第七章作业

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证明：

由 $\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})=0$ $\quad\quad$ $\vec{a}\times(-\vec{a})=0$

则左边 $= -{}^C({}^U\Omega_C)\times[({}^C({}^U\Omega_C)\times{}^CP_i)\times{}^CP_i]-({}^C({}^U\Omega_C)\times{}^CP_i)\times({}^CP_i\times{}^C({}^U\Omega_C))$

$= -{}^C({}^U\Omega_C)\times({}^CP_i\times({}^CP_i\times{}^C({}^U\Omega_C)))$

= 右边

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证明：

$$^CI=\begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{pmatrix}$$

$$P_C^TP_CI_3=\begin{pmatrix} x_c^2+y_c^2+z_c^2 & & \\ & x_c^2+y_c^2+z_c^2 & \\ & & x_c^2+y_c^2+z_c^2 \end{pmatrix}$$

$$P_CP_C^T=\begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}[x_c, y_c, z_c]=\begin{bmatrix} x_c^2 & x_cy_c & x_cz_c \\ x_cy_c & y_c^2 & y_cz_c \\ x_cz_c & y_cz_c & z_c^2 \end{bmatrix}$$

$$\therefore m(P_C^TP_CI_3-P_CP_C^T)=m\begin{bmatrix} y_c^2+z_c^2 & -x_cy_c & -x_cz_c \\ -x_cy_c & x_c^2+z_c^2 & -y_cz_c \\ -x_cz_c & -y_cz_c & x_c^2+y_c^2 \end{bmatrix}$$

现在质心 C 在 {A} 中坐标为 $P_C=[x_c, y_c, z_c]^T$

任一质点在 {C} 中坐标为 $[X_C, Y_C, Z_C]^T$，{A} 中为 $[X_A, Y_A, Z_A]^T$

则有：

$$\begin{cases} X_A = X_C + x_c \\ Y_A = Y_C + y_c \\ Z_A = Z_C + z_c \end{cases}$$

对惯量积：　　　$\rho(X_A, Y_A, Z_A)$，因为描述的是同一点？

$${}^{A}I_{xx} = \int_B (Y_A^2 + Z_A^2)\,\rho(X_C, Y_C, Z_C)\,dV$$

代入 $Y_A = Y_C + y_c$，$Z_A = Z_C + z_c$

$${}^{A}I_{xx} = \int_B \rho\left[(y_c^2 + z_c^2) + (Y_c^2 + Z_c^2) + 2(Y_c y_c + Z_c z_c)\right] dV$$

$$\int_B Y_c y_c \rho\, dV = y_c \int_B Y_c \rho\, dV = 0$$

$$= {}^{C}I_{xx} + m(y_c^2 + z_c^2)$$

同理

$${}^{A}I_{yy} = {}^{C}I_{yy} + m(x_c^2 + z_c^2)$$

$${}^{A}I_{zz} = {}^{C}I_{zz} + m(y_c^2 + z_c^2)$$

对惯量矩：

$${}^{A}I_{xy} = -\int_B \rho X_A Y_A\, dV$$

代入 $X_A = X_c + x_c$，$Y_A = Y_C + y_c$

$${}^{A}I_{xy} = -\int_B \rho(X_c Y_c + X_c y_c + Y_c x_c + x_c y_c)\,dV$$

$$= {}^{C}I_{xy} - m x_c y_c$$

同理

$${}^{A}I_{xz} = {}^{C}I_{xz} - m x_c z_c$$

$${}^{A}I_{yz} = {}^{C}I_{yz} - m y_c z_c$$

逐项代入可得

$${}^{A}I = {}^{C}I + m(P_c^T P_c I_3 - P_c P_c^T)$$

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解:

$$ {}^{0}_{1}R=\begin{bmatrix} c_1 & -s_1 & 0\\ s_1 & c_1 & 0\\ 0 & 0 & 1\end{bmatrix} \quad {}^{1}_{2}R=\begin{bmatrix} 1 & 0 & 0\\ 0 & 0 & 1\\ 0 & -1 & 0\end{bmatrix} \quad {}^{1}O_2=\begin{bmatrix} l_1\\ 0\\ 0\end{bmatrix} $$

① 连杆质心的位置矢量:

$$ {}^{1}P_{c_1}=l_1\hat{X}_1=[l_1,0,0]^T \qquad {}^{2}_{3}R=I_3 \qquad {}^{2}O_3=\begin{bmatrix}0\\0\\d_2\end{bmatrix} $$

$$ {}^{2}P_{c_2}=d_2\hat{Z}_2=[0,0,d_2]^T $$

② 惯性张量

$$ {}^{c_1}I_1=0 $$

$$ {}^{c_2}I_2=0 $$

③ 无力作用于末端执行器; 机器人基座保持不动

$$ \begin{cases} {}^{3}f_3=[0,0,0]^T \\ {}^{3}n_3=[0,0,0]^T \end{cases} ; \quad \begin{cases} {}^{0}\omega_0=[0,0,0]^T \\ {}^{0}\dot{\omega}_0=[0,0,0]^T \end{cases} \quad {}^{0}\dot{v}_0=g\hat{Y}_0=[0,g,0]^T $$ (考虑重力)

向外迭代 0→1

$$ {}^{1}\omega_1={}^{1}_{0}R\,{}^{0}\omega_0+\dot{\theta}_1\,{}^{1}\hat{Z}_1=[0,0,\dot{\theta}_1]^T $$

$$ {}^{1}\dot{\omega}_1={}^{1}_{0}R\,{}^{0}\dot{\omega}_0+{}^{1}_{0}R\,{}^{0}\omega_0\times\dot{\theta}_1\,{}^{1}\hat{Z}_1+\ddot{\theta}_1\,{}^{1}\hat{Z}_1=[0,0,\ddot{\theta}_1]^T $$

$$ {}^{1}\dot{v}_1={}^{1}_{0}R({}^{0}\dot{\omega}_0\times{}^{0}O_1+{}^{0}\omega_0\times({}^{0}\omega_0\times{}^{0}O_1)+{}^{0}\dot{v}_0)=[gs_1,gc_1,0]^T $$

则 $$ {}^{1}\dot{v}_{c_1}={}^{1}\dot{\omega}_1\times{}^{1}P_{c_1}+{}^{1}\omega_1\times({}^{1}\omega_1\times{}^{1}P_{c_1})+{}^{1}\dot{v}_1=\begin{bmatrix}-l_1\dot{\theta}_1^2+gs_1\\ l_1\ddot{\theta}_1+gc_1\\ 0\end{bmatrix} $$

则 $$ {}^{1}F_1=m_1\,{}^{1}\dot{v}_{c_1}=\begin{bmatrix}-m_1l_1\dot{\theta}_1^2+m_1gs_1\\ m_1l_1\ddot{\theta}_1+m_1gc_1\\ 0\end{bmatrix} $$

$$ {}^{1}N_1={}^{c_1}I_1\,{}^{1}\dot{\omega}_1+{}^{1}\omega_1\times{}^{c_1}I_1\,{}^{1}\omega_1=[0,0,0]^T $$

向外迭代 1→2

$$ {}^{2}\omega_2={}^{2}_{1}R\,{}^{1}\omega_1=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\cdot\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix}=\begin{bmatrix}0\\-\dot{\theta}_1\\0\end{bmatrix} $$

$$ {}^{2}\dot{\omega}_2={}^{2}_{1}R\,{}^{1}\dot{\omega}_1=[0,-\ddot{\theta}_1,0]^T $$

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$${}^2\dot{v}_2 = {}^2_1R\left({}^1\dot{\omega}_1 \times {}^1O_2 + {}^1\omega_1 \times ({}^1\omega_1 \times {}^1O_2) + {}^1\dot{v}_1\right) + 2\,{}^2\omega_2 \times \dot{d}_2\,{}^2\hat{Z}_2 + \ddot{d}_2\,{}^2\hat{Z}_2$$

$$[0, l_1\ddot{\theta}_1, 0]^T$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}\left(\begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix}\right) + {}^1\dot{v}_1\right) + 2\begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ \dot{d}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \ddot{d}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ gc_1 + l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -2\dot{\theta}_1\dot{d}_2 \\ 0 \\ \ddot{d}_2 \end{bmatrix}$$

$$= \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 - 2\dot{\theta}_1\dot{d}_2 \\ 0 \\ gc_1 + \ddot{d}_2 + l_1\ddot{\theta}_1 \end{bmatrix}$$

则 ${}^2\dot{v}_{C2} = {}^2\dot{\omega}_2 \times {}^2P_{C2} + {}^2\omega_2 \times ({}^2\omega_2 \times {}^2P_{C2}) + {}^2\dot{v}_2$

$$= \begin{bmatrix} 0 \\ -\ddot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 0 \\ -\dot{\theta}_1 \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ d_2 \end{bmatrix}\right) + {}^2\dot{v}_2$$

$$= \begin{bmatrix} -\ddot{\theta}_1 d_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\dot{\theta}_1^2 d_2 \end{bmatrix} + {}^2\dot{v}_2$$

$$= \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 - 2\dot{\theta}_1\dot{d}_2 - \ddot{\theta}_1 d_2 \\ 0 \\ gc_1 + \ddot{d}_2 - \dot{\theta}_1^2 d_2 + l_1\ddot{\theta}_1 \end{bmatrix}$$

则 ${}^2F_2 = m_2\,{}^2\dot{v}_{C2} = [-m_2l_1\dot{\theta}_1^2 + m_2gs_1 - 2\dot{\theta}_1\dot{d}_2m_2 - m_2\ddot{\theta}_1d_2 \quad 0 \quad m_2l_1\ddot{\theta}_1 + m_2gc_1 + m_2\ddot{d}_2 - m_2\dot{\theta}_1^2d_2]$

${}^2N_2 = [0, 0, 0]^T$

向内迭代 3→2

${}^2f_2 = {}^2_3R\,{}^3f_3 + {}^2F_2 = {}^2F_2$

$${}^2n_2 = {}^2N_2 + {}^2_3R\,{}^3n_3 + {}^2P_{C2} \times {}^2F_2 + {}^2O_3 \times {}^2_3R\,{}^3f_3 = \begin{bmatrix} 0 \\ d_2(-m_2l_1\dot{\theta}_1^2 + m_2gs_1 - 2\dot{\theta}_1\dot{d}_2m_2 - m_2\ddot{\theta}_1d_2) \\ 0 \end{bmatrix}$$

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$$\tau_2 = {}^2f_2^T\,{}^2\hat{Z}_2$$

$$= {}^2F_2^T \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= m_2 g c_1 + m_2\ddot{d}_2 - m_2\dot{\theta}_1^2 d_2 + m_2 l_1 \ddot{\theta}_1$$

向内迭代 2→1

$${}^1f_1 = {}^1_2R\,{}^2f_2 + {}^1F_1$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} {}^2f_{2x} \\ 0 \\ {}^2f_{2z} \end{bmatrix} + \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -m_2 l_1 \dot{\theta}_1^2 + m_2 g s_1 - 2\dot{\theta}_1 \dot{d}_2 m_2 - m_2 \ddot{\theta}_1 d_2 - m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_2 g c_1 + m_2 \ddot{d}_2 - m_2 \dot{\theta}_1^2 d_2 + m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

$${}^1n_1 = {}^1N_1 + {}^1_2R\,{}^2n_2 + {}^1P_{c1} \times {}^1F_1 + {}^1O_2 \times {}^1_2R\,{}^2f_2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ * \\ 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} * \\ * \\ 0 \end{bmatrix} + \begin{bmatrix} l_1 \\ 0 \\ 0 \end{bmatrix} \times \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} * \\ 0 \\ * \end{bmatrix} \right)$$

$$= \begin{bmatrix} 0 \\ 0 \\ -{}^2n_2\text{的}y \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1(m_1 l_1 \ddot{\theta}_1 + m_1 g c_1) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ l_1 \cdot {}^2f_2\text{的}z \end{bmatrix}$$

$$= \begin{bmatrix} 0, & 0, & \begin{matrix} -d_2(-m_2 l_1 \dot{\theta}_1^2 + m_2 g s_1 - 2\dot{\theta}_1 \dot{d}_2 m_2 - m_2 \ddot{\theta}_1 d_2) + l_1(m_1 l_1 \ddot{\theta}_1 + m_1 g c_1) \\ + l_1(m_2 g c_1 + m_2 \ddot{d}_2 - m_2 \dot{\theta}_1^2 d_2 + m_2 l_1 \ddot{\theta}_1) \end{matrix} \end{bmatrix}^T$$

$$\therefore \tau_1 = {}^1n_1^T\,{}^1\hat{Z}_1 = (m_1 l_1^2 + m_2(l_1^2 + d_2^2))\ddot{\theta}_1 + 2m_2 d_2 \dot{d}_2 \dot{\theta}_1 + m_2 l_1 \ddot{d}_2 + m_1 g l_1 c_1 + m_2 g(l_1 c_1 - d_2 s_1)$$

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证明：

$E_k = \frac{1}{2}\int |\vec{v}_c|^2 dm$, $\vec{v}_c = \vec{\omega}_c \times \vec{r}_c$

$$E_k = \frac{1}{2}\int |\vec{\omega}_c \times \vec{r}_c|^2 dm$$

$$= \frac{1}{2}\int (\vec{\omega}_c \times \vec{r}_c)\cdot(\vec{\omega}_c \times \vec{r}_c)\, dm$$

$(\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a}$

$$= \frac{1}{2}\int [\vec{r}_c \times (\vec{\omega}_c \times \vec{r}_c)]\cdot \vec{\omega}_c\, dm$$

$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$

$$= \frac{1}{2}\int [r_c^2\vec{\omega}_c - (\vec{r}_c\cdot\vec{\omega}_c)\vec{r}_c]\cdot\vec{\omega}_c\, dm$$

$$= \frac{1}{2}\int \vec{\omega}_c^T [r_c^2\vec{\omega}_c - (\vec{r}_c\cdot\vec{\omega}_c)\vec{r}_c]\, dm$$

标量

$$= \frac{1}{2}\int [r_c^2\vec{\omega}_c^T\vec{\omega}_c - (\vec{r}_c\cdot\vec{\omega}_c)\cdot\vec{\omega}_c^T\vec{r}_c]$$

$$(\vec{r}_c^T\vec{\omega}_c)(\vec{\omega}_c^T\vec{r}_c)$$

$$= \frac{1}{2}\int r_c^2\vec{\omega}_c^T\vec{\omega}_c - \vec{\omega}_c^T\vec{r}_c\cdot\vec{r}_c^T\vec{\omega}_c\, dm$$

$$= \frac{1}{2}\vec{\omega}_c^T \int r_c^2 - \vec{r}_c\cdot\vec{r}_c^T dm\ \vec{\omega}_c$$

$= I$

$$= \frac{1}{2}\vec{\omega}_c^{T}\, I\, \omega_c$$

即 $\frac{1}{2}\,{}^c\omega^T\,{}^cI\,{}^c\omega$

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$V_{c1} = l_1\dot\theta_1$

$P_{c2} = [l_1c_1 - d_2s_1,\ l_1s_1 + d_2c_1,\ 0]^T$ 在世界坐标系下

$\therefore V_{c2} = \dfrac{dP_{c2}}{dt} = \begin{bmatrix} -l_1s_1\dot\theta_1 - d_2c_1\dot\theta_1 - \dot d_2s_1 \\ l_1c_1\dot\theta_1 - d_2s_1\dot\theta_1 + \dot d_2c_1 \\ 0 \end{bmatrix}$

动：

$\therefore k_1 = \frac{1}{2}m_1V_{c1}^2 = \frac{1}{2}m_1l_1^2\dot\theta_1^2$ $\quad k = k_1 + k_2$

$k_2 = \frac{1}{2}m_2V_{c2}^TV_{c2} = \frac{1}{2}m_2[(l_1^2+d_2^2)\dot\theta_1^2 + 2l_1\dot\theta_1\dot d_2 + \dot d_2^2]$

势：

$u_1 = m_1gl_1s_1$

$u_2 = m_2g(l_1s_1 + d_2c_1)$ $\quad u = u_1 + u_2$

由 Lagrange 方程 $\dfrac{d}{dt}\dfrac{\partial k}{\partial \dot q} - \dfrac{\partial k}{\partial q} + \dfrac{\partial u}{\partial q} = \tau$

$\therefore \dfrac{\partial k}{\partial \dot q} = \begin{bmatrix} m_1l_1^2\dot\theta_1 + m_2(l_1^2+d_2^2)\dot\theta_1 + m_2l_1\dot d_2 \\ m_2l_1\dot\theta_1 + m_2\dot d_2 \end{bmatrix}$ (对 $\dot\theta_1$) (对 $\dot d_2$)

$\dfrac{\partial k}{\partial q} = \begin{bmatrix} 0 \\ m_2\dot\theta_1^2d_2 \end{bmatrix}$

$\dfrac{\partial u}{\partial q} = \begin{bmatrix} m_1gl_1c_1 + m_2g(l_1c_1 - d_2s_1) \\ m_2gc_1 \end{bmatrix}$

$\dfrac{d}{dt}\left(\dfrac{\partial k}{\partial \dot q}\right) = \begin{bmatrix} (m_1l_1^2 + m_2(l_1^2+d_2^2))\ddot\theta_1 + 2m_2d_2\dot d_2\dot\theta_1 + m_2l_1\ddot d_2 \\ m_2l_1\ddot\theta_1 + m_2\ddot d_2 \end{bmatrix}$

$\therefore \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} (m_1l_1^2 + m_2(l_1^2+d_2^2))\ddot\theta_1 + 2m_2d_2\dot d_2\dot\theta_1 + m_2l_1\ddot d_2 + m_1gl_1c_1 + m_2g(l_1c_1 - d_2s_1) \\ m_2l_1\ddot\theta_1 + m_2\ddot d_2 - m_2\dot\theta_1^2d_2 + m_2gc_1 \end{bmatrix}$