CECS 428

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1 The good subspace: Simple problems

To take advantage of a good subspace, you might ask the customer who is ordering your algorithm whether they really require the intractable problem to be solved in its full generality. If the customer can narrow the problem slightly, then it may be that it makes a once intractable problem efficiently solvable.

1.1 Clique

1.1.1 Application of the clique problem

In pharmacology, there is a graph based method for searching for a compound that will interact with a specific biochemical target. One of the ways that this is done is to generate a large library of graphs describing known molecules. Then, given a particular target, you generate a *complimentary* graph describing its structure; then you search for graphs in the library that have large cliques in common with the complimentary graph of the target. The resulting list of molecules from the library form a set of likely candidates for a molecule that will produce the desired action by bonding with the target.

1.1.2 Planar cliques

The clique problem asks whether there exists a clique of size k in a given graph $G = (V, E)$.

Kuratowski's Theorem: A graph is planar if and only if it does not contain an isomorphic copy of a $K_{3,3}$ nor a K_5.

Assume that we restrict our problem to only planar graphs G. Notice that if $k \geq 5$, then Kuratowski's Theorem simply states that the answer is NO. If $k < 5$, then simple searching every subset of k vertices to test for a clique is polynomial in the size of the graph. (There is an algorithm by Hopcroft and Tarjan to test whether any given graph is planar that runs in linear time!)

1.2 The Gray code

The Hamilton cycle question is defined to be: Given a graph G does there exists a simple tour that visits every vertex in G exactly once and returns to its starting point?

Obviously, there is a Hamiltonian cycle in C_n and K_n for any n.

Remember that Q_{n+1} is defined by taking two copies of Q_n and putting a 0 bit on the front of every node in the first copy and a 1 bit on the front of every node in the second copy. Then connect the two copies by connecting all vertices in the first copy with their corresponding vertices in the second copy.

1.2.1 Hamilton circuit application

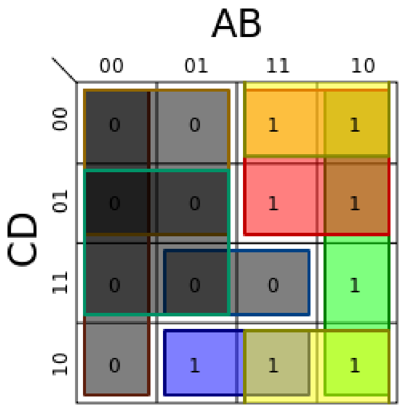
A **Gray code** is an interesting application of Hamilton circuits. Back in the days, computer equipment wasn't as reliable as it is nowadays. Bits were sometimes measured using analog signals. (In other words, the numbers from 0 to 7 would be measured as some number modulo 8.) If we just use the obvious numbering of the analog signals (0 corresponding to 0, 1 corresponding to 1, and so on), then a small error in signal strength might lead to a large error in the bit string representation of the number. For example, do we count the analog signal 3.51 as 3, 011, or 4, 100? Notice that a small error in the signal produces an error in *every* bit of the resulting bit string! Gray came up with a numbering scheme such that a small error in analog signal produces only a single bit of error in the resulting bit string. One possible numbering system is 0,1,3,2,6,7,5,4. Notice that adjacent numbers only differ by one bit. This numbering system corresponds to a Hamilton circuit of the graph Q_3. See if you can determine why...

Claim: A Hamilton cycle in Q_n always exists of $n \geq 2$.

Proof: Induction. The claim is clearly true for $n = 2$. Now assume that there exists a Hamilton cycle in Q_k. Consider the following cycle in Q_{k+1}: Take the Hamilton cycle given by the inductive assumption in the first copy of Q_k but do not traverse the final edge back to the initial vertex; instead, cross the edge that leads to the vertex in the second copy of Q_k. Now take the same path *in reverse* but do not cross the final edge; instead cross back to the first copy of Q_k and you are guaranteed to have made a Hamilton cycle. □

1.2.2 Gray code application

An interesting application of Gray codes are Karnaugh maps for simplifying boolean expressions for the purposes of, for example, designing simple boolean circuits from a potentially complex boolean function. The Karnaugh map (discovered in 1953), also known as the K-map, is a method to simplify boolean algebra expressions. The Karnaugh map reduces the need for extensive calculations by taking advantage of humans' pattern-recognition capability. It also permits the rapid identification and elimination of potential race conditions.



CD \ AB	00	01	11	10
00	0	0	1	1
01	0	0	1	1
11	0	0	0	1
10	0	1	1	1

f(A,B,C,D) = E(6,8,9,10,11,12,13,14)
F=AC'+AB'+BCD'+AD'
F=(A+B)(A+C)(B'+C'+D')(A+D')

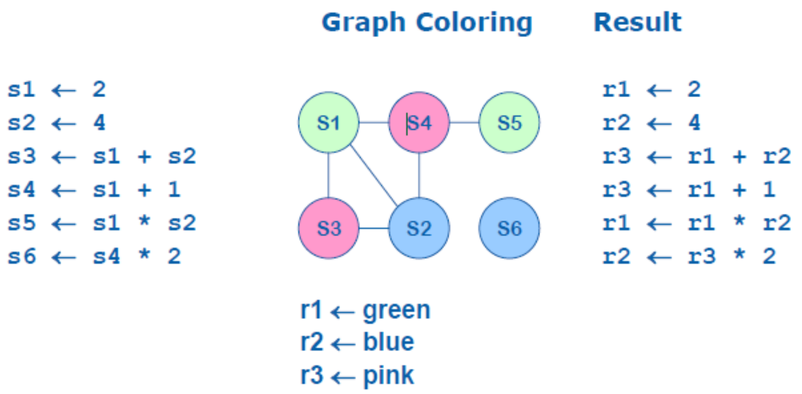
The required boolean results are transferred from a truth table onto a two-dimensional grid where the cells are ordered in Gray code, and each cell represents one combination of input conditions. Optimal groups of 1s or 0s are identified, which represent the terms of a canonical form of the logic in the original truth table. These terms can be used to write a minimal boolean expression representing the required logic.

1.3 Register Allocation

The k-coloring problem is defined as follows: Given an undirected graph G, does there exist a way to color the vertices of G using at most k colors in such a way that the two vertices on every edge are different colors? It is known that the k-coloring problem is NP-complete. (Beware: k is a paramter of the problem and *not* a constant.)

Consider the problem of designing a compiler. A given chip has a certain number of registers that stay "close" to the CPU, say k of them (where k may be around 32). One of the goals of the compiler is to act as a register allocator: an entity that determines which of the program's variables are kept in the registers as the program executes. Note that typically there will be many more variables in a given program than k, but only k can be kept on the CPU at any given time.

When formulated as a coloring problem, each node in the graph represents the live range of a particular value. A variable is defined to be *live* when the variable will be used again before it is overwritten. A *dead* variable is one that either (a) will never be used again or (b) will be overwritten during its next use. A live range is defined as the time when a given temporal variable (defined more thoroughly below) is alive. An edge between two nodes indicates that those two live ranges interfere with each other because their lifetimes overlap. In other words, they are both simultaneously active at some point in time, so they must be assigned to different registers. The resulting graph is thus called an interference graph.



```
s1 ← 2
s2 ← 4
s3 ← s1 + s2
s4 ← s1 + 1
s5 ← s1 * s2
s6 ← s4 * 2
```

```
r1 ← 2
r2 ← 4
r3 ← r1 + r2
r3 ← r1 + 1
r1 ← r1 * r2
r2 ← r3 * 2
```

r1 ← green
r2 ← blue
r3 ← pink

Chaitin (1982) made the following observation: It is possible to solve the register allocation problem if and only if the interference graph is k-colorable. The intuition behind this observation is that we can assign a distinct color to each register; because each temporal variable needs to be assigned a register when it is used, its color is its register assignment. If two adjacent nodes have the same color, then the compiler would be required to store these two variables simultaneously in the same register! If a graph is not k-colorable, then some temporal variables need to spill over into the L1 or L2 cache (or, further, if necessary) and cause a delay when their time comes.

In the top figure above, the variable s_1 is live until s_5 is assigned and after that point, it is no longer used. Thus, s_1 needs to be adjacent to s_2, s_3, s_4. s_2 is live until s_5 as well so it needs to be adjacent to s_3 and s_4. s_3, once assigned, is never used again. s_4 is used once more when assigning s_6 so it is live at the assignment of s_5. Neither s_5 nor s_6, once assigned, will ever be used again. Notice that if we changed s_5 to s_1, then this new value of s_1 would need to be its *own* temporal variable s_1' and would require its own node (which would appear in exactly the same spot as s_5 currently).

Even though the problem is likely intractable, there is an obvious fact that helps in certain cases: If there exists a node x in a graph G with fewer than k neighbors, then G is k-colorable if and only if $G - \{x\}$ is k-colorable. This leads to the following algorithm by Chaitin:

- Create a node stack as follows: Choose any node x with fewer than k neighbors. Push x on the stack. Then delete x and all its incident edges.
- Using the stack created above: Pop the next node x from the stack. Assign x a different color than all of the nodes surrounding that have already been colored.

It is clear that inductively, every time a color is assigned, it is a legal color and thus the graph is k-colored legally.

Interesting fact: Most interference graphs are sparse and do *not* look random in that they contain clumps of high-density nodes mixed with low. As of 2005, it was estimated that as many as 95% of all programs written in practice have *chordal* interference graphs; these can be colored optimally in linear time. Even many of the non-chordal graphs respond well to the chordal algorithm for coloring. (An undirected graph is chordal if every cycle with 4 or more nodes has a chord, that is, an edge not part of the cycle that connects two nodes on the cycle.)

2 The good subspace: Satisfiability I

2.1 Introduction

These problems are among the first to be shown to be computationally intractable. The Boolean Satisfiability Problem is as follows: Given a Boolean formula f, does there exist a satisying assignment for f?

An essential circuit design task is to check the functional equivalence of two circuits. Let C_A and C_B denote two circuits, both with inputs $x_1, x_2, \ldots, x_n$ and both with m outputs. The functions implemented by the two circuits are defined as follows: $f_A : \{0,1\}^n \to \{0,1\}^m$ and $f_B : \{0,1\}^n \to \{0,1\}^m$. Let $x \in \{0,1\}^n$ and define $f_A(x) = (f_A^{(1)}(x), f_A^{(2)}(x), \ldots, f_A^{(m)}(x))$ and $f_B(x) = (f_B^{(1)}(x), f_B^{(2)}(x), \ldots, f_B^{(m)}(x))$. Determining whether these two circuits compute the same function is equivalent to determining whether there exists a satisfying assignment to the following boolean formula:

$$\bigvee_{i=1}^{m} (f_A^{(i)}(x) \oplus f_B^{(i)}(x))$$

If there is a way to satisfy this, then there exists a way to make the two formulas different on one of the outputs.

One might think that simplifying the problem to only 3CNF formulas might help: The 3-SAT (3-CNF) Problem is as follows: Given a set of clauses, each of which contains exactly 3 literals, does there exist an assignment to the variables so that at least one literal in each clause is TRUE? Unfortuntately, the 3-SAT problem has also been shown to be intractable.

In the following, we will introduce further specializations that do help.

2.2 2-SAT

2-SAT: Instead of 3 variables per clause, what about 2? Let there be m clauses with variables $x_1, x_2, \ldots, x_n$. Create the graph G in the following way: If there is a clause $A \vee B$, then we place two directed arrows in G, namely $\neg A \to B$ and $\neg B \to A$. (Obviously, these mean that in order to make the entire statement TRUE, if A is FALSE, then B must be TRUE and vice versa.)

2.2.1 Example

$$x_0 \vee x_2, x_0 \vee \neg x_3, x_1 \vee \neg x_3, x_1 \vee \neg x_4,$$

$$x_2 \vee \neg x_4, x_0 \vee \neg x_5, x_1 \vee \neg x_5, x_2 \vee \neg x_5,$$

$$x_3 \vee x_6, x_4 \vee x_6, x_5 \vee x_6$$

2.2.2 Algorithm

Claim: For any x_i, if x_i and $\neg x_i$ are in the same connected component, then there cannot be a satisfying assignment for the variables.

Proof: Either x_i must get assigned to TRUE or FALSE. If x_i is assigned to TRUE, then the path from x_i to $\neg x_i$ implies that x_i must be FALSE, a contradiction. If x_i is assigned to FALSE, then then path from $\neg x_i$ to x_i implies that x_i must be TRUE, a contradiction. Thus, x_i cannot be assigned in such a way as to make all implications true and there is no satisfying assignment. □

Claim: If $\forall i, x_i$ and $\neg x_i$ are in different strongly connected components, then there exists a satisfying assignment that can be computed quickly.

Proof: Perform Tarjan's strongly connected component algorithm on G, and then topologically sort the strongly connected components. Choose any strongly connected component C with out-degree 0 (there must exist one), and let all unassigned literals in C be TRUE; then remove C from the SCC DAG. Continue this until all variables have been assigned.

By the definition of the algorithm and the fact that no variable and its negation can be in the same SCC, we will never simultaneously assign a variable and its negation to the same truth value. Is it possible to ever have a path from TRUE to FALSE? Every time we assign a literal to TRUE, is it possible for there to exist a path from that literal to a FALSE one? If not, then we are done.

Let the first point in time when we are assigning a literal l_1 to TRUE and there exists a path from l_1 to a FALSE literal l_2 be time t. Either these literals are both being assigned simultaneously at time t or not.

- Assume that the assignments are being assigned simultaneously. Then $\neg l_2$ is currently being assigned to TRUE in the same connected component as l_1. This implies that l_2 cannot be in the same connected component as l_1 because l_2 cannot be in the same connected component as $\neg l_2$. However, this implies that, before the assignment, there exists a path $l_1 \leadsto l_2$ where l_2 is unassigned and the connected component from l_1 points to l_2's component. This implies that the connected component from l_1 does not have out-degree 0 and we have violated the steps of the algorithm.

- Assume that the assignments are not simultaneous.

  If l_1 was assigned strictly before l_2, then there existed a point in the algorithm where a literal assigned to TRUE pointed to an unassigned literal which is impossible if we follow the steps of the algorithm.

  Now assume that l_2 was assigned strictly before l_1. Recalling that $l_1 \leadsto l_2 \Rightarrow \neg l_2 \leadsto \neg l_1$ by the construction of the graph. But this implies that there was a point where a TRUE literal $\neg l_2$ was assigned when there was a path leading to an unassigned literal $\neg l_1$. Again, this is not possible for the same reason as above.

Thus, we never assign a TRUE literal that leads to a FALSE literal and our algorithm never makes a contradiction. This works in linear time! □

3 The good subspace: Satisfiability II

3.1 Linear 3-SAT

Consider a 3-SAT instance with the following special locality property. Suppose there are n variables in the Boolean formula, and that they are numbered $1, 2, \ldots, n$ in such a way that each clause involves variables whose numbers are within ± 10 of each other. Does this restriction allow the problem to be solved more efficiently?

Solution: We will build a function that, given a linear 3-SAT instance with variables x through y (where $x < y$), determines all possible assignments for the variables x through $x+9$ and $y-9$ through y such that there exists a satisfying assignment to the instance given the assignments for these 20 variables. You can think about what gets returned from this function as a list of assignments for 20 variables. There are at most 2^{20} of these total, a constant number.

If $y - x < 200$, just perform a brute force search to determine if top and bottom assignments exist and return them if they do, NULL if they don't.

Define the middle variable to be $m = \lfloor \frac{x+y}{2} \rfloor$. Make the following definitions.

1. Let the set of clauses that contain at least one variable strictly less than $m - 10$ be called s_1.

2. Let the set of clauses that contain at least one variable strictly greater than $m + 10$ be called s_2.

3. Let the set of all other clauses be called s_3.

Note that $s_1 \cup s_2 \cup s_3$ makes up the entire problem instance and also that s_1, s_2, and s_3 are pairwise disjoint. Finally we note that at most 10 variables can overlap between s_1 and s_3 ($m-10$ up to $m-1$) and between s_2 and s_3 ($m+10$ down to $m+1$).

We first claim that the satisfying assignments for the set s_3 can be enumerated in time $O(|s_3|)$. Note that this is the size of the clause and not the number of variables in the clause. The number of variables that can occur in s_3 is bounded above by a constant: 21. Thus, we simply try all 2^{21} possible satisfying assignments and keep track of those that make the entire clause set s_3 TRUE.

We recursively call the function on s_1 and s_2. This yields a solution list for the top 10 and bottom 10 variables only for each of the first and second clause sets. We now match these solutions with the "top and bottom" assignments from the middle that we have already calculated. Even though it seems as if there is a great to deal to check, there is actually at most a constant amount of additional checking to be done. We can now tell which of the assignments of the 20 variables that consist of the top 10 from s_2 and the bottom 10 from s_1 match up to form a possible satisfying assignment for the problem instance as a whole. This list of assignments is then returned.

Let $T(n)$ be the running time for an instance of this problem of size n. Then we get the recursion

$$T(n) = T(|s_1|) + T(|s_2|) + O(|s_3|) + O(1) \Rightarrow$$

$$\exists c' > 0 \text{ such that eventually } T(n) \leq T(|s_1|) + T(|s_2|) + c'|s_3|$$

We claim that $T(n) = O(n) \Rightarrow \exists c > 0$ such that eventually $T(n) \leq cn$. Inductively assume that the result is true for all values strictly less than n. Then we get that

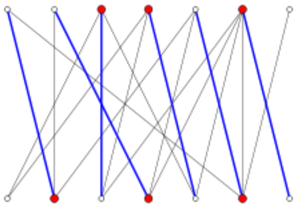
$$T(n) \leq T(|s_1|) + T(|s_2|) + c'|s_3| \leq c|s_1| + c|s_2| + c'|s_3| =$$

$$cn + (c' - c)|s_3| \leq cn \text{ if } c \geq c'$$

4 The good subspace: Vertex cover in bipartite graphs

In the vertex cover problem, we are given a connected, undirected graph $G = (V, E)$. We would like to find the subset $W \subseteq V$ with the least number of vertices such that $\{v_1, v_2\} \in E \Rightarrow v_1 \in W \vee v_2 \in W$. A simple application of the vertex cover problem is the placement of police officers in a city or security guards in a museum. You need an authority figure to cover any given street/hallway in the museum. The following will show that if you have some say over the design of the city/museum, you may be able to efficiently guard it using minimal resources.

Konig-Egervary Theorem: In a bipartite, undirected graph, the cardinality of a maximum matching is equal to the minimum cardinality of a vertex cover.



Proof: First, let us observe that the cardinality of *any* vertex cover is greater than or equal to the cardinality of *any* matching. Note that any vertex cover is required to place at least one vertex in every possible match. Thus, it suffices to find a vertex cover of cardinality equal to the maximum matching and we are done.

Let the graph be $G = (V, E)$ and let $L \cup R = V$ such that $L \cap R = \emptyset$ and $\forall \{x, y\} \in E$, x and y are not both in L and not both in R. Create a maximum flow problem as follows: Let edges go from the source to vertices in L and edges go from vertices in R to the sink all have weight 1. Let the weights of the edges E have weight ∞ and point from L to R.

Given a max flow, let the corresponding cut be given by S and T where S represents the vertices on the source side of the minimum cut and T represents the vertices on the sink side.

Let the "potential" vertex cover be

$$(L \cap T) \cup (R \cap S)$$

We now need to prove two separate claims.

Claim: $(L \cap T) \cup (R \cap S)$ forma a vertex cover.

Proof: Let $\{x, y\} \in E$ be any edge. Assume WLOG that $x \in L$ and $y \in R$. Assume that neither x nor y was chosen for the vertex cover. Then $x \in L \Rightarrow x \notin T \Rightarrow x \in S$ and $y \in R \Rightarrow y \notin S$. If $x \in S$, then it is possible to push flow from the source to x. But because every in E has weight ∞, this implies that $y \in S$, a contradiction. □

Claim: $|(L \cap T) \cup (R \cap S)|$ is equal to the size of the maximum matching.

Proof: Note that because each of the edges in G has infinite capacity, we will never cut an edge of G. Thus the only edges that are cut are those that involve either the source or the sink and a vertex of G.

Note that the size of the maximum matching is equal to the magnitude of the maximum flow through the network.

If an edge from the source to a vertex in L is cut, then the vertex is clearly on the sink side of the minimum cut; thus, there are $|L \cap T|$ of these edges that are cut. Also, if an edge from a vertex in R to the sink is cut, then the vertex is is on the source side of the minimum cut; thus, there are $|R \cap S|$ of these edges that are cut. The flow through the minimum cut is equal to the maximum flow of the network and therefore the maximum flow is $|L \cap T| + |R \cap S|$.

So finally, we notice that

$$|L \cap T| + |R \cap S| = |(L \cap T) \cup (R \cap S)|$$

and we're done. □

This completes the proof of the main theorem. □

So, given this theorem, we only need to find the cardinality of a maximum matching in the bipartite graph. But this is a standard application of network flow!

5 Fixed parameter tractability: 0-1 knapsack problem, Vertex cover problem

To take advantage of fixed paramter tractability, we ask the customer if there are any parameters of the problem that can be considered to be "small" enough so that they can be essentially ignored in the running time calculations. For example, if we are given an algorithm that solves a particular problem with two parameters, x and y, and the running time is $O(3^y x^2)$, then this is an exponential time algorithm. However, if the customer declares that all instances for which he is running your algorithm will keep the parameter y to very small values, then your algorithm may be good enough, as it is only exponential in y and not x.

In general, if y is a fixed parameter and x is not, we want a running time that is $O(f(y)x^k)$ for some constant k and some function f (that may be exponential or larger).

5.1 0-1 knapsack problem

A thief robs a bank and can carry a maximum of W pounds in his sack before the sack breaks. Assume that there are n items and that the weights of each item are given by w_i and that each comes with a profit of p_i. What items should we steal?

Note that a recursive relation for $P(i, c)$ (This represents the profit assuming you are only allowed to choose from objects 1 through i and you have capacity c remaining in your sack.) is as follows.

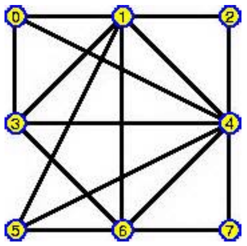
$$P(i,c) = \begin{cases} 0, & \text{if } i = 0 \text{ or } c = 0 \\ P(i-1,c), & \text{if } w_i > c \\ max\{P(i-1,c), P(i-1,c-w_i)+p_i\}, & \text{if } w_i \le c \text{ and } i > 0 \end{cases}$$

Example:

Object	*Weight*	*Profit*
1	3	4
2	5	6
3	5	5
4	1	3
5	4	5

Doing the standard dynamic programming algorithm yields a running time of $O(nW) = O(2^{\log W} n)$. If the value of W is defined to be a "small" parameter, then this becomes a tolerable running time.

5.2 Vertex cover problem



In the vertex cover problem, we are given an undirected graph $G = (V, E)$. We would like to find the subset $W \subseteq V$ with the least number of elements such that $\{v_1, v_2\} \in E \Rightarrow v_1 \in W \vee v_2 \in W$. Let $k = |W|$ and $n = |V|$. We claim that VCP is fixed parameter tractable in k. In other words, if you are guaranteed that your city/museum is guardable by a small force, then it is possible to efficiently determine where they should be placed.

This result isn't obvious. Checking every possible subset of k will yield a running time of at least $\binom{n}{k} = \Omega(n^k)$ and we want $O(f(k)n^{O(1)})$.

VERTEX-COVER(G,C)

```
1. If $|C| = k$ and $G$ still contains an edge, return $\emptyset$.
2. If $G$ contains no edges, then return $C$.
3. Pick any edge $\{v, w\}$.
4. Let $G' \equiv G$ with vertex $v$ and all edges incident to $v$ removed.
5. Let $C' \equiv C \cup \{v\}$.
6. If VERTEX-COVER(G',C')$\neq \emptyset$, then return $C'$.
7. Let $G'' \equiv G$ with vertex $w$ and all edges incident to $w$ removed.
8. Let $C'' \equiv C \cup \{w\}$.
9. If VERTEX-COVER(G",C")$\neq \emptyset$, then return $C''$.
10. Return $\emptyset$.
```

Why does this work? No matter what edge we pick, at least one vertex of the edge must be in the minimal-sized cover; otherwise, there will be an uncovered edge. Essentially, we are checking every possible vertex cover based on this observation. This takes time $O(2^k n)$ because the full recursion tree has depth at most k.

6 Fixed parameter tractability and good subspace: Planar independent set

The independent set problem is: given a planar graph $G = (V, E)$ and a positive integer k, does there exists a set $V' \subseteq V$ such that $|V'| \geq k$ and the vertices in V' are pairwise nonadjacent?

There is an obvious application to scheduling logistics. Let a graph consist of vertices that correspond to jobs that need to be completed. Let there be an edge between any two vertices if and only if the two jobs that the vertices represent require a mutually unsharable resource... Clearly, the maximum number of jobs that can be completed corresponds to the maximum independent set in this graph.

Task	Start	Duration
A	1	5
B	2	2
C	5	3
D	4	7

What happens if we declare that the graph is planar?

Euler's Formula: In any planar graph, $|F| + |V| - |E| = 2$.

Proof: We will use mathematical induction on the number of edges. If the number of edges is 0, then $|V| = |F| = 1$ and the formula is true. Assume that the formula is true for all planar graphs with $k - 1$ edges, and assume that G has $|E| = k$. If G is a tree, then $|V| = k + 1$ and $|F| = 1$ so the formula holds. If G is not a tree, then there is a cycle in G. Choose any edge e in the cycle and remove e from G. This will reduce the number of edges *and faces* of G by exactly 1. By the inductive hypothesis, the formula holds for $G - \{e\}$ and must therefore hold for G as well. □

Claim: In any connected, undirected planar graph with $|V| \geq 3$, $|E| \leq 3|V| - 6$.

Proof: Let's ask the question: In a planar graph G, what is the maximum number of edges that G can possibly have, given a fixed number of vertices such that $|V| \geq 2$. Clearly, if there exists a cycle in G with more than 3 edges, it is possible to add an edge to G without changing the number of vertices; so we can assume that all cycles in G have exactly 3 edges. (Note that this includes cycles that bound the "infinite" face as well.) Thus, the graph with the maximum number of edges given a fixed $|V|$ assumes that all cycles have exactly 3 edges and every face is therefore a triangle.

We can count the number of edge-face pairs (where an edge-face pair is defined to be a pair consisting of an edge and a face where the edge borders the face) in two different ways. Each edge borders exactly 2 faces and each face has exactly 3 edges. Thus, we have $2|E| = 3|F|$. Plugging this into Euler's Formula:

$$\frac{2}{3}|E| + |V| - |E| = 2 \Rightarrow |E| = 3|V| - 6$$

Because this represent the graph with the maximum number of edges given a fixed $|V|$, we must have that $|E| \leq 3|V| - 6$ for every planar graph G. □

Claim: In any connected, undirected planar graph, there exists at least one vertex with degree 5 or less.

Proof: Notice that by the Handshaking Theorem and the above claim, we have that

$$2|E| = \sum_{v \in V} deg(v) \leq 6|V| - 12 \Rightarrow \frac{\sum_{v \in V} deg(v)}{|V|} \leq 6 - \frac{12}{|V|} < 6$$

So the average degree of a planar graph is strictly less than 6. This implies that there must exist some vertex with degree 5 or less. □

Consider the following function for the planar independent set problem on a given graph $G = (V, E)$ with parameter k: $IS(G = (V, E), S)$

1. If G has no edges, then if $|V| + |S| \geq k$ return $V \cup S$ else return $\emptyset$.
2. If $|S| \geq k$, then return S.
3. Let v be the vertex of smallest degree in G. For each $x \in N(v) \cup \{v\}$,
   (a) Construct G' by deleting x and $N(x)$ (and the appropriate edges) from G.
   (b) If $IS(G', S \cup \{x\}) \neq \emptyset$, then return $IS(G', S \cup \{x\})$.
4. Return $\emptyset$.

Pick the vertex $v \in V$ with smallest degree (bounded above by 5, by the above claim). Either v is going into the independent set or one of its at most 5 neighbors will be. In each branching case, delete the corresponding vertex together with all its adjacent edges and vertices from G. Thus, obtain a smaller graph G' in each case, and recursively search for an independent set in each G'. (Obviously, the base cases where the graph is empty or is very small are trivial to handle separately.)

This algorithm must find a large independent set because from the set $\{v\} \cup N(v)$, *exactly* one vertex has to be in an optimal independent set. Since the parameter in each branch decreases by one, we obtain a search tree of size at most 6^k. Using a suitable edge list representation of G, pick a vertex and generating G' can easily be done in linear time in the size of G. Therefore, this algorithm runs in time $O(6^k(|V| + |E|))$ and is fixed parameter tractable in the size of the independent set itself.

7 Amortized analysis: The Pancake Stack

Amortized analysis is the method of determining the *worst case average cost per operation.* Though there are a few different ways of learning amortized analysis, we will focus on the most powerful, the potential method. We assume that we perform n operations numbered 1 through n. We define a potential function $\phi(i)$ (for i between 0 and n) for the data structure[1] that satisfies the following conditions:

1. $\phi(0) = 0$
2. $\phi(i) \geq 0$ for all i

Let $c(i)$ be the actual cost of the ith operation. Then we define the *amortized cost* of the ith operation to be $a(i) = c(i) + \phi(i) - \phi(i-1)$. We now claim that the total amortized cost of all operations is an upper bound for the actual cost. To show this:

$$\sum_{i=1}^{n} a(i) = \sum_{i=1}^{n} c(i) + \sum_{i=1}^{n} (\phi(i) - \phi(i-1)) = \sum_{i=1}^{n} c(i) + \phi(n) - \phi(0) \Rightarrow$$

[1] Note that the potential function depends on the configuration of the data structure at time i and *not* the time that has elapsed.

$$\sum_{i=1}^{n} a(i) \geq \sum_{i=1}^{n} c(i)$$

(Note that because the potential function is defined the way it is, this conclusion is true independent of the value of n.) The potential function intuitively measures how scared you are that you will have to do a bunch of work during the next operation.

7.1 The Quickly Popped Stack

Consider a stack that implements the operations push, pop, and multipop. (Multipop pops all of the elements off the stack all at once.) What is the worst case average amount of time per operation that we do? Define a potential function ϕ that is equal to the number of items stored in the stack. (Verify that ϕ is legal.) Assume that it takes unit time to push or pop a single item on or off the stack. Let s represent the size of the stack. Then let's analyze the possible operations:

1. Push: $a(i) = c(i) + \Delta\phi = 1 + ((s+1) - s) = 2$
2. Pop: $a(i) = c(i) + \Delta\phi = 1 + ((s-1) - s) = 0$
3. Multipop: $a(i) = c(i) + \Delta\phi = s + (0 - s) = 0$

So it takes constant average worst case time to update this data structure!

8 Amortized analysis: Dynamic tables, Binary counters

8.1 Dynamic tables

Consider database storage. A simple method of allocating memory for an integer database with a size we don't know in advance is to start with a single integer register. In general, when we try to add an integer to the end of the database and discover that we have run out of memory, we simply ask the operating system for double the memory that we are currently using. When the database is $< \frac{1}{4}$ full[2], we cut it in half. That way, our database is always guaranteed to be about half utilized. Of course, if we generate a new table, we have to copy the old values into the new table. So some operations are going to be very expensive, but most do not seem to be...

Assume that it takes unit cost to copy an integer into memory or delete an integer from memory. Let ϕ, the potential of the table, be equal to $2(t - \frac{s}{2})$ if $t \geq \frac{s}{2}$ and $\frac{s}{2} - t$ if $t < \frac{s}{2}$ (where s is the size of the table and t is equal to the number of integers stored in the table). (Verify that ϕ is legal.)

Let's calculate the amortized cost per operation for time i.

[2] Why shouldn't we use $\frac{1}{2}$ here? There's a reason. Think about it.

1. Assume that we add an integer to the table and it does not cause us to double the table size. Then
$$a(i) = c(i) + \Delta\phi \leq 3$$

2. Assume that we delete an integer from the table and it does not cause us to halve the table size. Then
$$a(i) = c(i) + \Delta\phi \leq 3$$

3. Assume that we add an integer to the table and it does cause us to double the table size. Then the potential of the old data structure was $2(s-\frac{s}{2}) = s$ and the potential of the new data structure is $2(s+1-s) = 2$. So we get
$$a(i) = c(i) + \Delta\phi = s + 1 + (2 - s) = 3$$

4. Assume that we delete an integer from the table and it does cause a halving of the table size. Then the potential of the old data structure was $(\frac{s}{2} - \frac{s}{4}) = \frac{s}{4}$ and the potential of the new data structure is $\frac{s}{4} - (\frac{s}{4} - 1) = 1$. So we get
$$a(i) = c(i) + \Delta\phi = \frac{s}{4} - 1 + (1 - \frac{s}{4}) = 0$$

So the average worst case effort to update this data structure is constant!

8.2 The binary counter

Assume that we have a binary register initially containing any number and we are allowed to perform the "increment by 1" operation. Assume that it takes unit cost to flip a bit (from 0 to 1 or from 1 to 0). What is the average worst case time to increment the counter n times? Let the potential function ϕ be define to be the number of 1 bits in the register. (Verify that ϕ is legal.) To calculate the amortized cost per increment, let s be the number of 1 low order bits in the register and let t be the total number of 1 bits in the register. Then

$$a(i) = c(i) + \Delta\phi = (s + 1) + ((t - s + 1) - t) = 2$$

So again we get a worst case average constant time per operation!

9 Binomial Heaps

9.1 Application

The Large Hadron Collider (LHC) is the world's largest and most powerful particle collider, built by the European Organization for Nuclear Research (CERN) from 1998 to 2008. Its aim is to allow physicists to test the predictions of different theories of particle physics and high-energy physics, and particularly prove or disprove the existence of the theorized Higgs particle and of the large family of new particles predicted by supersymmetric theories. The Higgs particle was confirmed by data from the LHC in 2013.

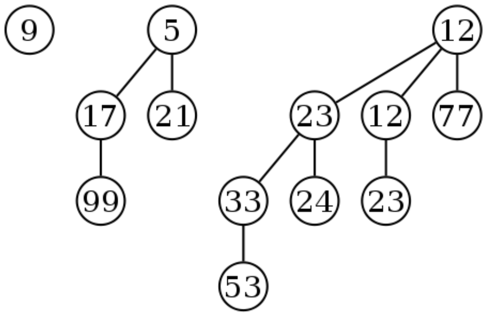
Approximately 600 million times per second, particles collide within the Large Hadron Collider (LHC). By 2012 data from over 300 trillion LHC proton-proton collisions had been analyzed, LHC collision data was being produced at approximately 25 petabytes per year, and the LHC Computing Grid had become the world's largest computing grid (as of 2012), comprising over 170 computing facilities in a worldwide network across 36 countries. The Data Centre processes about one petabyte of data every day - the equivalent of around 210,000 DVDs. The centre hosts 10,000 servers with 90,000 processor cores. Some 6000 changes in the database are performed every second. The Grid runs more than one million jobs per day. At peak rates, 10 gigabytes of data may be transferred from its servers every second.

The basic goal of the grid is to sift through the data produced by the machine and save for later analysis the collision data for those that produce interesting physics.

9.2 The problem

Assume that a person is feeding you distinct numbers (measurements, say) and he will feed you a total of n numbers (where n is large). Every so often, he wants you to output the k smallest numbers you have seen thus far (where $k = o(n)$ is a number he will tell you at that moment) *with the caveat that once you tell him a measurement, he never wants to hear about it again.* If you implement this with a standard heap, you will need to insert all n numbers (for a total time of $\Theta(n \log n)$) and then delete k numbers every so often. Can we do better?

9.3 The structure



A binomial heap B_0 is a single node. B_k is defined to be the heap that occurs when you *link* two B_{k-1} trees together. You choose the root of one of the B_{k-1} trees to be the root of the B_k tree and then make the root of the other B_{k-1} tree the child of the other. (Of course, you can make these max-heaps or min-heaps.) A binomial heap is an ordered collection of binomial trees such that (i) smaller size binomial trees come before larger ones and (ii) there is only a single binomial tree of any given size in a given binomial heap.

There are several operations one can perform on a binomial heap. For example, it is fairly easy to

1. *find the minimum/maximum* node in the heap. Just search the tops of all the trees in the heap.
2. *unite* two binomial heaps. To do so, perform the following steps.
   (a) Arrange the binomial trees of each heap together in order of size. (Note that there might be two trees of the same size next to each other when we do so. The next step takes care of this.)
   (b) Starting from the left (the smallest binomial trees first), if there are two trees next to each other with the same size, merge them together by performing the link operation. If you wind up with three binomial trees of the same size, merge the latter two. Continue merging to the right until you are done.
3. *inserting* a node to a binomial heap H. Simply create a single-node binomial heap and unite it with H.
4. *decreasing/increasing a key* in a binomial heap. Just decrease/increase the value in the appropriate node in the tree and let it percolate upwards.
5. *delete the node with the minimum/maximum key* from a binomial heap H. To do so, search the tree roots for the minimum/maximum key. Once found, remove that tree from H. Form a new binomial heap H' by simply deleting the root (the minimum node). Now unite H and H'. To *delete an arbitrary key*, just replace the key with the value $\pm\infty$ and let the $\pm\infty$ percolate to the root of the binomial tree. Then delete the node with the minimum key.

We need to show the following things about binomial heaps:

1. The number of nodes in B_k is 2^k.
2. The number of binomial trees in a heap of size n is $O(\log n)$.
3. Insert, decrease/increase-key, and delete both take $O(\log n)$ actual time because any arbitrary union operation only takes $O(\log n)$ time.

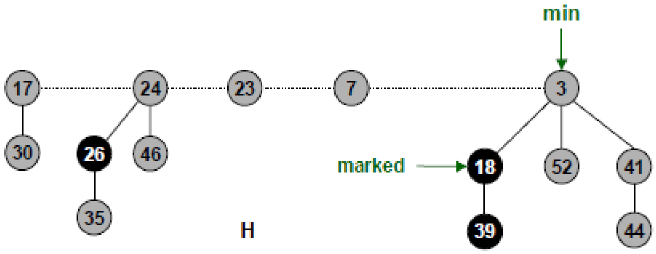
If we use the potential function $\phi(B)$ = the number of trees in the binomial heap, we can see that insertion takes $O(1)$ amortized time (using essentially the same analysis as the binary counter), an improvement over the $O(\log n)$ regular heap. So all of the insertions in the problem we have above can be done in $O(n)$ amortized time, a great improvement.

9.4 Application

One clear application of the binomial heap is discrete event simulation. If you are, for example, modeling the action of cars on a network of roads, the best way of doing so is to keep track of the next event that occurs to each car. In other words, a car will continue along its pathway until it either gets near the back of another car or approaches a stop light or stop sign, in which case it will slow down. However, we need not pay attention to the car *until that happens.* Thus, we can calculate using the laws of physics exactly when the next time we need to pay attention to the car is and store that "event" in a heap. The event simulator can just keep track of the next event that needs to be dealt with and assume that everything else is running smoothly. What happens if an event causes new events to occur or old events to be updated such as a new car appearing on a roadway in front of another car? We can increase a key and/or decrease as necessary to indicate that an event occurs sooner (i.e. that a car needs to slow down sooner than we expect because someone cut him off) or later (i.e. that a car can slow down later than we expect because the car in front turned off the road).

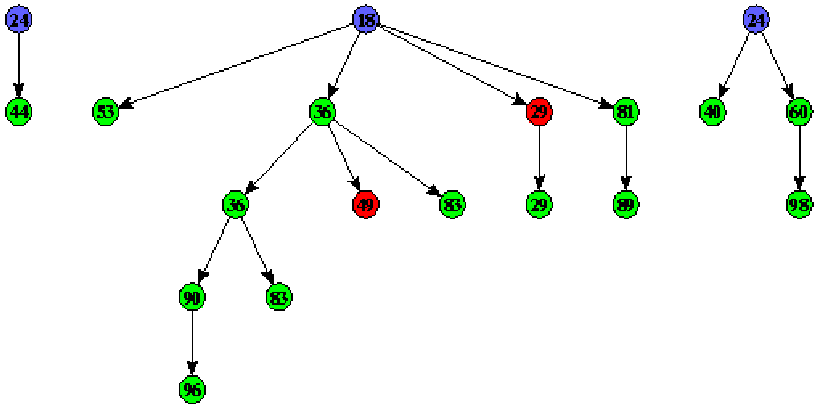
10 Fibonacci Heaps I

10.1 Fibonacci heaps



(Fredman and Tarjan 1986) Fibonacci heaps have the following properties: The roots of the trees are kept in a circular doubly linked list. There is a separate pointer that points to the min/max value in the list. Each node is either *marked* or not[3]. Each root has a *rank/degree*, defined to be the number of children it has.

[3] Please be aware that some of the nodes that are marked in the pictures below are not marked correctly.

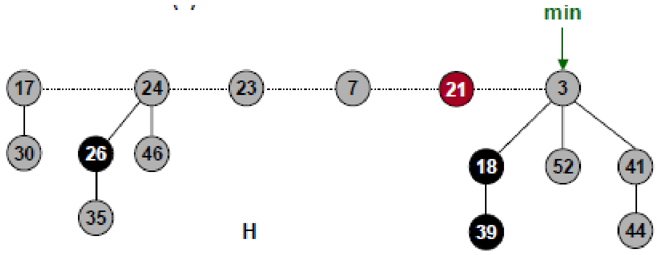


We define a potential function for a Fibonacci heap H to be $\phi(H) = c(t(H) + 2m(H))$ where $t(H)$ represents the number of trees in H, $m(H)$ represents the number of marked nodes in H, and $c > 0$ is a constant to be determined later. We will analyze the amortized cost of each function as we go.

1. To *insert* a node into the heap, make a B_0 and insert it into the doubly linked list to the left of the min pointer (and update the min/max pointer if necessary). Nodes are initially unmarked.

   Amortized cost: $a(i) = c(i) + \Delta\phi = O(1) + c = O(1)$

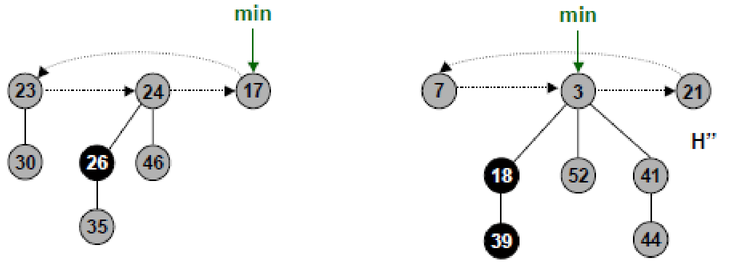
   Insert the node 21 into the tree to get the following result.



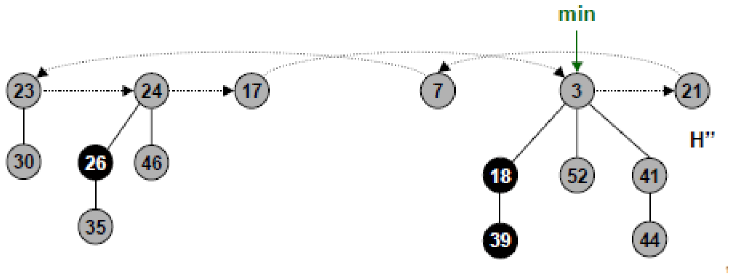
2. To *union* two heaps together, just concatenate the two circularly linked lists together and update the min/max pointer.

   Amortized cost: $a(i) = c(i) + \Delta\phi = O(1) + 0 = O(1)$

   The union of the following two heaps...



...is



3. To *delete the min/max*, delete the min/max from the appropriate tree and add the root's children to the tree list. Consolidate the roots of the trees by scanning and merging from smallest to largest so that no two trees have the same degree: To do this, initialize an initially empty array indexed by $0, 1, 2, \ldots$. This array will represent, for a given index i, the tree with root that has i children. Note that there is only allowed to be one of these each at a time. Scan from left to right and, for each root, either the root will be the unique root with its number of children or there will be another already in the array. If there is another already there, make the appropriate (min/max-valued) root a child of the first and continue the check.

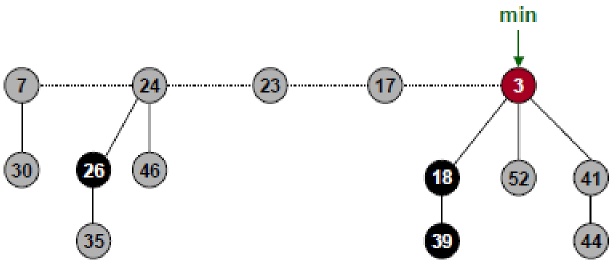
   Amortized cost: $a(i) = c(i) + \Delta\phi = O(D(n) + t(H)) + c(t(H') - t(H))$ where $D(n)$ is the maximum degree of a vertex in a Fibonacci heap of size n (the number of nodes in the heap before the deletion) and H' is the newly created heap structure: there is at most $O(D(n))$ work to add the children of the deleted node into the circularly linked list. To consolidate the trees, there can be at most $O(D(n) + t(H))$ work because the work is proportional to the size of the circularly linked root list prior to the merge and this size is at most $D(n) + t(n) - 1$.

   We claim that $t(H') \leq D(n) + 1$ because no two trees are allowed to have the same degree so $c(t(H') - t(H)) \leq c(D(n) + 1 - t(H))$. This implies that

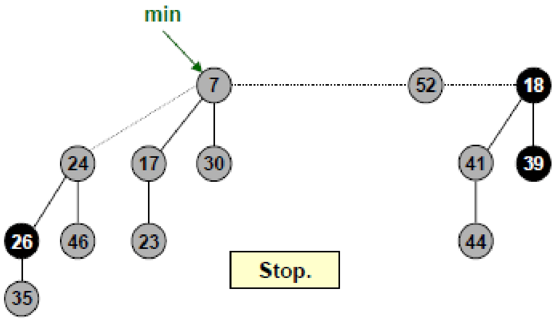
$$a(i) = O(D(n)+t(H))+c(t(H')-t(H)) \leq O(D(n)+t(H))+c(D(n)+1-t(H))$$

We will now assume that c is greater than or equal to the constant hidden by O notation so that $a(i) = O(D(n))$. Below, we will figure out a bound on $D(n)$...

Delete the minimum of the following heap...



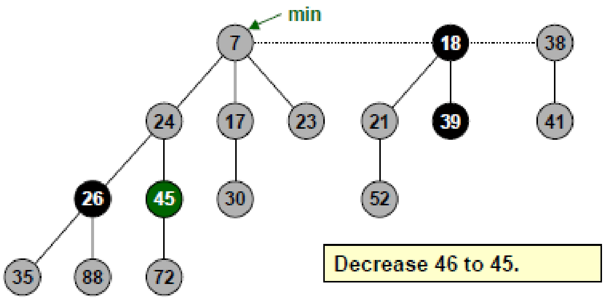
...to get



4. To *decrease/increase a given key*, there are 3 cases we need to consider. The min/max pointer is updated every time a new tree is added to the circularly linked root list. If a tree is added to the circularly linked root list, it becomes unmarked.

   (a) The heap property is not violated. In this case, just decrease/increase the key.

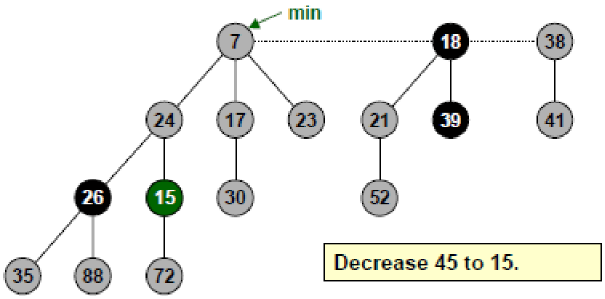
   Amortized cost: $a(i) = c(i) + \Delta\phi = O(1) + 0 = O(1)$



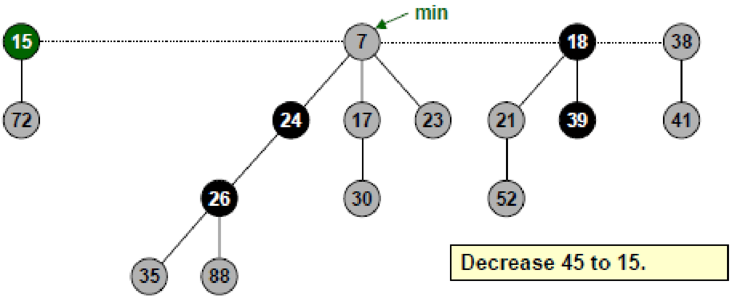
(b) The heap property is violated and the parent of the changed node x is unmarked. Mark x's parent. Cut off the tree rooted at x and add it to the circularly linked root list.

Amortized cost: $a(i) = c(i) + \Delta\phi = O(1) + c(1 + 2) = O(1)$

Consider the following example where the 45 key is decreased to 15. The heap goes from this...



...to this:

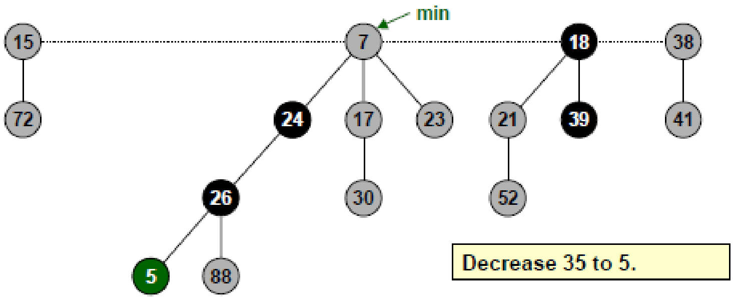


(c) The heap property is violated and the parent of the changed node x is marked. Cut off the tree rooted at x and add it to the circularly linked root list. Let the new value of x be x's former parent $p[x]$. Continue doing the following until x is unmarked: Cut off the tree rooted at x, add it to the circularly linked root list, unmark it, and let

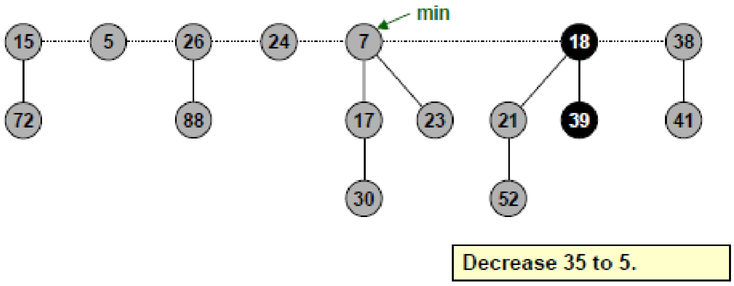
x become $p[x]$. (We are basically trimming all nodes upward toward the root until we get to one that is not marked.)

Amortized cost: Let r be the number of times we needed to unmark a node. Then $a(i) = c(i) + \Delta\phi = O(r) + cr(1-2)$. Again, we let c be greater than or equal to the constant hidden by the O notation to get $a(i) = O(1)$.

Consider the following example where the 35 key is decreased to 5. The heap goes from this...



...to this:



5. To *delete a key*, decrease/increase its value to ±∞ and then delete the min/max element.

   Amortized cost: To decrease/increase a key has an amortized cost of $O(1)$ and the amortized cost for delete is $O(D(n))$.

11 Fibonacci Heaps II

11.1 Fibonacci rabbits

Assume that there is a breed of immortal rabbits. A mature pair of these rabbits makes a new pair of baby rabbits every month (but it takes the full month to produce them). Baby rabbits mature after a single month. Assume that we go to an initially rabbit-free island and drop a pair of baby rabbits from a helicopter. Let $f(n)$ be the number of pairs of rabbits during the nth month. Then $f(n)$ is the function that represents the **Fibonacci numbers**.

Let $f(0) = 0, f(1) = 1$, and $f(n) = f(n-1) + f(n-2)$. Then

$$f(n) = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$$

The proof uses mathematical induction. Before you do anything, find the two solutions to the following equation: $x^2 = x + 1$.

11.2 The final analysis

The goal is now to get some idea of what size $D(n)$ is.

Lemma: Let x be any node in the Fibonacci heap and let $y_1, \ldots, y_k$ be the children of x in the order that they were linked to x. Then degree$(y_i) \geq 0\delta_{i1} + (i-2)(1-\delta_{i1})$ (where δ_{ij} is 1 if $i = j$ and 0 if $i \neq j$).

Proof Obviously, the degree of y_1 is greater than or equal to 0, and the inequality holds if $i = 1$.

Now assume that $i > 1$. Notice that when y_i is linked to x, $y_1, \ldots, y_{i-1}$ are already linked to x. Thus because two nodes are linked only if they have the same degree y_i was linked to x with degree $i-1$. It is a general rule that if two children are removed from a node, then it must become its own root (the first removed child causes the parent to become marked and the next causes it to be removed). Thus at most one child could have been removed from y_i. □

Lemma: In a Fibonacci heap with n nodes, the maximum degree $D(n)$ of any node in any tree is $O(\log n)$.

Proof: Let size(x) be the size of the subtree rooted at x (including the node x itself). Let s_k be the minimum size of a tree rooted at any degree k node; notice that $s_0 = 1$ and $s_1 = 2$. Then we have

$$s_k = 1 + \sum_{i=1}^{k} size(y_i)$$

(where the y_i are the children of the root added in order)

$$1 + \sum_{i=1}^{k} size(y_i) = 2 + \sum_{i=2}^{k} size(y_i) \geq 2 + \sum_{i=2}^{k} s_{deg(y_i)} \geq$$

$$2 + \sum_{i=2}^{k} s_{i-2} = 2 + \sum_{i=0}^{k-2} s_i$$

The (adjusted) Fibonacci sequence is defined to be $f_0 = 1, f_1 = 2, f_k = f_{k-1} + f_{k-2}$. Two easy facts to show via induction are that (i) $f_k = \Omega(\varphi^k)$ and (ii) $f_k = 2 + \sum_{i=0}^{k-2} f_i$ for $k \geq 2$. Note that the recursion for s_k and f_k match up and therefore they are the same. Thus, $s_k = \Omega(\varphi^k)$. Finally,

$$n \geq s_{D(n)} = \Omega(\varphi^{D(n)}) \Rightarrow D(n) = O(\log_\varphi n)$$

□

11.3 Applications

Consider Prim's algorithm for the minimum spanning tree or Dijkstra's algorithm for single source shortest paths:

1. Maintain a heap for the vertices.
2. Put s in the heap where s is either (i) the start vertex (Dijkstra) or (ii) the initial vertex (Prim) with a key of 0.
3. Repeatedly delete the minimum key in the heap and mark it "scanned." For each neighbor w of the deleted vertex v, if w is not in the heap and not scanned, add it to the heap with key value (i) Dijkstra: $key(v) + |(v, w)|$ or (ii) Prim: $|(v, w)|$. If w is already in the heap, decrease its key value to be the minimum of $key(w)$ and the value given.

Using the classical heap data structure or the binomial heaps, we get a running time of $O(|E| \log |V|)$. Can we do better? Note that decrease key runs very fast. So Dijkstra and Prim's algorithms, given above, now run in time $O(|E| + |V| \log |V|)$, a major improvement. This is asymptotically the fastest known single-source shortest-path algorithm for arbitrary directed graphs with unbounded nonnegative weights though there are faster known minimum spanning tree algorithms.

12 Disjoint Sets I

12.1 Description

To represent a set, we simply add a node with the appropriate key value. We let $p[x]$ represent the parent of the node x. If x is a root, then we let $p[x] = x$. For sets, we are really only interested in (i) creating a set (MAKE-SET), (ii) unioning two sets (UNION), and (iii) given an element, determining which set it is in (FIND). (The sets are named using the roots.) We will implement the UNION and FIND operations using two heuristic methods.

Union by rank: The *rank* is an upper bound on the height of a node within the tree: Whenever a new tree is created, the rank of the root is equal to 0. When unioning two sets, if the ranks are different, then the smaller rank becomes a child of the larger rank and no ranks change. When two ranks are equal, we arbitrarily choose one to become the parent and increment its rank by 1.

Path compression: Whenever we conduct a *find* on an element, we make all children on the find path point directly to the root (thereby making the path to the root shorter for all the elements on the path).

- What happens if you use union by rank but not path compression? Build a tree as follows. Starting from a single node, to build stage i, take two copies of stage $i - 1$ and union them together. We claim that building a tree with n nodes takes $\Theta(n)$ operations. However, the height of the tree is $\Theta(\log n)$. We can now do m FIND operations on the lowest elements to get $\Omega(m \log n)$ behavior.

- What happens if you use path compression but not union by rank? It is clear that if you do not use union by rank, then you can build a full binary tree in linear time in n, the number of elements. First perform a FIND on the outermost elements, then, with the remaining structure, perform a FIND repeatedly/recursively on the outermost elements of each binary tree. How many operations will this take? Let $T(n)$ be the number of operations used to perform these operations. Then we have that (because you need to perform the 2 FINDs and then each subsequent FIND on the smaller trees)

$$T(n) \geq 2(\sum_{i=1}^{\log n} T(n/2^i) + \log n - 1) = 2\log n - 2 + 2\sum_{i=1}^{\log n} T(n/2^i) =$$

$$2\log n - 2 + 2\sum_{i=0}^{\log n - 1} T(2^i)$$

We claim that $T(n) = \Omega(n \log n)$. To show this, let $S(i) = T(2^i)$ and let $k = \log n$. Then we have that

$$S(k) \geq 2(k-1) + 2\sum_{i=0}^{k-1} S(i)$$

We claim that eventually $S(k) \geq ck2^k$ for constant $c > 0$. Assume that this is true for all integers smaller than k. Then we know that if k is large (in the analysis below, we can let c^* be any constant we wish because k is assumed to be large)

$$S(k) \geq 2(k-1) + 2\sum_{i=0}^{k-1} ci2^i = 2(k-1) + c(k-1)2^k + 2\sum_{i=1}^{k-2} ci2^i \geq$$

$$c(k-1)2^k + c^* + \sum_{i=1}^{k-2} c2^{i+1} \geq c(k-1)2^k + c2^k = ck2^k$$

and this implies the result.

13 Disjoint Sets II

13.1 Analysis

13.1.1 Preliminary definitions and lemmas

Let $f^{(i)}(x) = f(f(f(\dots f(x))))$ where the function is applied i times. Define the following functions.

$$A_k(j) = \begin{cases} j+1 & \text{if } k = 0 \\ A_{k-1}^{(j+1)}(j) & \text{if } k \geq 1 \end{cases}$$

This function grows *extremely* fast and is monotonically increasing with both k and j. Let

$$\alpha(n) = \min\{k : A_k(1) \geq n\}$$

Let's compute $A_4(1)$.

$$A_4(1) = A_3^{(2)}(1) = A_3(A_3(1))$$

$$A_3(1) = A_2^{(2)}(1) = A_2(A_2(1))$$

$$A_2(1) = A_1^{(2)}(1) = A_1(A_1(1))$$

$$A_1(1) = A_0^{(2)}(1) = A_0(A_0(1)) = A_0(2) = 3 \Rightarrow A_2(1) = A_1(3) = 7$$

$$\Rightarrow A_3(1) = A_2(A_2(1)) = A_2(7) = 2047 \Rightarrow A_4(1) = A_3(2047)$$

So how big is this? The Ackermann function grows *much* faster than exponential so this number is larger than 3^{2047}. (The number of particles in the known universe is roughly 10^{82}.) So for all practical purposes $\alpha(n) \leq 4$.

Properties of the rank:

1. The value of $rank[x]$ is initially 0 and increases through time until $x \neq p[x]$; from that point onwards, $rank[x]$ does not change.

2. For all nodes x, $rank[x] \leq rank[p[x]]$ with strict inequality if $x \neq p[x]$. As we follow the path from any node to the root, the node ranks strictly increase.

3. Every node has rank at most $n - 1$ where n is the number of elements.

We need to define two functions on nodes x such that x is not a root and $rank[x] \geq 1$.

1. Let the level of a node x be defined to be

   $$level(x) = \max\{k : rank[p[x]] \geq A_k(rank[x])\}$$

   So $level(x)$ is the greatest k for which A_k when applied to x's rank is no greater than x's parent's rank.

Lemma: $0 \leq level(x) < \alpha(n)$

Proof: Note the following facts:

$$rank[p[x]] \geq rank[x] + 1 = A_0(rank[x]) \Rightarrow 0 \leq level(x)$$

$$A_{\alpha(n)}(rank[x]) \geq A_{\alpha(n)}(1) \geq n > rank[p[x]]$$

$A_y(rank[x])$ is a monotonically increasing function of y and we have by the above that $A_y(rank[x]) > rank[p[x]]$ when $y = \alpha(n)$. Thus, the maximum y for which $A_y(rank[x]) \leq rank[p[x]]$ must be strictly less than $\alpha(n) \Rightarrow level(x) < \alpha(n)$. □

2. The second function is referred to as the iteration.

$$iter(x) = \max\{i : rank[p[x]] \geq A^{(i)}_{level(x)}(rank[x])\}$$

So $iter(x)$ is the largest number of times we can apply $A_{level(x)}$, applied initially to x's rank, before we get to a value greater than x's parent's rank.

Lemma: $1 \leq iter(x) \leq rank[x]$

Proof: Note the following facts:

$$rank[p[x]] \geq A_{level(x)}(rank[x]) = A^{(1)}_{level(x)}(rank[x]) \Rightarrow 1 \leq iter(x)$$

$$A^{(rank[x]+1)}_{level(x)}(rank[x]) = A_{level(x)+1}(rank[x]) > rank[p[x]] \Rightarrow iter(x) \leq rank[x]$$

□

Fact: As long as level(x) remains unchanged, iter(x) must either increase or remain unchanged.

13.1.2 The potential function

Let the potential of a given node x after q operations be

$$\phi_q(x) = \begin{cases} \alpha(n)rank[x] & \text{if } x \text{ is a root or rank[x]=0} \\ (\alpha(n) - level(x))rank[x] - iter(x) & \text{if } x \text{ is not a root and rank[x]} \geq 1 \end{cases}$$

The potential of the entire data structure is the scaled sum of the potentials of its nodes: $\phi_q = c \sum_{\text{nodes } x} \phi_q(x)$ where c is a positive constant that we will choose later.

Lemma: $0 \leq \phi_q(x) \leq \alpha(n)rank[x]$ for all nodes x.

Proof: Because $level(x) \geq 0$ and $iter(x) \geq 1$, we know that

$$(\alpha(n) - level(x))rank[x] - iter(x) < \alpha(n)rank[x] \Rightarrow \phi_q(x) \leq \alpha(n)rank[x]$$

Because $level < \alpha(n)$ and $iter \leq rank[x]$, we have that

$$(\alpha(n) - level(x))rank[x] - iter(x) \geq 0 \Rightarrow \phi_q(x) \geq 0$$

□

Potential Lemma: Let x be a node that is not a root, and suppose that the qth operation is either a UNION or a FIND. Then

1. $\phi_q(x) \leq \phi_{q-1}(x)$
2. If rank[x]$\geq$ 1 and iter(x) or level(x) changes during the qth operation, then $\phi_q(x) < \phi_{q-1}(x)$.

Proof: Because x is not a root, the qth operation cannot change rank[x].

Note that if rank[x]=0, then $\phi_q(x) = \phi_{q-1}(x) = 0$. So we assume from now on that rank[x]$\geq$ 1.

- Assume that level(x) is unchanged. If both level(x) and iter(x) are unchanged, then $\phi_q(x) = \phi_{q-1}(x)$. If iter(x) increases, then it must increase by at least 1 so $\phi_q(x) < \phi_{q-1}(x)$.
- Assume that level(x) changes. Note that level(x) is monotonically increasing. The term $(\alpha(n) - level(x))rank[x]$ therefore decreases by at least rank[x]. Because level(x) increased, the value of iter(x) might drop but because $1 \leq iter(x) \leq rank[x]$, the drop can be by at most $rank[x] - 1$. Thus, $\phi_q(x) < \phi_{q-1}(x)$ in this case as well.

□

14 Disjoint Sets III

14.0.3 Amortized time per operation

Amortized time to MAKE-SET: $a(q) = c(q) + \Delta\phi = 1 + 0 = O(1)$.

Amortized time to UNION: $c(q) = O(1)$. Assume that the UNION makes node y the parent of node x. The nodes with potentials that may change during the operation are x, y, and y's children prior to the link.

1. y's children: By the Potential Lemma, the potential for any node that is not a root cannot increase for a UNION operation. Thus $\Delta\phi \leq 0$ in this case.
2. x: Because x was a root prior to the operation, $\phi_{q-1}(x) = \alpha(n)rank[x]$. If, prior to the operation, rank[x]=0, then clearly the potential does not change because x's rank does not change. Otherwise, x becomes a non-root and $\phi_q(x) = (\alpha(n) - level(x))rank[x] - iter(x) < \alpha(n)rank[x] = \phi_{q-1}(x)$. So x's potential actually decreases; $\Delta\phi \leq 0$ in this case also.
3. y: Because y was a root prior to the operation, $\phi_{q-1}(y) = \alpha(n)rank[y]$. Either y's rank is left alone (leaving a potential change of 0) or it increases by 1 (which increases y's potential by $\alpha(n)$).

Thus, the total potential amortized increase in potential $\Delta\phi$ is $O(\alpha(n))$.

Amortized time to FIND-SET: Assume that there are s nodes on the path up to the root of the set. Then $c(q) = O(s)$.

Note that no node's potential increases due to the cost of the operation (because of the Potential Lemma and the fact that the root's potential doesn't change).

Claim: At least $\max\{0, s - (\alpha(n) + 2)\}$ nodes have their potential decrease by at least 1.

Proof: Let us refer to a node x on the search path that satisfies the following a *special* node: (a) $rank[x] > 0$ and (b) x is followed somewhere higher up on the path by y such that y is not a root and $level(y) = level(x)$ just before the operation. We claim that there are at most $\alpha(n) + 2$ nodes on the find path that are *not* special: the first node on the find path (if it has rank 0), the last node on the find path (the root because nothing follows it along the path), and the last node w on the path for which $level(w) = k$ (for $k = 0, 1, \ldots, \alpha(n) - 1$).

Notationally, let anything with a prime on it denote its value *after* the FIND-SET operation has been performed and without the prime denotes *before*. Let x be a special node and let y be any node that makes x special.

$$\begin{aligned}
rank[p[y]] &\geq A_{level(y)}(rank[y]) \\
& \text{by the definition of level} \\
&\geq A_{level(y)}(rank[p[x]]) \\
& \text{because } A \text{ and rank are monotonically increasing} \\
&= A_{level(x)}(rank[p[x]]) \\
& \text{because } level(x) = level(y) \\
&\geq A_{level(x)}(A^{(iter(x))}_{level(x)}(rank[x])) \\
& \text{by the definition of iter} \\
&= A^{(iter(x)+1)}_{level(x)}(rank[x])
\end{aligned}$$

We know that $rank[p'[y]] = rank[p'[x]]$ because x and y will have the same parent after the path compression.

We must have $rank[p'[y]] \geq rank[p[y]]$ because nodes must point to the root after the path compression, and $rank'[x] = rank[x]$ because x is not a root.

Thus, after path compression, we will have

$$rank[p'[x]] = rank[p'[y]] \geq rank[p[y]] \geq A^{(iter(x)+1)}_{level(x)}(rank[x]) = A^{(iter(x)+1)}_{level(x)}(rank'[x])$$

by the above.

Thus, either $iter(x)$ or $level(x)$ must have changed during the qth operation. By the Potential Lemma, x's potential must have decreased by at least one. □

Thus, $\Delta\phi \leq -c(s - (\alpha(n) + 2)) \Rightarrow$

$$a(q) = c(q) + \Delta\phi \leq O(s) - c(s - (\alpha(n) + 2))$$

Now we choose c to dominate the constant in the O-notation yielding an asymptotic running time of $O(\alpha(n))$.

14.1 Applications

1. Equivalence classes: if it takes a long time to check a relation, every time it is found that a is related to b, union their sets)

2. The online undirected connected components problem: You are given edges one at a time and told to output the connected components.

3. Fast maze generation

   (a) Create a large array of all the squares in an array

   (b) Have another array list of all of the adjacent squares in grid and initialize to true (implying is wall)

   (c) Start knocking down walls by randomly choosing from the adjacent square list and making them false (implying no wall)

   (d) Once wall knocked down, place two elements into same equivalence class.

   (e) Once the start and end element in the same equivalence class - done

   (f) Better - all elements in same equivalence class - more detours

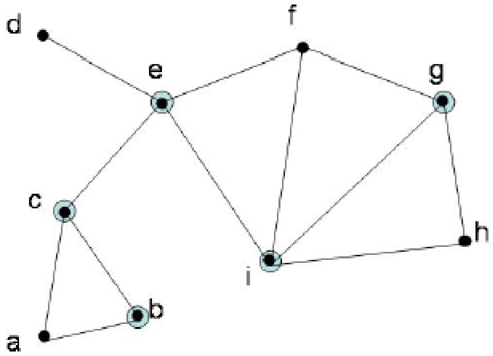
15 Simple approximations: 2-approximations

15.1 Vertex cover

The vertex cover problem is as follows: Given an undirected graph $g = (V, E)$, find a minimal-sized subset $C \subseteq V$ such that for every edge $\{v_1, v_2\} \in E$, either $v_1 \in C$ or $v_2 \in C$.

You can think of a *matching* in an undirected graph G as a set of monogamous marriages where two vertices can be married if and only if there exists and edge between them. A *maximal matching* is a matching that cannot be made larger by additional marriages.

It is easy to find a maximal matching. While there are edges remaining to be considered, add any edge to the set of marriages and delete the vertices on the edge and any edges that are attached to those vertices.



Claim: Any maximal matching corresponds to a vertex cover by collecting all married vertices into the cover.

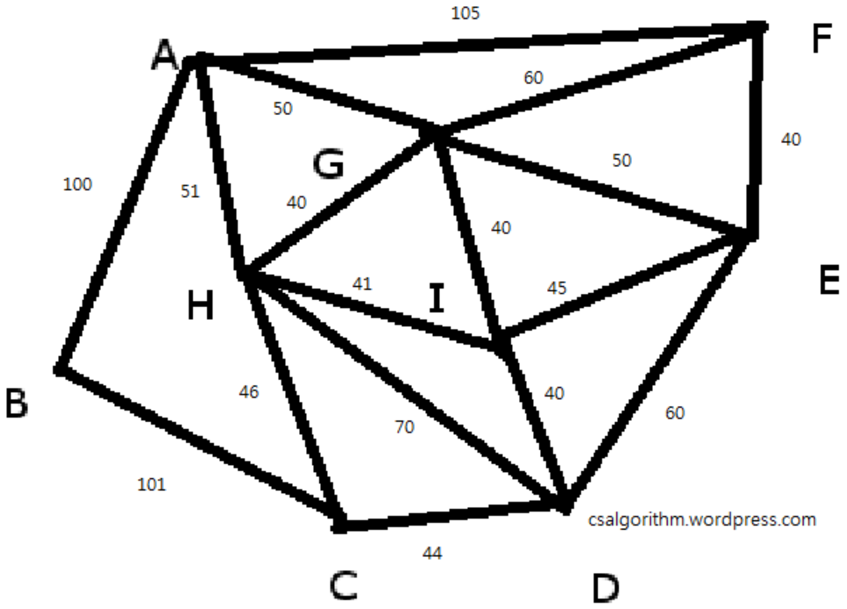
Proof: Is it possible to have an edge that is not covered? If so, that would be a potential marriage. Because the matching is maximal, this is not possible. □

Claim: Any maximal matching is at most twice the size of the optimal vertex cover.

Proof: We will show that any maximal matching is at most twice the size of any vertex cover. Notice that every marriage made by the maximal matching must have at least one representative in any vertex cover. Thus, any vertex cover must have size at least equal to the number of marriages. □

Notice then that the simple greedy algorithm of finding a maximal matching yields a 2-approximation to an extremely difficult problem, vertex cover.

15.2 Metric traveling salesman problem



The traveling salesman problem is as follows: A salesman want to visit each of n cities exactly once each, minimizing total distance traveled, and returning to the starting point.

The traveling salesman problem is known to be an extremely difficult problem to solve exactly similar to the vertex cover problem. We will find out later that you cannot even approximate this problem to within any constant factor.

The metric traveling salesman problem is the same as the traveling salesman problem with one additional caveat: The distances involved must obey the metric rules. In other words, (a) $d(c, c) = 0$, (b) for any two cities c_1, c_2, we must have $d(c_1, c_2) = d(c_2, c_1)$, (c) for any three cities c_1, c_2, c_3, we must have that $d(c_1, c_3) \leq d(c_1, c_2) + d(c_2, c_3)$ (triangle inequality).

Consider the following algorithm for solving the MTSP. Create a minimum spanning tree for the graph. Starting at any arbitrary city, perform a DFS of the tree. Any time a city is visited for the first time place it on a list. Once the list is complete, place the original city on the end of the list and the list is now your tour!

Claim: The length of the tour is less than twice the weight of the minimum spanning tree.

Proof: Consider an ant that moves along the DFS and the distance he travels. If he travels from c_1 to c_2 along the MST/DFS path, then by the triangle inequality, it is clearly at least as efficient to travel directly from c_1 to

c_2 (i.e. the tour's path). Thus, the tour has length less than or equal to the total DFS distance which is equal to twice the weight of the MST. □

Claim: The optimal tour distance is greater than or equal to the weight of the minimum spanning tree.

Proof: Consider the optimal tour itself. Deleting a single edge yields a spanning tree which must have weight at least equal to the minimum spanning tree. □

Putting these two claims together shows that the simple greedy algorithm yields a 2-approximation to the MTSP.

15.3 Inapproximability of the TSP

If $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial-time approximation algorithm with approximation ratio ρ for the general traveling salesman problem: Assume that there is such an algorithm. Given a graph $G = (V, E)$, form the graph G' which assigns the cost of 1 to an edge if it is in E and $\rho|V|+1$ otherwise. Does the original graph have a Hamilton cycle or not? Clearly, the answer is YES iff this new cityscape has an accepting tour (of length $\leq \rho|V|$)...

16 More simple approximations: Scheduling jobs, bin-packing

16.1 Scheduling jobs

This is one of the first approximation analyses ever done (Graham, 1966). In this model, there is a set of n jobs and m identical machines. Each job j_i must be processed without interruption for a time $p_i > 0$ on one of the m machines, each of which can process at most one job at a time. How should we schedule the jobs so as to minimize the time spent in the factory?

16.1.1 Example

With 3 machines, schedule the following jobs: 5, 11, 7, 3, 6, 13, 9, 50, 1, 4

16.1.2 Analysis

Consider the following algorithm called *list scheduling* (or LS, for short). We are given a list of the jobs in some arbitrary order. Whenever a machine becomes available, the next job on the list is immediately sent to that machine for processing.

Define the starting time of job j_i using this algorithm as s_i for $i \in [1, n]$. Let the job with the latest completion time be j_x. Then no machine can be idle at any time prior to s_x; otherwise, j_x would have been scheduled there.

Define the completion time of the list scheduling algorithm to be C^{LS} and define the latest completion time of the optimal scheduling to be C^{OPT}. Note the following.

- Note that $C^{OPT} \geq s_x + p_x \geq p_x$ because job j_x must complete on some machine.
- Note that $C^{OPT} \geq \frac{1}{m}\sum_{j=1}^{n} p_j$ because there must be at least one machine that takes at least as long as the average machine processing time.
- As we observed above, the start time of j_x must occur at the earliest possible moment. If we remove j_x from the job list, then the average machine time spent on jobs *other than* j_x is $\frac{1}{m}\sum_{i \neq x} p_i$. So some machine must be idle at least as early as that; hence, $s_x \leq \frac{1}{m}\sum_{i \neq x} p_i$.

Now we notice the following string of implications:

$$C^{LS} = s_x + p_x \leq \frac{1}{m}\sum_{i \neq x} p_i + p_x$$

$$= \frac{1}{m}\sum_{i=1}^{n} p_i + (1 - \frac{1}{m})p_x$$

$$\leq C^{OPT} + (1 - \frac{1}{m})C^{OPT} = (2 - \frac{1}{m})C^{OPT}$$

Thus, list scheduling does at least as well as $2 - \frac{1}{m}$ times the best possible schedule.

16.2 Bin packing

You are given a sequence of n items $a_1, a_2, \ldots, a_n$ each with a size $s(a_i) \in (0, 1]$ and are asked to pack them into a minimum number of unit-capacity bins. (Applications: loading trucks subject to weight limitations, packing television commercials into the breaks, cutting the correct amount of some material, say lumber or paper, so as to create a given set of objects)

16.2.1 Example

Consider the following weights: .1, .1, .2, .2, .3, .4, .5, .6, .7, .9

The optimal packing is: .1, .9 : .3, .7 : .4, .6 : .1, .2, .2, .5

16.2.2 Next fit

One of the simplest algorithms for packing is called Next Fit (NF). The NF algorithm functions as follows: If the item to be packed fits into the currently open bin, then pack it in and go on to the next item. Otherwise, close the current bin and open a new bin.

For a given bin-packing algorithm A, let $B(A)$ be defined to be the number of bins that algorithm A uses (assuming the problem instance is clear).

Notice that with the possible exception of the final open bin (in the case where only one bin is used), the average amount of utilized space per bin must be strictly more than one-half via NF. To see this, notice that if an item of size s cannot be packed in a bin with current level l, then we must have that $s + l > 1$. Another way to phrase this is

$$\forall j \in [1..\lfloor B(NF)/2 \rfloor], l_{2j-1} + l_{2j} > 1$$

If we sum all of these equations together, we get that

$$\sum_{i=1}^{n} l_i > \lfloor \frac{B(NF)}{2} \rfloor \Rightarrow$$

The right side is an integer so we get

$$\lceil \sum_{i=1}^{n} l_i \rceil - 1 \geq \lfloor \frac{B(NF)}{2} \rfloor \Rightarrow$$

$$\lceil \sum_{i=1}^{n} l_i \rceil - 1 \geq \frac{B(NF) - 1}{2}$$

But remember that $B(OPT) \geq \lceil \sum_{i=1}^{n} l_i \rceil$. Thus, we must have that $B(NF) \leq 2B(OPT) - 1$.

Consider the following sequence of weights (where N is a very large number):

$$(\frac{1}{N+1}, 1 - \frac{1}{2(N+1)}, \frac{1}{N+1}, 1 - \frac{1}{2(N+1)}, \ldots, \frac{1}{N+1}, 1 - \frac{1}{2(N+1)}, \frac{1}{N+1})$$

where the $\frac{1}{N+1}$ is repeated $N+1$ times and the $1 - \frac{1}{2(N+1)}$ is repeated only N times. The optimal arrangement would be to pack all of the small items into the same bin and the rest of the items into their own separate bins for a total of $N + 1$ total bins. NF puts every item into its own bin for a total of $2N + 1$. Thus, we get that

$$B(NF) = 2N + 1 = 2(N + 1) - 1 = 2B(OPT) - 1$$

Combining this result with the one above, $B(NF) = 2B(OPT) - 1$.

16.2.3 Inapproximability

The set partition problem is NP-complete: Given a set of numbers $s_1, s_2, \ldots, s_n$, can they be partitioned into 2 sets such that the sum of the numbers in one set is equal to the sum of the numbers in the other.

Assume that you are given a $\frac{3}{2} - \epsilon$ polynomial-time approximation for bin-packing. In other words, assume that there exists a polynomial time algorithm that will yield a solution to the bin-packing problem such that the number of bins used is no more than $\frac{3}{2} - \epsilon$ times the number of bins used by the optimal solution.

Take the instance of the set partition problem given by $s_1, s_2, \dots, s_n$. Create the corresponding bin-packing problem given by the following item sizes: $\frac{2s_1}{\sum_{i=1}^n s_i}, \frac{2s_2}{\sum_{i=1}^n s_i}, \dots, \frac{2s_n}{\sum_{i=1}^n s_i}$. If the answer to the set partition problem is NO, then the answer to the bin-packing problem (independent of the algorithm used) will be at least 3. On the other hand, if the answer to the set partition problem is YES, then the optimal answer to the bin-packing problem will be exactly 2 and the $\frac{3}{2} - \epsilon$ polynomial-time approximation will yield an answer of $(\frac{3}{2} - \epsilon)2 < 3$ bins.

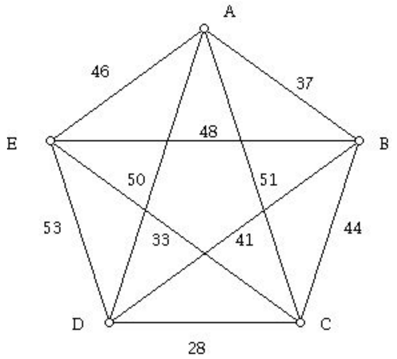
Thus, if a polynomial-time $\frac{3}{2} - \epsilon$ polynomial-time approximation to the bin-packing problem exists, $P = NP$ because we can use it to solve the set partition problem in polynomial time.

17 Intermediate approximations: Metric k-center problem

17.1 Definitions

The metric k-center problem is defined on the complete undirected graph $K_n = G = (V, E)$. Let the shortest distance between vertices v_i and v_j in G be d_{ij}. The problem is to find a subset of the vertices S such that $|S| = k$ and so that the longest distance of a node from the nearest node in S is minimized. So imagine each node in the complete graph is a city and edges represent the shortest distance between these cities. We have funds to build exactly k emergency centers. The k-center problem with triangle inequality is to place our k emergency centers such that no one has to go too far to get to their closest center. More formally, the goal is to minimize over all subsets $S \subseteq V$ of size k the following expression:

$$\max_{j \in V} \min_{i \in S} d_{ij}$$



Because we are doing the *metric* k-center problem, we assume that (a) $d_{ii} = 0$ (b) $d_{ij} = d_{ji}$ and (c) $d_{ij} \leq d_{ik} + d_{kj}$.

The metric k-center problem is NP-hard (Kariv and Hakimi, 1979). Applications of this include placement of supply centers for the military, placement of hospitals, data backups, etc.

The *square* of a graph $G = (V, E)$ is defined to be $G^2 = (V, E^2)$ where there exists an edge between two vertices x and y in G^2 if and only if there exists a path of length 1 or 2 between x and y in G.

An *independent set* of a graph is defined to be a set of vertices none of which shares a mutual edge with any other vertex in the set. A *dominating set* of a graph is defined to be a set of vertices such that every vertex in the graph is either in the dominating set or shares an edge with a vertex in the dominating set.

17.2 Algorithm

Consider the following algorithm: Initially, think of the cities as having no edges between them. Order the edges from least weight to most so that $w(e_1) \leq w(e_2) \leq \ldots \leq w(e_m)$ where $m = \binom{n}{2}$. We will add these edges in order to the empty graph, and, at some point, we will have a dominating set of k vertices. After this point, we have our k centers and adding heavier edges will not help: To see why, assume that we have a dominating set of k vertices and we are thinking about adding edge e_i to the graph. There is already a way to get from any city to an emergency center in time at most $w(e_{i-1})$ because the k centers form a dominating set; thus, adding an additional edge of higher weight gains us nothing.

Our problem is that the dominating set problem (the problem of finding a dominating set of size k or smaller in an arbitrary undirected graph) is also NP-complete. This is a problem...

Define G_i to be the graph that consists of adding edges $e_1, \ldots, e_i$ to the set of cities devoid of edges. Now, we can use the following algorithm: Starting from $i = 1, 2, \ldots$, compute a maximal independent set (not a *maximum* independent set, which is NP-hard to compute) L_i for the graph G_i^2. As soon as $|L_i| \leq k$, call the set of vertices in L_i the k centers (and if there are strictly less than k, choose any artbitrary others until there are k).

Let the index of the graph that gets returned by the algorithm above be x (so that G_x^2 is the first graph for which we find a maximal independent set of size less than or equal to k).

17.3 Analysis

Claim: Let $H = (V, E)$ be an undirected graph, and let I^2 be an independent set of H^2. Let $dom(H)$ denote the size of a minimum cardinality dominating set in H. (Note that $dom(H)$ is NP-hard to compute.) Then $|I^2| \leq |dom(H)|$.

Proof: Let D be a minimum cardinality dominating set in H. For any vertex $d \in D$, its neighborhood in H forms a clique in H^2. Thus, H^2 contains at most

$|D|$ cliques spanning all the vertices. But any independent set in H^2 can contain only at most one vertex from any given clique. Thus $|I^2| \leq |D|$. □

Claim: Let c^* be the cost of an optimal solution to a given instance of the k-center problem on $G = (V, E)$. Then $w(e_x) \leq c^*$.

Proof: Note that it is not possible for every edge in G to have strictly smaller weight than c^*. If so, then there would need to be some node for which the fastest way to the nearest center is not via the edge directly there. However, this violates the triangle inequality.

Let x' be the smallest value such that $w(e_{x'}) \geq c^*$. If we can show that $x \leq x'$, then we are done because the edges are being added in increasing order. We only need to check that *any* maximal independent set in $G^2_{x'}$ has size less than or equal to k; if this is true, then because x was defined as being the first we find with an independent set in G^2_x with size $\leq k$, it must have come at least as early as x'.

We claim that $G_{x'}$ has a dominating set with size $\leq k$. Note that $w(e_{x'}) \geq c^*$. Remember that the shortest path from a node to its corresponding center must be via the edge directly there by the triangle inequality. If we have already added edges at least as large as the optimal answer c^*, then every node has its optimal direct route to its optimal center and there are exactly k centers.

By the above lemma, we have that for any independent set I^2 of $G^2_{x'}$, $|I^2| \leq |dom(G^2_{x'})| \leq k$. □

Claim: The above algorithm returns a solution with cost at most twice the optimal.

Proof: Note that any maximal independent set in a graph is necessarily a dominating set. So the maximal independent set L_x that was returned by the algorithm is a dominating set in the graph G^2_x. This implies that every vertex in G_x is at most 2 edges away from a center. But by the above lemma, $w(e_x) \leq c^*$ which implies that every edge in G_x is less than or equal to c^*.

So consider the worst case distance from a given vertex v to v_k. With our solution, at most two edges need to be traversed, both of which are of length at most c^*; this implies that the worst case distance our solution yields is at most $2c^*$ and thus our algorithm has at most twice the optimal cost. □

17.4 Inapproximability

Claim: Approximating the metric k-center problem in polynomial time within a constant factor strictly less than 2 is not possible unless $P = NP$.

Proof: Assume that it is possible to approximate the metrix k-center problem to within $2 - \epsilon$ for some $\epsilon > 0$.

Assume that you are given an undirected graph $G = (V, E)$ and look for a dominating set of size k or less. Construct the following instance of the k-center problem. Let $V' = V$ and let G' be the complete undirected graph on V'. Let the weight of an edge $e \in E'$ be such that $e \in E \Rightarrow w(e) = 1$ and $e \notin E \Rightarrow w(e) = 2$.

Use the polynomial-time approximation algorithm to get a solution to the above instance. Note that if the answer to the approximation is less than 2 (namely, 1), then a dominating set of size k exists in the original graph, but if the answer to the approximation is 2, then there does not. $\square$

18 Randomized approximations: Maximum satisfiability

The Maximum Satisfiability problem (MAXSAT) is the problem where you are given a list of clauses (each clause is a list of literals) and the goal is to return the largest possible number of simultaneously satisfied clauses. (We assume that there are no clauses where a variable and its negation both appear and that a literal appears at most once in any clause.) To solve this problem exactly is NP-complete.

We will solve this problem in a unique way: We will create 2 algorithms that work well with some probability. However, we *guarantee* that at least one of them will always produce an answer in which at least $\frac{3}{4}$ of the optimal number of clauses in the instance are satisfied. But as we don't know which of the algorithms does, in fact, produce this answer, we simply run them both and choose the algorithm with the higher number of satisfied clauses!

18.1 Example

$$x_1, x_2, \neg x_3 : x_3, \neg x_2 : \neg x_1 : \neg x_3, x_1, \neg x_2, x_4 : x_4 : \neg x_1, \neg x_2, \neg x_3, \neg x_4$$

18.2 Random assignments

Assume that you are given an instance of MAXSAT with n clauses and m variables such that every clause has at least k literals. Assign every variable to true with probability $\frac{1}{2}$. Then the probability that a given clause is satisfied is at least $\alpha_k \equiv 1 - 2^{-k}$ and the expected number of satisfied clauses is at least $n(1 - 2^{-k})$.

Now, if the expected number of satisfied clauses is at least $n(1 - 2^{-k})$, then there must exist some assignment for the variables such that at least that many clauses is actually satisfied. We will implicitly use this fact multiple times below.

Unfortunately, if there are lots of clauses with a single literal, this is only an approximation ratio of $\frac{1}{2}$.

18.3 Linear programming

Consider the following linear program (where S_j^+ represent the set of clauses where the variable x_j appears positively and similarly for S_j^-; recall that by our assumption $S_j^+ \cap S_j^- = \emptyset$). z_i should represent a lower bound on the probability that clause i is satisfied.

Maximize $\sum_{i=1}^{n} z_i$ subject to

1. $\forall r \in [1, n], \sum_{i \in S_r^+} x_i + \sum_{i \in S_r^-} (1 - x_i) \geq z_r$

2. $\forall r \in [1, m], 0 \leq x_r \leq 1$ and $\forall r \in [1, n], 0 \leq z_r \leq 1$

Randomized rounding will independently set the variable x_i to TRUE with probability equal to the value of x_i that comes out of the linear program.

We claim that $\sum_{i=1}^{n} z_i$ is a lower bound for the expected number of satisfiable clauses. Note that clause r can only be made TRUE if and only if at least one of the literals within it is made TRUE. The probability that clause r is satisfied is therefore upper bounded by the sum of the probabilities that at least one literal within it is made TRUE; this is the purpose of the first set of constraints. Now, summing the probabilities that each clause is satisfied yields a lower bound on the expected number of satisfied clauses.

Define $\beta_k \equiv 1 - (1 - \frac{1}{k})^k$.

Lemma: Let c_r be a clause with k literals. The probability that c_r is satisfied by randomized rounding is at least $\beta_k z_r$.

Proof: Consider the constraint $\sum_{i \in S_r^+} x_i + \sum_{i \in S_r^-} (1 - x_i) \geq z_r$ from the linear program. It is clear that the only way for c_r to remain unsatisfied is if every $x_i \in S_r^+$ is rounded to 0 and every $x_i \in S_r^-$ is rounded to 1. Because each variable is rounded independently, this occurs with probability

$$\prod_{i \in S_r^+} (1 - x_i) \prod_{i \in S_r^-} x_i$$

Thus, our goal is to show that

$$1 - \prod_{i \in S_r^+} (1 - x_i) \prod_{i \in S_r^-} x_i \geq \beta_k z_r$$

Define the Lagrangian function

$$L(x, \gamma) = 1 - \prod_{i \in S_r^+} (1 - x_i) \prod_{i \in S_r^-} x_i + \gamma \left(\sum_{i \in S_r^+} x_i + \sum_{i \in S_r^-} (1 - x_i) - z_r \right)$$

We look to find $\min_x \max_{\gamma \leq 0} L(x, \gamma)$: Notice that if the constraint is not satisfied, then the maximum will be ∞; whereas, if the constraint is satisfied, then the maximum will occur when $\gamma = 0$ or the constraint is tight and the maximum of the Lagrangian is exactly equal to $1 - \prod_{i \in S_r^+} (1 - x_i) \prod_{i \in S_r^-} x_i$. When we minimize over x, we get exactly what we want. By the KKT conditions (no, we are not getting into these),

$$\min_x \max_{\gamma \leq 0} L(x, \gamma) = \max_{\gamma \leq 0} \min_x L(x, \gamma)$$

So, we can minimize over x first.

$$\frac{\partial}{\partial x_t \in S_r^+} \left[1 - \prod_{i \in S_r^+} (1 - x_i) \prod_{i \in S_r^-} x_i + \gamma \left(\sum_{i \in S_r^+} x_i + \sum_{i \in S_r^-} (1 - x_i) - z_r \right) \right] = 0 \Rightarrow$$

$$\prod_{i\in S_r^+ - \{x_t\}} (1-x_i) \prod_{i\in S_r^-} x_i + \gamma = 0$$

But because this is true for every $x_t \in S_r^+$, we must have that all variables in S_r^+ are equal. The same reasoning yields that all variables in S_r^- are equal. Let the value of the variables in S_r^+ all be v^+ and similarly for v^-; we have that

$$(v^+)^{|S_r^+|-1}(1-v^-)^{|S_r^-|} = -\gamma \text{ and } (v^+)^{|S_r^+|}(1-v^-)^{|S_r^-|-1} = -\gamma \Rightarrow$$

$$v^+ = 1 - v^-$$

Now we can maximize with respect to γ to get

$$\sum_{i\in S_r^+} x_i + \sum_{i\in S_r^-}(1-x_i) - z_r = 0 \Rightarrow$$

Because there are k literals in c_r,

$$kv^+ = z_r \Rightarrow v^+ = \frac{z_r}{k}$$

This implies that the minimum of $1 - \prod_{i\in S_r^+}(1-x_i)\prod_{i\in S_r^-} x_i$ is

$$1 - (1 - \frac{z_r}{k})^k$$

Now, let's examine the function

$$f(z_r) = 1 - (1-\frac{z_r}{k})^k - [1-(1-\frac{1}{k})^k]z_r$$

We know that $f(0) = f(1) = 0$.

$$f'(z_r) = (1-\frac{z_r}{k})^{k-1} - [1-(1-\frac{1}{k})^k]$$

Note that $f'(0) > 0$ and $f''(z_r) < 0$ everywhere. Thus, the function looks like a frowny face and the two sides intersect at the x-axis. Hence for $z_r \in [0,1], f(z_r) \geq 0 \Rightarrow \forall z_r \in [0,1]$,

$$1-(1-\frac{z_r}{k})^k \geq [1-(1-\frac{1}{k})^k]z_r$$

□

Now, by the above lemma, if we sum over all j, we get a lower bound on the number of satisfied clauses using randomized rounding:

$$\sum_{j=1}^{n}[1-(1-\frac{1}{k})^k]z_j \sim (1-\frac{1}{e})\sum_{j=1}^{n} z_j$$

and this is clearly within a factor of $1-\frac{1}{e} \approx .632$ times the optimal.

18.4 Putting them together

Lemma: $\forall k \geq 1, \alpha_k + \beta_k \geq \frac{3}{2}$

Proof: Let $f(k) = \alpha_k + \beta_k = 1 - 2^{-k} + 1 - (1 - \frac{1}{k})^k$. Note that $f(1) = \frac{3}{2}$ and

$$f'(k) = 2^{-x} \ln 2 + (1 - \frac{1}{k})^k (\frac{1}{1-k} - \ln(1 - \frac{1}{k})) > 0$$

□

Now, what would happen if we flipped a fair coin and, if the coin came up heads, decided to use the first algorithm and tails for the second? For each clause, if the number of literals in the clause is k, then the probability that it gets satisfied is

$$\frac{1}{2}\alpha_k + \frac{1}{2}\beta_k z_r \geq \frac{\alpha_k + \beta_k}{2} z_r \geq \frac{3}{4} z_r$$

Thus, when we sum over all of the clauses, we get that the expected number of satisfied clauses is at least $\frac{3}{4}$ optimal. But that implies that one of the two methods must yield a $\frac{3}{4}$ approximation at least! Do both methods and pick the one that yields the higher number of satisfied clauses.

19 Simple Las Vegas Algorithms: Randomized quicksort

A *Las Vegas* algorithm is an algorithm that uses probability. It will always get the right answer but, depending on the randomness, may take a long time to terminate. These algorithms are sometime known as Robin Hood algorithms because they take from the rich (longer running times) and give to the poor (shorter running times).

Consider quicksort on any instance using a pivot chosen randomly (i.e. assuming all permutations of n distinct numbers are equally likely). We know that the worst case happens when we get back luck and the pivot is chosen as the largest or smallest number remaining in the list and that leads to a running time of $O(n^2)$. But now let's figure out what happens in the average case.

Let X_{ij} be the random indicator variable that counts the number of times x_i and x_j get compared. Note that two elements only get compared when one of them is the pivot, and afterwards, because the pivot is in the correct position, they can never be compared again. Therefore, X_{ij} is an indicator variable for every i and j.

The total number of comparisons during the quicksort algorithm is

$$X = \sum_{i=1}^{n} \sum_{j=i+1}^{n} X_{ij} \Rightarrow E[X] = \sum_{i=1}^{n} \sum_{j=i+1}^{n} E[X_{ij}]$$

So assume that $i < j$. Denote the ith smallest element in the array by e_i and the jth smallest element by e_j. If the pivot we choose is between e_i and e_j, then these two will end up in difference sublists and wil never get compared. If the pivot is *exactly* e_i or e_j, then we do compare them. Finally, if the pivot chosen is either larger or smaller than both, then they both end up in the same sublist and we will need to pick another pivot.

This is a kind of dart game for each i and j. If the dart lands between them, you win. If the dart hits one of them, you lose. If the dart lands outside the target, you shoot again. There are $j - i + 1$ numbers between i and j so the probability that you lose the game is $\frac{2}{j-i+1}$ (because you must hit the target eventually). So $E[X_{ij}] = \frac{2}{j-i+1} \Rightarrow$

$$E[X] = \sum_{i=1}^{n} 2(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n-i+1})$$

Let us know consider the integral $\int_1^n \frac{1}{x} dx = \ln n$. If we estimate this integral by rectangles, then we get that

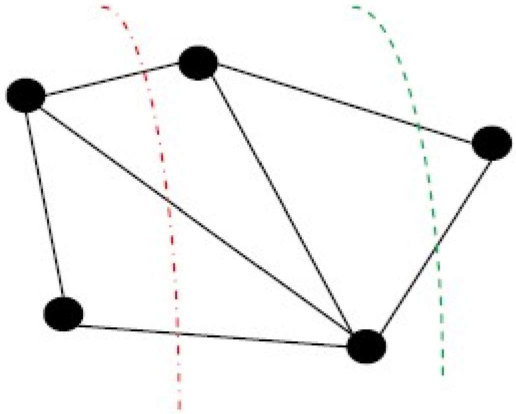
$$\ln n = \int_1^n \frac{1}{x} dx \geq (\frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}) \geq (\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n-i+1}) \Rightarrow$$

$$E[X] = \sum_{i=1}^{n} 2(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n-i+1}) \leq 2n \ln n$$

This implies that the *expected* time for quicksort assuming a random pivot is chosen every time is $O(n \ln n)$ (which is not that exciting), but the constant hidden by the O-notation is 2!

20 Simple Monte Carlo Algorithms: Karger's algorithm

The problem of finding a minimum cut in a connected unweighted undirected graph G is to determine the smallest number of edge cuts that it takes to disconnect G. Note that this problem can be solved by running a fast maximum flow algorithm $\Theta(n)$ times. The fastest known deterministic max cut algorithm (which we will not do here) takes time $O(nm \log \frac{n^2}{m})$.



Definition: Let $G = (V, E)$ be a multigraph without self-loops. Given some $\{x, y\} \in E$, we can *contract* the edge $\{x, y\}$ as follows: (a) replace x and y with a new vertex w (b) if $\{v_x, x\} \in E$ for $v_x \neq y$, then create the edge $\{v_x, w\}$; similarly, if $\{v_y, y\} \in E$ for $v_y \neq x$, then create the edge $\{v_y, w\}$ (c) remove all self-loops. Let $|V| = n$ and $|E| = m$.

(Karger) Consider the following randomized algorithm: While there are more than 2 nodes in G, choose an edge $\{x, y\}$ uniformly at random from G and contract it. Output the remaining edges as the minimum cut.

Claim: The probability that this algorithm terminates with a minimum cut is at least $\frac{1}{\binom{|V|}{2}}$.

Proof: It is clear that if there are exactly 2 vertices in G, then the output of the algorithm is correct with probability 1.

Let $n = |V|$. Let C be a minimal cut. Note that $|C|$ must be less than or equal to the minimum degree of any vertex in G. Note that by the Handshaking Theorem, $2|E| = \sum_{v \in |V|} deg(v) \geq n|C| \Rightarrow |C| \leq \frac{2|E|}{n}$. Thus, the probability that a randomly selected edge is in C is $\leq \frac{2}{n}$. We perform $n - 2$ removals...

$$P(e_1 \notin C) \geq 1 - \frac{2}{n}, P(e_2 \notin C | e_1 \notin C) \geq 1 - \frac{2}{n-1}, \ldots,$$

$$P(e_{n-2} \notin C | e_1, e_2, \ldots, e_{n-3} \notin C) \geq 1 - \frac{2}{n - (n-3)} = \frac{1}{3}$$

Taking the product of all these, the probability that we get the right answer is lower bounded by $\frac{2}{n(n-1)}$. □

Claim: Given that it is possible to implement a single iteration of Karger's algorithm in time $O(n^2)$ time using appropriate data structures, it is possible to determine the minimum cut in G with any arbitrary probability $\frac{1}{n^c}$ for any constant $c > 0$ in time $O(n^4 \log n)$.

Proof: Run the algorithm $x\binom{n}{2}$ times for some x to be determined. Then the probability that none of the runs yields the minimum cut is

$$1-(1-\frac{1}{\binom{n}{2}})^{x\binom{n}{2}} \geq 1-e^{-x}$$

If we let $x = c \ln n$, then this probability is $\frac{1}{n^c}$. □

Note that this algorithm runs faster than the fastest known deterministic algorithm!

21 Intermediate Monte Carlo: Karger's II

21.1 An improvement

Definition: A *concave* function is a function such that the second derivative is everywhere negative or, equivalently, functions f such that $\forall x, y, f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$.

Lemma: Concave functions such that $f(0) \geq 0$ are subadditive. In other words, for any concave function f such that $f(0) \geq 0$, $f(a) + f(b) \geq f(a+b)$.

Proof: In the definition, let $y = 0$. Then you have that $f(tx) = f(tx + (1-t)0) \geq tf(x) + (1-t)f(0) \geq tf(x)$. Now notice that for any a and b, we get

$$f(a)+f(b) = f(\frac{a}{a+b}(a+b))+f(\frac{b}{a+b}(a+b)) \geq \frac{a}{a+b}f(a+b)+\frac{b}{a+b}f(a+b) = f(a+b)$$

□

What happens if, instead of simply running the algorithm multiple times from the beginning, you do the following: From a multigraph G, if G has c_{base} vertices or less (where $c_{base} > 0$ is a constant that we will determine later), brute force the minimum cut via any max flow algorithm. Otherwise, make two copies of G, run Karger's algorithm on each copy until there are $\frac{n}{\sqrt{2}} + c$ vertices left in each (where $c > 0$ is a constant that we will determine later), then recursively continue on each copy.

The intuition behind doing this is that the highest probability of error occurs near the end of the contraction process. Thus, we want as many leaf nodes in the recursion tree as possible. However, at beginning of the algorithm it is unlikely that we make a significant error so there is no reason do redo the initial computations multiple times.

Let $T(n)$ be the running time of this recursive algorithm on a graph with n vertices. Then we have that

$$T(n) = 2T(\frac{n}{\sqrt{2}} + c) + \Theta(n^2)$$

Claim: For any constant $c > 0$ and if $c_{base} \geq 2c + 1$,

$$T(n) = \Theta(n^2 \log n)$$

Proof: First, we will show that $T(n) = O(n^2 \log n)$.

Induction. Assume that $\exists r_1 > 0$ and r_2, r_3, r_4, r_5 (constants) such that eventually

$$T(x) \leq r_1 x^2 \log x + r_2 x^2 + r_3 x \log x + r_4 \log x + r_5 \text{ if } x < n$$

Our goal is to show that

$$T(n) \leq r_1 n^2 \log n + r_2 n^2 + r_3 n \log n + r_4 \log n + r_5$$

In order to push the induction through (so that the recursion relation yields a smaller value upon being applied), we require that $\frac{n}{2} + c < n \Rightarrow n > 2c$ so that $c_{base} \geq 2c + 1$. So we get that $\exists d > 0$ such that eventually

$$T(n) \leq 2T(\frac{n}{\sqrt{2}} + c) + dn^2 \leq$$

(inductively)

$$r_1(c + \frac{n}{\sqrt{2}})^2 \log(c + \frac{n}{\sqrt{2}}) + r_2(c + \frac{n}{\sqrt{2}})^2 +$$

$$r_3(c + \frac{n}{\sqrt{2}} \log(c + \frac{n}{\sqrt{2}}) + r_4 \log(c + \frac{n}{\sqrt{2}}) + r_5 + dn^2 \leq$$

(by the Lemma above: $\log(c + \frac{n}{\sqrt{2}}) \leq \log c + \log n - \frac{1}{2}$)

$$\frac{1}{2} r_1 n^2 \log n + (\frac{1}{2} r_2 + \frac{1}{2} r_1 \log c - \frac{r_1}{4} + d) n^2 + (\sqrt{2} c r_1 + \frac{r_3}{\sqrt{2}}) n \log n +$$

$$(\sqrt{2} r_1 c \log c - \frac{c r_1}{\sqrt{2}} + \sqrt{2} c r_2 - \frac{r_3}{2\sqrt{2}} + \frac{r_3 \log c}{\sqrt{2}}) n +$$

$$(c^2 r_1 + c r_3 + r_4) \log n + (\log c(c^2 r_1 + c r_3 + r_4) + \frac{1}{2}(-c^2 r_1 + 2c^2 r_2 - c r_3 - r_4 + 2 r_5)$$

Though this looks nastier than before, we can note that the high order term is only $\frac{1}{2} r_1 n^2 \log n$ and adding lower order terms will never eventually exceed $T(n) = r_1 n^2 \log n + r_2 n^2 + r_3 n \log n + r_4 \log n + r_5$ independent of the values of the r_i; we're done.

Now, we will show that $T(n) = \Omega(n^2 \log n)$. Notice that because $T(n)$ is clearly an increasing function of n, $\exists d' > 0$ such that eventually

$$T(n) = 2T(\frac{n}{\sqrt{2}} + c) + \Theta(n^2) \geq 2T(\frac{n}{\sqrt{2}}) + d' n^2$$

But this implies that $T(n) = \Omega(n^2 \log n)$ by the Master Theorem. □

We are now in a position to choose the constants c and c_{base}. The probability that the algorithm succeeds at every step is roughly $\frac{1}{2}$, but we would like to guarantee that it is at least $\frac{1}{2}$. We have that the probability of success works out as follows (almost exactly as before):

$$P(e_1 \notin C) \geq 1 - \frac{2}{n}, P(e_2 \notin C | e_1 \notin C) \geq 1 - \frac{2}{n-1}, \ldots,$$

$$P(e_{(n-(\frac{n}{\sqrt{2}}+c))} \notin C | e_1, e_2, \ldots e_{(n-(\frac{n}{\sqrt{2}}+c-1))} \notin C) \geq 1 - \frac{2}{\frac{n}{\sqrt{2}}+c-1}$$

When you take the product of all of these, you wind up with

$$\frac{(\frac{n}{\sqrt{2}}+c-3)(\frac{n}{\sqrt{2}}+c-2)}{n(n-1)}$$

Notice that if $c \geq 3$, then this is expression is clearly at least $\frac{1}{2}$. Thus, we choose $c = 3$ and the probability that the algorithm fails during one mini-contraction process is bounded above by $\frac{1}{2}$. By the restriction on c_{base} that we found above, we choose $c_{base} = 7$.

Thus the probability of success $P(n)$ on a graph with n vertices can be calculated recursively as follows (interpreted as one minus the probability that both fail):

$$P(n) = 1 - \left(1 - \frac{1}{2}P(\frac{n}{\sqrt{2}})\right)^2$$

Claim: $P(n) = \Omega(\frac{1}{\log n})$
Proof:

$$P(n) = \Omega(\frac{1}{\log n}) \Leftrightarrow \exists c > 0, P(n) \geq \frac{c}{\log n}$$

Induction. Assume that for every $k < n$, $P(k) \geq \frac{c}{\log k}$. Our goal is to show that $P(n) \geq \frac{c}{\log n}$. Then we have

$$P(n) \geq 1 - \left(1 - \frac{1}{2}P(\frac{n}{\sqrt{2}})\right)^2 \geq 1 - \left(1 - \frac{1}{2}\frac{c}{\log \frac{n}{\sqrt{2}}}\right)^2 \geq$$

$$\frac{c}{\log \frac{n}{\sqrt{2}}} - \frac{c^2}{4(\log \frac{n}{\sqrt{2}})^2} \geq \frac{c}{\log \frac{n}{\sqrt{2}}} \geq \frac{c}{\log n}$$

Clearly this is true for any $c > 0$. □

By the same reasoning as above, we can run this algorithm for $O(\log^2 n)$ iterations to get a failure rate of at most $\frac{1}{n^c}$ for arbitrary $c > 0$. This yields a total running time of $O(n^2 \log^3 n)$ for essentially the same result as before.

22 Online problems: The file cabinet I

22.1 Defining the problem

Suppose we have a filing cabinet containing n labeled but unsorted files. We receive a sequence of requests $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_m)$ to access certain files. Each request is the label of a given file. After a request to access a given file x is given, we must locate the file and remove it from the cabinet. Assume that the only way this can be done is by flipping through the files from the beginning until the file is located. If we need to flip through i files to find the one we want, then we incur a cost of i (and to attempt to find a file that does not exist incurs a cost of n). Once we find a file and remove it, we can replace it anywhere in the cabinet at no cost. On the other hand, we can also move a file without explicitly searching for it: If we decide that we want to move a file from position j to position i (where $i < j$; you assume that you always want to move a file closer to the front, never the back), then it costs $j - i$ to do so.

Our first question will be: "Are these extra movements necessary for optimal performance?" Is it always necessary to do prep work in order to get the best possible performance? Consider the following 3-element list: (x_1, x_2, x_3) (where x_1 is at the front of the cabinet). Assume that the element requests are x_3, x_2, x_3, x_2 and *you know that these requests are coming in this order.* What is the optimal sequence of moves if you know what is coming? The requests can be serviced as follows:

1. Move x_2 to the front (cost 1)
2. Move x_3 to the second position (cost 1)
3. Search for x_3 (cost 2)
4. Search for x_2 (cost 1)
5. Search for x_3 (cost 2)
6. Search for x_2 (cost 1)

This works out to a cost of 8.

Claim: Any algorithm that is not allowed to use paid exchanges must use at least a cost of 9 to perform the searches.

Proof: The first search for x_3 is required to take at least a cost of 3. Now either we move x_3 in front of x_2 or we don't.

- If we move x_3 in front of x_2, then the cost of searching for x_2 becomes 3. The second searches for x_2 and x_3 must take a total of at least cost 3 and therefore the total cost of the requests are at least 9.
- If we do not move x_3 in front of x_2, then the cost of search for x_2 is only at least 2, but then the following search for x_3 is 3 and the last search for x_2 is at least 1; therefore the total cost of the requests are at least 9.

$\square$

22.2 Definition

Definition: Let $A(\sigma)$ be the cost that an algorithm A uses to process an online problem instance σ with initial data size n. We say that A is α-competitive for some constant α iff

$$\exists\beta(n)\forall\sigma A(\sigma) \leq \alpha OPT(\sigma) + \beta(n)$$

where β can grow with n, the initial size of the data, but cannot depend on the online request sequence. (You might think of β as the cost of preprocessing the data before the online part of the problem starts.) α is called the *competitive ratio* for algorithm A. We allow the algorithm OPT to know the request sequence σ in advance; in other words, OPT is offline. The competitive ratio is (basically) the ratio of how the algorithm does against an oblivious adversary as opposed to how an optimal algorithm would do against an oblivious adversary, the ratio of online to offline (because an optimal algorithm would be the one that had the entire problem in advance).

22.3 Lower bound

We will now look to prove a lower bound on the competitive ratio for algorithms that solve the online cabinet problem. Recall that the competitive ratio is the ratio of how well a given algorithm performs as compared with an optimal offline algorithm. Recall that an algorithm A is α-competitive for some constant α iff

$$\exists\beta(n)\forall\sigma A(\sigma) \leq \alpha OPT(\sigma) + \beta(n)$$

Assume that you are given any algorithm A that solves the cabinet problem. We will attempt to find a lower bound for the value of α, its competitive ratio. To do this, we will use an upper bound for $A(\sigma)$ and a lower bound for $OPT(\sigma)$, pushing the two together and making α as small as possible.

To get an upper bound on $A(\sigma)$, we can assume that the adversary that is choosing σ is offline; in other words, he can choose σ as the algorithm progresses rather than making a static choice at the beginning. An optimal offline adversary can clearly choose the final object in A's list every single time regardless of any of A's attempts to optimize. Thus there must $\exists\sigma, A(\sigma) = |\sigma|n$, independent of A, which will serve as an upper bound for $A(\sigma)$.

We now concentrate on determining a lower bound for $OPT(\sigma)$. Assume that the adversary has chosen his value for σ as cleverly as he can. We now ask ourselves how low the optimal offline algorithm can possibly be *guaranteed* to get? Consider the following behavior by some algorithm R: At the beginning of the algorithm, using paid exchanges, R randomizes the ordering of the n elements of the list. This is accomplished in time $O(n^2)$ and is therefore worked into the $\beta(n)$ value. Then R searches for the various elements as they are requested in σ without doing any movements whatsoever. What is the total expected cost of R? The total expected cost is

$$|\sigma|\frac{1}{n}\sum_{i=1}^{n} i = |\sigma|\frac{(n+1)}{2}$$

Thus, there must exist some ordering of the elements that an offline algorithm can create that will yield at most this cost. Thus, $\forall\sigma, OPT(\sigma) \leq |\sigma|\frac{(n+1)}{2}$.

Plugging in the two bounds from above, we get

$$\frac{A(\sigma)}{OPT(\sigma)} \geq \frac{|\sigma|n}{|\sigma|\frac{(n+1)}{2}} = \frac{2n}{n+1}$$

is a lower bound for the competitive ratio α.

23 Online problems: The file cabinet II

23.1 Move to front upper bound

The Move To Front (MTF) algorithm works as follows: After accessing an element, always move it to the front of the file cabinet (which is free). Do not change the relative order of any other items.

The following argument was given by Sandy Irani in 1995.

23.1.1 Definitions

Define the following cost function for a pair of items $x_i \neq x_j$ and any algorithm A:

$$A_{ij}(t,\sigma) \equiv \begin{cases} 1 + \frac{1}{n-1} & \text{if } x_i \text{ appears before } x_j \text{ in } A\text{'s list and } \sigma_t = x_j \\ \frac{1}{n-1} & \text{if } x_j \text{ appears before } x_i \text{ in } A\text{'s list and } \sigma_t = x_j \\ 0 & \text{otherwise} \end{cases}$$

Intuitively, $A_{ij}(t,\sigma)$ is $1 + \frac{1}{n-1}$ if A needs to step over x_i for a request to x_j at time t.

Denote $A(\sigma)$ to be the total cost of servicing the request sequence σ.

Let $A^p_{ij}(t) \equiv \begin{cases} 1 & \text{if } (x_i, x_j) \mapsto (x_j, x_i) \text{ via paid exchange between } \sigma_{t-1} \text{ and } \sigma_t \\ 0 & \text{otherwise} \end{cases}$

23.1.2 The proof

Claim: $\forall\sigma$ and for any algorithm A,

$$A(\sigma) = \frac{1}{2} \sum_{t=1}^{|\sigma|} \sum_{1 \leq i,j \leq n, i \neq j} [A_{ij}(t,\sigma) + A^p_{ij}(t) + A_{ji}(t,\sigma) + A^p_{ji}(t)]$$

Proof: First, notice that the A^p terms simply sum the paid exchanges that occur during the course of the algorithm. We therefore ignore their effect in the analysis afterwards and assume that no paid exchanges occur.

Assume that $\sigma_t = x_k$. Then we have that

$$\sum_{1 \leq i,j \leq n, i \neq j} A_{ij}(t,\sigma) = \sum_{1 \leq i \leq n, i \neq k} A_{ik}(t,\sigma)$$

In this sum, there is a contribution of $\frac{1}{n-1}$ for every element in the list other than x_k (for a total of exactly 1) and a contribution of exactly 1 for every element x_i such that x_i appears before x_k in the list. This works out to exactly the cost for accessing x_k. Because moving x_k after it has already been searched for is a free operation and the summation for the second term is exactly equal to the first, we are done. □

Definition: For notational convenience, we will let

$$\varphi^{A}_{\{i,j\}}(t,\sigma) \equiv A_{ij}(t,\sigma) + A^{p}_{ij}(t) + A_{ji}(t,\sigma) + A^{p}_{ji}(t)$$

Then we have that

$$A(\sigma) = \frac{1}{2}\sum_{t=1}^{|\sigma|}\sum_{1\le i,j\le n, i\neq j}\varphi^{A}_{\{i,j\}}(t,\sigma)$$

Definition: Say that two items x_i and x_j are in the *wrong order* at some time t if the item that is requested at time t appears after the other (so that the algorithm is charged to step over the offending item).

Fix any two distinct elements, x_i and x_j. Notice that if $\sigma_t \neq x_i$ and $\sigma_t \neq x_j$, then the only charges we incur in $\varphi^{A}_{\{i,j\}}(t,\sigma)$ are paid exchanges between x_i and x_j Thus, in the future, we ignore those terms such that $\sigma_t \neq x_i$ and $\sigma_t \neq x_j$. Let τ represent the ordered vector of times such that either $\sigma_t = x_i$ or $\sigma_t = x_j$. In other words if $k \in [1, |\tau|]$, then σ_{τ_k} is either x_i or x_j. We will assume that all paid exchanges that occur in terms we ignore are gathered into the τ terms.

Lemma: For every $k \in [1, |\tau| - 1]$, we either have the following inequality hold *or* MTF has the ordering wrong at time τ_k and correct at time τ_{k+1}:

$$\varphi^{MTF}_{\{i,j\}}(\tau_k,\sigma) + \varphi^{MTF}_{\{i,j\}}(\tau_{k+1},\sigma) \le \frac{2n}{n+1}\left(\varphi^{OPT}_{\{i,j\}}(\tau_k,\sigma) + \varphi^{OPT}_{\{i,j\}}(\tau_{k+1},\sigma)\right)$$

Proof: Notice that if MTF gets the order of $\{i, j\}$ correct, then because it does no paid exchanges, it has minimum possible φ value. So if both τ_k and τ_{k+1} represent times when MTF has the order correct, then the worst case ratio winds up being 1.

Assume then that τ_{k+1} is a time when MTF has the order wrong: assume WLOG that $\sigma_{\tau_{k+1}} = x_i$ and (x_j, x_i) is the ordering at time τ_{k+1}. Notice that if MTF has the order wrong at time τ_{k+1}, then at time τ_k, we know that $\sigma_{\tau_k} = x_j$. There are two cases to consider.

- OPT has the initial ordering (x_i, x_j) at time τ_k. Between τ_k and τ_{k+1}, OPT is then required to either perform a paid exchange or leave the ordering as is and incur the extra charge for stepping over x_i to get to x_j – either way this yields a total cost of at least $1 + \frac{2}{n-1}$.
- OPT has the initial ordering (x_j, x_i) at time τ_k. Just before τ_k, OPT is again required to either perform a paid exchange (which does not pay off so we assume that he does not do this) or leave the ordering as is and incur the extra charge for stepping over x_j to get to x_i – this yields a total cost of at least $1 + \frac{2}{n-1}$.

Thus, OPT must spend at least $1 + \frac{2}{n-1}$ on this value of k. The worst case for MTF is to get the ordering wrong both times, yielding a total cost of at most

$2 + \frac{2}{n-1}$ on this value of k. Thus, the ratio is

$$\frac{\varphi^{MTF}_{\{i,j\}}(\tau_k, \sigma) + \varphi^{MTF}_{\{i,j\}}(\tau_{k+1}, \sigma)}{\varphi^{OPT}_{\{i,j\}}(\tau_k, \sigma) + \varphi^{OPT}_{\{i,j\}}(\tau_{k+1}, \sigma)} \leq \frac{2 + \frac{2}{n-1}}{1 + \frac{2}{n-1}} = \frac{2n}{n+1}$$

We have covered the cases where MTF has the ordering correct at both times and wrong at time τ_{k+1}. The only case remaining is the case where MTF is correct at time τ_{k+1} and wrong at time τ_k, but that is specifically the case we chose to exclude. □

Claim: $\forall i \neq j \in [1, n], \forall t \in [1, \sigma - 1]$,

$$\sum_{t=1}^{|\sigma|} \varphi^{MTF}_{\{i,j\}}(t, \sigma) \leq O(1) + \frac{2n}{n+1} \sum_{t=1}^{|\sigma|} \varphi^{OPT}_{\{i,j\}}(t, \sigma)$$

Proof: Consider writing the sum for MTF out term by term. It is clear that by the above lemma, the inequality will hold for every possible run of incorrect orderings for MTF including the possibility that occurs if the run is not of even length and is preceded by a correct ordering; to see this, consider applying the lemma from the end of the request sequence and working back to the beginning. Note then that the leftovers are all correctly positioned orderings for MTF and it is clear that if MTF receives only correctly positioned elements, then the worst case ratio reverts to 1 (as in the above lemma).

The only request run that is not covered by the above lemma would be a run of incorrect orderings for MTF of odd length that is *not* preceded by a correct ordering. Note, however, that this can only happen at most once at the beginning of the request sequence, and we will only need to ignore at most a constant number of terms in the worst case. □

Claim: $\exists \beta(n), \forall \sigma, MTF(\sigma) \leq \beta(n) + \frac{2n}{n+1} OPT(\sigma)$

Proof: By the above claim, we know that

$$\sum_{t=1}^{|\sigma|} \varphi^{MTF}_{\{i,j\}}(t, \sigma) \leq O(1) + \frac{2n}{n+1} \sum_{t=1}^{|\sigma|} \varphi^{OPT}_{\{i,j\}}(t, \sigma) \Rightarrow$$

$$\frac{1}{2} \sum_{1 \leq i,j \leq n, i \neq j} \sum_{t=1}^{|\sigma|} \varphi^{MTF}_{\{i,j\}}(t, \sigma) \leq \frac{1}{2} \sum_{1 \leq i,j \leq n, i \neq j} \left(O(1) + \frac{2n}{n+1} \sum_{t=1}^{|\sigma|} \varphi^{OPT}_{\{i,j\}}(t, \sigma) \right) \Rightarrow$$

$$MTF(\sigma) \leq \beta(n) + \frac{2n}{n+1} OPT(\sigma)$$

□

This implies that the simple algorithm MTF is $\frac{2n}{n+1}$-competitive for the file cabinet problem. In other words, no algorithm is going to be faster than $\frac{2n}{n+1}$ times faster than MTF for any sequence of requests.

But MTF *meets* this lower bound, and is therefore asymptotically optimal for the cabinet problem.

24 Online problems: Buying at the right price

24.1 Setup and definitions

Assume that you are attempting to sell a single item to an adversary. Assume that you know in advance that there is a lower bound price of m and an upper bound price of M for the item. Obviously, this is not realistic, but assume that you have valid historical estimates of these values. We will study in a model where the adversary is offering prices within the range $[m, M]$ for the item. We will also assume that there are n periods in which (a) the adversary offers his price and (b) you choose to buy the item or not. Note that the adversary does not need to strictly increase his price; in fact, the adversary will only offer you the going price for the object in the general marketplace; this is almost guaranteed to fluctuate. After n rounds have passed, we will assume that the adversary is only going to give you m and that you will be forced to sell at that price. At what price should you sell your item?

Definition: Let $\varphi = \frac{M}{m}$ be the *global fluctuation ratio.*

Your goal will be to minimize the ratio of the maximum price offered over the n rounds to the price that you actually sell for. This is, in some sense, the "Dammit, if I had only known..." ratio (referred to below as the *competitive ratio*). If this ratio is 1 (its lowest possible value), then you have managed to sell at exactly the right time; on the other hand, if the ratio is large, it means that your sell price, the denominator, was way too low compared to the maximum price that either was offered to you or would have been offered to you had you waited longer. The maximum this ratio can ever get it φ. Clearly, the goal of the adversary will be to get this ratio as close to φ as possible.

We say that an online algorithm in this setting is c-competitive if its competitive ratio is upper bounded by c.

What powers are we assigning to the adversary? The adversary is allowed to know your algorithm in advance, but should you make random choices, he is not allowed to know in advance the outcome of your coin flips.

24.2 The deterministic seller

Assume that you are a deterministic seller and have no access to randomness. Your goal is to choose the lower bound for a price p that, if the seller reaches the price p, then you will sell the item at that price. How should you choose p?

Remember that the goal is to minimize the competitive ratio. Given that your adversary knows your value of p, he will act to maximize this ratio. He therefore has two choices, either he will offer exactly p sometime during the n rounds or he will not.

- If the adversary offers p sometime during the n rounds, he will do it early and then, in order to maximize your ratio, he will offer M sometime later (basically, just to piss you off) though you will have already accepted the offer of p. Therefore, the competitive ratio will be $\frac{M}{p}$.

- On the other hand, if the adversary never offers p, you will never accept any offer he makes and will wind up with m. To maximize your competitive ratio (i.e. to annoy you as much as possible), the adversary will at some point offer $p - \epsilon$ for some very small $\epsilon > 0$ and this will be his highest price.

Your goal is to make sure that the adversary does not have a clear choice between these two. Thus, set the two competitive ratios equal to each other:

$$\frac{M}{p} = \frac{p - \epsilon}{m} \Rightarrow p \approx \sqrt{mM} \text{ as } \epsilon \to 0$$

This algorithm is therefore $\frac{M}{\sqrt{Mm}} = \sqrt{\frac{M}{m}} = \sqrt{\varphi}$ competitive.

24.3 The probabilistic seller

Now, we assume that the seller is allowed to make probabilistic choices. The adversary, once again, is aware of the algorithm, but does not know in advance the flips of the coin that the seller will make during the n stages (assumed to be large). The adversary is required to choose the prices he will offer during the n stages *before* any randomization is done.

The algorithm the seller uses will be as follows: Split the interval $[m, M]$ into $2^k - 1$ slices. We can now imagine that the interval is $[1, 2^k]$. Choose a random value for $i \in \{1, \dots, k\}$. Accept the first price that is greater than or equal to 2^i.

Now, the question becomes: how does the adversary annoy you? We claim that his optimal algorithm is to choose some $j_{max} \in \{0, 1, \dots, k - 1\}$ and, starting from $j = 0$ and working upwards to j_{max}, offer 2^j, then make the final offer $2^{j_{max}+1} - \epsilon$ for a very small $\epsilon > 0$. The reason for this choice is that as long as you are within a given $[2^j, 2^{j+1})$ interval, you want to choose the highest price possible because it increases the offline cost while leaving the online cost exactly the same; see the two cases below. Assume the seller randomly chooses x for his threshold, meaning that he will accept any offer $\geq 2^x$.

1. $j_{max} < x$: Then the seller does not get his price. The ratio in this case works out to
$$\frac{2^{j_{max}+1} - \epsilon}{1}$$
Note that the numerator is as high as it can possibly be because of the form of the final offer.

2. $j_{max} \geq x$: The seller does get his price of 2^x, but he could have received as much as $2^{j_{max}+1} - \epsilon$. Thus the ratio in this case works out to
$$\frac{2^{j_{max}+1} - \epsilon}{2^x}$$
In this case, the numerator is as large as it can be and the denominator is as small as it can be, given that the seller gets his price.

Now the adversary looks to maximize the expected competitive ratio. The numerator is always $2^{j_{max}+1} - \epsilon$ (the maximum offline price). We can solve for the expected denominator (the expected online price) as follows[4]:

$$\frac{1}{k}(\sum_{x=1}^{j_{max}} 2^x + \sum_{x=j_{max}+1}^{k} 1) = \frac{1}{k}(2^{j_{max}+1} + k - j_{max} - 2) \Rightarrow$$

The expected competitive ratio is therefore roughly

$$\frac{2^{j_{max}+1} - \epsilon}{\frac{1}{k}(2^{j_{max}+1} + k - j_{max} - 2)} \text{ as } \epsilon \to 0$$

This is minimized when $j_{max} = k - 2 + \frac{1}{\ln 2}$. This implies that the competitive ratio is $kf(k)$ where $f(k) \to 1$ as $k \to \infty$. Note that the global fluctuation ratio is $\varphi = \frac{2^k}{1}$ and therefore this randomized algorithm is almost $\log_2 \varphi$ competitive.

25 Online problems: Keeping a portfolio I

25.1 The model

Financial agents study and use a large variety of market strategies. Some of these are almost static strategies, typically used by mutual fund managers who select and buy some portfolio and then hold it for a relatively long time. Such strategies rely on the tendency of securities to increase in value as a result of natural economic forces. In contrast, some financial agents use aggressive strategies that buy and sell securities very frequently, sometimes even many times during one day. Such strategies primarily attempt to take advantage of security price fluctuations and are called market timing strategies.

Our model will be as follows: Assume that you have a store of cash and there is a single stock that you are interested in. You start with 1 in cash, and the stock starts its price at $p_1 = 1$. During the first night, the stock price is multiplied by some value x_1 and will have value $p_2 = x_1$ the following day. During the second night, the stock price is multiplied by some value x_2 and will have value $p_3 = x_1 x_2$ on the second day. In general, during the jth night, the stock is muliplied by x_j and the price of the stock on the following day is $p_{j+1} = \prod_{i=1}^{j} x_i$.

Once per day, you may invest whatever percentage of your remaining cash into the stock as you like, and realize the results in cash the following day. Clearly, the price of the stock will be $\prod_{i=1}^{n-1} x_i$, and the optimal offline return can easily be calculated to be

$$\phi = \prod_{i=1}^{n-1} \max\{1, x_i\}$$

[4] Note that this will not *exactly* yield the maximum expected competitive ratio because $E[\frac{1}{X}] \neq \frac{1}{E[X]}$, but it is a reasonable approximation.

where n is the number of days (including the first and last); note that there are only $n-1$ changes in price. We will assume that the online player knows the value of ϕ but does not know the values of the x_i's until they arrive during the night. The adversary, having foreknowledge of your algorithm, can make up the values of the x_i however he wishes subject to the constraint that the optimal offline return stays ϕ. We will assume that $\phi > 1$; otherwise, why play the game at all if there is no hope of making money?

25.2 The algorithm

Think of $r_i(\phi)$ as the online ratio that you hope to get if there are i days left in the game (including the current day) assuming that an optimal offline player could make a profit ratio of ϕ. Similarly, think of $b_i(\phi)$ as the buy ratio given those same assumptions. In other words, on day $n-i+1$, if you buy stock with a fraction $b_i(\phi)$ of your remaining cash, you hope to make a profit ratio of $r_i(\phi)$ with the cash you spend.

Define the expressions $r_i(\phi)$ and $b_i(\phi)$ for $1 < i \leq n$ and $\phi > 1$ as follows:

- $r_2(\phi) \equiv \phi$ and $b_2(\phi) = 1$.
- $r_i(\phi) \equiv \max_{0 \leq b \leq 1} \min_{x \leq \phi} (bx + (1-b)) r_{i-1}(\min\{\phi, \frac{\phi}{x}\})$ and $b_i(\phi)$ is the value of b that achieves the maximum in the expression for $r_i(\phi)$.

If $\phi \leq 1$, then we define $r_i(\phi) = 1$ and $b_i(\phi) = 0$ for all valid i.

Assume that you use the following online algorithm: Given that you are keeping track of the price changes on each day, it is always within your power on day j to (theoretically) compute $b_{n-j+1}(\frac{\phi}{\prod_{i=1}^{j-1} \max\{1, x_i\}})$. Use this fraction of whatever cash you have left to purchase stock. We will spend the remainder of this section proving that this algorithm makes sense. If $\phi \leq 1$, then the check is simple so assume that $\phi > 1$ in the following.

Notice that on day j, the profit ratio that an optimal offline player could make *on the remaining days* is exactly $\frac{\phi}{\prod_{i=1}^{j-1} \max\{1, x_i\}}$.

In what follows, let d be the number of days left in the game.

- If $d = 2$, then there is only one more day left in the game and the price will only change once. Therefore, the optimal offline player has the same knowledge as the optimal online player. Because $\phi \geq 1$, it is always in the player's favor to purchase as much of the stock as possible (100%, making the buy ratio 1). The profit ratio will be exactly ϕ. Thus $r_2(\phi) = \phi$ and $b_2(\phi) = 1$.
- If $d > 2$, then concentrate on what happens the first night. Assume that the adversary chooses a value x for the first night's fluctuation and you choose a buy ratio b. Note that if $x < 1$ and the stock decreases in value, then the optimal offline ratio *from tomorrow onwards* is still ϕ; on the other hand, if the stock increases in value $x \geq 1$, then the optimal offline

ratio changes to $\frac{\phi}{x}$. If $r_{d-1}(\min\{\phi, \frac{\phi}{x}\})$ is the ratio you make by playing the algorithm for the remaining days, then your cumulative ratio is

$$(bx + 1 - b)r_{d-1}(\min\{\phi, \frac{\phi}{x}\})$$

Now the way you should compute the value of b is to assume that the adversary does the worst case thing to you, which is to minimize your cumulative ratio over x. However, note that x cannot exceed ϕ. Thus, your cumulative ratio (the ratio you make by playing this algorithm over d days) is

$$\max_{0 \leq b \leq 1} \min_{x \leq \phi} (bx + (1 - b))r_{d-1}(\min\{\phi, \frac{\phi}{x}\})$$

which is exactly the definition of $r_d(\phi)$.

If this reminds you of dynamic programming with continuous parameters, you are correct.

25.3 Computational example

Let's assume that the adversary has picked out the following x vector for a game where $n = 3$:

$$x = (.75, 2)$$

Then we have that $\phi = 2$ because an optimal offline player could make 2 times the amount he started with if he invests 100% of his cash overnight on the second night and nothing on the first. In order to compute the optimal online quantities, we start with the final day.

1. On the final day ($d = 2$), $\phi = \max\{1, 2\} = 2$. So $r_2(2) = 2$ and $b_2(2) = 1$. Thus, we should invest all our money on the final day.

2. On the next to last day ($d = 3$), $\phi = \max\{1, .75\} \max\{1, 2\} = 2$. We look to compute the following quantity:

   $$r_3(2) = \max_{0 \leq b \leq 1} \min_{x \leq 2} (bx + (1 - b))r_2(\min\{2, \frac{2}{x}\})$$

   We already know that $r_2(2) = 2$ and $r_2(\frac{2}{x}) = \frac{2}{x}$ so there are two possibilities. Either $x < 1$ or $x \geq 1$.

   - If $x < 1$, then $\frac{2}{x} > 2$ and we are evaluating

     $$\min_{x<1} (bx + (1 - b))r_2(2) = \min_{x<1} 2(bx + (1 - b)) = 2(1 - b)$$

   - If $x \geq 1$, then $\frac{2}{x} \leq 2$ and we are evaluating

     $$\min_{1 \leq x \leq 2} (bx + (1 - b))r_2(\frac{2}{x}) = \min_{1 \leq x \leq 2} 2b + \frac{2(1 - b)}{x} = 2b + (1 - b) = b + 1$$

We are now looking to evaluate

$$\max_{0\leq b\leq 1} \min\{2(1-b), b+1\}$$

The two of these are equal when $2(1-b) = b+1 \Rightarrow b_3(2) = \frac{1}{3}$ and the maximum of the minimum occurs at this point. Plugging in this value of b, the ratio $r_3(2) = \frac{4}{3}$.

Thus, we can make $\frac{4}{3}$ of our money with no risk by following this investment strategy.

Note, however, that if $n > 3$, working backwards is not feasible! A computational package is necessary for getting these values for higher n.

26 Online problems: Keeping a portfolio II

26.1 How much does it make?

Claim: This makes money. Another way of saying this is that $\phi > 1 \Rightarrow r_d(\phi) > 1$.

Proof: Assume that $\phi > 1$. We will prove this claim by induction on d. If $d = 2$, then this is trivial. If $d > 2$, then

$$r_d(\phi) = \max_{0\leq b\leq 1} \min_{x\leq\phi}(bx + (1-b))r_{d-1}(\min\{\phi, \frac{\phi}{x}\})$$

Now either the expression $\min\{\phi, \frac{\phi}{x}\} > 1$ or not. If so, then note that if $b = 0$, then $bx + 1 - b = 1$ and thus by the inductive hypothesis

$$> \max_{0\leq b\leq 1} \min_{x\leq\phi}(bx + (1-b)) \geq 1$$

On the other hand, if $\min\{\phi, \frac{\phi}{x}\} \leq 1$, then $r_{d-1}(1) = 1$ and $x \geq \phi$; if we let $b = 1$, then $bx + (1-b) \geq \phi > 1$. □

Claim: For any $\phi > 1$ and $n \geq 2$, the cumulative ratio

$$r_n(\phi) \leq \frac{1}{1-(1-\frac{1}{\phi})^{n-1}}$$

Proof: Consider an adversary that chooses a single random day to increase the price by a factor of ϕ and decrease the price by a factor of $\epsilon > 0$ on every other day. (Because we are restricting the actions of the adversary, the cumulative ratio we wind up with will be an upper bound on the actual cumulative ratio.) Of course, once this adversary does the single increase, he cannot ever increase again and the game is effectively over. Thus, we can write out a recurrence relation as before:

$$r'_n(\phi) = \max_{0\leq b\leq 1} \min\{b\phi + 1 - b, (b\epsilon + 1 - b)r'_{n-1}(\phi)\}$$

and letting $\epsilon \to 0$

$$r'_n(\phi) = \max_{0 \le b \le 1} \min\{b\phi + 1 - b, (1-b)r'_{n-1}(\phi)\}$$

Once again, $r'_2(\phi) = \phi$. The optimal choice of b will wind up equating both arguments in the minimum. Thus, we get

$$b\phi + 1 - b = (1-b)r'_{n-1}(\phi)$$

We now solve for b to get

$$b = \frac{r'_{n-1}(\phi) - 1}{r'_{n-1}(\phi) + \phi - 1} \Rightarrow$$

$$(1-b)r'_{n-1}(\phi) = \frac{\phi r'_{n-1}(\phi)}{r'_{n-1}(\phi) + \phi - 1} \Rightarrow$$

$$r'_n(\phi) = \frac{1}{\frac{1}{\phi} + (1 - \frac{1}{\phi})\frac{1}{r'_{n-1}(\phi)}}$$

Now we proceed by induction on n. When $n = 2$, $r'_2(\phi) = \phi$ and the result is trivially true. We have by induction that

$$r'_{n-1}(\phi) \le \frac{1}{1 - (1 - \frac{1}{\phi})^{n-2}}$$

Then notice that to make a fraction large, we make the denominator small. So to get an upper bound on $r'_n(\phi)$, we need a lower bound on $\frac{1}{\phi} + (1 - \frac{1}{\phi})\frac{1}{r'_{n-1}(\phi)}$ and thus an upper bound on $r'_{n-1}(\phi)$, which we have inductively. Plugging the appropriate values in yields the conclusion directly. □

Now we can notice that if ϕ is large enough, then

$$(1 - \frac{1}{\phi})^{n-1} = (1 - \frac{1}{\phi})^{\frac{\phi(n-1)}{\phi}} \approx e^{-\frac{n-1}{\phi}}$$

and therefore

$$r_n(\phi) \approx \frac{1}{1 - e^{-\frac{n-1}{\phi}}}$$

We now have the following:

- $\phi = \omega(n) \Rightarrow e^{-(n-1)/\phi} \approx 1 - \frac{n-1}{\phi} \Rightarrow r_n(\phi) \approx \frac{\phi}{n-1}$
- $\phi = \Theta(n) \Rightarrow r'_n(\phi) \approx \frac{1}{1-e^{-c}}$ where $c > 0$ is some constant.
- $\phi = o(n) \Rightarrow r'_n(\phi) \to 1$ as $n \to \infty$

This implies that even though this strategy is money-making, the amount that is being made is quite small.

27 Splay Trees I: Introduction

27.1 Description

A *splay tree* is a binary search tree that adjusts itself whenever necessary. The explicit balance of the tree is never actually calculated anywhere (as opposed to red/black trees, AVL trees, and most other known self-balancing trees) and there may occasionally be operations that take a long time, but over time, the tree behaves exactly as if it were balanced (even though at any particular time, it may be wildly unbalanced).

Inserts, searches, and deletes are performed the same way as a binary search tree. Every time a node is inserted/found, we splay it to the root. (If we have an unsuccessful search, then splay the final node on the search. On a delete, splay the parent of the erased node.) Therefore, the nodes that are accessed the most frequently are closest to the root and the tree becomes somewhat balanced in the process. Let us assume that node R is the node we are accessing. Then there are 3 cases to consider. Note that in each case, R is moved to the top of the relevant tree. We continue splaying until R has reached the root.

1. Node R's parent is the root. Perform a singular splay: rotate R around its parent.

2. Homogeneous configuration: Node R is the left child of its parent Q, and Q is the left child of its parent P. (or both right) Perform a homogeneous splay. First, rotate Q about P, then R about Q.

3. Heterogeneous configuration: Node R is the right child of its parent Q, and Q is the left of its parent P. (or reversed) Perform a heterogeneous splay. First, rotate R about P, then R about P.

To find out how a splay tree operates, insert the following numbers into an initially empty splay tree: 37,12,65,43,21,11,2,35,27
Delete 43 from the tree. To find out how badly off-balance a splay tree can get, try inserting the numbers from 1 through 10 in order.

Interesting historical note: The first researchers to examine the online behavior of the file cabinet and MTF were Sleator and Tarjan, the same guys who came up with and analyzed the splay tree. Can you see the similarities between the two data structures and settings?

28 Splay Trees II: Analysis

28.1 Analysis

To perform an amortized analysis of splay trees, we will need to assign a potential function ϕ to the data structure. For any given node i, let $w(i)$ be an arbitrarily positive weight assigned to that node. For any node x, let $r(x) = \log_2 |S(x)|$ (where $S(x)$ represents the sum of the weights of the nodes of the subtree with x as the root). Then we define ϕ of the data structure as being $\sum_{\text{nodes } x} r(x)$. (Verify that ϕ is legal assuming that $\forall i, w(i) \geq 1$.) As an exercise, find the potential of both of the trees you found above. (The unbalanced tree potential should be higher than the balanced tree...) We will assume that it takes unit time to perform a single rotation.

Lemma For any two real numbers $x, y > 0$,

$$\frac{\log_2(x) + \log_2(y)}{2} \leq \log_2\left(\frac{x+y}{2}\right)$$

Proof: Let $f(x) = \log_2\left(\frac{x+y}{2}\right) - \frac{\log_2(x)+\log_2(y)}{2}$ for constant $y > 0$. Notice that $\frac{d}{dx}f(x) = \frac{x-y}{2x(x+y)}$. So f is decreasing if $x < y$ and increasing for $x > y$. Thus f reaches its minimum when $x = y$. □

Lemma: For any three nodes a, b, c (not even necessarily in the same binary trees),

$$|S(a)| + |S(b)| \leq |S(c)| \Rightarrow r(a) + r(b) \leq 2(r(c) - 1)$$

Proof: By the above lemma,

$$\begin{aligned}
\frac{r(a)+r(b)}{2} &= \frac{\log_2|S(a)| + \log_2|S(b)|}{2} \\
&\leq \log_2\left(\frac{|S(a)| + |S(b)|}{2}\right) \\
&\leq \log_2\left(\frac{|S(c)|}{2}\right) \\
&= r(c) - 1
\end{aligned}$$

□

Consider the case where x is a child of the root. Let the name of the root be y. What is the amortized cost of splaying x upwards? Note that the actual cost of splaying x to the root is 1. Let x' and y' be the new positions of x and y after the rotation[5]. Notice that the potential of all other nodes remains constant.

$$a(i) = c(i) + \Delta\phi = 1 + (r(x') + r(y') - r(x) - r(y))$$

We claim that $r(y') \leq r(x') = r(y)$ (obviously) and so

$$a(i) \leq 1 + r(x') - r(x) \leq 1 + 3(r(x') - r(x))$$

Consider the case where x is in homogeneous position with y and z (z is the parent of y is the parent of x). What is the amortized cost of splaying x upwards? The actual cost is 2. Notice that the potential of all nodes other than x, y, and z remains constant.

$$a(i) = c(i) + \Delta\phi = 2 + (r(x') + r(y') + r(z') - r(x) - r(y) - r(z))$$

[5] We will keep the prime notation for simplicity throughout this analysis.

Notice that $r(y') \leq r(x') = r(z)$ and $r(x) \leq r(y)$ and so

$$a(i) \leq 2 + r(x') + r(z') - 2r(x)$$

Notice that $|S(x)| + |S(z')| \leq |S(x')|$. By the lemma,

$$a(i) \leq 2 + r(x') - 2r(x) + (2(r(x') - 1) - r(x)) = 3(r(x') - r(x))$$

Finally, consider the case where x is in heterogeneous position with y and z (same situation as above). What is the amortized cost of moving x upwards? The actual cost is 2.

$$a(i) = c(i) + \Delta\phi = 2 + (r(x') + r(y') + r(z') - r(x) - r(y) - r(z))$$

Note that $r(x') = r(z)$ and $r(y) \geq r(x)$. So

$$2 + (r(x') + r(y') + r(z') - r(x) - r(y) - r(z)) \leq 2 + r(y') + r(z') - 2r(x)$$

We note that $|S(y')| + |S(z')| \leq |S(x')|$ and again use the lemma to get

$$a(i) \leq 2(r(x') - r(x))$$

Note that for each operation except when x becomes the root, $a(i) \leq 3(r(x') - r(x))$ and at the root $a(i) \leq 3(r(x') - r(x)) + 1$. To get the total amortized cost of a splay operation, we need to sum over all the positions of x. But this is a telescoping sum! So the final result is less than or equal to $1 + 3(r(\text{root}) - r(x)) \leq 3r(\text{root}) + 1$.

Now, we are allowed to make the weights on the nodes anything we like. Let the weight of each node be 1. Then $r(\text{root}) = \log_2 |S(\text{root})| = \log_2 n$ and each splay to the root takes only $O(\log n)$ amortized time!

29 Splay Trees III: Information

29.1 Shannon information

If you are sending a character over a line, the amount of information you get from the character should depend on the frequency that the character is sent. Intuitively, the information function should measure the surprise you feel at receiving the given character. Let $c(x)$ be a character that is sent with probability x. The information function I should have the following properties:

1. I should depend on x.
2. $I(x) + I(y) = I(xy)$
3. $I'(x)$ is defined everywhere on $(0, 1]$ from the left hand side (because of $x = 1$)
4. $I'(1)$ is negative and finite.

Let x be any character that is sent with probability strictly less than 1. First, note that by rule 2, we must have that $I(1) = 0$. Then

$$\begin{aligned}
I'(x) &= \lim_{h\to 0^+} \frac{I(x) - I(x-h)}{h} \\
&= \lim_{h\to 0^+} \frac{I(x) - I(x) - I(1-\frac{h}{x})}{h} \\
&= \lim_{h\to 0^+} -\frac{I(1-\frac{h}{x})}{h} \\
&= \lim_{h\to 0^+} \frac{1}{x} I'(1 - \frac{h}{x}) \\
&= \frac{I'(1)}{x} \\
\Rightarrow \quad I(x) = I'(1)\log_2 x + C
\end{aligned}$$

Letting $x = 1$, we get that $C = 0$ so that the amount of information contained in a character that appears with probability x is $I(x) = I'(1)\log_2 x$.

Note that the value of $I'(1)$ changes the base of the log depending on how you measure the information. You can measure information in bits, trits, digits, etc. From now on, we will assume that the base of the log is 2 so that we are always measuring in bits.

The *entropy* of a sequence of bits is the average quantity of information that you get per character in the sequence. We denote it by

$$H(\{x_1, x_2, \ldots, x_n\}) = \sum_{i=1}^{n} x_i I(x_i)$$

Another way of looking at the entropy is this: The entropy measures the smallest theoretical code that can be created for a given set of character frequencies (in the sense that any other uniquely decipherable code must have at least this many bits per codeword on average). What are the best and worst cases for sending information over a line using only two characters? We use the entropy to compare how good a code is: the closer the average codeword size is to the entropy, the better the code.

A *uniquely decipherable* code is one that has no ambiguity: a given set of characters always has a unique decoding. The following code is not uniquely decipherable: $\{0, 1, 01\}$.

Most and least efficient: What are the minimum and maximum entropy of n-character uniquely decipherable codes? Clearly, the minimum entropy is 0. This entropy arises if one of the characters appears with probability 1 and the rest 0. This corresponds to the "perfect" model, one we always predict with absolute certainty. To find the maximum entropy, note that we are looking to maximize the function $g(p) = -\sum_{i=1}^{n} p_i \log_2 p_i$. Introduce the Lagrangian

$f(p, \alpha) = g(p) + \alpha(\sum_{i=1}^n p_i - 1) = 0$. So we get that for every i,

$$\frac{\partial f}{\partial p_i} = -\log_2 p_i - \frac{1}{\ln 2} + \alpha = 0 \Rightarrow$$

$$\forall i, j, p_i = p_j \Rightarrow p_i = \frac{1}{n} \Rightarrow H(p) = \log_2 n$$

Notice that this corresponds to the number of bits necessary to distinguish n distinct objects. What does this analyze say about, for example, the standard encoding of characters in an average novel? □

Jensen's inequality: Let f be a function such that $\forall x_1, x_2, t$,

$$tf(x_1) + (1-t)f(x_2) \geq f(tx_1 + (1-t)x_2)$$

In other words, f is *concave up*; this is the same thing as stating that f has nonnegative second derivative everywhere in its domain. Then for any weights $t_1, \ldots, t_n$ such that $\sum t_i = 1$ and points $x_1, \ldots, x_n$,

$$\sum t_i f(x_i) \geq f(\sum t_i x_i)$$

with equality iff $x_1 = x_2 = \ldots = x_n$.

Proof: Mathematical induction. It is obviously true if $n = 1, 2$. For $k+1$, we have the following

$$f(\sum_{i=1}^{k+1} t_i x_i) = f(t_1 x_1 + (1-t_1)\sum_{i=2}^{k+1} \frac{t_i}{1-t_1} x_i) \leq$$

$$t_1 f(x_1) + (1-t_1) f(\sum_{i=2}^{k+1} \frac{t_i}{1-t_1} x_i)$$

Now note that $\sum_{i=2}^{k+1} \frac{t_i}{1-t_1} = 1$ and there are exactly k terms in the sum. Thus,

$$t_1 f(x_1) + (1-t_1) f(\sum_{i=2}^{k+1} \frac{t_i}{1-t_1} x_i) \leq t_1 f(x_1) + \sum_{i=2}^{k+1} (1-t_1) \frac{t_i}{1-t_1} f(t_i) =$$

$$\sum_{i=1}^{k+1} t_i f(x_i)$$

Note that the if and only if part of the inequality follows from the induction as well. □

Gibbs' inequality: Let p_i, q_i be any 2 probability distributions (for $1 \leq i \leq n$). Then

$$-\sum_{i=1}^n p_i \log_2 p_i \leq -\sum_{i=1}^n p_i \log_2 q_i$$

with equality if and only if $p_i = q_i$ for all i.

Proof: The second derivative of $f(x) = -\log_2 x$ is everywhere positive and it is therefore concave up. So we have that by Jensen's inequality

$$\sum_{i=1}^{n} p_i(-\log_2 \frac{q_i}{p_i}) \geq -\log_2(\sum_{i=1}^{n} p_i \frac{q_i}{p_i}) = 0$$

with equality iff $\forall i, j, \frac{q_i}{p_i} = \frac{q_j}{p_j} \Rightarrow \forall i, \frac{q_i}{p_i} = 1 \Rightarrow q_i = p_i$. □

Definition: A *prefix* code is a code such that no codeword is the prefix of any other codeword. For example, the following is *not* a prefix code: $\{01, 10, 101\}$. The following is a prefix code: $\{01, 10, 11\}$.

Claim: Any prefix code is uniquely decipherable, but not every uniquely decipherable code is a prefix code.

Proof: Consider the following code: $\{0, 01, 11\}$. We claim that this is a uniquely decipherable code that is not a prefix code. Obviously, it is not a prefix code. To prove that it is uniquely decipherable, notice that a 1 followed by a 0 implies that there is a word break in between. Where do the other breaks occur? Determine the 10 breaks. Now, for each of those breaks, if the number of 1's before the break is even, we must have been using the third codeword; otherwise, the second codeword was used during the initial entry into the 1 run. All other codewords are now uniquely determined.

To show that any prefix code is uniquely decipherable, consider any code that is not uniquely decipherable. Can it be a prefix code? No; consider the shortest sequence of codewords that is not uniquely decipherable. This implies that there are two possible ways to decipher the sequence. The first codeword in each sequence must be different; otherwise, we could make the sequence shorter. But whichever codeword is shorter must be a prefix of the longer. □

Kraft inequality for prefix codes: There exists a prefix code for a set of codeword lengths $l_1 = l(x_1), l_2 = l(x_2), \ldots, l_n = l(x_n)$ where $x_i \in C$ if and only if $\sum_{i=1}^{n} 2^{-l(x_i)} \leq 1$.

Proof: First, we show the positive direction. Assume that you are given positive integers $l_1, \ldots, l_n$ such that $\sum_{i=1}^{n} 2^{-l_i} \leq 1$. Our goal will be to show that there exists a prefix code C such that $x_1, \ldots, x_n \in C$. Order the lengths so that WLOG we have $l_1 \leq l_2 \leq \ldots \leq l_n$. Start with a full binary tree of height l_n. Note that this tree has 2^{l_n} leaves.

We will repeatedly perform a pruning of this tree. Starting from the left of the tree, gather $2^{l_n - l_1}$ consecutive leaves together and find their unique lowest common ancestor x_1. Let x_1 be the first codeword node. Now gather the next $2^{l_n - l_2}$ consecutive leaves together and find their unique lowest common ancestor. Note that it cannot possibly be related (i.e. ancestor nor descendant) to x_1. Assign x_2 to this ancestor. Continue this process until x_n has been assigned. We know x_n can be assigned because we assumed that $\sum_{i=1}^{n} 2^{-l_i} \leq 1 \Rightarrow \sum_{i=1}^{n} 2^{l_n - l_i} \leq 2^{l_n}$.

To find the codeword corresponding to the given node, start from the root and traverse the tree downwards until you reach the appropriate x_i. On the way, if you make a left, write down a 0; if you make a right, write down a 1. The binary string that is on the page at the time you reach x_i is its codeword. We claim that no codeword is the prefix of any other codeword: for that to occur, it would need to be the case that x_i is the ancestor of x_j for some i and j, but by construction, this cannot occur.

Now, we show the other direction. Assume that you are given a prefix code $x_1, x_2, \ldots, x_n \in C$. Our goal will be to show that $\sum_{i=1}^{n} 2^{-l(x_i)} \leq 1$. Consider the following process.

1. Flip a fair coin. Append a 0 to what is already on the page; 1 otherwise. (Obviously, if there is nothing on the page, just write down the appropriate bit to start off.)

2. Check if what is written on the page is a valid codeword. If so, you win the game. Otherwise, flip the coin again and append the result. Continue with this process until you either win the game or have a bit string of length longer than the longest possible codeword.

What is the probability of winning this game? The probability of winning this game is equal to the sum of the probabilities that you reach one of the codewords. Note that because the code is a prefix code, the probability of reaching x_i or x_j is the sum of the probabilities of reaching x_i and x_j. (This would not be true if x_i was a prefix of x_j.) This probability, which must be less than or equal to 1 is equal to exactly $\sum_{i=1}^{n} 2^{-l(x_i)}$. □

30 Splay Trees IV: Asymptotic static optimality

<u>Kraft inequality for uniquely decipherable codes</u>: There exists a uniquely decipherable code for a set of codeword lengths $l_1 = l(x_1), l_2 = l(x_2), \ldots, l_n = l(x_n)$ where $x_i \in C$ if and only if $\sum_{i=1}^{n} 2^{-l(x_i)} \leq 1$.

<u>Proof</u>: Because every prefix code is uniquely decipherable, it is clearly that given $l_1, l_2, \ldots, l_n$, we can construct a uniquely decipherable code.

Assume that you are given a uniquely decipherable code. Let C be the set of all codewords. Then, if $l(x)$ is the length of codewords x in bits, our goal is to show that $\sum_{x \in C} 2^{-l(x)} \leq 1$.

Note that $\sum_{x \in X} 2^{-l(x)} = \sum_{j=1}^{T} w_j 2^{-j}$ where w_j is the number of codewords of length j in C and T denotes the length of the maximal codeword length in C. Now notice that

$$\left(\sum_{j=1}^{T} w_j 2^{-j} \right)^s = \sum_{k=s}^{Ts} \frac{N_k}{2^k}$$

where $N_K = \sum_{i_1 + \ldots + i_s = k} w_{i_1} \ldots w_{i_s}$ is the total number of messages whose coded representation is of length k.

For an example, consider the code $\{0, 10, 1100, 1101, 1111\}$. Then

$$\frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^4} + \frac{1}{2^4} =$$

$$\frac{1}{2^1}+\frac{1}{2^2}+\frac{3}{2^4}$$

$$(\frac{1}{2^1}+\frac{1}{2^2}+\frac{3}{2^4})^2 =$$

(via FOIL multiplication)

$$\frac{1}{2^2}+\frac{1}{2^3}+\frac{2}{2^5}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{3}{2^6}+\frac{3}{2^5}+\frac{3}{2^6}+\frac{9}{2^8}=$$

$$\frac{1}{2^2}+\frac{2}{2^3}+\frac{1}{2^4}+\frac{5}{2^5}+\frac{6}{2^6}+\frac{9}{2^8}$$

Note that in particular, given two consecutive choices for codewords, there is only one way to form a bit sequence of length 4, 6 ways to form a bit sequence of length 6, 9 ways to form a bit sequence of length 8, etc.

Because the code is uniquely decipherable, to every sequence of k bits there corresponds at most one possible message; thus, $N_k \leq 2^k \Rightarrow$

$$\sum_{k=s}^{Ts} \frac{N_k}{2^k} \leq \sum_{k=s}^{Ts} 1 = Ts - s + 1 \leq Ts$$

Now

$$\left(\sum_{j=1}^{T} w_j 2^{-j}\right)^s = \sum_{k=s}^{Ts} \frac{N_k}{2^k} \leq Ts \Rightarrow$$

$$\sum_{j=1}^{T} w_j 2^{-j} \leq (Ts)^{\frac{1}{s}}$$

Because this must hold for every s, we can let $s \to \infty \Rightarrow$

$$\sum_{j=1}^{T} w_j 2^{-j} \leq \lim_{s\to\infty} (Ts)^{\frac{1}{s}} = 1$$

and the result follows. □

Shannon's Noiseless Coding Theorem Part I: The average length of a codeword of any uniquely decipherable code must be at least as large as the entropy of the distribution of the code. More explicitly, if we let l_i be the length of a codeword x_i that appears with probability p_i, then $\sum p_i l_i \geq \sum -p_i \log_2 p_i$.

Proof: Take $x_1, \ldots, x_n$ and $p_1, \ldots, p_n$ as in the statement, suppose the x_i have been encoded in such a way that the code is uniquely decipherable, and let l_i be the codeword for x_i. Then by Kraft's inequality $\sum 2^{-l_i} \leq 1$. Let $C = \frac{1}{\sum 2^{-l_i}}$ so that $C2^{-l_1}, C2^{-l_2}, \ldots, c2^{-l_n}$ is a probability distribution, and can play the role of $\{q_i\}$ in Gibbs' inequality, which yields

$$\sum_{i=1}^{n} p_i \log_2 p_i \geq \sum_{i=1}^{n} p_i \log_2 (c2^{-l_i}) = \sum_{i=1}^{n} p_i (\log_2 C - l_i) = \log_2 C - \sum_{i=1}^{n} p_i l_i$$

Now put back the minus signs and remember that $\frac{1}{C} \leq 1 \Rightarrow C \geq 1 \Rightarrow \log_2 C \geq 0 \Rightarrow$

$$\sum_{i=1}^{n} p_i l_i \geq -\sum_{i=1}^{n} p_i \log_2 p_i + \log_2 C \geq -\sum_{i=1}^{n} p_i \log_2 p_i$$

□

30.1 Static optimality and online solutions

Assume that we are given an large alphabet of keys and a sequence of searches for these keys. We will also assume that each key is accessed at least once; otherwise, just remove it from the alphabet. Consider the set of all binary search trees that, at each node (including internal nodes), we hold a record corresponding to the key in the node. There is at least one such binary search tree in this set that is "optimal" in the sense that the total number of nodes accessed is minimized for the given sequences of searches. Aymptotically, how many node accesses does any given sequence *require* assuming we are allowed to use the best possible binary search tree when we do our searches?

Think of the binary search tree as a code for transmitting bits from a code for the purposes of sending the key sequence from one place to another. Because the tree is not a prefix code, you might imagine adding an extra character that means "end codeword" so that the receiver of the bits knows when a codeword has finished being sent. This makes the code uniquely decipherable. The size of a given key/codeword is equal to one more than its depth in the tree. Assume that the keys are numbered $k_1, k_2, \ldots, k_n$ and let the depth of key k_i be $d(k_i)$. Let the frequency that k_i appears in the sequence be $f(k_i)$; thus, the total number of keys in the sequence is $\sum_{i=1}^{n} f(k_i) \equiv T$. Then, Shannon's Noiseless Coding Theorem yields that the asymptotically shortest possible sequence of bits for this code sequence has total length

$$\sum_{i=1}^{n} f(k_i)(d_{\text{opt}}(k_i) + 1) \geq \sum_{i=1}^{n} -f(k_i) \log_2 \frac{f(k_i)}{T} = \sum_{i=1}^{n} f(k_i) \log_2 \frac{T}{f(k_i)}$$

Unfortunately, the optimal static tree is difficult to find (though not impossible, given the entire sequence of searches) and there is the added problem that, in practice, you cannot possibly count on having the entire message to analyze in advance so that some kind of online solution is necessary. Is there an easier way to transmit the message? Assume that we start with a binary search tree of any kind that contains all of the relevant keys. Think of this binary tree as operating like a splay tree: upon request of a key's codeword, simply transmit the correct codeword (the key's current position in the splay tree) followed by an "end codeword" character; each time a key is searched for in the splay tree, its codeword changes (probably) based on its being splayed to the root. How many bits/operations will this strategy take?

Recall in the analysis of the splay tree that the maximum amortized running time of any splay tree operation on node x was at most $1 + 3(r(\text{root}) - r(x))$

where the weights that we assigned to each node were arbitrary numbers 1 or greater. Assume that we are given any splay tree populated by the keys k_i. Assign the weight of node k_i to be $f(k_i)$ (as opposed to 1). Then note the following facts.

$$r(\text{root}) = \log_2 |S(\text{root})| = \log_2 \sum_{i=1}^{n} f(k_i) = \log_2 T$$

$$\forall i, r(k_i) = \log_2 |S(k_i)| \geq \log_2 f(k_i)$$

These facts imply that for any node k_i, any splay operation (ssarch, in particular) takes amortized time

$$1 + 3(r(\text{root}) - r(k_i)) \leq 1 + 3 \log_2 \frac{T}{f(k_i)} = O(\log_2 \frac{T}{f(k_i)})$$

Thus, the entire sequence of operations takes amortized time at most (because the constant in the O-notation does not vary with any parameter)

$$\sum_{i=1}^{n} f(k_i) O(\log_2 \frac{T}{f(k_i)}) = O(\sum_{i=1}^{n} f(k_i) \log_2 \frac{T}{f(k_i)})$$

Comparing this to result above, we conclude that the use of a splay tree for performing *any nontrivial sequence* of searches is at most a constant factor worse than the *best possible* static tree!

31 Future lectures

1. Christofides algorithm
2. Approximating the permanent/monomer-dimer problem
3. Randomized competitiveness against oblivious adversaries and Yao's principle: pg. 120-121, randomized definition, randomized paging algorithm