved after approximately 4 time constants or $4\tau = 640\ \mu s$. Note: the capacitor approaches V_0 asymptotically but this value is not ever reached exactly. The question was originally posed as: when does the capacitor's voltage to reach V_0 should have been written instead as "when does the capacitor's voltage to reach within ≈ 2 % of V_0?" . If a student provided a response "V_0 is never reached, or V_0 is reached as $t\to\infty$ they receive full credit.

2 Problem

For circuit B in the appendix:

- Determine the current profile $i(t)$ in response to a step-input of V_0 volts using the impedance method.
- Determine the voltage profile $v_{\text{ind}}(t)$ across the inductor in response to a step-input of V_0 volts using the impedance method.

Solution

Since the resistor and inductor are in series the total impedance of the circuit is given by

$$Z_{\text{circuit}} = \frac{V(s)}{I(s)} = R + Ls \tag{15}$$

The transfer function from voltage to current is then

$$G_{V\to I}(s) = \frac{I(s)}{V(s)} = \frac{1}{Z_{\text{circuit}}} = \frac{1}{Ls+R} = \frac{1/L}{s+(R/L)} \tag{16}$$

In the time domain, a step-input of V_0 is $v(t) = V_0 H(t)$ which has Laplace transform $V(s) = V_0/s$. Then,

$$I(s) = G_{V\to I}(s)V(s) = \frac{1/L}{s+(R/L)}\frac{V_0}{s} = \frac{V_0/L}{s(s+(R/L))} \tag{17}$$

Let $a = (R/L)$ then

$$I(s) = \frac{V_0}{L}\frac{1}{s(s+a)} \tag{18}$$

which is in the form of row 14. The inverse Laplace transform is

$$i(t) = \frac{V_0}{L}\frac{1}{a}(1-e^{-at}) \tag{19}$$

$$i(t) = \frac{V_0}{L}\frac{L}{R}(1-e^{-(R/L)t}) \tag{20}$$

$$\implies i(t) = \frac{V_0}{R}(1-e^{-(R/L)t}) \tag{21}$$

At the initial time, $t_0 = 0$, the current is $i(0) = 0$ and as time increases the current reaches V_0/R. The electrical time constant is $\tau = 1/a = L/R$. The impedance across the inductor is

$$Z_{\text{inductor}} = Ls = \frac{V_{\text{inductor}}(s)}{I(s)} \tag{22}$$

We can obtain a transfer from from V to V_{inductor} by dividing the total circuit impedance and the capacitor impedance:

$$G(s)_{V\to V_{\text{inductor}}} = \frac{Z_{\text{inductor}}}{Z_{\text{circuit}}} = \frac{V_{\text{inductor}}/I(s)}{V(s)/I(s)} \tag{23}$$

$$= \frac{Ls}{Ls+R} \tag{24}$$

$$= \frac{s}{s+(R/L)} \tag{25}$$

The response to the step input is

$$V_{\text{cap}} = G(s)_{V\to V_{\text{inductor}}}V(s) \tag{26}$$

$$= \frac{s}{s+(R/L)}\frac{V_0}{s} \tag{27}$$

$$= V_0\frac{1}{s+(R/L)} \tag{28}$$

Taking the inverse Laplace transform

$$\implies v_{\text{inductor}}(t) = V_0 e^{-(R/L)t} \tag{29}$$

3 Problem

For circuit C in the appendix:

- Derive the transfer function
$$G_{V \to V_{\text{output}}} = \frac{V_{\text{out}}(s)}{V(s)}$$
from input voltage $V(s)$ to output voltage $V_{\text{out}}(s)$ (across the parallel RC connection)
- State the natural frequency and damping ratio of the circuit in terms of R, C, and L.
- Given C = 10E-6 Farads and L = 1E-3 Henries select R (units of ohms) so the circuit is critically damped and state the natural frequency of the circuit in Hz.

Solution

The capacitor and resistor are joined in parallel and their combined impedance is

$$Z_{RC} = \frac{Z_C Z_R}{Z_C + Z_R} = \frac{(1/(Cs))R}{(1/(Cs)) + R} = \frac{R}{RCs + 1} \tag{30}$$

Further, this impedance is in series with the inductor. Thus,

$$Z_{\text{circuit}} = Z_L + Z_{CR} = Ls + \frac{R}{RCs+1} = \frac{Ls(RCs+1)+R}{RCs+1} = \frac{RLCs^2 + Ls + R}{RCs+1}$$

Since V_{out} is measured across the parallel RC connection then, by definition,

$$Z_{RC} = V_{\text{out}}(s)/I(s)$$

and, as always, for the total circuit impedance

$$Z_{\text{circuit}} = V(s)/I(s)$$

Dividing the two quantities gives the desired transfer function

$$G_{V \to V_{\text{output}}} = \frac{V_{\text{out}}}{V(s)} = \frac{Z_{RC}}{Z_{\text{circuit}}} \tag{31}$$

$$= \frac{R}{RLCs^2 + Ls + R} \tag{32}$$

$$= \left[\frac{1}{LC}\right] \frac{1}{s^2 + (1/(RC))s + (1/(LC))} \tag{33}$$

Comparing the denominator of the above atransfer function to the damped harmonic oscillator we see that

$$\omega_n^2 = 1/(LC)$$

and

$$2\zeta\omega_n = 1/(RC) \tag{34}$$

$$\tag{35}$$

from which we can can conclude that the natural frequency of the circuit is

$$\omega_n = \sqrt{1/(LC)}$$

and the damping ratio is

$$2\zeta\sqrt{1/(LC)} = 1/(RC) \tag{36}$$

$$\zeta = \frac{\sqrt{LC}}{2RC} \tag{37}$$

For the circuit to be critically damped, $\zeta = 1$, the relation must hold

$$2RC = \sqrt{LC} \implies R_{\text{crit}} = \sqrt{LC}/(2C)$$

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C = 10E-6;
L = 1E-3;


R = sqrt(L*C)/(2*C)
wn = sqrt(1/(L*C))
wn_hz = wn/(2*pi)
zeta = sqrt(L*C)/(2*R*C)

R =
     5

wn =
       10000

wn_hz =
   1.5915e+03

zeta =
     1
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The required resistor value is $R = 5\Omega$ and the frequency is $\omega_n = 1,591.5$ Hz.

4 Problem

For circuit D in the appendix. Assume that all resistors, capacitors, and inductors have the same value R, C, and L, respectively. Find the total equivalent impedance of the circuit, Z_{circuit}. (Hint: simplify the circuit step-by-step, similar to this example [Link])

Solution

Refer to the solutions sketched on the following page. The first element of the circuit is a group of four resistors. Grouping each pair of resistors in series gives an impedance of $2R$.

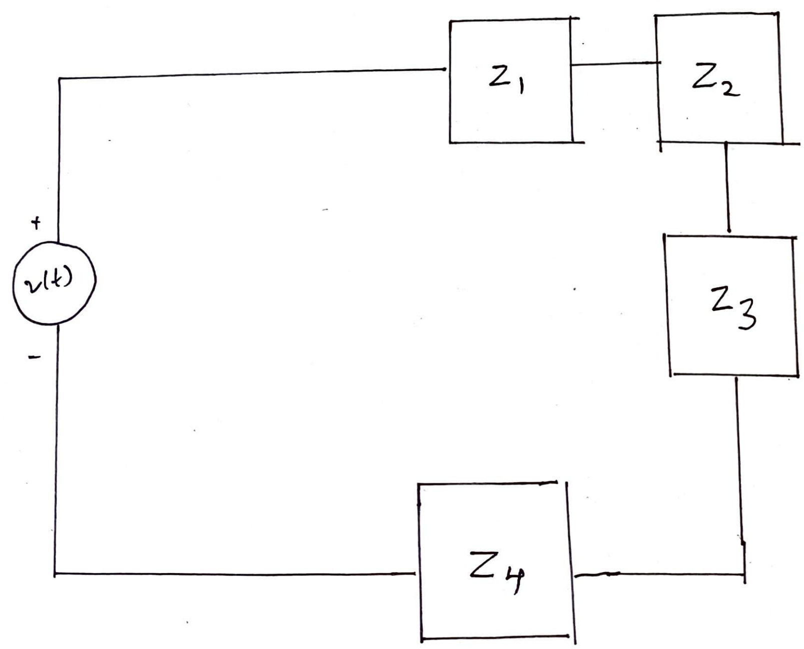
Further grouping the parallel arrangement gives the total impedance of the four resistors as $Z_1 = (2R)^2/4R = R$. The impedance of the capacitor is $Z_2 = 1/(Cs)$. The triplet of inductors in parallel has an impedance of $(1/(Ls) + 1/(Ls) + 1/(Ls))^{-1} = (3/(Ls))^{-1} = Ls/3$ and the two capacitors in series have an impedance of $1/(Cs) + 1/(Cs) = 2/(Cs)$. Grouping the inductors and capacitors gives the impedance of

$$Z_3 = \left(\frac{3}{Ls} + \frac{Cs}{2}\right)^{-1} = \left(\frac{6 + CLs^2}{2Ls}\right)^{-1} = \frac{2Ls}{CLs^2 + 6}$$

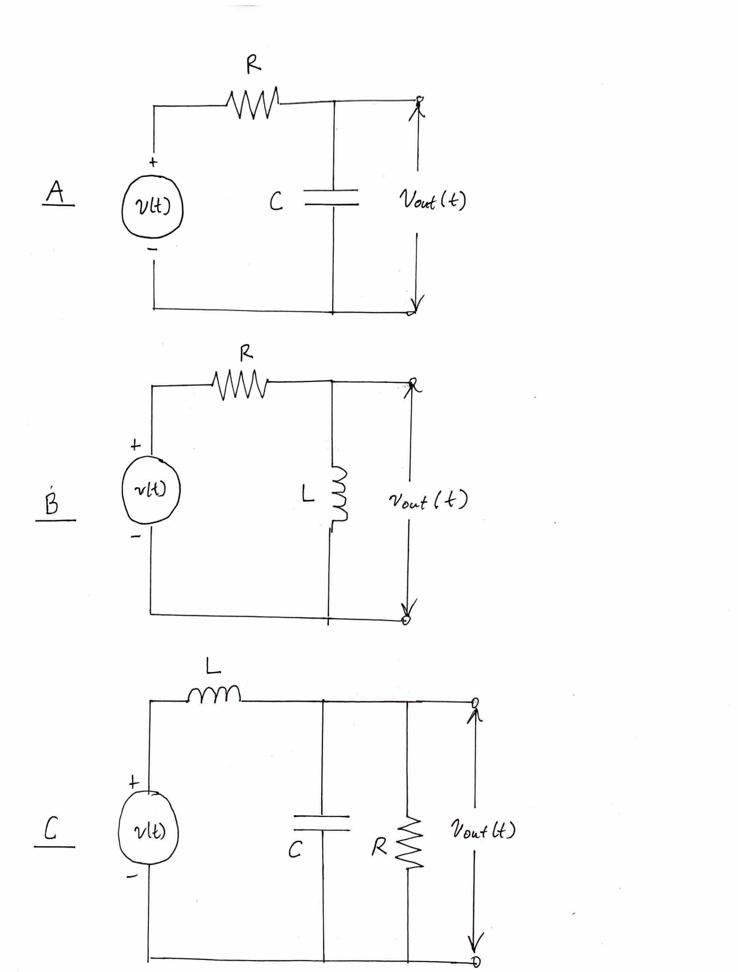
and $Z_4 = R$. Thus the total impedance is

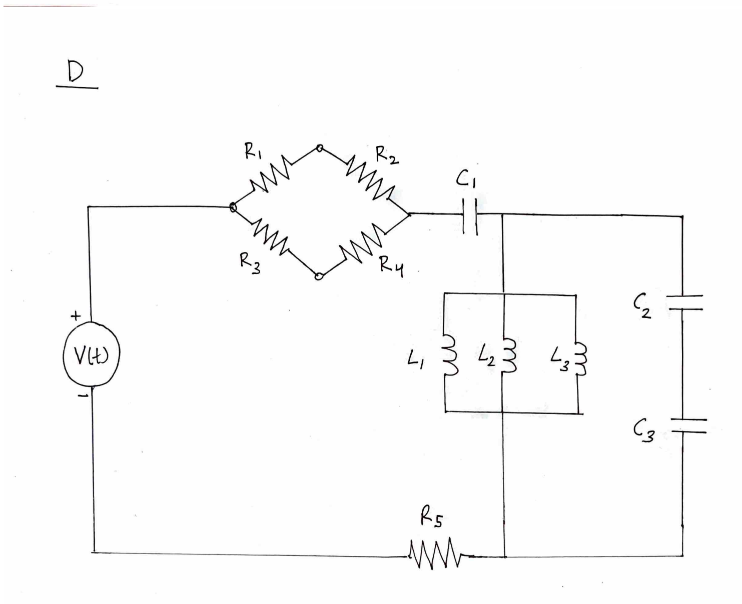
$$\begin{aligned}
Z_{\text{circuit}} &= Z_1 + Z_2 + Z_3 + Z_4 \\
\implies Z_{\text{circuit}} &= R + \frac{1}{Cs} + \frac{2Ls}{CLs^2 + 6} + R \\
&= \frac{2R(CLs^2 + 6)(Cs) + (CLs^2 + 6) + 2Ls(Cs)}{(CLs^2 + 6)(Cs)} \\
&= \frac{2RCs(CLs^2 + 6) + (CLs^2 + 6) + 2CLs^2}{C^2Ls^3 + 6Cs} \\
&= \frac{(2RLC^2s^3 + 12RCs) + (CLs^2 + 6) + 2CLs^2}{C^2Ls^3 + 6Cs} \\
&= \frac{(2RLC^2)s^3 + (3CL)s^2 + 12RCs + 6}{C^2Ls^3 + 6Cs}
\end{aligned}$$

The circuit has third-order dynamics.



Appendix


A
R
v(t)
C
Vout(t)
B
R
v(t)
L
Vout(t)
C
L
v(t)
C
R
Vout(t)


D
R1
R2
R3
R4
C1
V(t)
L1
L2
L3
C2
C3
R5